Toroidalization of Morphisms

by Steven Dale Cutkosky

1. Statement of the problem and known results.

Throughout these lectures, unless explicitely stated otherwise, k is an algebraically closed field of characteristic zero. A variety is an open subset of an irreducible proper k-variety.

A *toroidal structure* on a nonsingular variety X is a SNC divisor D_X. $p \in D_X$ is an n-point if p is on (exactly) n components of D_X. If $p \in X$, regular parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{X,p}$ (or in $\hat{\mathcal{O}}_{X,p}$) are *permisssible parameters* for D_X at p if there exists l (with $0 \le l \le n$) such that $x_1 \cdots x_l = 0$ is a local equation of D_X at p.

A nonsingular subvariety V of X is a *possible center* for D_X if $V \subset D_X$ and V makes SNCs with D_X. The blow up $\Phi : X_1 \to X$ of a possible center is called a possible blow up. $D_{X_1} = \Phi^{-1}(D_X)$ is then a toroidal structure on X_1.

Recall that $f : X \to Y$ is *toroidal* (with respect to D_Y and D_X) if $f : (X, D_X) \to (Y, D_Y)$ is locally formally isomorphic to a morphism of toric varieties ([KKMS], [AK]).

The "toroidalization conjecture" of [AKMW] is:

Conjecture 1.1 *Suppose that $f : X \to Y$ is a dominant morphism of nonsingular varieties. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \Phi \downarrow & & \downarrow \Psi \\ X & \xrightarrow{f} & Y \end{array}$$

where Φ, Ψ are products of blow ups of nonsingular centers, and there exist SNC divisors D_{Y_1} on Y_1 and D_{X_1} on X_1 such that f_1 is toroidal with respect to D_{Y_1} and D_{X_1}.

A stronger version of this (which is also stated in [AKMW]) we will call the "strong toroidalization conjecture". It is stated as follows:

Conjecture 1.2 *Suppose that $f : X \to Y$ is a dominant morphism of nonsingular varieties. Further suppose that there is a SNC divisor D_Y on Y such that $D_X = f^{-1}(D_Y)$ is a SNC divisor on X which contains the singular locus of the map f. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \Phi \downarrow & & \downarrow \Psi \\ X & \xrightarrow{f} & Y \end{array}$$

where Φ, Ψ are products of possible blow ups for the preimages of D_X and D_Y respectively, and f_1 is toroidal with respect to $D_{Y_1} = \Psi^{-1}(D_Y)$ and $D_{X_1} = \Phi^{-1}(D_X)$.

The characteristic zero assumption on our base field k is necessary in these conjectures. The conjecture even fails in positive characteristic for morphisms of curves (where all blowups are trivial). A simple example is

$$t = x^p + x^{p+1}$$

over a field of characteristic p. We have that $t = x^p(1+x)$, but $(1+x)^{\frac{1}{p}} \notin k[[x]]$.

The case where there is an "easy" proof of the conjecture is when Y is a curve.

Theorem 1.3 *Suppose that $f : X \rightarrow Y$ is a morphism from an n-fold to a curve. Then f has a toroidalization*

$$\begin{array}{lll} X_1 & & \\ \Phi \downarrow & \searrow & \\ X & \stackrel{f}{\rightarrow} & Y. \end{array}$$

Proof. Let $D_Y = f(\mathrm{sing}(f))$. Embedded resolution of hypersurface singularities implies there exists a sequence of possible blow ups $\Phi : X_1 \rightarrow X$ such that $D_{X_1} := (f \circ \Phi)^{-1}(D_Y)$ is a SNC divisor on X_1. Suppose that $p \in D_{X_1}$ and $q = (f \circ \Phi)(p)$. Let $t_q \in \mathcal{O}_{Y,q}$ be a regular parameter. There exist permissible parameters $x_1, \dots, x_n$ in $\mathcal{O}_{X_1,p}$ such that

$$t_q = x_1^{a_1} \cdots x_l^{a_l} u$$

where $u \in \mathcal{O}_{X_1,p}$ is a unit. Set $\overline{x}_1 = x_1 u^{\frac{1}{a_1}} \in \hat{\mathcal{O}}_{X_1,p}$. Then

$$t_q = \overline{x}_1^{a_1} \cdots x_l^{a_l}.$$

From now on, we will suppose that $f : X \rightarrow Y$ is a morphism of nonsingular varieties with toroidal structures D_Y and $D_X = f^{-1}(D_Y)$ such that $\mathrm{sing}(f) \subset D_X$.

The character of the toroidalization problem is completely different when Y is not a curve. The essential problem is that we must then blow up above Y to toroidalize.

Example 1.4 *In general, if dim $Y \geq 2$, we must blow up above Y to toroidalize.*

Consider the morphism $f : X = \mathbf{A}^2 \rightarrow Y = \mathbf{A}^2$ with toroidal structures $D_Y = \{u = 0\}$ and $D_X = \{x = 0\}$, defined by

$$\begin{array}{c} u = x^a \\ v = P(x) + x^b y \end{array}$$

where P is a polynomial of degree less than b with zero constant term. Thus f is not toroidal. (To have a toroidal form, we would have to change variables to get permissible parameters related by a form $u = x^a, v = y$). Further, suppose that $\Phi : X_1 \to X$ is a sequence of blow ups of points, and $p \in X_1$ is a 1-point which maps to the origin of X. Then there are permissible parameters x_1, y_1 in $\hat{\mathcal{O}}_{X_1,p}$ such that

$$\begin{array}{c} x = x_1^m \\ y = \sum_{i=1}^r \alpha_i x_1^i + x_1^r y_1. \end{array}$$

Substituting into u and v, we find that

$$\begin{array}{c} u = x_1^{am} \\ v = P(x_1^m) + x_1^{mb}(\sum_{i=1}^r \alpha_i x_1^r) + x_1^{mb+r} y_1 \end{array}$$

which is not toroidal.

The cases where the (strong) toroidalization conjecture is known to be true are:

1. $\dim(Y) = 1$, $\dim(X)$ arbitrary.
2. $\dim(X) = \dim(Y) = 2$ [AkK], [CP1], [AKMW], [Mat].
3. Local monomialization (locally along a possibly non Noetherian valuation) [C1], [C2], [C5]. The full theorem is stated in Theorem 6.1 of these lectures. From 3, we infer the following theorem:

Theorem 1.5([C1],[C2], [C5]) *Suppose that $f : X \to Y$ is a dominant morphism of proper varieties. Then there exists a commutative diagram*

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\to} & Y_1 \\ \Phi \downarrow & & \downarrow \Psi \\ X & \stackrel{f}{\to} & Y. \end{array}$$

such that f_1 is toroidal, and Φ, Ψ are locally products of blow ups of nonsingular centers. The morphisms Φ, Ψ and f_1 satisfy existence of the valuative criterion for properness, but in general uniqueness fails (so that these maps are in general not separated).

From Case 3 we also obtain the proof of "Local strong factorization" (conjectured by Abhyankar [Ab4]). Case 3 reduces the proof of the conjecture to the case of a toroidal mapping and a "toroidal" valuation. This is proven in dimension 3, by Christensen [Ch], and extended to arbitrary dimension by Karu [K]. A proof in the style of Christensen's original proof (using determinants and elementary linear algebra) is given in [CS].

Local monomialization along a valuation could possibly be true in positive characteristic. It is certainly true for morphisms of curves, and for morphisms of n-folds to curves in dimensions where resolution of singularities is true. Good progress on this problem is made for morphisms of surfaces in [CP2].

4. $\dim(X)=3$, $\dim(Y)=2$ [C3] (strong toroidalization)
5. $\dim(X)=\dim(Y)=3$, f birational [C6] (toroidalization), [C7] (strong toroidalization)

   From 5, we reduce the "strong factorization" conjecture for birational morphisms of proper 3-folds to the case of toroidal morphisms, so we see that "strong factorization" of birational morphisms of proper 3-folds will follow from the Oda conjecture on "strong factorization" of torodial varieties [O].

We further obtain a new proof of "weak factorization" of birational morphisms of proper 3-folds. Case 5 reduces "weak factorization" to the case of toroidal morphisms, which is solved in [D2] (dimension 3), [Mo], [W1], [AMR1], [AMR2] (arbitrary dimension). "Weak factorization" is proven in all dimensions (using geometric invariant theory) in [W2], [AKMW], [W3].

2. Toroidalization of morphisms of surfaces. In this section, we will suppose that $f:X\rightarrow Y$ is a dominant morphism of nonsingular surfaces with toroidal structures D_Y and $D_X=f^{-1}(D_Y)$ such that $\mathrm{sing}(f)\subset D_X$.

The proof that we give is from [AkK].

Suppose that $p\in D_X$. The following are the possible toroidal forms for f at p. Let $q=f(p)$. There exist permissible parameters u,v in $\mathcal{O}_{Y,q}$ and x,y in $\hat{\mathcal{O}}_{X,p}$ such that one of the following forms hold.

q a 1-point and p a 1-point

$$u=x^a, v=y.$$

q a 2-point and p a 1-point

$$u=x^a, v=x^b(\alpha+y)$$

with $0\neq\alpha\in k$.

q a 2-point and p a 2-point

$$u=x^ay^b, v=x^cy^d$$

with $ad-bc\neq 0$.

We will say that f is strongly prepared at p if one of the following forms hold.

q a 1-point

$$u = x^a, v = P(x) + x^b y \tag{1}$$

or q a 2-point

$$u = (x^a y^b)^t, v = P(x^a y^b) + x^c y^d \tag{2}$$

with $ad - bc \neq 0$, $\gcd(a, b) = 1$.

Theorem 2.1 *f is strongly prepared.*

Proof. Suppose that $p \in D_X$. Let u, v be permissible parameters at $q = f(p)$, $x, y \in \hat{\mathcal{O}}_{X,p}$ be permissible parameters such that

$$u = x^a$$

if p is a 1-point, and

$$u = (x^a y^b)^t$$

with $\gcd(a, b) = 1$ if p is a 2-point.

If p is a 1-point, there exists a unit $\delta \in \hat{\mathcal{O}}_{X,p}$ such that

$$u_x v_y - u_y v_x = \delta x^e.$$

If p is a 2-point, there exists a unit $\delta \in \hat{\mathcal{O}}_{X,p}$ such that

$$u_x v_y - u_y v_x = \delta x^e y^f.$$

Expand v as a series $v = \sum a_{ij} x^i y^j$ with $a_{ij} \in k$.

If p is a 1-point, then $a x^{a-1} v_y = \delta x^e$, from which it follows that (1) holds and f is strongly prepared at p.

Suppose that p is a 2-point. Then

$$\begin{aligned} u_x v_y - u_y v_x &= \sum t(aj - bi) a_{ij} x^{at+i-1} y^{bt+j-1} \\ &= \delta x^e y^f. \end{aligned}$$

This implies that

$$v = \sum_{aj - bi = 0} a_{ij} x^i y^j + \epsilon x^{e+1-at} y^{f+1-bt}$$

where ϵ is a unit series. It follows that (2) holds and f is strongly prepared at p.

The final step of the proof of toroidalization of morphisms of surfaces is the following theorem:

Theorem 2.2 *There exists a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\rightarrow} & Y_1 \\ \downarrow & & \downarrow \\ X & \stackrel{f}{\rightarrow} & Y \end{array}$$

such that the vertical arrows are products of (possible) blow ups of points and f_1 is toroidal.

For E a component of D_X, $p \in E$ a 1-point, define

$$A(f,p) = \min\{b - \mathrm{ord}_x(v)\} = A(f,E)$$

where the minimum is over permissible parameters u, v at $q = f(p)$. Here $\mathrm{ord}_x(v)$ is the largest power of x which divides v in $\hat{\mathcal{O}}_{X,p}$. If $A(f,p) > 0$, define

$$C(f,p) = \min\{(b - \mathrm{ord}_x(v), \mathrm{ord}_x(v) + a)\} = C(f,E)$$

where the minimum is in the lex order over permissible parameters u, v at $q = f(p)$.

Now define

$$A(f) = \max\{A(f,E) \mid E \text{ is a component of } D_X\}.$$

If $A(f) > 0$, define

$$C(f) = \max\{C(f,E) \mid E \text{ is a component of } D_X \text{ such that } A(f,E) > 0\}.$$

Lemma 2.3 *Let $\Psi_1 : Y_1 \to Y$ be the blow up of $q \in D_Y$. Let $f_1 : X \to Y_1$ be the induced rational map.*

1. Suppose that $p \in f^{-1}(q)$ is a 1-point and f_1 is a morphism at p. Then

$$A(f_1,p) \le A(f,p).$$

If $A(f_1,p) = A(f,p) > 0$, then $(C, f_1, p) < C(f,p)$.

2. Suppose that $p \in X$ is such that f_1 is not a morphism at p. Let $\Psi_1 : X_1 \to X$ be the blow up of p and $E = \Psi_1^{-1}(p)$. Let $\overline{f}_1 = f \circ \Psi_1 : X_1 \to Y$. Then

$$A(\overline{f}_1, E) = 0 \text{ or } A(\overline{f}_1, E) < A(f).$$

By iteration of Lemma 2.3, we reduce to $A(f) = 0$ in the proof of Theorem 2.2.

We now list the prepared forms with $A(f) = 0$, with respect to suitable permissible parameters x, y for $p \in D_X$ and u, v for $q = f(p)$.

q a 1-point and p a 1-point:

$$u = x^a, v = x^b(\alpha + y)$$

with $\alpha \in k$.

q a 1-point and p a 2-point:

$$u = x^a y^b, v = x^c y^d$$

with $ad - bc \neq 0$.

q a 2-point and p a 1-point:

$$u = x^a, v = x^b(\alpha + y)$$

with $0 \neq \alpha \in k$.

q is a 2-point and p is a 2-point:

$$u = x^a y^b, v = x^c y^d$$

with $ad - bc \neq 0$.

If E is a component of D_X and $p \in E$ is a 1-point, define

$$I(f, p) = b - a = I(f, E).$$

Define

$$I(f) = \max\{I(f, E) \mid E \text{ is a component of } D_X\}.$$

Lemma 2.4 *Suppose that $A(f) = 0$. Let $\Psi_1 : Y_1 \to Y$ be the blow up of a 1-point $q \in D_Y$. Let $f_1 : X \to Y_1$ be the induced rational map.*

1. *Suppose that $p \in f^{-1}(q)$ is a 1-point, f_1 is a morphism at p and $f_1(p)$ is a 1-point. Then:*
   a. *If $I(f, p) > 0$ then $I(f_1, p) < I(f, p)$.*
   b. *If $I(f, p) \leq 0$ then f_1 is toroidal at p.*

2. Suppose that $p \in f^{-1}(q)$ is a 1-point and f_1 is not a morphism at p. Let $\Psi_1; X_1 \to X$ be the blow up of p and $E = \Psi_1^{-1}(p)$. Let $\overline{f}_1 = f \circ \Psi_1; X_1 \to Y$. Then

$$I(f,p) < I(\overline{f}_1, E) \leq 0.$$

Now by successive application of Lemma 2.4, we reduce to the case $A(f) = 0$ and $I(f) \leq 0$. Finally, by further application of Lemma 2.4, we prove Theorem 2.2.

3. Toroidalization of morphisms from 3-folds to surfaces. In this lecture we maintain our assumption that $f : X \to Y$ is a dominant morphism of nonsingular varieties with toroidal structures defined by SNC divisors D_Y and $D_X = f^{-1}(D_Y)$ such that $\mathrm{sing}(f) \subset D_X$.

We restrict to the case where $\dim(Y) = 2$. Initially, we allow $n = \dim(X)$ to be arbitrary.

With these assumptions, we say that f is *strongly prepared* at $p \in D_X$ if there exist permissible parameters u, v in $\mathcal{O}_{Y,q}$ (where $q = f(p)$) and $x_1, \ldots, x_n$ in $\hat{\mathcal{O}}_{X,p}$ such that $x_1 \cdots x_l = 0$ is a local equation of D_X at p, and one of the following forms hold:

1. $u = 0$ is a local equation of D_X,

$$u = (x_1^{a_1} \cdots x_l^{a_l})^m, v = P(x_1^{a_1} \cdots x_l^{a_l}) + x_1^{b_1} \cdots x_l^{b_l}$$

or

2. $u = 0$ is a local equation of D_X,

$$u = (x_1^{a_1} \cdots x_l^{a_l})^m, v = P(x_1^{a_1} \cdots x_l^{a_l}) + x_1^{b_1} \cdots x_l^{b_l} x_{l+1}$$

or

3. $uv = 0$ is a local equation of D_X,

$$u = x_1^{a_1} \cdots x_l^{a_{l-1}}, v = x_2^{b_2} \cdots x_l^{b_l}.$$

In all of these cases $\gcd(a_1, \ldots, a_l) = 0$ and P is a series. In Case 1 we have

$$\mathrm{rank} \begin{pmatrix} a_1 & \cdots & a_l \\ b_1, & \cdots & b_l \end{pmatrix} = 2.$$

Recall that in the case when $n = \dim(X) = 2$ (and $\dim(Y) = 2$), Theorem 2.1 implies that f is strongly prepared. However, the situation is much more complicated if X has higher dimension.

We will say that f is *prepared* if conditions 1 or 2 above in the definition of strongly prepared always hold.

Example 3.1 *In general, if $n = dim(X) \geq 3$, then f is not strongly prepared.*

Define a germ of a morphism $f : X \rightarrow Y$ from a 3-fold to a surface by

$$u = x^a, v = x^c F$$

where $a \geq 2$, $c \geq 0$, $r \geq 1$ and

$$F = x^r z + h(x, y)$$

where $h(x, y)$ is an arbitrary series with $h(0, 0) = 0$. Define toroidal structures $D_Y = \{u = 0\}$, $D_X = \{x = 0\}$. We compute the Jacobian matrix

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} = \begin{pmatrix} ax^{a-1} & 0 & 0 \\ (c+r)x^{c+r-1}z + \frac{\partial x^c h}{\partial x} & x^c \frac{\partial h}{\partial y} & x^{c+r} \end{pmatrix}.$$

We see that the ideal of sing(f), which is obtained from the 2×2 minors of $J(f)$, is

$$\sqrt{I_2(J(f))} = \sqrt{(x^{a+c-1}\frac{\partial h}{\partial y}, x^{a+c+r-1})} = (x).$$

Thus sing$(f) \subset D_X = f^{-1}(D_Y)$.

Theorem 2.2 for morphisms of surfaces does generalize, but the proof is much harder.

Theorem 3.2 ([C3] if $\dim(X) = 3$, [CK] for arbitrary dimension) *Suppose that Y is a surface, and $f : X \rightarrow Y$ is strongly prepared. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\rightarrow} & Y_1 \\ \Phi_1 \downarrow & & \downarrow \Psi_1 \\ X & \stackrel{f}{\rightarrow} & Y \end{array}$$

such that Φ_1 and Ψ_1 are products of possible blow ups, f_1 is toroidal.

At least in the case when X is a 3-fold, it is possible to construct a prepared morphism.

Theorem 3.3 ([C3]) *Suppose that X is a 3-fold and Y is a surface. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1 & f_1 & \\ \Phi_1 \downarrow & \searrow & \\ X & \stackrel{f}{\rightarrow} & Y \end{array}$$

such that Φ_1 is a product of possible blow ups and f_1 is prepared.

Now, since prepared implies strongly prepared, Theorems 3.2 and 3.3 immediately imply

Theorem 3.4 ([C3]) *Strong toroidalization is true for dominant morphisms of 3-folds to surfaces*

Comments on the proof of Theorem 3.3

Suppose that $p \in D_X$, $q = f(p) \in D_Y$. Then we can see from the Jacobian matrix of f that there are permissible parameters u, v in $\mathcal{O}_{Y,q}$, x, y, z in $\hat{\mathcal{O}}_{X,q}$ and a series P such that one of the following forms hold.

1. p is a 1-point
$$u = x^a, v = P(x) + x^b F_p$$
where $x \not| F_p$, F_p has no terms which are monomials in x.
2. p is a 2-point
$$u = (x^a y^b)^m, v = P(x^a y^b) + x^c y^d F_p$$
where $\gcd(a, b) = 1$, $x \not| F_p$, $y \not| F_p$, and $x^c y^d F_p$ has no terms which are monomials in $x^a y^b$.
3. p is a 3-point
$$u = (x^a y^b z^c)^m, v = P(x^a y^b z^c) + x^d y^e z^f F_p$$
where $\gcd(a, b, c) = 1$, $x \not| F_p$, $y \not| F_p$, $z \not| F_p$, and $x^d y^e z^f F_p$ has no terms which are monomials in $x^a y^b z^c$.

For $p \in D_X$, define
$$\nu(p) = \mathrm{mult}(F_p).$$
It can be shown that $\nu(p)$ is an invariant of p. Set $S_r(X) = \{p \in D_X \mid \nu(p) \geq r\}$.

In Example 3.1, write
$$h(x, y) = h_0(x) + y x^m h_1(x, y)$$
where $x \not| h_1$. Suppose that $m < r$. Then we have
$$\begin{aligned} u &= x^a \\ v &= x^c h_0(x) + x^{c+m}(y h_1(x, y) + x^{r-m} z) \\ &= P(x) + x^{c+m} F_p. \end{aligned}$$

Thus

$$\nu(p) = \min\{r - m + 1, 1 + \mathrm{ord}(h_1)\}.$$

Let

$$S_r(X) = \{p \in D_X \mid \nu(p) \geq r\}.$$

$S_r(X)$ is constructible but may not be Zariski closed, as is shown in the following example.

Example 3.5 *In general, $S_r(X)$ is not Zariski closed.*

Define a germ of a morphism $f : X \to Y$ from a 3-fold to a surface at a point $p \in X$ by

$$u = xy, v = x^2 y.$$

Define toroidal structures $D_Y = \{uv = 0\}$, $D_X = \{xy = 0\}$. We have $\nu(p) = 0$.

At 1-points p_1 on the surface $x = 0$, there are regular parameters x, y_1, z with $y = y_1 + \alpha$ for some $0 \neq \alpha \in k$. Set $\overline{x} = x(y_1 + \alpha)$. We then have permissible parameters $\overline{x}, \overline{y}_1, z$ at p_1 such that

$$\begin{array}{ll} u & = \overline{x} \\ v & = \overline{x}^2 (y_1 + \alpha)^{-1} \\ & = \alpha^{-1}\overline{x}^2 + \overline{x}^2 \overline{y}_1. \end{array}$$

Thus $\nu(p_1) = 1$.

Other important invariants in the proof are $\gamma(p)$ and $\tau(p)$. $\gamma(p)$ is defined by

$$\gamma(p) = \begin{cases} \text{mult } F_p(0, y, z) & \text{if } p \text{ is a 1-point} \\ \text{mult } F_p(0, 0, z) & \text{if } p \text{ is a 2-point} \end{cases}$$

Suppose that $p \in X$ is a 1-point. We have an expression

$$\begin{array}{ll} u & = x^a \\ v & = P(x^a) + x^b F_p \\ F_p & = \sum_{i+j+k \geq r} a_{ijk} x^i y^j z^k \end{array}$$

at p, where $\nu(p) = r$. Define

$$\tau(p) = \max\{j + k \mid \text{ there exists } a_{ijk} \neq 0 \text{ with } i + j + k = r\}.$$

Suppose that $p \in X$ is a 2-point. We have an expression

$$\begin{array}{ll} u & = (x^a y^b)^m \\ v & = P(x^a y^b) + x^c y^d F_p \\ F_p & = \sum_{i+j+k \geq r} a_{ijk} x^i y^j z^k \end{array}$$

at p, where $\nu(p) = r$. Define

$$\tau(p) = \max\{k \mid \text{ there exists } a_{ijk} \neq 0 \text{ with } i + j + k = r\}.$$

τ measures the independence of the leading form of F_p from local equations of D_X.

Lemma 3.6 *f is prepared if and only if*

1. *If p is a 1-point then $\nu(p) \le 1$.*
2. *If p is a 2-point then $\gamma(p) \le 1$.*
3. *If p is a 3-point then $\nu(p) = 0$.*

We see that Example 3.5 is prepared. It satisfies 1 and 2 of Lemma 3.6.

Lemma 3.7 *Suppose that $\nu(p) = r$, $\Phi : X_1 \to X$ is the blow up of p and $p_1 \in \Phi^{-1}(p)$. Then*

$$\nu_{f\circ\Phi}(p_1) \le r + 1.$$

If $\nu_{f\circ\Phi}(p_1) = r + 1$ then there are restrictions on γ and τ.

Example 3.8 *ν can increase after blowing up a point.*

Recall the morphism of Example 3.5,

$$u = xy, v = x^2y,$$

with toroidal structures $D_Y = \{uv = 0\}$, $D_X = \{xy = 0\}$. We have $\nu(p) = 0$. Extend x, y to permissible parameters x, y, z at p. Let $\Phi : X_1 \to X$ be the blow up of p. Suppose that $p_1 \in \Phi^{-1}(p)$ is a 1-point with regular parameters x_1, y_1, z_1 defined by

$$x = x_1, y = x_1(y_1 + \alpha), z = x_1z_1$$

with $0 \neq \alpha \in k$. Then

$$u = x_1^2(y_1 + \alpha), v = x_1^3(y_1 + \alpha).$$

Set

$$\overline{x}_1 = x_1(y_1 + \alpha)^{\frac{1}{2}}, \overline{y}_1 = (y_1 + \alpha)^{-\frac{1}{2}} - \alpha^{-\frac{1}{2}}.$$

Then $\overline{x}_1, \overline{y}_1, z_1$ are permissible parameters at p_1, with

$$u = \overline{x}_1^2, v = \alpha^{-\frac{1}{2}}\overline{x}_1^3 + \overline{x}_1^3\overline{y}_1.$$

Thus $\nu(p_1) = 1$.

Lemma 3.9 *Suppose that $C \subset \overline{S}_r(X)$ (= Zariski closure of $S_r(X)$) is a nonsingular curve which makes SNCs with D_X. Then either*

1. *C is r-big:*

$$F_p \in \hat{\mathcal{I}}^r_{C,p} \text{ for all } p \in C$$

or

2. *C is r-small:*

$$F_p \in \hat{\mathcal{I}}^{r-1}_{C,p} - \hat{\mathcal{I}}^r_{C,p} \text{ for all } p \in C.$$

The invariants ν, γ and τ behave reasonably well under possible blow ups of such curves.

Definition 3.10 *Suppose that $r \geq 2$. $\overline{A}_r(X)$ holds if*

1. *$\nu(p) \leq r$ if $p \in X$ is a 1-point or a 2-point.*
2. *If $p \in X$ is a 1-point and $\nu(p) = r$ then $\gamma(p) = r$.*
3. *If $p \in X$ is a 2-point and $\nu(p) = r$, then $\tau(p) > 0$.*
4. *$\nu(p) \leq r - 1$ if $p \in X$ is a 3-point.*

$\overline{A}_r(X)$ is stable under blow ups of points. The proof of Theorem 3.3 is by descending induction on r. $\overline{A}_1(X)$ holds if and only if f is prepared (by Lemma 3.6). We must blow up points and curves which are r-big and r-small. There are a couple of stubborn cases which require blow ups of appropriately general curves.

4. Toroidalization of morphisms from 3-folds to 3-folds.

In this lecture we maintain our assumption that $f : X \to Y$ is a dominant morphism of nonsingular varieties with toroidal structures defined by SNC divisors D_Y and $D_X = f^{-1}(D_Y)$ such that $\text{sing}(f) \subset D_X$.

We restrict to the case where $\dim(X) = \dim(Y) = 3$.

We say that f is *prepared* (for D_Y and D_X) if for all $q \in D_Y$ and $p \in f^{-1}(q)$ there exist permissible parameters u, v, w in $\mathcal{O}_{Y,q}$ and x, y, z in $\hat{\mathcal{O}}_{X,p}$ such that u, v have a toroidal form in terms of x, y, z (or a related condition holds).

Suppose that $f : X \to Y$ is prepared. Then D_X is *cuspidal for f* if f is toroidal in a neighborhood of all components of D_X which do not contain a 3-point, and in a neighborhood of all 2-curves of D_X which do not contain a 3-point.

Theorem 4.1([C7]) *Suppose that dim(X) = dim(Y) = 3. Then there exists a commutative diagram of morphisms*

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\to} & Y_1 \\ \Phi\downarrow & & \downarrow\Psi \\ X & \stackrel{f}{\to} & Y \end{array}$$

such that Φ and Ψ are products of possible blow ups, f_1 is prepared for $D_{Y_1} = \Phi^{-1}(D_Y)$ and $D_{X_1} = \Psi^{-1}(D_X)$, and D_{X_1} is cuspidal for f_1.

The significance of the prepared condition is that now we can read off of the Jacobian matrix of f nice local forms for f. Suppose that f is prepared, $p \in D_X$, $q = f(p)$, and u, v, w are permissible parameters at p such that u, v are toroidal forms at p. Then there are permissible parameters x, y, z in $\hat{\mathcal{O}}_{X,p}$ such that one of the following cases hold:

1. q is a 2-point or a 3-point, p is a 1-point and

$$u = x^a, v = x^b(\alpha + y), w = g(x, y) + x^c z$$

where $0 \neq \alpha \in k$ and g is a series.

2. q is 2-point or a 3-point, p is a 2-point and

$$u = x^a y^b, v = x^c y^d, w = g(x, y) + x^e y^f z$$

where $\mathrm{rank}((a, b), (c, d)) = 2$ and g is a series.

3. q is a 2-point or a 3-point, p is a 2-point and

$$u = (x^a y^b)^k, v = (x^a y^b)^t(\alpha + z), w = g(x^a y^b, z) + x^c y^d$$

where $0 \neq \alpha \in k$, $a, b, k, t > 0$, $\gcd(a, b) = 1$, g is a series and $\mathrm{rank}((a, b), (c, d)) = 2$.

4. q is a 2-point or a 3-point, p is a 3-point and

$$u = x^a y^b z^c, v = x^d y^e z^f, w = g(x, y, z) + N$$

where $\mathrm{rank}((a, b, c), (d, e, f)) = 2$, g is a series in monomials $M = x^{\overline{a}} y^{\overline{b}} z^{\overline{c}}$ in x, y, z such that $\mathrm{rank}((a, b, c), (d, e, f), (\overline{a}, \overline{b}, \overline{c})) = 2$, and $N = x^{a'} y^{b'} z^{c'}$ is such that

$$\mathrm{rank}((a, b, c), (d, e, f), (a', b', c')) = 3.$$

5. q is a 1-point, p is a 1-point and

$$u = x^a, v = y, w = g(x, y) + x^c z$$

where g is a series.

6. q is a 1-point, p is a 2-point and

$$u = (x^a y^b)^k, v = z, w = g(x^a y^b, z) + x^c y^d$$

with $a, b, k > 0$, $\gcd(a, b) = 1$, g is a series and $\mathrm{rank}((a, b), (c, d)) = 2$.

At first sight, the prepared forms for morphisms of 3-folds appear to be similar to those of prepared forms of morphisms of n-folds to surfaces, which we are able to toroidalize in Theorem 3.3. However, a little experimentation shows that the situation when the base Y is a 3-fold is much more complex. The essential problem is that prepared forms are not stable under possible blow ups above Y when Y is a 3-fold.

However, we are able to accomplish toroidalization in the case when f is birational.

Theorem 4.2 ([C6], toroidalization; [C7], strong toroidalization) *Strong toroidalization is true for birational morphisms $f : X \to Y$ of 3-folds.*

Outline of proof of Theorem 4.2

By Theorem 4.1, we may assume that f is prepared, and D_X is cuspidal for f. These conditions are preserved throughout the construction.

We define the τ-invariant of a 3-point $p \in X$. Since f is prepared, $f(p) = q$ is a 2-point or a 3-point. There are permissible parameters u, v, w in $\mathcal{O}_{Y,q}$ and x, y, z in $\hat{\mathcal{O}}_{X,p}$ such that $xyz = 0$ is a local equation of D_X, $uv = 0$ or $uvw = 0$ is a local equation of D_Y and there is an expression

$$\begin{aligned} u &= x^a y^b z^c \\ v &= x^d y^e z^f \\ w &= \textstyle\sum_{i \geq 0} \alpha_i M_i + N \end{aligned} \tag{3}$$

with $\alpha_i \in k$, $M_i = x^{a_i} y^{b_i} z^{c_i}$, $N = x^g y^h z^i$,

$$\mathrm{rank} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = 2, \det \begin{pmatrix} a & b & c \\ d & e & f \\ a_i & b_i & c_i \end{pmatrix} = 0 \text{ for all } i,$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.$$

If q is a 3-point, then

$$w = \text{unit series } N$$

if and only if f is toroidal at p. In this case define $\tau_f(p) = -\infty$.

Otherwise, define

$$H_p = \mathbf{Z}(a,b,c) + \mathbf{Z}(d,e,f) + \sum_i \mathbf{Z}(a_i,b_i,c_i),$$

$$A_p = \begin{cases} \mathbf{Z}(a,b,c) + \mathbf{Z}(d,e,f) + \mathbf{Z}(a_0,b_0,c_0) & \text{if } q \text{ is a 3-point} \\ & \text{(we have } w = \text{(unit series)}M_0) \\ \mathbf{Z}(a,b,c) + \mathbf{Z}(d,e,f) & \text{if } q \text{ is a 2-point.} \end{cases}$$

Now define

$$\tau_f(p) = |H_p/A_p|.$$

We define

$$\tau_f(X) = \max\{\tau_f(p) \mid p \in X \text{ is a 3-point}\}.$$

Theorem 4.3 *Suppose that f is prepared for D_Y and $D_X = f^{-1}(D_Y)$, and D_X is cuspidal for f. Further suppose that $\tau_f(X) = -\infty$. Then f is toroidal.*

We have that $\tau_f(X) \geq 0$ or $\tau_f(X) = -\infty$. The proof of Theorem 4.2 is by descending induction on $\tau_f(X)$. In our proof of Theorem 4.2 we may thus assume that $\tau = \tau_f(X) \neq -\infty$ (so that $\tau \geq 0$).

Step 1. There exist sequences of blow ups of 2-curves

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\rightarrow} & Y_1 \\ \downarrow & & \downarrow \\ X & \stackrel{f}{\rightarrow} & Y \end{array}$$

such that f_1 is prepared, D_{X_1} is cuspidal for f_1, $\tau_f(X_1) \leq \tau$, and $\tau_{f_1}(p) = \tau$ implies that $f_1(p)$ is a 2-point. We use the concept of 3-point relation in this step.

Step 2. In this step we construct a commutative diagram of morphisms

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\rightarrow} & Y_1 \\ \Phi\downarrow & & \downarrow\Psi \\ X & \stackrel{f}{\rightarrow} & Y \end{array}$$

such that

1. Φ and Ψ are products of possible blow ups.
2. $\tau(X_1) = \tau$, and if $p \in X$ is a 3-point such that $\tau_f(p) = \tau$ then $f_1(p)$ is a 2-point.

3. D_{X_1} is cuspidal for f_1.
4. f_1 is τ-very-well prepared.

Step 2 is the most difficult step technically.

We will not give the complete definition of τ-very-well prepared, which uses the concept of 2-point relation, and requires the preliminary definitions of quasi-well prepared and well prepared.

By virtue of the result of this step, we can assume that f is τ-very-well prepared. We will also assume that $\tau > 0$. The case when $\tau = 0$ is actually a little easier, but the definition is a bit different.

We now summarize some of the properties of a τ-very-well-prepared morphism.

There exists a finite, distinguished set of nonsingular algebraic surfaces $\Omega(\overline{R}_i)$ in Y, with a SNC divisor F_i on $\Omega(\overline{R}_i)$ such that the intersection graph of F_i is a tree.

Suppose that $p \in X$ is a 3-point with $\tau_f(p) = \tau$ (so that $q = f(p)$ is 2-point). Then the following conditions hold.

1. The expression (3) has the form

$$w = \gamma M_0 \tag{4}$$

   where γ is a unit series, $M_0^e = u^{\overline{a}}v^{\overline{b}}$, with $\overline{a}, \overline{b}, e \in \mathbf{Z}$, $e > 1$, and $\gcd(a, b, e) = 1$. Observe that we cannot have both $\overline{a} < 0$ and $\overline{b} < 0$, since M_0, u, v are all monomials in x, y, z.
2. Suppose that V is the curve in Y with local equations $u = w = 0$ (or $v = w = 0$) at q. Then V is a possible center for D_Y and there exists a commutative diagram of morphisms

$$\begin{array}{ccc} X_1 & \stackrel{f_1}{\rightarrow} & Y_1 \\ \Phi_1 \downarrow & & \downarrow \Psi_1 \\ X & \stackrel{f}{\rightarrow} & Y \end{array} \tag{5}$$

   where Ψ_1 is the blow up of V (possibly followed by blow ups of some special 2-points), such that f_1 and $\overline{f} = \Psi_1 \circ f_1 : X_1 \rightarrow Y$ are prepared, $\tau_{f_1}(X_1) \leq \tau$ and Φ_1 is toroidal at 3-points $p_1 \in \Phi_1^{-1}(p)$. Further, f_1 is τ-very-well prepared.
3. There exists a surface $\Omega(\overline{R}_i)$ such that
   a. $f(p) = q \in \Omega(\overline{R}_i)$.
   b. The w of (4) gives a local equation $w = 0$ of $\Omega(\overline{R}_i)$ at q.
   c. $uv = 0$ is a local equation of F_i (on the surface $\Omega(\overline{R}_i)$) at q.

The necessity of several different surfaces $\Omega(\overline{R}_i)$ arises because of the possibility that there may be several 3-points p_j with $\tau_f(p_j) = \tau$ which map to q, and require different w in their expressions (4). We require that the surfaces $\Omega(\overline{R}_i)$ intersect in a controlled way.

The first step in the construction of a τ-very well prepared morphism is the construction of a morphism such that for all 3-points p with $\tau_f(p) = \tau$, an expression (4) holds for some possibly formal w.

Step 3. We construct a commutative diagram

$$\begin{array}{ccc} X_n & \stackrel{f_n}{\to} & Y_n \\ \downarrow & & \downarrow \\ X & \stackrel{f}{\to} & Y \end{array}$$

such that $\tau_{f_n}(X_n) < \tau$. By induction on τ, we then obtain the proof of Theorem 4.2.

We fix an index i of the surfaces $\Omega(\overline{R}_i)$. A curve E on Y is good if it is a component of F_i, and if j is such that $E \cap \Omega(\overline{R}_j) \neq \emptyset$, then E is a component of F_j.

In our construction we begin with $i = 1$, and blow up a good curve V on Y, by a morphism (5). Part of the definition of τ-very-well prepared implies the existence of a good curve. Suppose that $p \in X$ is a 3-point with $\tau_f(X) = \tau$ and $q = f(p) \in V$. Suppose that $p_1 \in \Phi_1^{-1}(p)$ is a 3-point. Set $q_1 = f_1(p_1)$. If V has local equations $u = w = 0$ at q, then q_1 has regular parameters u_1, v, w_1 with

$$u = u_1 w_1, w = w_1 \tag{6}$$

or

$$u = u_1, w = u_1(w_1 + \alpha) \tag{7}$$

and $\alpha \in k$.

If (6) holds then q_1 is a 3-point. Since $e > 1$, we have

$$\tau_{f_1}(p_1) = |H_p/A_p + M_0\mathbf{Z}| < |H_p/A_p| = \tau.$$

If (7) holds, then $e > 1$ implies $\alpha = 0$. Thus f_1 has the form (3), (4) at p_1 with $(\overline{a}, \overline{b}, e)$ changed to $(\overline{a} - e, \overline{b}, e)$. If V has local equations $v = w = 0$, then f_1 has the form (3), (4) at p_1 with $(\overline{a}, \overline{b}, e)$ changed to $(\overline{a}, \overline{b} - e, e)$.

We have SNC divisors $\Phi_1^{-1}(F_i)$ on the surfaces $\Phi_1^{-1}(\Omega(\overline{R}_i)))$. If there are no 3-points p_1 in X_1 satisfying 1, 2 and 3 of Step 2 for $\Phi_1^{-1}(\Omega(\overline{R}_1))$, then we increase i to 2.

Otherwise, there exists a good curve on Y_1 for the SNC divisor $\Phi^{-1}(F_1)$ on the surface $\Phi_1^{-1}(\Omega(\overline{R}_1)))$. We continue to iterate, blowing up good curves. If we always have a 3-point satisfying 1, 2 and 3 for the preimage of $\Omega(\overline{R}_1)$, then we eventually obtain a form (4) with both $\overline{a} < 0$ and $\overline{b} < 0$ which is impossible.

We then continue this algorithm for the preimages of all of the surfaces $\Omega(\overline{R}_i)$. The algorithm terminates in the construction of a morphism with a drop in τ as desired.

Open problems. We conclude this section with a list of open problems on toroidalization.

1. Prove (strong) toroidalization for arbitrary dominant morphisms of 3-folds.

   By Theorem 4.1 we can assume that f is prepared. Much of the proof of Theorem 4.2 works in the case when f is not birational.
2. Suppose that $f : X \rightarrow Y$ is a dominant morphism from an n-fold to a surface Y. Prove that there exists a commutative diagram

$$\begin{array}{ccc} X_1 & f_1 & \\ \Phi_1 \downarrow & \searrow & \\ X & \stackrel{f}{\rightarrow} & Y \end{array}$$

   such that Φ_1 is a product of possible blow ups and f_1 is (strongly) prepared. Recall that Theorem 3.2 now implies that f can be toroidalized.
3. Prove the toroidalization conjecture in all dimensions.

5. Valuations in Algebraic Geometry. Suppose that K is an algebraic function field of dimension (transcendence degree n) over a ground field k. A valuation of K is a homomorphism

$$\nu : K^{\times} \rightarrow \Gamma_{\nu}$$

of the multiplicative group of K onto a totally ordered abelian group, such that $\nu(a) = 0$ if $a \in k^{\times}$. We formally extend ν to K by defining $\nu(0) = \infty$. Associated to a valuation ν we have a valuation ring

$$V_{\nu} = \{f \in K \mid \nu(f) \geq 0\}.$$

V_{ν} is a local ring with maximal ideal $m_{\nu} = \{f \in K \mid \nu(f) > 0\}$. V_{ν} contains k.

If $A \subset B$ are local rings with respective maximal ideals m_A and m_B, we say that B dominates A if $m_B \cap A = m_A$.

The connection of valuation theory to algebraic geometry is explained by the following lemma.

Lemma 5.1 *Suppose that X is a projective variety with function field $K = k(X)$, and ν is a valuation of k. Then there exists a unique point $p \in X$ such that V_ν dominates the local ring $\mathcal{O}_{X,p}$.*

This lemma is of course true (by definition) on any proper k-variety. The point p (which may not be closed) is called the *center* of ν on X.

We define a locally ringed space Σ_K, which we call the Zariski-Riemann manifold of K. The points of Σ_K are the valuation rings of K. We define a topology on Σ_K by taking basic open sets to be

$$\overline{U} = \{V \in \Sigma_K \mid V \text{ dominates } \mathcal{O}_{X,p} \text{ for some } p \in U\},$$

where U is an open subset of a proper k-variety X, with function field $k(X) = K$.

The local ring $\mathcal{O}_{\Sigma_K,p}$ of a point $p = V_p \in \Sigma_K$ is the valuation ring V_p. For $\overline{U} \subset \Sigma_K$ an open set, we define

$$\Gamma(\overline{U}, \mathcal{O}_{\Sigma_K}) = \cap_{V_p \in \overline{U}} V_p.$$

Theorem 5.2 *Suppose that X is a proper variety with function field K. Then the mapping $\pi_X : \Sigma_K \to X$ defined by $\pi_X(V) = p$ if V dominates $\mathcal{O}_{X,p}$ is continuous.*

If K has dimension 1, then the valuation rings of K are local Dedekind domains which are essentially of finite type over k. Thus, when K has dimension 1, Σ_K is the (unique) nonsingular projective curve with function field K.

However, if K has dimension ≥ 2, then there are valuation rings V of K which are not Noetherian. Still, Σ_K is in fact always quasi-compact ([Z2], [ZS]).

There are three main invariants associated to a valuation ring V ([Z1], [ZS]). They are:

1. The *dimension* of V is the transcendence degree of V/m_V over k (this number is always finite, although in general V/m_V is not a finitely generated extension of k).
2. The *rank* of V is the length n of the sequence of prime ideals

$$0 = p_1 \subset \cdots \subset p_n = m_V$$

in V. This is also the number of isolated subgroups of Γ_V.

3. The *rational rank* of V is the dimension of the vector space $\Gamma_V \otimes \mathbf{Q}$. The rational rank is always finite.

We will denote the respective invariants by $\dim(V)$, $\mathrm{rank}(V)$ and $\mathrm{rrank}(V)$. We have ([Ab3], [ZS])

$$\dim(V) + \mathrm{rrank}(V) \leq \mathrm{trdeg}_k(K)$$

and

$$\mathrm{rank}(V) \leq \mathrm{rrank}(V).$$

Valuation rings in dimension 2.

There are 4 types of valuation rings in dimension 2 ([Z1], [MS], [ZS], [C4]). They are:

1. *V is one dimensional.* $\Gamma_V = \mathbf{Z}$, $\dim(V) = 1$ and $\mathrm{rrank}(V) = \mathrm{rank}(V) = 1$.
2. *V is discrete, zero dimensional of rank 1.* $\Gamma_V = \mathbf{Z}$, $\dim(V) = 1$ and $\mathrm{rrank}(V) = \mathrm{rank}(V) = 1$.
3. *V is discrete, zero dimensional of rank 2.* $\Gamma_V = \mathbf{Z} \oplus \mathbf{Z}$ with the lex order, $\dim(V) = 0$ and $\mathrm{rrank}(V) = \mathrm{rank}(V) = 2$.
4. *V is non-discrete of dimension zero.* We have $\dim(V) = 0$, $\mathrm{rank}(V) = 1$, and $\mathrm{rrank}(V) = 1$ or 2.

We now give characteristic examples of these types. All valuations of K can be obtained by these constructions. Suppose that X is a surface with $k(X) = K$ and $p \in X$ is a nonsingular (closed) point. Let x, y be regular parameters in $A = \mathcal{O}_{X,p}$. For simplicity, we assume that k is algebraically closed.

1. *V is one dimensional.* $V = \mathcal{O}_{X,E}$ where X is a normal surface with $k(X) = K$, and E is a codimension 1 subvariety.
2. *V is discrete, zero dimensional of rank 1.* Embed A into a power series ring $k[[t]]$, by mapping x to t and y to a power series $P(t)$ which is transcendental over $k[t]$. Then $V = k[[t]] \cap K$.
3. *V is discrete, zero dimensional of rank 2.* For $f \in A$, we can factor $f = x^n g(x, y)$ in $\hat{A} = k[[x, y]]$, so that $x \not| g$. Define $\nu(f) = (n, \mathrm{ord}\ g(0, y))$.
4. *V is non-discrete of dimension zero.*

   a. $\mathrm{rrank}(V) = 2$. Choose $\tau \in \mathbf{R}$ which is irrational. Define $\nu(x) = 1$, $\nu(y) = \tau$. If $f \in A$, expand

$$f = \sum a_{ij} x^i y^k$$

in $k[[x, y]]$ where $a_{ij} \in k$. Define

$$\nu(f) = \min\{i + \tau j \mid a_{ij} \neq 0\}.$$

Since τ is irrational, there is a unique monomial in f which attains this minimum. This property implies that ν is a valuation. The value group of ν is the ordered subgroup $\mathbf{Z} + \mathbf{Z}\tau$ of $\mathbf{R}$.

b. rrank$(V) = 1$. This is the really interesting case. Let S be the field of "formal" series

$$f = \sum_{\rho} a_{\alpha_\rho} t^{alpha_\rho}$$

where $\alpha_\rho \in \mathbf{R}$ increase monotonically with ρ, $a_{\alpha_\rho} \in k$ are nonzero and the sum is over all ordinal numbers $\rho \leq \sigma$ for some fixed ordinal number σ. S has a valuation defined by $\nu(g(t)) = \text{ord }(g(t))$. We embed A in S by mapping x to t and y to some $P(t) \in S$. For $f \in A$, $\nu(f) = \text{ord }(f(t, P(t))$. Any subgroup of $\mathbf{Q}$ can be obtained as a value group Γ_V by this construction.

Local uniformization.

An algebraic local ring R of an algebraic function field K is the local ring $\mathcal{O}_{X,p}$ of a point on a variety X with $k(X) = K$. A monoidal transform of an algebraic local ring R in an inclusion $R \subset R_1$ where $R_1 = \mathcal{O}_{X_1,p_1}$ is a local ring of the blow up $\pi_1 : X_1 \to X$ of a nonsingular subvariety of X, and $\pi_1(p_1) = p$. If V is a valuation ring which dominates R, then there is a unique point $p_1 \in X_1$ whose local ring R_1 is dominated by V (p_1 is the center of V on X_1). We say that $R \to R_1$ is a monoidal transform along V.

Theorem 5.3 ([Z3]) *Suppose that k is a field of characteristic zero, K is an algebraic function field over k, and ν is a valuation of K which dominates an algebraic local ring R of K. Then there exists a sequence of monodial transforms $R \to R'$ along ν such that R' is a regular local ring.*

The first proof of resolution in dimension 3 used local uniformization.

Corollary 5.4 ([Z1], [Z4]) *Resolution of singularities is true in characteristic zero for varieties of dimension ≤ 3.*

[Z4] was published in 1944, about 20 years before Hironaka's characteristic zero proof [H] of resolution in all dimensions.

The first proof of resolution of surfaces and of resolution of 3-folds in positive characteristic was by local uniformization ([Ab1], [Ab2]). There has recently been progress on local uniformization in positive characteristic (including [Kuhl2], [Sp], [T]).

6. Local Monomialization.

Suppose that $R \subset S$ is a local homomorphism of local rings essentially of finite type over a field k, and that V is a valuation ring of the quotient field K of S, such that V dominates S. Then we can ask if there are sequences of monodial transforms $R \rightarrow R'$ and $S \rightarrow S'$ along V such that V dominates S', S' dominates R' and $R' \rightarrow S'$ is a "monomial mapping",

$$\begin{array}{ccccc} R' & \rightarrow & S' & \subset V \\ \uparrow & & \uparrow & \\ R & \rightarrow & S & & . \end{array} \tag{8}$$

We completely answer this question in the affirmative when k is a field of characteristic zero.

Theorem 6.1 ([C1], [C3], [C5]) *Suppose that k is a field of characteristic zero, $K \rightarrow K^*$ is a (possibly transcendental) extension of algebraic function fields over k, and that ν^* is a valuation of K^* which is trivial on k. Further suppose that R is an algebraic local ring of K and S is an algebraic local ring of K^* such that S dominates R and ν^* dominates S. Then there exist sequences of monoidal transforms $R \rightarrow R'$ and $S \rightarrow S'$ along ν^* such that R' and S' are regular local rings, S' dominates R', there exist regular parameters $(y_1, \dots, y_n)$ in S', $(x_1, \dots, x_m)$ in R', units $\delta_1, \dots, \delta_m \in S'$ and an $m \times n$ matrix (c_{ij}) of nonnegative integers such that (c_{ij}) has rank m, and*

$$x_i = \prod_{j=1}^{n} y_j^{c_{ij}} \delta_i \tag{9}$$

for $1 \leq i \leq m$.

We make a few comments and observations.

1. In the case when $K = k$, so that R is just the field k, Theorem 6.1 specializes to the local uniformization theorem Theorem 5.3.
2. Since k has characteristic zero in Theorem 6.1, and (c_{ij}) has rank m, we can obtain a toroidal form of $R' \rightarrow S'$, by choosing regular parameters $\overline{y}_1, \dots, \overline{y}_n$ in $\hat{S}'$ so that

$$x_i = \prod_{j=1}^{n} \overline{y}_j^{c_{ij}}.$$

3. The question of the existence of a diagram (8) makes sense over fields of positive characteristic. Certainly it is true when R has dimension 1, and S has dimension ≤ 2 (or in any dimension of S where good resolution theorems hold).
4. Some applications of Theorem 6.1 were given in Section 1 to local toroidalization and local factorization. Applications to the ramification theory of general valuations are given in [CP2], and [CG].
5. The proof of Theorem 6.1 actually gives a very special form to the matrix (c_{ij}) which depends on the rank and rational rank of ν and ν^*, which we call "strong local monomialization".

   For simplicity, assume that K^* is finite over K. Let $r = \text{rank}\ \nu = \text{rank}\ \nu^*$, $s = \text{rrank}\ \nu = \text{rrank}\ \nu^*$. In the valuation ring V^* of ν^*, let

$$0 = P_0 \subset \cdots \subset P_r = m_V$$

be the chain of prime ideals. Then $V_{P_i}/(P_{i-1})_{P_i}$ for $1 \le i \le r$ are rank 1 valuation rings of rational rank s_i, where $s_1 + \cdots s_r = s$. These are the "composite" valuation rings of V. The (square) matrix $C = (c_{ij})$ has the block form

$$C = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}. \tag{10}$$

Each A_i has the block form

$$A_i = \begin{pmatrix} B_i & \\ & I \end{pmatrix} \tag{11}$$

where B_i is an $s_i \times s_i$ matrix and I is an appropriate identity matrix.

6. The first case where local monomialization along a valuation is open is when K and K^* have dimension 2 over an algebraically closed field k of positive characteristic. Local monomialization for all of the cases of valuations in the classification given in Section 5 can be proven to be true fairly easily, except for the last case 4 b, where V is a nondiscrete valuation ring of rank 1. This case is studied in [CP2]. An example is given where "strong monomialization" (stated in 5. above) fails. Good local forms are constructed along valuations in general, which are shown to give strong monomialization for defectless ([ZS], [Kuhl1]) extensions.

   The less restrictive question of local monomialization along nondiscrete valuations in an extension K^*/K with defect remains open.

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