Periodic Orbits of Coupled Oscillators Near Resonance

Carmen Chicone *
Department of Mathematics
University of Missouri
Columbia, MO 65211
USA
carmen@chicone.math.missouri.edu

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Abstract

A persistence theorem for fixed points of a parameterized family of maps is specialized to give a method for detecting the existence of persistent periodic solutions of perturbed systems of differential equations, in particular, systems of weakly coupled planar oscillators. An application is make to the problem of synchronization of two detuned and weakly inductively coupled van der Pol oscillators.

1 Introduction

Coupled oscillators arise as mathematical models of many physical systems. While the full range of dynamical behavior is exhibited by systems of differential equations of this type, periodic orbits are perhaps of primary importance. The purpose of this paper is to describe a geometric approach to proving the

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existence and stability of periodic orbits for coupled oscillators in case the unperturbed oscillators are in resonance.

To orient the reader, we will consider some examples and then formulate the precise problem that we intend to solve. The prototypical example is the single periodically forced oscillator. Although there are many specific examples, the perturbed harmonic oscillator, rotor-pendulum, Duffing, and van der Pol oscillators, given respectively by,

$$\begin{aligned} \ddot{x} + \omega^2 x &= \epsilon q(t, x, \dot{x}), \\ \ddot{\theta} + \sin\theta &= \epsilon q(t, \theta, \dot{\theta}), \\ \ddot{x} + \omega^2 x - \lambda x^3 &= \epsilon q(t, x, \dot{x}), \\ \ddot{x} + \lambda(x^2 - 1)\dot{x} + \omega^2 x &= \epsilon q(t, x, \dot{x}), \end{aligned}$$

where λ, ω, and ϵ denote parameters, and q denotes a function periodic in time, are often encountered in the applications. A basic problem is to determine the existence and stability of periodic solutions for these differential equations under the assumption that ϵ is small.

A natural generalization of the forced oscillator is an array of coupled oscillators. Models of mechanical systems, for example models of rotating machinery, often reduce to coupled rotors and springs. A typical example, studied in [13][15][53], is a model for an unbalanced rotor interacting with an elastic support that is given by

$$\begin{aligned} \ddot{\theta} + m_2\dot{\theta} + \mu\ddot{z}\sin\theta + \sin\theta &= -m_1, \\ \ddot{z} + c\dot{z} + \mu(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) + \omega^2 z &= 0. \end{aligned} \tag{1}$$

For mechanical systems and for electrical systems, for example Josephson junctions, arrays of coupled pendula often occur. Many possible couplings are reasonable, but we mention only two: spring coupling

$$\ddot{\theta}_i + \sin\theta_i = \epsilon(\theta_{i+1} - \theta_{i-1}), \tag{2}$$

where $i = 1 \ldots n$ and, for example, θ_0 is identified with θ_n and θ_{n+1} with θ_1—the pendula are coupled in a ring; and diffusive coupling

$$\ddot{\theta}_i + \sin\theta_i = \epsilon(\theta_{i+1} - 2\theta_i + \theta_{i-1}). \tag{3}$$

In models of biological and electrical systems, for example in signal synchronization problems, where there is some dissipation, arrays of coupled limit cycle oscillators often occur, especially van der Pol oscillators,

$$\ddot{x}_i + (x_i^2 - 1)\dot{x}_i + \omega^2 x_i = \epsilon g_i(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n). \tag{4}$$

All of these examples are expressed compactly as first order systems of differential equations of the form

$$\dot{\boldsymbol{u}} = F(\boldsymbol{u}) + \epsilon G(t, \boldsymbol{u}, \epsilon) \tag{5}$$

where $\boldsymbol{u}$ is a vector variable and ϵ is a small parameter. In this paper, we will assume that the unperturbed system

$$\dot{\boldsymbol{u}} = F(\boldsymbol{u}) \tag{6}$$

has an invariant manifold $\mathcal{A}$, called a period manifold, consisting of periodic solutions all of which have a common period η, and that either G is constant with respect to its first argument, or that the function $t \mapsto G(t, \boldsymbol{u}, \epsilon)$ is periodic with period $\tau(\epsilon) > 0$ independent of $\boldsymbol{u}$. In the nonautonomous case, where G is a nonconstant periodic function, we will assume $\tau(0)$, the period of G at $\epsilon = 0$ is in resonance with η, that is, there are relatively prime integers m and n such that $m\tau(0) = n\eta$. The problem we will address is the following: Determine the existence of periodic solutions of (5) for small nonzero values of ϵ. We will not discuss in detail the important problem of the stability type of the perturbed periodic solutions. However, the methods of this paper can be used in many cases to determine this stability.

The plan of the paper is the following. In § 2 we briefly outline a general theory of the continuation of resonant periodic orbits. In § 3 this theory will be specialized to the case of coupled planar oscillators. Finally, in § 4, the theory will be applied to systems of two identical limit cycle oscillators. In particular, a system of two coupled van der Pol oscillators studied in [12] is reconsidered. The analysis of this system in [12] contains an error that is corrected here.

This paper includes a lengthy bibliography. It is intended to serve as an invitation to the many possible applications of the theory of coupled oscillators. Hopefully, the present paper will stimulate one possible avenue of research.

2 Continuation Theory

In order to analyze the continuation (persistence) of periodic solutions of system (6) into the perturbed equation (5), we will outline a theory proposed in [14] where the reader is referred for more details.

For coupled oscillators such as (2), which contain angular variables, the natural phase space is the cylinder $\mathbf{S}^1 \times \mathbf{R}$, while for systems such as (4), the natural phase space is $\mathbf{R}^2$. To accommodate both cases, we will view $\boldsymbol{u}$ as a coordinate on a manifold M consisting of a cross product of Euclidean spaces and tori. Let $t \mapsto \boldsymbol{u}(t, \xi, \epsilon)$ denote the solution of (5) with initial condition $\boldsymbol{u}(0, \xi, \epsilon) = \xi$ for $\xi \in M$. In the nonautonomous case, where the perturbation G is a nonconstant periodic function, there is a natural Poincaré section Σ consisting of the entire phase space M. Naturally associated to this manifold is the mth order (parametrized) Poincaré map defined by $\mathcal{P}^m(\xi, \epsilon) = \boldsymbol{u}(m\tau(\epsilon), \xi, \epsilon)$; it corresponds to a strobe that illuminates the orbit after m cycles of the perturbation. In the autonomous case, where G is a constant with respect to time, there is a codimension one submanifold Σ of M that is transverse to the period manifold $\mathcal{A}$ of the unperturbed system, and there is a Poincaré map defined on some neighborhood of the manifold $\mathcal{Z} := \mathcal{A} \cap \Sigma$ by $\mathcal{P}^m(\xi, \epsilon) = \boldsymbol{u}(mT(\xi, \epsilon), \xi, \epsilon))$ where $T(\xi, \epsilon)$ is the time of first return of the orbit starting at $\xi \in \Sigma$ and $mT(\xi, 0)$ is the common period of the orbits on $\mathcal{A}$. Here, the existence and smoothness of T is an immediate consequence of the Implicit Function Theorem, the main tool that will be used in all of the analysis to follow. Also, by our assumption about the period manifold $\mathcal{A}$, the function $\boldsymbol{z} \mapsto mT(\boldsymbol{z}, 0)$ is constant on $\mathcal{Z}$.

Of course, in both cases, a fixed point of the map $\xi \mapsto \mathcal{P}^m(\xi, \epsilon)$ corresponds to a periodic orbit of (5). In the nonautonomous case, if m is the smallest such integer for which ξ is a fixed point, then we say ξ is the initial point of a subharmonic of order m.

In order to unify the two cases we are considering, we let $\mathcal{Z} = \mathcal{A}$ in the nonautonomous case, and we note that in both cases, $\mathcal{Z}$ is a submanifold of Σ consisting entirely of fixed points of the unperturbed order m Poincaré map, defined by $p^m(\xi) := \mathcal{P}^m(\xi, 0)$. Our continuation theory is a method that can be used to decide if any of the fixed points on $\mathcal{Z}$ survive after perturbation. More precisely, we say a point $\boldsymbol{z} \in \mathcal{Z}$, and therefore the unperturbed periodic orbit of (6) with initial point $\boldsymbol{z}$, is continuable (or that it persists) if there is a continuous curve $\epsilon \mapsto \gamma(\epsilon)$ in Σ such that $\gamma(0) = \boldsymbol{z}$ and $\mathcal{P}^m(\gamma(\epsilon), \epsilon) \equiv \gamma(\epsilon)$.

Here, $\gamma(\epsilon) \in \Sigma$ is the initial point of a periodic solution of (5).

In order to apply the method of [14], namely Lyapunov-Schmidt reduction to the Implicit Function Theorem, the fixed-point manifold (resonance manifold) $\mathcal{Z}$ must satisfy a nondegeneracy condition relative to the unperturbed Poincaré map. To specify this condition, consider $\boldsymbol{z} \in \mathcal{Z}$ and the derivative $Dp^m(\boldsymbol{z})$ viewed as a linear transformation of the tangent space $\mathcal{T}_{\boldsymbol{z}}\Sigma$. The base point stays fixed because p^m is the identity on $\mathcal{Z}$. Moreover, every vector in $\mathcal{T}_{\boldsymbol{z}}\Sigma$ that is tangent to the submanifold $\mathcal{Z}$ is fixed by $Dp^m(\boldsymbol{z})$, or, as we will say, every such vector is in the kernel of the infinitesimal displacement

$$\mathcal{D}(\boldsymbol{z}) = Dp^m(\boldsymbol{z}) - I \tag{7}$$

that is also a linear transformation of $\mathcal{T}_{\boldsymbol{z}}\Sigma$. The manifold $\mathcal{Z}$ is called *normally nondegenerate* if the kernel of the infinitesimal displacement is exactly the tangent space $\mathcal{T}_{\boldsymbol{z}}\mathcal{Z} \subset \mathcal{T}_{\boldsymbol{z}}\Sigma$. Equivalently, $\mathcal{Z}$ is normally nondegenerate, if for each $\boldsymbol{z} \in \mathcal{Z}$, the dimension of the kernel of the infinitesimal displacement at $\boldsymbol{z}$ is equal to the dimension of the manifold $\mathcal{Z}$.

Suppose $\mathcal{Z}$ has dimension d, and that it is a normally nondegenerate submanifold of Σ. In this case, the range of the infinitesimal displacement at each point in $\mathcal{Z}$ has codimension d. Thus, for $\boldsymbol{z} \in \mathcal{Z}$, there is a vector space complement $\mathcal{S}(\boldsymbol{z})$, to the range $\mathcal{R}(\boldsymbol{z})$ of $\mathcal{D}(\boldsymbol{z})$. We let $s(\boldsymbol{z})$ denote the projection of $\mathcal{T}_{\boldsymbol{z}}M$ to $\mathcal{S}(\boldsymbol{z})$ relative to this direct sum splitting $\mathcal{R}(\boldsymbol{z}) \oplus \mathcal{S}(\boldsymbol{z})$ of $\mathcal{T}_{\boldsymbol{z}}\Sigma$. By choosing local coordinates, both $\mathcal{Z}$ and $\mathcal{S}(\boldsymbol{z})$ may be identified with $\mathbf{R}^d$.

Let $\boldsymbol{z} \in \mathcal{Z}$ and consider the curve in M given by $\epsilon \mapsto \mathcal{P}^m(\boldsymbol{z}, \epsilon)$. This curve passes through $\boldsymbol{z}$ at $\epsilon = 0$. Its tangent vector at $\epsilon = 0$, which may be identified with the partial derivative $\mathcal{P}^m_\epsilon(\boldsymbol{z}, 0)$, is in $\mathcal{T}_{\boldsymbol{z}}\Sigma$. We define the bifurcation function $\mathcal{B}$ to be the map, from $\mathcal{Z}$ to the complement $\mathcal{S}$ of the range of the infinitesimal displacement, given by

$$\mathcal{B}(\boldsymbol{z}) = s(\boldsymbol{z})\mathcal{P}^m_\epsilon(\boldsymbol{z}, 0). \tag{8}$$

In local coordinates $\mathcal{B} : \mathbf{R}^d \to \mathbf{R}^d$. We will say $\boldsymbol{z} \in \mathcal{Z}$ is a simple zero of the bifurcation function provided $\mathcal{B}(\boldsymbol{z}) = 0$ and the derivative $D\mathcal{B}(\boldsymbol{z})$ is invertible.

A result in [14] is the following continuation theorem:

Theorem 2.1 *If $\mathcal{Z}$ is a normally nondegenerate fixed-point submanifold of the unperturbed system (6) relative to a Poincaré section, and if $\boldsymbol{z} \in \mathcal{Z}$ is a* *simple zero of the associated bifurcation function (8), then the unperturbed periodic orbit of (6) with initial point $\boldsymbol{z}$ is continuable in the system (5).*

The reader will notice that Theorem 2.1 is not really a theorem about differential equations. In fact, it is a theorem about parametrized families of maps represented here by the family $\xi \mapsto \mathcal{P}(\xi, \epsilon)$ parametrized by ϵ.

To use Theorem 2.1 as a practical tool, we must be able to compute $\mathcal{P}_\epsilon^m(\boldsymbol{z}, 0)$ for $\boldsymbol{z} \in \mathcal{Z}$. To do this, we note that

$$\mathcal{P}_\epsilon^m(\boldsymbol{z}, 0) = mkF(\boldsymbol{z}) + \boldsymbol{u}_\epsilon(mT, \boldsymbol{z}, 0) \tag{9}$$

where k, the "detuning", is given by $\tau_\epsilon(0)$ or by $T_\epsilon(\boldsymbol{z}, 0)$ (in the second case k is a function of $\boldsymbol{z}$), and T is given by $\tau(0)$, the unperturbed period of G in the nonautonomous case, or by $T(\boldsymbol{z}, 0)$, the time of first return to the Poincaré section, in the autonomous case. Thus, if $t \mapsto W(t)$ is the solution of the second variational initial value problem

$$\dot{W} = DF(\boldsymbol{u}(t, \boldsymbol{z}, 0))W + G(t, \boldsymbol{u}(t, \boldsymbol{z}, 0), 0), \quad W(0) = 0,$$

then

$$\mathcal{P}_\epsilon^m(\boldsymbol{z}, 0) = mkF(\boldsymbol{z}) + W(m\eta). \tag{10}$$

In effect, $W(t) = \boldsymbol{u}_\epsilon(t, \boldsymbol{z}, 0)$ with $W(0) = 0$ because $\boldsymbol{u}(0, \boldsymbol{z}, \epsilon) \equiv \boldsymbol{z}$.

3 Planar Coupled Oscillators

While Theorem 2.1 can be specialized in several directions, we will consider in this section the specialization to the case of coupled planar oscillators. In this case, the normal nondegeneracy of period manifolds and the projection into a complement of the range of the associated infinitesimal displacement can be investigated with integrals, given by certain averages over the unperturbed orbits, that have very specific geometric interpretations.

To interpret the system (5) as a system of coupled oscillators, we will suppose that $\boldsymbol{u} = (u_1, \ldots, u_N)$, where each component u_i is in $\mathbf{S}^1 \times \mathbf{R}$ or in $\mathbf{R}^2$. In particular, in local coordinates, $\boldsymbol{u}$ is represented by a point in $\mathbf{R}^{2N}$. We will denote the manifold consisting of the product of these cylinders and Euclidean spaces by M. For each $i = 1, \ldots, N$, we will assume the corresponding "planar" system has the form

$$\dot{u}_i = f_i(u_i) + \epsilon g_i(t, \boldsymbol{u}, \epsilon)$$

with the functions F and G appearing in (5) given by

$$F(\boldsymbol{u}) = (f_1(u_1), \ldots, f_N(u_N)), \quad G(t, \boldsymbol{u}, \epsilon) = (g_1(t, \boldsymbol{u}, \epsilon), \ldots, g_N(t, \boldsymbol{u}, \epsilon)). \tag{11}$$

Here, the components u_j, with $j \neq i$, of the argument $\boldsymbol{u}$ of g_i can be considered external inputs to the "planar" system. Also, we will interpret F as a vector field on $\mathbf{R}^{2N}$, and define, in addition, the vector fields F_i and $F_i^\perp$, for $i = 1 \ldots, N$, by

$$F_i(\boldsymbol{u}) := (0, \ldots, 0, f_i(u_i), 0, \ldots, 0), \qquad F_i^\perp(\boldsymbol{u}) := (0, \ldots, 0, f_i^\perp(u_i), 0, \ldots, 0)$$

where these vector fields have nonzero entries only in the ith component and where $\perp$ denotes the operation of positive rotation in the plane through $\pi/2$ radians.

Our basic assumption is that each unperturbed planar system is an oscillator, that is, for each $i = 1, \ldots, n$, the corresponding unperturbed planar system has a periodic orbit Γ_i with period $\eta_i > 0$. There are two cases: G autonomous and G nonautonomous. Although the analysis is essentially the same in both cases, the second case is technically simpler. We will do the analysis for this case first.

3.1 The nonautonomous case

Our analysis begins with the assumption that the unperturbed oscillators are all resonant with the periodic perturbation G. Recall that the period τ of G is a function of ϵ. As a notational convenience, we define $T := \tau(0)$, and we say that the unperturbed oscillators are in resonance with the periodic perturbation if there are relatively prime integers m and n_i such that, for each $i = 1, \ldots, N$, the following relation holds: $mT = n_i\eta_i$.

Since the time dependence is periodic, the system is best considered as an autonomous system on the cylinder $\mathbf{S}^1 \times M$ where the first factor is given by the time variable considered as an angular variable modulo τ. In particular, we view (5) as

$$\begin{aligned} \dot{\boldsymbol{u}} &= F(\boldsymbol{u}) + \epsilon G(s, \boldsymbol{u}, \epsilon), \\ \dot{s} &= 1 \text{ modulo } \tau(\epsilon). \end{aligned} \tag{12}$$

The $(N+1)$-dimensional period manifold is given by

$$\mathcal{A} := \mathbf{S}^1 \times \Pi_{i=1}^N \Gamma_i. \tag{13}$$

This case is easier than the autonomous case because the periodic time dependence supplies a natural Poincaré section $\Sigma := M$ and a natural Poincaré map

$$\mathcal{P}^m(\xi, \epsilon) = \boldsymbol{u}(m\tau(\epsilon), \xi, \epsilon), \quad \xi \in \Sigma. \tag{14}$$

Here, the fixed point manifold is given by

$$\mathcal{Z} := \mathcal{A} \cap \Sigma = \Pi_{i=1}^{N} \Gamma_i. \tag{15}$$

It is an N-dimensional submanifold of the $2N$-dimensional manifold Σ.

At each point $\boldsymbol{z} \in \mathcal{Z}$, the vectors $F_1(\boldsymbol{z}), F_2(\boldsymbol{z}), \ldots, F_N(\boldsymbol{z})$ generate a subspace we call $\mathcal{E}^{\tan}(\boldsymbol{z})$ that is tangent to $\mathcal{Z}$. This space is complemented in the tangent space $\mathcal{T}_{\boldsymbol{z}}\Sigma$ by the subspace, we call $\mathcal{E}^{\mathrm{nor}}(\boldsymbol{z})$, that is generated by $F_1^{\perp}(\boldsymbol{z}), F_2^{\perp}(\boldsymbol{z}), \ldots, F_N^{\perp}(\boldsymbol{z})$. We will use the splitting

$$\mathcal{T}_{\boldsymbol{z}}\Sigma = \mathcal{E}^{\tan}(\boldsymbol{z}) \oplus \mathcal{E}^{\mathrm{nor}}(\boldsymbol{z}) \tag{16}$$

in the analysis to follow. In particular, we will determine the infinitesimal displacement and the bifurcation function relative to this splitting.

The infinitesimal displacement is computed using the first variational equation. In effect, for $\boldsymbol{z} \in \mathcal{Z}$, the infinitesimal displacement is given by

$$\mathcal{D}(\boldsymbol{z}) = Dp^m(\boldsymbol{z}) - I = \boldsymbol{u}_\xi(mT, \boldsymbol{z}, 0). \tag{17}$$

The function $t \mapsto \boldsymbol{u}_\xi(t, \boldsymbol{z}, 0)$ is the principal fundamental matrix solution at $t = 0$ of the variational equation

$$\dot{W} = DF(\boldsymbol{u}(t, \boldsymbol{z}, 0))W. \tag{18}$$

To take advantage of the fact that the unperturbed system decouples into N planar oscillators during the computation of the solution of (18), we will use that fact that the first variational equation for a planar system is solvable in quadratures. For this, and to avoid confusion with (18), we will consider an arbitrary plane autonomous system given by

$$\dot{u} = f(u), \tag{19}$$

and let $t \mapsto u(t, \zeta)$ denote its solution with the initial condition $u(0, \zeta) = \zeta$. Also, we will consider the principal fundamental matrix solution, $t \mapsto \Phi(t)$, at $t = 0$ of the first variational equation for (19), and let ζ denote a variable on the plane.

Our first observation is standard: $\Phi(t)f(\zeta) = f(u(t,\zeta))$. To see this, note that $\Phi(0)f(\zeta) = f(\zeta)$ and

$$\frac{d}{dt}f(u(t,\zeta)) = Df(u(t,\zeta))f(u(t,\zeta)).$$

Thus, $t \mapsto f(u(t,\zeta))$ and $t \mapsto \Phi(t)f(\zeta)$ must be identical: they are both solutions of the same initial value problem.

Since the vector functions f and $f^\perp$ are linearly independent at each point in the plane, there are two real valued functions $t \mapsto a(t,\zeta)$ and $t \mapsto b(t,\zeta)$ such that

$$\Phi(t)f^\perp(\zeta) = a(t,\zeta)f(u(t,\zeta)) + b(t,\zeta)f^\perp(u(t,\zeta)). \tag{20}$$

We will soon find formulas for a and b in terms of f. However, with the definitions just given, the fundamental matrix $\Phi(t)$, represented as a linear transformation from $\mathbf{R}^2$, with the basis $\{f(\zeta),\, f^\perp(\zeta)\}$, to $\mathbf{R}^2$, with the basis $\{f(u(t,\zeta)),\, f^\perp(u(t,\zeta))\}$, is given by the matrix

$$\Phi(t) = \begin{pmatrix} 1 & a(t,\zeta) \\ 0 & b(t,\zeta) \end{pmatrix}. \tag{21}$$

The geometric interpretations of a and b are easy to obtain. In fact, suppose Γ is a periodic solution of (19) with period η, that $\zeta \in \Gamma$, and let $t \mapsto \sigma(t)$ denote the solution of the initial value problem

$$\dot{u} = f^\perp(u), \qquad u(0) = \zeta.$$

The orbit given by σ defines an orthogonal trajectory of (19) that is a section for (19) near $\zeta \in \Gamma$. If h denotes the corresponding Poincaré map and r the corresponding return time function, then

$$h(\sigma(t)) = u(r(\sigma(t)), \sigma(t)).$$

After differentiation with respect to t, we find that

$$h'(\zeta)f^\perp(\zeta) = \frac{d}{dt}r(\sigma(t))\bigg|_{t=0} f(\zeta) + a(T,\zeta)f(\zeta) + b(T,\zeta)f^\perp(\zeta).$$

In other words, $b(T,\zeta) = h'(\zeta)$, the derivative of the Poincaré map at ζ (the characteristic multiplier of Γ), and $a(T,\zeta) = -\frac{d}{dt}r(\sigma(t))\Big|_{t=0}$, the negative of the derivative of the time of first return to σ at ζ.

The identification of a and b in quadratures is the content of the following fundamental result. The statement of the theorem uses κ to denote the signed scalar curvature, *div* for the divergence of a plane vector field $f = (f^1, f^2)$, namely $\operatorname{div} f = f^1_x + f^2_y$, and $\operatorname{curl} f = f^2_x - f^1_y$. Also, we let $\|f\| = \langle f, f\rangle^{1/2}$ denote the usual norm in $\mathbf{R}^2$.

Theorem 3.1 (Diliberto's Theorem) *Suppose $t \mapsto u(t,\zeta)$ is the solution of the differential equation (19) with initial condition $u(0,\zeta) = \zeta$. If $f(\zeta) \neq 0$, then the principal fundamental matrix solution $t \mapsto \Phi(t)$ at $t = 0$ of the first variational equation*

$$\dot{W} = Df(u(t,\zeta))W$$

is such that

$$\Phi(t)f(\zeta) = f(u(t,\zeta)), \quad \Phi(t)f^{\perp}(\zeta) = a(t,\zeta)f(u(t,\zeta)) + b(t,\zeta)f^{\perp}(u(t,\zeta)).$$

where

$$b(t,\zeta) := \frac{\|f(\zeta)\|^2}{\|f(u(t,\zeta))\|^2}\, e^{\int_0^t \operatorname{div} f(u(t,\zeta))\,ds}, \tag{22}$$

$$a(t,\zeta) := \int_0^t (2\kappa(s,\zeta)\|f(u(s,\zeta))\| - \operatorname{curl} f(u(s,\zeta)))b(s,\zeta)\,ds. \tag{23}$$

The integral formulas (23) and (22) for $a(t,\zeta)$ and $b(t,\zeta)$ seem to have been first obtained by S. P. Diliberto [23], although his formula for $a(t,\zeta)$ incorrectly omits the multiple 2 of the curvature term. Also, the proof given below is new.

Proof. If

$$t \mapsto a(t,\zeta)f(u(t,\zeta)) + b(t,\zeta)f^{\perp}(u(t,\zeta))$$

is the solution of the first variational equation with initial value $f^{\perp}(\zeta)$, then $a(0,\zeta) = 0$, $b(0,\zeta) = 1$, and

$$\begin{aligned} &a(t,\zeta)Df(u(t,\zeta))f(u(t,\zeta)) + a'(t,\zeta)f(u(t,\zeta)) \\ &\qquad + b(t,\zeta)Df^{\perp}(u(t,\zeta))f(u(t,\zeta)) + b'(t,\zeta)f^{\perp}(u(t,\zeta)) \\ &= a(t,\zeta)Df(u(t,\zeta))f(u(t,\zeta)) + b(t,\zeta)Df(u(t,\zeta))f^{\perp}(u(t,\zeta)). \end{aligned} \tag{24}$$

After taking the inner product of both sides of (25) with $f^\perp(u(t,\zeta))$, we obtain schematically

$$b'\|f\|^2 = b(\langle Df \cdot f^\perp, f^\perp\rangle - \langle Df^\perp \cdot f, f^\perp\rangle).$$

Since $f^\perp = Jf$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have

$$\langle Df^\perp \cdot f, f^\perp\rangle = \langle JDf \cdot f, Jf\rangle = \langle Df \cdot f, f\rangle$$

and

$$b'\|f\|^2 = b(\langle Df \cdot f^\perp, f^\perp\rangle + \langle Df \cdot f, f\rangle - 2\langle Df \cdot f, f\rangle).$$

An easy computation now shows

$$b' = b \operatorname{div} f - b\frac{d}{dt}\ln\|f\|^2.$$

By solving this differential equation, we obtain the formula (22) for b given in the statement of the theorem.

From (25), taking the inner product this time with $f(u(t,\zeta))$, we obtain

$$\begin{aligned} a'\|f\|^2 &= b(\langle Df \cdot f^\perp, f\rangle - \langle Df^\perp \cdot f, f\rangle) \\ &= b(\langle f^\perp, (Df)^* f\rangle - \langle JDf \cdot f, f\rangle) \\ &= b(\langle f^\perp, (Df)^* f\rangle + \langle f^\perp, Df \cdot f\rangle) \\ &= b(\langle f^\perp, 2Df \cdot f\rangle + \langle f^\perp, ((Df)^* - (Df))f\rangle) \qquad (25) \end{aligned}$$

where $*$ denotes the transpose. Again, a computation shows

$$\begin{aligned} \langle f^\perp, 2Df \cdot f\rangle &= 2\kappa\|f\|^3, \\ \langle f^\perp, ((Df)^* - (Df))f\rangle &= -\|f\|^2 \operatorname{curl} f \end{aligned}$$

where $\kappa = \kappa(t,\zeta)$ is the (signed) scalar curvature of the curve $t \mapsto u(t,\zeta)$. Taken together, these formulas yield the expression (23) for a given in the statement of the theorem. □

Corollary 3.2 *With the hypotheses and notation of Diliberto's Theorem, if* $\zeta \in \mathbf{R}^2$, *then* $b(t, u(\theta,\zeta)) = b(t+\theta,\zeta)/b(\theta,\zeta)$ *and*

$$a(t, u(\theta,\zeta)) = [a(t+\theta,\zeta) - a(\theta,\zeta)]/b(\theta,\zeta).$$

If, in addition, ζ *lies on a periodic orbit with period* η, *then* $b(\eta, u(\theta,\zeta)) = b(\eta,\zeta)$, $b(\theta+\eta,\zeta) = b(\theta,\zeta)b(\eta,\zeta)$, *and* $a_t(t,\zeta) = q(t)b(t,\zeta)$ *where* q *is an* η*-periodic function.*

Proof. The proof is a straight forward computation using variable changes in the appropriate integrals, the flow property $u(t, u(\theta, \zeta)) = u(t+\theta, \zeta)$, and, for the second statement of the corollary, the η-periodicity of $t \mapsto u(t, \zeta)$. □

The next theorem, the main result of this section, uses the following notation. For $f = (f^1, f^2)$ and $g = (g^1, g^2)$, vectors in $\mathbf{R}^2$, we define $f \wedge g := f^1 g^2 - f^2 g^1 = \langle g, f^\perp \rangle$. Also, A and B denote the $N \times N$ diagonal matrix functions given by

$$A(t, \boldsymbol{z}) := \begin{pmatrix} a_1(t, z_1) & 0 & \cdots & & 0 \\ 0 & a_2(t, z_2) & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & a_N(t, z_N) \end{pmatrix},$$

$$B(t, \boldsymbol{z}) := \begin{pmatrix} b_1(t, z_1) & 0 & \cdots & & 0 \\ 0 & b_2(t, z_2) & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & b_N(t, z_N) \end{pmatrix} \qquad (26)$$

where $\boldsymbol{z} = (z_1, \ldots, z_N)$, with each component z_i in $\mathbf{R}^2$ or $\mathbf{S}^1 \times \mathbf{R}$, and the functions a_i, b_i correspond to (22) and (23) relative to each component of the coupled system.

Theorem 3.3 *Consider the coupled oscillator (12) with components (11) and let A and B be the matrix functions defined in (26). Also, suppose that the unperturbed system has the period manifold $\mathcal{A}$ given by (13), and the fixed point manifold $\mathcal{Z}$ given by (15). If the $2N \times N$ matrix*

$$\begin{pmatrix} A(mT, \boldsymbol{z}) \\ B(mT, \boldsymbol{z}) - I \end{pmatrix}$$

has rank N at each point $\boldsymbol{z} \in \mathcal{Z}$, then $\mathcal{A}$ is normally nondegenerate. In this case, the bifurcation function is, up to a normalization, the function $\mathcal{B}(\boldsymbol{z}) = (\mathcal{B}_1(\boldsymbol{z}), \ldots, \mathcal{B}_N(\boldsymbol{z}))$ where, for each $i = 1, \ldots, N$,

$$\mathcal{B}_i(\boldsymbol{z}) = a_i(mT, z_i)\mathcal{M}_i(\boldsymbol{z}) + (1 - b_i(mT, z_i))(\mathcal{N}_i(\boldsymbol{z}) + m\tau'(0)) \qquad (27)$$

with

$$\mathcal{M}_i(\boldsymbol{z}) := \int_0^{mT} \frac{1}{\|f_i(u_i(t, z_i, 0))\|^2} \Big(\frac{1}{b_i(t, z_i)} f_i(u_i(t, z_i, 0)) \wedge g_i(t, \boldsymbol{u}(t, \boldsymbol{z}, 0), 0) \Big)\, dt,$$

$$\mathcal{N}_i(\boldsymbol{z}) := \int_0^{mT} \frac{1}{\|f_i(u_i(t,z_i,0))\|^2}\Big(\langle f_i(u_i(t,z_i,0)), g_i(t,\boldsymbol{u}(t,\boldsymbol{z},0),0)\rangle - \frac{a_i(t,z_i)}{b_i(t,z_i)} f_i(u_i(t,z_i,0)) \wedge g_i(t,\boldsymbol{u}(t,\boldsymbol{z},0),0)\Big)\,dt. \tag{28}$$

Proof. To prove the first assertion of the theorem about normal nondegeneracy, we must compute the kernel of the infinitesimal displacement function (17) over the fixed point manifold. We use Diliberto's Theorem to represent the infinitesimal displacement with respect to the bases of the tangential and normal subspaces of the splitting (16). In fact, the obvious computation shows that the principal fundamental matrix solution of the variational equation (18), in block matrix form relative to $\mathcal{E}^{\tan} \oplus \mathcal{E}^{\text{nor}}$ is given by

$$\Psi(t,\boldsymbol{z}) = \begin{pmatrix} I & A(t,\boldsymbol{z}) \\ 0 & B(t,\boldsymbol{z}) \end{pmatrix} \tag{29}$$

where I denotes the $N \times N$ identity of $\mathcal{E}^{\tan}(\boldsymbol{z})$. It follows that the infinitesimal displacement is given by

$$\mathcal{D}(\boldsymbol{z}) = \begin{pmatrix} 0 & A(mT,\boldsymbol{z}) \\ 0 & B(mT,\boldsymbol{z}) - I \end{pmatrix}. \tag{30}$$

The period manifold is normally nondegenerate provided the kernel of the infinitesimal displacement at each point $\boldsymbol{z} \in \mathcal{Z}$ is exactly the corresponding tangent space $\mathcal{T}_{\boldsymbol{z}}\mathcal{Z} = \mathcal{E}^{\tan}(\boldsymbol{z})$. But, in view of (30), this condition is equivalent to the rank condition in the statement of the theorem.

Consider, for fixed $\boldsymbol{z} \in \mathcal{Z}$, the map $\lambda : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ given by

$$(x,y) \mapsto (I - B(mT,\boldsymbol{z}))x + A(mT,\boldsymbol{z})y.$$

Here, we are viewing the tangent space $\mathcal{E}^{\tan}(\boldsymbol{z}) \oplus \mathcal{E}^{\text{nor}}(\boldsymbol{z})$ as $\mathbf{R}^N \times \mathbf{R}^N$ with the isomorphism given by the choice of the bases

$$F_1, F_2, \ldots, F_N \quad \text{and} \quad F_1^{\perp}, F_2^{\perp}, \ldots, F_N^{\perp}.$$

Note that the range $\mathcal{R}$ of the infinitesimal displacement, in our coordinate representation, is exactly the kernel of λ. In effect, the tangent vector (x,y) is in $\mathcal{R}$ provided that there is some $u \in \mathbf{R}^N$ with $Au = x$ and $(B - I)u = y$. Using the fact that the diagonal matrices A and $B - I$ commute, it follows immediately that $\lambda(x,y) = 0$. By the rank condition, equivalent to the normal nondegeneracy, the map λ is surjective.

The subspace $\mathcal{R}(\boldsymbol{z})$ of $\mathcal{E}^{\mathrm{tan}}(\boldsymbol{z}) \oplus \mathcal{E}^{\mathrm{nor}}(\boldsymbol{z})$ has a vector space complement $\mathcal{S}(\boldsymbol{z})$. Moreover, λ is a coordinate representation of the projection $\pi_{\mathcal{S}}$ of the direct sum $\mathcal{R}(\boldsymbol{z}) \oplus \mathcal{S}(\boldsymbol{z})$ onto $\mathcal{S}(\boldsymbol{z})$. To see this, let $v = (r,s)$ denote a point in the direct sum, and note that $\lambda(v) = \lambda \circ \pi_{\mathcal{S}}(v)$. Also, note that the restriction of λ to $\mathcal{S}$ is an isomorphism.

We will complete the proof by using λ to project the coordinate representation of $\mathcal{P}_\epsilon(\boldsymbol{z},0)$ in $\mathcal{E}^{\mathrm{tan}}(\boldsymbol{z}) \oplus \mathcal{E}^{\mathrm{nor}}(\boldsymbol{z})$ to $\mathcal{S}(\boldsymbol{z})$.

To compute $\mathcal{P}_\epsilon^m(\boldsymbol{z},0)$, recall the definition of the Poincaré map (14), and note that

$$\mathcal{P}_\epsilon^m(\boldsymbol{z},0) = mkF(\boldsymbol{z}) + \boldsymbol{u}_\epsilon(mT,\boldsymbol{z},0)$$

where $k := \tau'(0)$. Here, the function $t \mapsto \boldsymbol{u}_\epsilon(t,\boldsymbol{z},0)$ is the solution of the second variational initial value problem

$$\dot{\boldsymbol{w}} = DF(\boldsymbol{u}(t,\boldsymbol{z},0))\boldsymbol{w} + G(t,\boldsymbol{u}(t,\boldsymbol{z},0),0), \quad \boldsymbol{w}(0) = 0. \tag{31}$$

By variation of constants relative to the fundamental matrix given in (29), we compute

$$\boldsymbol{u}_\epsilon(mT,\boldsymbol{z},0) = \Psi(mT,\boldsymbol{z}) \int_0^{mT} \Psi^{-1}(t,\boldsymbol{z}) G(t,\boldsymbol{u}(t,\boldsymbol{z},0),0)\, dt.$$

Also, in block matrix form, we have

$$\Psi^{-1}(t,\boldsymbol{z}) = \begin{pmatrix} I & -A(s,\boldsymbol{z})B^{-1}(s,\boldsymbol{z}) \\ 0 & B^{-1}(s,\boldsymbol{z}) \end{pmatrix}, \qquad G = \begin{pmatrix} G^{\mathrm{tan}} \\ G^{\mathrm{nor}} \end{pmatrix}.$$

To finish the computation, define

$$\begin{pmatrix} \mathcal{N}(\boldsymbol{z}) \\ \mathcal{M}(\boldsymbol{z}) \end{pmatrix} = \begin{pmatrix} \int_0^{mT} G^{\mathrm{tan}}(t,\boldsymbol{u}(t,\boldsymbol{z},0),0) - A(t,\boldsymbol{z})B^{-1}(t,\boldsymbol{z})G^{\mathrm{nor}}(t,\boldsymbol{u}(t,\boldsymbol{z},0),0)\, dt \\ \int_0^{mT} B^{-1}(t,\boldsymbol{z})G^{\mathrm{nor}}(t,\boldsymbol{u}(t,\boldsymbol{z},0),0)\, dt \end{pmatrix},$$

and note that the obvious computation, taking into account the coordinate representations of G^{tan} and G^{nor}, yields the expressions for $\mathcal{N}$ and $\mathcal{M}$ that are given in the statement of the theorem.

We now have

$$\mathcal{P}_\epsilon^m(\boldsymbol{z},0)=\begin{pmatrix} I & A(mT,\boldsymbol{z})\\ 0 & B(mT,\boldsymbol{z})\end{pmatrix}\begin{pmatrix}\mathcal{N}(\boldsymbol{z})\\ \mathcal{M}(\boldsymbol{z})\end{pmatrix}+mk\begin{pmatrix}\mathbf{1}\\ 0\end{pmatrix}$$

where $\mathbf{1}$ is the coordinate representation (the vector with all components 1) of $F(\boldsymbol{z})$. Finally, after applying the projection λ, we obtain the bifurcation function in the statement of the theorem. □

3.2 The Autonomous case

In this subsection we consider the coupled oscillator system (5) with the autonomous components

$$F(\boldsymbol{u})=(f_1(u_1),\ldots,f_N(u_N)),\qquad G(t,\boldsymbol{u},\epsilon)=(g_1(\boldsymbol{u},\epsilon),\ldots,g_N(\boldsymbol{u},\epsilon)).\quad (32)$$

Our analysis begins with the assumption that the unperturbed oscillators are in resonance. In particular, for $i=1,\ldots,N$, we suppose that the ith unperturbed planar system has a periodic orbit Γ_i with period η_i. Moreover, we suppose that there are relatively prime integers n_i, for each $i=1,\ldots,N$, and a number $T>0$, such that, for each i, we have $T=n_i\eta_i$.

The N-dimensional period manifold $\mathcal{A}$ is the product of the periodic orbits:

$$\mathcal{A}=\Pi_{i=1}^N\Gamma_i.\quad (33)$$

A convenient ($(2N-1)$-dimensional) Poincaré section Σ is obtained by choosing an integral manifold of the involutive distribution

$$F_2,\ldots,F_N,F_1^\perp,F_2^\perp,\ldots,F_N^\perp$$

at some point of $\mathcal{A}$. Essentially, this amounts to viewing the periodic orbit Γ_1 as being transverse to Σ while the product $\Pi_{i=2}^N\Gamma_i$ is contained in Σ.

The Poincaré map is given by

$$\mathcal{P}(\xi,\epsilon)=\boldsymbol{u}(\tau(\xi,\epsilon),\xi,\epsilon),\quad \xi\in\Sigma\subset M\quad (34)$$

where τ denotes the time of first return to Σ. The existence and smoothness of τ on a neighborhood of the $(N-1)$-dimensional fixed point manifold $\mathcal{Z}\subset M$ given by

$$\mathcal{Z}:=\mathcal{A}\cap\Sigma=\Pi_{i=2}^N\Gamma_i\quad (35)$$

is an easy consequence of the Implicit Function Theorem.

Note that the tangent space $\mathcal{T}_{\boldsymbol{z}}\Sigma = \mathcal{E}_a^{\text{tan}}(\boldsymbol{z}) \oplus \mathcal{E}_a^{\text{nor}}(\boldsymbol{z})$ where $\mathcal{E}_a^{\text{tan}}(\boldsymbol{z})$ at each $\boldsymbol{z} \in \Sigma$ is defined to be the span of $F_2, \ldots, F_N$ at $\boldsymbol{z}$, while $\mathcal{E}_a^{\text{nor}}(\boldsymbol{z})$ is defined to be the span of $F_1^{\perp}, F_2^{\perp}, \ldots, F_N^{\perp}$ at $\boldsymbol{z}$. The subscript "a", for autonomous, is meant to distinguish the direct summands from the similar subspaces defined for the nonautonomous case.

The infinitesimal displacement function $\mathcal{D}(\boldsymbol{z})$ at $\boldsymbol{z} \in \mathcal{Z}$ is the linear transformation of $\mathcal{T}_{\boldsymbol{z}}\Sigma$ given by

$$\mathcal{D}(\boldsymbol{z}) = Dp(\boldsymbol{z}) - I = F(\boldsymbol{z})\tau_{\xi}(\boldsymbol{z}, 0) + \boldsymbol{u}_{\xi}(T, \boldsymbol{z}, 0) - I \tag{36}$$

where $t \mapsto \boldsymbol{u}_{\xi}(t, \boldsymbol{z}, 0)$ is the fundamental matrix solution $\Psi(t, \boldsymbol{z})$ of the first variational equation (18).

The block matrix form of $\Psi(t, \boldsymbol{z})$ relative to the basis

$$F_1(\boldsymbol{z}), \ldots, F_N(\boldsymbol{z}), F_1^{\perp}(\boldsymbol{z}), \ldots, F_N^{\perp}(\boldsymbol{z}) \tag{37}$$

is given by (29). Here, we are using the basis

$$F(\boldsymbol{z}), F_2(\boldsymbol{z}), \ldots, F_N(\boldsymbol{z}), F_1^{\perp}(\boldsymbol{z}), \ldots, F_N^{\perp}(\boldsymbol{z}). \tag{38}$$

The change of coordinates, from the basis (37) to the basis (38), is given by

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \tag{39}$$

where each block is $N \times N$, and $Q = I + Q_0$ with

$$Q_0 = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ -1 & 0 & \ldots & 0 \\ \vdots & & & \\ -1 & 0 & \ldots & 0 \end{pmatrix}. \tag{40}$$

Thus, the fundamental matrix Ψ, in the new coordinates, in block matrix form relative to $\mathcal{E}_a \oplus \mathcal{E}_a^{\text{tan}} \oplus \mathcal{E}_a^{\text{nor}}$, where $\mathcal{E}_a$ is spanned by F, is given by

$$\Psi(t, \boldsymbol{z}) = \begin{pmatrix} I & QA(t, \boldsymbol{z}) \\ 0 & B(t, \boldsymbol{z}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & C(t, \boldsymbol{z}) \\ 0 & I & \widehat{A}(t, \boldsymbol{z}) \\ 0 & 0 & B(t, \boldsymbol{z}) \end{pmatrix} \tag{41}$$

where the $1 \times N$ matrix function C has components $(a_1, 0 \ldots, 0)$, and the $(N-1) \times N$ matrix $\widehat{A}$ is given by

$$\widehat{A}(t, \boldsymbol{z}) = \begin{pmatrix} -a_1(t, z_i) & a_2(t, z_i) & 0 & & \cdots & 0 \\ -a_1(t, z_i) & 0 & a_3(t, z_i) & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ -a_1(t, z_i) & 0 & & & \cdots & a_N(t, z_i) \end{pmatrix}. \tag{42}$$

Theorem 3.4 *Consider the coupled oscillator (5) with components (32), let B denote the $N \times N$ diagonal matrix function defined in (26) and $\widehat{A}$ the $(N-1) \times N$ matrix function (42). Also, suppose that the unperturbed system has the period manifold $\mathcal{A}$ given by (33), and the fixed point manifold $\mathcal{Z}$ given by (35). If the $(2N-1) \times N$ matrix*

$$\begin{pmatrix} \widehat{A}(T, \boldsymbol{z}) \\ B(T, \boldsymbol{z}) - I \end{pmatrix}$$

has rank N at each point $\boldsymbol{z} \in \mathcal{Z}$, then $\mathcal{A}$ is normally nondegenerate. In case the matrix $B(T, \boldsymbol{z}) - I$ is invertible, the bifurcation function is, up to a normalization, the function $\mathcal{B}(\boldsymbol{z}) = (\mathcal{B}_2(\boldsymbol{z}), \ldots, \mathcal{B}_N(\boldsymbol{z}))$ where, for each $i = 2, \ldots, N$,

$$\begin{aligned} \mathcal{B}_i(\boldsymbol{z}) \;=\; & (b_i(T, z_i) - 1)^{-1}(a_i(T, z_i)\mathcal{M}_i(\boldsymbol{z}) + (1 - b_i(T, z_i))\mathcal{N}_i(\boldsymbol{z})) \\ & -(b_1(T, z_i) - 1)^{-1}(a_1(T, z_i)\mathcal{M}_1(\boldsymbol{z}) + (1 - b_1(T, z_i))\mathcal{N}_1(\boldsymbol{z})) \end{aligned} \tag{43}$$

with $\mathcal{M}_i$ and $\mathcal{N}_i$, for $i = 1, \ldots, N$, given by (28).

In the statement of the theorem, the explicit bifurcation function is given for the case of N hyperbolic limit cycle oscillators. However, it is clear that our coupled oscillator system can be normally nondegenerate when some of the limit cycles are replaced by periodic orbits such that the derivative of the return time at the periodic orbit is not zero. This replacement can be used for nonhyperbolic limit cycles and for periodic orbits in a period annulus where the period function is monotone at the resonant periodic orbit. However, the rank condition can not be satisfied if none of the resonant orbits are hyperbolic. This is one reason why systems of coupled pendula are more difficult to analyze than systems of coupled limit cycle oscillators—for pendula, the period manifolds are normally *degenerate*. As a result, to find the periodic orbits of a perturbed coupled pendulum system, higher order methods must be used. Of course, in all the cases of normal nondegeneracy, there is a bifurcation function defined on $\mathcal{Z}$ with range in $\mathbf{R}^{N-1}$ whose components are given by a combination of the functions $\mathcal{M}_i$, $\mathcal{N}_i$, a_i, and b_i. The formula for these components can be easily computed in each case once the nonzero elements of the matrices $\widehat{A}$ and $B - I$ are known.

The proof of Theorem 3.4 follows.

Proof. To prove the first assertion of the theorem about normal nondegeneracy, we must compute the kernel of the infinitesimal displacement function over the fixed point manifold. Using (41), proceeding as in the proof of Theorem 3.3, and taking into account the fact that we are only interested in the vectors represented in the second two summands of the splitting $\mathcal{E}_a \oplus \mathcal{E}_a^{\tan} \oplus \mathcal{E}_a^{\mathrm{nor}}$; that is, the vectors in $\mathcal{T}_{\boldsymbol{z}}\Sigma$, we find that the block matrix form of the infinitesimal displacement relative to $\mathcal{E}_a^{\tan} \oplus \mathcal{E}_a^{\mathrm{nor}}$ is given by

$$\mathcal{D}(\boldsymbol{z}) = \begin{pmatrix} 0 & \widehat{A}(T, \boldsymbol{z}) \\ 0 & B(T, \boldsymbol{z}) - I \end{pmatrix}. \tag{44}$$

The period manifold $\mathcal{A}$ is normally nondegenerate provided the kernel of the infinitesimal displacement at each point $\boldsymbol{z} \in \mathcal{Z}$ is exactly the corresponding $(N-1)$-dimensional tangent space $\mathcal{T}_{\boldsymbol{z}}\mathcal{Z} = \mathcal{E}_a^{\tan}(\boldsymbol{z})$. But, in view of (44), this condition is equivalent to the rank condition in the statement of the theorem.

The partial derivative $\mathcal{P}_\epsilon(\boldsymbol{z}, 0)$ is computed from the second variational equation just as in the proof of Theorem 3.3 by

$$\begin{aligned} \mathcal{P}_\epsilon(\boldsymbol{z}, 0) &= F(\boldsymbol{z})\tau_\epsilon(\boldsymbol{z}, 0) + \boldsymbol{u}_\epsilon(T, \boldsymbol{z}, 0) \\ &= F(\boldsymbol{z})\tau_\epsilon(\boldsymbol{z}, 0) + \Psi(T, \boldsymbol{z}) \int_0^T \Psi^{-1}(t, \boldsymbol{z}) G(\boldsymbol{u}(t, \boldsymbol{z}, 0), 0)\, dt. \end{aligned} \tag{45}$$

The fundamental matrix Ψ is exactly the same as before, while the function G is formally the same, only here it is autonomous. Thus, with $\mathcal{M}$ and $\mathcal{N}$ defined by (28), the second summand of the last equation is given, in block form relative to the basis (37), by

$$\begin{pmatrix} I & A(T, \boldsymbol{z}) \\ 0 & B(T, \boldsymbol{z}) \end{pmatrix} \begin{pmatrix} \mathcal{N}(\boldsymbol{z}) \\ \mathcal{M}(\boldsymbol{z}) \end{pmatrix}.$$

This summand relative to the basis (38), obtained by applying the change of coordinates (39), is given by

$$\begin{pmatrix} Q\mathcal{N}(\boldsymbol{z}) + QA(T,\boldsymbol{z})\mathcal{M}(\boldsymbol{z}) \\ \mathcal{M}(\boldsymbol{z}) + (B(T,\boldsymbol{z}) - I)\mathcal{M}(\boldsymbol{z}) \end{pmatrix}. \tag{46}$$

Note that the block form of this vector relative to the splitting associated with (38) separates the first displayed block into its first component, corresponding to $\mathcal{E}_a(\boldsymbol{z})$, and its remaining $(N-1)$ components, corresponding to $\mathcal{E}_a^{\tan}(\boldsymbol{z})$. Since $\mathcal{P}_\epsilon(\boldsymbol{z}, 0)$ is tangent to Σ, this first component must be the negative of the vector

$$(\tau_\epsilon(\boldsymbol{z}, 0), 0, 0) \in \mathcal{E}_a(\boldsymbol{z}) \oplus \mathcal{E}_a^{\tan}(\boldsymbol{z}) \oplus \mathcal{E}_a^{\mathrm{nor}}(\boldsymbol{z})$$

corresponding to $F(\boldsymbol{z})\tau_\epsilon(\boldsymbol{z}, 0)$. This implies that

$$\mathcal{P}_\epsilon(\boldsymbol{z}, 0) = \begin{pmatrix} 0 \\ \widehat{\mathcal{N}}(\boldsymbol{z}) \\ \mathcal{M}(\boldsymbol{z}) \end{pmatrix} + \begin{pmatrix} 0 \\ \widehat{A}(T,\boldsymbol{z})\mathcal{M}(\boldsymbol{z}) \\ (B(T,\boldsymbol{z}) - I)\mathcal{M}(\boldsymbol{z}) \end{pmatrix} \tag{47}$$

where $\widehat{\mathcal{N}}(\boldsymbol{z})$ is the $(N-1)$ vector

$$(\mathcal{N}_2(\boldsymbol{z}) - \mathcal{N}_1(\boldsymbol{z}), \mathcal{N}_3(\boldsymbol{z}) - \mathcal{N}_1(\boldsymbol{z}), \ldots, \mathcal{N}_N(\boldsymbol{z}) - \mathcal{N}_1(\boldsymbol{z})).$$

As in the proof of Theorem 3.3, the projection to the complement of the range of the infinitesimal displacement has a local coordinate representation as a map $\mathcal{E}_a^{\tan} \oplus \mathcal{E}_a^{\mathrm{nor}} \to \mathbf{R}^{N-1}$ defined by $(x, y) \mapsto x - \widehat{A}(B - I)^{-1}y$. An easy computation, shows the second summand of (47) projects to zero, and the first summand projects to the bifurcation function in the statement of the theorem. □

An analysis of autonomous coupled oscillators similar to the analysis of this subsection is given in [12]. However, the projection to the complement of the range of the infinitesimal displacement claimed on page 435 of the cited paper is in error. Nonetheless, the first two examples presented in [12] on pages 436–441 are correct. In effect, the projection taken there degenerates to the correct projection in the examples because $A(t, \boldsymbol{z}) \equiv 0$. However, the analysis of the final example, a system of coupled van der Pol oscillators, is incorrect. The correct approach to the analysis of this system is outlined in the next section.

4 Mutual Synchronization of Coupled Oscillators

All of the above analysis can be viewed as a methodology for determining the near resonant mutual synchronization of a system of coupled oscillators. However, as an illustration of the method as it applies to mathematical models where a particular parameter is distinguished, we will consider the problem of the mutual synchronization of two coupled oscillators with small nonlinear coupling strength and one detuning parameter. The ideas of this section can be generalized in many different directions; the reader is invited to do so.

A typical mathematical model, adapted from [41][p. 448], that arises in the theory of synchronization of electric circuits is given by two inductively coupled van der Pol oscillators:

$$\begin{aligned} \ddot{x}_1 + (x_1^2 - 1)\dot{x}_1 + x_1 &= \epsilon Q_1 \ddot{x}_2, \\ \ddot{x}_2 + (x_2^2 - 1)\dot{x}_2 + \omega^2 x_2 &= \epsilon Q_2 \ddot{x}_1. \end{aligned} \tag{48}$$

We suppose that the uncoupled oscillators are nearly in $(1:1)$ resonance; that is, the linearized frequency ω of the second oscillator is detuned as a function of the perturbation parameter, but in tune at $\epsilon = 0$. In particular, we will assume

$$\omega = 1 + \ell\epsilon + O(\epsilon^2).$$

The number ℓ is a detuning parameter.

We are interested in the $(1:1)$ synchronization domain. To give a precise definition of this set, consider the period manifold $\mathcal{A}$ obtained as the cross product of the hyperbolic limit cycles of the two identical unperturbed oscillators. This invariant torus persists for small perturbations, and the perturbed periodic solutions, if they exist, lie on it. Since the flow of the coupled system restricted to $\mathcal{A}$ has no fixed points, periodic solutions on $\mathcal{A}$ can be classified by their number of meridional and longitudinal wraps on the torus. The $(1:1)$ *synchronization domain* is, for fixed Q_1 and Q_2, the set of points in the (ω, ϵ)-parameter space such that the corresponding system has a periodic solution with exactly one meridional and exactly one longitudinal wrap. The general problem is to determine the boundary of the synchronization domain; it determines the detunings for a given coupling strength that correspond to synchronization of the *coupled* oscillators.

As an application of the continuation analysis of this paper, we will show how to determine the tangents to the boundary of the synchronization domain at the point where the boundary meets the frequency axis. For the $(1:1)$ synchronization domain of (48), this is the point $(\omega, \epsilon) = (1, 0)$.

In order to apply the continuation analysis, we consider (48) in phase coordinates. In particular, the system given by

$$\begin{aligned} \dot{x}_1 &= y_1, \qquad \dot{y}_1 = -x_1 + (1 - x_1^2)y_1 - \epsilon Q_1(x_2 + (x_2^2 - 1)y_2), \\ \dot{x}_2 &= y_2, \qquad \dot{y}_2 = -x_2 + (1 - x_2^2)y_2 - \epsilon(2\ell x_2 + Q_2(x_1 + (x_1^2 - 1)y_1)) \end{aligned} \tag{49}$$

is $O(\epsilon)$ equivalent to (48). Of course, by the results of our first order theory, it suffices to consider only the $O(\epsilon)$ approximation of the system. As we have seen, an unperturbed orbit continues into the full perturbed system provided the bifurcation function, that depends only on the $O(\epsilon)$ terms of the perturbation, has a simple zero.

We will also consider a generalization of (49) consisting of two identical planar oscillators of the form

$$\begin{aligned} \dot{u}_1 &= f(u_1) + \epsilon g_1(u_1, u_2), \\ \dot{u}_2 &= f(u_2) + \epsilon(\ell h(u_2) + g_2(u_1, u_2)), \end{aligned} \tag{50}$$

where the oscillator

$$\dot{u} = f(u) \tag{51}$$

has a hyperbolic limit cycle Γ with period η, and the functions g_1 and g_2 do not depend on the detuning parameter ℓ. System (49) has the form of (50) with $u_i = (x_i, y_i)$, $i = 1, 2$.

Let $\xi \in \Gamma$ and note that the set $\{\xi\} \times \Gamma$, contained in the period manifold $\Gamma \times \Gamma$, is exactly the fixed point manifold $\mathcal{Z}$ for (50) in a Poincaré section constructed as in the discussion following equation (33). The associated bifurcation function (43) is the scalar function $\boldsymbol{z} \mapsto \mathcal{B}_2(\boldsymbol{z})$. Here, $\mathcal{Z}$ is parametrized by the curve $\theta \mapsto (\xi, u(\theta, \xi))$ with $\theta \in [0, \eta)$, where $t \mapsto u(t, \zeta)$ denotes the solution of (51) with the initial condition $u(0, \zeta) = \zeta$, $\zeta \in \mathbf{R}^2$. In effect, a point $\boldsymbol{z} \in \mathcal{Z}$ is given by $\boldsymbol{z} = (z_1, z_2) = (\xi, u(\theta, \xi))$ for some $\theta \in [0, \eta)$. Moreover, using Corollary 3.2, it follows that the simple zeros of the bifurcation function are exactly the same as the simple zeros of the normalized bifurcation function

$$\boldsymbol{z} \mapsto a(\eta, u(\theta, \xi))\mathcal{M}_2(\boldsymbol{z}) - a(\eta, \xi)\mathcal{M}_1(\boldsymbol{z}) + (1 - b(\eta, \xi))(\mathcal{N}_2(\boldsymbol{z}) - \mathcal{N}_1(\boldsymbol{z})). \tag{52}$$

Here, the adjective "normalized" refers to the bifurcation function obtained from (43) for system (50) with the factor $(b(\eta,\xi)-1)^{-1}$ removed.

Relative to the detuning ℓ, the normalized bifurcation function, when viewed as a function of θ, decomposes into a sum of the following form $\mathcal{B}^{\rm loc}(\theta)=\ell\mathcal{J}(\theta)+\mathcal{I}(\theta)$ where the coefficient functions $\mathcal{J}$ and $\mathcal{I}$ have the following interpretations: $\mathcal{J}$ is the normalized bifurcation function of (50) in case $\ell=1$ and the functions g_1 and g_2 both vanish; $\mathcal{I}$ is the normalized bifurcation function of (50) in case h vanishes.

Proposition 4.1 *If $\ell=1$ and the functions g_1, g_2 both vanish, then the normalized bifurcation function $\mathcal{J}$ for (50) is constant. Moreover, if $T(\epsilon)$ denotes the period of the limit cycle of the associated differential equation $\dot u = f(u)+\epsilon h(u)$, then $\mathcal{J}$ vanishes if and only if $T'(0)=0$.*

Proof. The function $\mathcal{J}$, obtained from (52), is given by

$$\begin{aligned}\mathcal{J}(\theta) &= a(\eta,u(\theta,\xi))\int_0^\eta \frac{c(t+\theta)}{b(t,u(\theta,\xi))}\,dt+(1-b(\eta,\xi))\int_0^\eta d(t+\theta)\,dt\\ &\quad -(1-b(\eta,\xi))\int_0^\eta \frac{a(t,u(\theta,\xi))c(t+\theta)}{b(t,u(\theta,\xi))}\,dt, \qquad (53)\end{aligned}$$

where

$$\begin{aligned}c(s) &:= \frac{1}{||f(u(s,\xi))||^2}(f\wedge h)(u(s,\xi)),\\ d(s) &:= \frac{1}{||f(u(s,\xi))||^2}(\langle f,h\rangle)(u(s,\xi)).\end{aligned}$$

The functions c and d are both η periodic. Thus, for example,

$$\int_0^\eta d(t+\theta)\,dt=\int_\theta^{\eta+\theta} d(t)\,dt=\int_0^\eta d(t)\,dt,$$

and we see that the second summand of the right hand side of (53) is constant. Using Corollary 3.2, we find that the first summand is equal to

$$(a(\eta+\theta,\xi)-a(\theta,\xi))\int_\theta^{\eta+\theta}\frac{c(t)}{b(t,\xi)}\,dt, \qquad (54)$$

and the third summand is equal to

$$(1-b(\eta,\xi))\Big\{\int_\theta^{\eta+\theta}\frac{a(t,\xi)c(t)}{b(t,\xi)}\,dt-a(\theta,\xi)\int_\theta^{\eta+\theta}\frac{c(t)}{b(t,\xi)}\,dt\Big\}. \tag{55}$$

Using Corollary 3.2 and the periodicity of c, an easy computation shows that the first derivative with respect to θ of the difference of the functions defined by (54) and (55) is zero. This proves the first statement of the proposition.

For the second statement of the proposition, consider an orthogonal section S in the plane at a point ξ on the limit cycle of the system (51). Since this limit cycle is hyperbolic, it continues for sufficiently small $\epsilon \neq 0$ with period $T(\epsilon)$ such that $T(0)=\eta$. In particular, if $t\mapsto u(t,\zeta,\epsilon)$ denotes the solution of the perturbed system $\dot{u}=f(u)+\epsilon h(u)$, then there is a curve $\epsilon\mapsto\beta(\epsilon)$, with range in S, such that $\beta(0)=\xi$, and $\beta(\epsilon)=u(T(\epsilon),\beta(\epsilon),\epsilon)$. The derivative with respect to ϵ at $\epsilon=0$ of each side of the last equation is expressed in terms of the quantities a, b, $\mathcal{M}$, and $\mathcal{N}$ by using Diliberto's Theorem, a computation with the second variational equation similar to the computation following equation (31), and the fact that there is a scalar λ such that $\beta'(0)=\lambda f^\perp(\xi)$. After computing these derivatives, we obtain

$$\begin{aligned}\lambda f^\perp(\xi) &= T'(0)f(\xi)+\lambda[a(\eta,\xi)f(\xi)+b(\eta,\xi)f^\perp(\xi)]\\ &\quad+[\mathcal{N}(\xi)+a(\eta,\xi)\mathcal{M}(\xi)]f(\xi)+b(\eta,\xi)\mathcal{M}(\xi)f^\perp(\xi).\end{aligned}$$

Using the linear independence of f and $f^\perp$, we find that

$$\lambda=\lambda b(\eta,\xi)+b(\eta,\xi)\mathcal{M}(\xi),\quad 0=T'(0)+\lambda a(\eta,\xi)+J(\xi)+b(\eta,\xi)\mathcal{N}(\xi)$$

where $\mathcal{J}(\xi)=a(\eta,\xi)\mathcal{M}(\xi)+(1-b(\eta,\xi))\mathcal{N}(\xi)$. If we solve for λ in the first displayed equation and substitute the solution into the second displayed equation, then, after an easy calculation with the resulting expression, we find that $T'(0)=(b(\eta,\xi)-1)^{-1}\mathcal{J}(\xi)$. $\square$

We will show how Proposition 4.1 can be applied to a system, such as the model (48), where there are two parameters: ω, a parameter that adjusts the frequency of one of the oscillators; and ϵ, a parameter that adjusts the strength of the perturbation. For this, consider a path γ given by $\epsilon\mapsto(\omega(\epsilon),\epsilon)$ in the parameter space such that the two oscillators are resonant at $\epsilon=0$. In system (48), the oscillators are in $(1:1)$ resonance provided $\omega(0)=1$.

In general, the normalized bifurcation function has the form $\mathcal{B}^{\text{loc}}(\theta) = \ell\mathcal{J}(0) + \mathcal{I}(\theta)$ where ℓ is a function of the first derivative of $\omega(\epsilon)$ at $\epsilon = 0$, and $\mathcal{J}(0)$ is the constant value of $\mathcal{J}(\theta)$. For system (49), $\ell = \omega'(0)$, and the function h appearing in (50) is given by $h(x_2, y_2) := -2x_2$. The zeros of $\mathcal{B}^{\text{loc}}$ correspond to the continuable periodic solutions of the original system.

Since $\mathcal{I}$ is η periodic; it has a finite maximum and minimum. Also, adjusting ℓ simply translates the graph of $\mathcal{B}^{\text{loc}}$ in the vertical direction. Thus, there are maximum and minimum values of the detuning ℓ that bound the set of detunings corresponding to the existence of periodic solutions of the coupled system. These values are easily related to the maximum and minimum slopes, at $\epsilon = 0$, of the various paths $\gamma(\epsilon)$ in the parameter space that correspond to synchronization.

For our model (48), a simple computation shows that the tangents to the boundary of the synchronization domain at $(\omega, \epsilon) = (1, 0)$ are given by the two lines

$$\epsilon = \frac{1}{\ell_1}(\omega - 1), \qquad \epsilon = \frac{1}{\ell_2}(\omega - 1)$$

where

$$\ell_1 = \min_{0 \le \theta < \eta} -\frac{\mathcal{I}(\theta)}{\mathcal{J}(0)}, \qquad \ell_2 = \max_{0 \le \theta < \eta} -\frac{\mathcal{I}(\theta)}{\mathcal{J}(0)}.$$

The precise values of ℓ_1 and ℓ_2 can be approximated by numerical integration. We also remark that $\mathcal{I}$ is given in quadrature by the normalized bifurcation function (52) with $\ell = 0$.

An interesting open problem is to determine *analytically* the number of zeros of the bifurcation function for (49), corresponding to the number of perturbed periodic solutions, as the detuning ℓ varies in the interval (ℓ_1, ℓ_2). This can be done by determining the relative extrema of $\mathcal{I}$. In fact, there will be a saddle node bifurcation of the zeros of $\theta \mapsto \mathcal{J}(0)\ell + \mathcal{I}(\theta)$ as ℓ is adjusted so that one of these relative extreme points passed through zero. In this regard, careful numerical approximations of the bifurcation function, or approximations of the extent of the synchronization domains in the parameter space, would also be very interesting.

In the generic case, system (50) has $\mathcal{J}(0) \neq 0$ and $\ell_1 \neq \ell_2$, thus the synchronization domain is an open "tongue" as it approaches the ω-axis. If $\mathcal{J}(0) = 0$, the entrainment domain at the resonant point $(\omega, \epsilon) = (1, 0)$ can consist of a single point, in case $\mathcal{I}$ has no zeros, or, it can contain an open neighborhood of the resonant point in case $\mathcal{I}$ has simple zeros.

The period manifold $\mathcal{A}$ of (50) is a normally hyperbolic attractor for the unperturbed flow. Thus, it persists as an attracting invariant torus $\mathcal{A}(\epsilon)$ for sufficiently small ϵ. Our analysis determines (in the generic case) the perturbations corresponding to entrainment to a periodic orbit and the perturbations corresponding to entrainment to a quasiperiodic motion on $\mathcal{A}(\epsilon)$. The different cases correspond to whether or not the bifurcation function has zeros. It should also be clear from topological considerations that stable and unstable periodic orbits on $\mathcal{A}(\epsilon)$, for $\epsilon \neq 0$, alternate on this invariant torus relative to their intersections with the simple closed curve given by $\mathcal{A}(\epsilon) \cap \Sigma$.

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