Graded algebras associated to surface singularities

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1 Introduction

Suppose that k is an algebraically closed field of characteristic zero, and (R, m, k) is a normal complete local ring of dimension 2. We restrict to characteristic zero as most of the results in this article are false in positive characteristic (as is shown in [8], [7] and [6]). Suppose that $\pi : X \rightarrow \mathrm{spec}(R)$ is a resolution of singularities. We can write the reduced exceptional divisor as

$$\pi^{-1}(m)_{\mathrm{red}} = E_1 + \cdots + E_r$$

where $E_1, \ldots, E_r$ are the irreducible components. Since X is normal, each local ring $\mathcal{O}_{X,E_i}$ is a (rank 1) discrete valuation ring with valuation ν_i. To each ν_i and $n \in \mathbf{N}$ there is an associated valuation ideal in R,

$$I_n(\nu_i) = \{f \in R \mid \nu_i(f) \geq n\}.$$

The ideals $I_n(\nu_i)$ are m-primary.

To $\underline{n} = (n_1, \ldots, n_r) \in \mathbf{N}^r$ we associate a divisor with exceptional support,

$$D_{\underline{n}} = n_1E_1 + \cdots + n_rE_r.$$

Then

$$H^0(X, \mathcal{O}_X(-D_{\underline{n}})) = \Gamma(X, \mathcal{O}_X(-D_{\underline{n}})) = I_{n_1}(\nu_1) \cap \cdots \cap I_{n_r}(\nu_r).$$

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We can associate a graded R-algebra

$$T_X = \bigoplus_{\underline{n}\in\mathbf{N}^r} \Gamma(X, \mathcal{O}_X(-D_{\underline{n}}))$$

to our resolution $\pi : X \to \mathrm{spec}(R)$.

In general, T_X is not a finitely generated R-algebra. We have the following necessary and sufficient condition for finite generation of T_X for all resolutions $\pi : X \to \mathrm{spec}(R)$ of R.

Theorem 1.1 *T_X is a finitely generated R-algebra for all resolutions $\pi : X \to spec(R)$ if and only if R has a rational singularity.*

We will prove Theorem 1.1 in Section 3. Section 2 is a survey of results from [7] which are required for the proof of Theorem 1.1.

The reader should observe that there exist examples of particular resolutions of non-rational singularities $\pi : X \to \mathrm{spec}(R)$ for which T_X is a finitely generated R-algebra. Simple examples are given by taking cones over plane curves. That is, take $R = \mathbf{C}[[x, y, z]]/(F)$ where F is a homogeneous polynomial defining a nonsingular projective curve of positive genus in the projective plane. The blow up $\pi : X \to \mathrm{spec}(R)$ of the maximal ideal m of R is a resolution of singularities, and the reduced exceptional divisor is an integral curve E, isomorphic to the projective plane curve $F = 0$. Since $-E$ is relatively ample, $\mathcal{O}_X(-nE)$ is generated by global sections for n sufficiently large, and it follows that T_X is a finitely generated R-algebra.

In section 4, we will give an example of a singularity R such that T_X is not a finitely generated R-algebra for any resolution $X \to \mathrm{spec}(R)$.

By these examples, and Theorem 1.1, we see that the condition of T_X being finitely generated for a particular resolution is a rather delicate condition, although the condition that T_X is finitely generated for all resolutions $X \to \mathrm{spec}(R)$ actually characterizes R as a rational singularity.

In the final Section 5, we survey results on Hilbert functions and Poincaré series associated to T_X, obtained in [7] and [6].

2 A combinatorial analysis of T_X

An initial understanding of T_X can be understood through linear algebra and combinatorics. We recall some of the analysis of [7].

The intersection matrix $(E_i \cdot E_j)_{1\le i,j\le r}$ is negative definite (Lemma 14.1 of [11]).

Let us consider the lattice $\mathbf{E} := \oplus_{i=1}^r \mathbf{Z}E_i$ and the semigroup

$$\mathbf{E}^+ := \{D \in \mathbf{E} \ / \ \mathcal{O}_X(-D) \text{ is nef, i.e. } (D \cdot E_i) \le 0 \text{ for } 1 \le i \le r\} \qquad (1)$$

Let $\mathbf{E}_{\mathbf{Q}} := \oplus_{i=1}^r \mathbf{Q}E_i$, and let $\mathbf{E}_{\mathbf{Q}}^+$ be the rational convex polyhedral cone in $\mathbf{E}_{\mathbf{Q}}$ generated by $\mathbf{E}^+$, i.e. $\mathbf{E}_{\mathbf{Q}}^+ = \oplus_{D\in\mathbf{E}^+}\mathbf{Q}_{\ge 0}D$. Then $\mathbf{E}_{\mathbf{Q}}^+$ is a cone contained in $\oplus_{i=1}^{r}\mathbf{Q}_{\geq 0}E_i$ ([11] p. 238). Therefore, it is a strongly convex cone, i.e. $\mathbf{E}_{\mathbf{Q}}^{+} \cap (-\mathbf{E}_{\mathbf{Q}}^{+}) = \{0\}$.

Suppose that $D = \sum a_i E_i$ is a divisor with exceptional support.

Proposition 2.1 *([15], Theorem 7.7) There exists a unique effective* $\mathbf{Q}$-*divisor* $B = \sum b_i E_i$ *such that*

(i) $\Delta = D + B$ *is in* $\mathbf{E}_{\mathbf{Q}}^{+}$, *that is* $(\Delta \cdot E_i) \leq 0$ *for* $1 \leq i \leq r$.

(ii) $(\Delta \cdot E_i) = 0$ *if* E_i *is a component of* B.

We will call Δ the *Zariski* $\mathbf{Q}$-*divisor associated to* D.

Proposition 2.2 *([4], Proposition 1) Among the divisors* $D' \in \mathbf{E}^{+}$ *such that* $D' \geq D$ *there is a minimal one* $\overline{D}$. *It can be computed applying the following algorithm: Let* $\hat{D}_1 := D$ *and, for* $i \geq 1$, *let* $\overline{D} := \hat{D}_i$ *if* $\hat{D}_i \in \mathbf{E}^{+}$ *or else* $\hat{D}_{i+1} := \hat{D}_i + E_{j_i}$ *where* E_{j_i} *is such that* $(\hat{D}_i \cdot E_{j_i}) > 0$.

We will call $\overline{D}$ *Laufer divisor associated to* D, since the previous algorithm is a generalization of Laufer's construction of the fundamental cycle ([10] prop. 4.1). Note that $\overline{D}$ is the unique divisor in $\mathbf{E}^{+}$ such that

$$H^0(X, \mathcal{O}_X(-D)) = H^0(X, \mathcal{O}_X(-\overline{D})).$$

Given two $\mathbf{Q}$-divisors D_1, D_2, let us denote $D_1 \leq D_2$ if $D_2 - D_1$ is effective.

Lemma 2.3 *(Lemma 2.3 [7]) The following holds:*

(i) $D \leq \Delta \leq \overline{D}$.

(ii) For $n \in \mathbf{N}$, $n\Delta$ *is the Zariski* $\mathbf{Q}$-*divisor associated to* nD.

(iii) Choose an integer s *such that* $s\Delta$ *is an integral divisor. Suppose that* n *is a natural number, and* $n = as + b$ *with* $0 \leq b < s$. *Then the natural inclusion*

$$\mathcal{O}_X(-as\Delta - bD) \to \mathcal{O}_X(-nD)$$

induces an isomorphism of global sections

$$H^0(X, \mathcal{O}_X(-as\Delta - bD)) \cong H^0(X, \mathcal{O}_X(-nD)).$$

For $1 \leq i \leq r$, let $\Delta_i \in \mathbf{E}_{\mathbf{Q}} = \oplus_{i=1}^{r}\mathbf{Q}E_i$ be defined by the condition $(\Delta_i \cdot E_j) = -\delta_{ij}$, so that $\mathbf{E}_{\mathbf{Q}}^{+} = \oplus_{i=1}^{r}\mathbf{Q}_{\geq 0}\Delta_i$.

Given $\underline{n} \in \mathbf{Q}^r$, let $D_{\underline{n}} = \sum_{i=1}^{r} n_i E_i \in \mathbf{E}_{\mathbf{Q}}$. For $\underline{n} \in \mathbf{N}^r$, let $\Delta_{\underline{n}}$ (resp. $\overline{D_{\underline{n}}}$) be the Zariski $\mathbf{Q}$-divisor (resp. Laufer divisor) associated to $D_{\underline{n}}$.

Definition 2.4 *(Definition 4.1 [7]) Let* S *be a subset of* $\{1,\dots,r\}$. *For* $\underline{n}\in\mathbf{Q}^r_{\geq 0}$, *let* $\Delta^S_{\underline{n}}$ *be the orthogonal projection of* $D_{\underline{n}}$ *on* $\{E_i\}^{\perp}_{i\in S}$, *i.e.* $\Delta^S_{\underline{n}}\in\oplus^r_{i=1}\mathbf{Q}E_i$ *is defined by*

$$\text{Supp }(\Delta^S_{\underline{n}}-D_{\underline{n}})\subseteq\bigcup_{i\in S}E_i,\qquad(\Delta^S_{\underline{n}}\cdot E_i)=0\ \text{ for } i\in S. \tag{2}$$

We define

$$\sigma_S:=\{\underline{n}\in\mathbf{Q}^r_{\geq 0}\ /\ \Delta^S_{\underline{n}}-D_{\underline{n}}\geq 0,\ \ \Delta^S_{\underline{n}}\in\mathbf{E}^+_{\mathbf{Q}}\}. \tag{3}$$

Let $\mathbf{F}_S$ be the face of $\mathbf{E}^+_{\mathbf{Q}}$ orthogonal to $\{E_i\}_{i\in S}$, that is $\mathbf{F}_S=\oplus_{j\notin S}\mathbf{Q}_{\geq 0}\Delta_j$. Note that

$$\sigma_S=\{\underline{n}\in\mathbf{Q}^r_{\geq 0}\ /\ \left(D_{\underline{n}}+(\bigoplus_{i\in S}\mathbf{Q}_{\geq 0}E_i)\right)\cap\mathbf{F}_S\neq\emptyset\}.$$

Given $S\subseteq\{1,\dots,r\}$, let $S=S_1\cup\ldots\cup S_k$ be the partition of S determined by the connected components of $\cup_{i\in S}E_i$, and let

$$e_S=s.c.m.\left\{|det(E_i\cdot E_j)_{i,j\in S_l}|\right\}_{l=1,\dots,k}. \tag{4}$$

Let $J:=\{1,\dots,r\}\setminus S$.

Theorem 2.5 *(Theorem 4.2 [7]) The following holds:*

(i) Given $\underline{n}\in\mathbf{Q}^r_{\geq 0}$, *let* $\underline{n}_J=(n_j)_{j\in J}$ *be its projection. Then,*

$$\Delta^S_{\underline{n}}=\sum_{j\in J}l^S_j(\underline{n}_J)\Delta_j=D_{\underline{n}}+\sum_{i\in S}b^S_i(\underline{n})E_i \tag{5}$$

where $l_j(\underline{n}_J)$ *(resp.* $b^S_i(\underline{n})$*) are linear functions on* $\underline{n}_J$ *(resp.* $\underline{n}$*) with coefficients in* $1/e_S\,\mathbf{Z}$.

(ii) For $\underline{n}\in\mathbf{N}^r$, $\underline{n}\in\sigma_S$ *if and only if* $\Delta^S_{\underline{n}}$ *is the Zariski* $\mathbf{Q}$*-divisor* $\Delta_{\underline{n}}$ *associated to* $D_{\underline{n}}$.

(iii) For any $S\subset\{1,\dots,r\}$ *(strictly contained),* σ_S *is a strongly convex rational polyhedral cone of dimension* r. *The set* Σ *consisting of all* σ_S*'s and its faces is a fan, which is a subdivision of* $\mathbf{Q}^r_{\geq 0}$.

Corollary 2.6 *(Corollary 4.3 [7]) The fan* Σ *consisting of the cones* σ_S *in Definition 2.4 and its faces satisfies the following property: For each* $\sigma\in\Sigma$, *each of the coefficients of the Zariski* $\mathbf{Q}$*-divisor, that is, of the function*

$$\sigma\cap\mathbf{Z}^r\to\mathbf{E}_{\mathbf{Q}}\cong\mathbf{Q}^r,\qquad\underline{n}\mapsto\Delta_{\underline{n}}$$

is a linear function of $\underline{n}$ *with coefficients in* $\mathbf{Q}$. *Moreover, if* $\sigma\subseteq\sigma_S$ *then the coefficients are in* $\frac{1}{e_S}\mathbf{N}$.

Next we will subdivide the fan Σ in order to have a certain periodicity for the coefficients of the function $\underline{n} \mapsto \overline{D_{\underline{n}}} - \Delta_{\underline{n}}$ (Theorem 2.12). The subdivision we will give consists of rational convex polyhedral sets. By a *rational convex polyhedral set in* $\mathbf{Q}^r$, or more simply a *polyhedral set*, we mean a set of the form

$$P = \{\underline{n} \in \mathbf{Q}^r \ / \ L_i(\underline{n}) \geq b_i, \quad 1 \leq i \leq m\} \tag{6}$$

where $m \in \mathbf{N}$ and, for $1 \leq i \leq m$, L_i is an integral linear form on $\mathbf{Q}^r$ and $b_i \in \mathbf{Z}$. We define the *cone associated to* P to be

$$\sigma_P := \{\underline{n} \in \mathbf{Q}^r \ / \ L_i(\underline{n}) \geq 0, \quad 1 \leq i \leq m\}.$$

A subset Q of P is called a *face of* P if there exist a integral linear form L on $\mathbf{Q}^r$ and $b \in \mathbf{Z}$ such that

$$P \subseteq \{\underline{n} \in \mathbf{Q}^r \ / \ L(\underline{n}) \geq b\} \quad \text{and} \quad Q = P \cap \{\underline{n} \in \mathbf{Q}^r \ / \ L(\underline{n}) = b\}.$$

Definition 2.7 *(Definition 4.4 [7]) An abstract complex of polyhedral sets in* $\mathbf{Q}^r$ *is a finite set* $\mathcal{P} = \{P_\gamma\}_{\gamma \in \Lambda}$ *of polyhedral sets in* $\mathbf{Q}^r$ *such that* P *has dimension* r *for all* $P \in \mathcal{P}$. *Given a fan* Σ *in* $\mathbf{Q}^r$, *an abstract complex of polyhedral sets* $\mathcal{P} = \{P_\gamma\}_{\gamma \in \Lambda}$ *is a subdivision of* Σ *with the same associated cones if*

(i) $\cup_{\gamma \in \Lambda} P_\gamma$ *is the support of* Σ.

(ii) For each $P_\gamma \in \mathcal{P}$, *there exists* $\sigma \in \Sigma$ *such that* $P_\gamma \subseteq \sigma$.

Definition 2.8 *(Definition 4.5 [7]) Let* S *be a subset of* $\{1, \ldots, r\}$ *and* $J = \{1, \ldots, r\} \setminus S$. *For any pair* (α, β) *where*

$$\alpha : T \to \frac{1}{e_S}\mathbf{N} \qquad \beta : J \setminus T \to \frac{1}{e_S}\mathbf{N}$$

are maps defined on some subset T *of* J *and its complement, we define the polyhedral set in* $\mathbf{Q}^r$

$$P_S(\alpha, \beta) := \{\underline{n} \in \sigma_S \ / \ l_j^S(\underline{n}) = \alpha(j) \ \text{ for } j \in T, \quad l_j^S(\underline{n}) \geq \beta(j) \ \text{ for } j \in J \setminus T\} \tag{7}$$

where $l_j^S(\underline{n}) = l_j^S(\underline{n}_J)$ *are the linear functions with coefficients in* $1/e_S\mathbf{Z}$ *in theorem 3.2 (i). We will simply denote* $P_S(\alpha, \beta)$ *by* $P(\alpha, \beta)$ *if there is no possible confusion on* S.

The cone associated to $P(\alpha, \beta)$ is

$$\sigma_{P(\alpha,\beta)} = \{\underline{n} \in \sigma_S \ / \ l_j^S(\underline{n}) = 0 \ \text{ for } j \in T\} = \sigma_S \cap \sigma_{S \cup T}$$

which is a face of σ_S.

Lemma 2.9 *(Lemma 4.6 [7]) Let S be a subset of $\{1,\dots,r\}$ and $J=\{1,\dots,r\}\setminus S$. Suppose that an assignment*

$$\alpha \mapsto \alpha^c$$

is given to each map $\alpha : T \to \frac{1}{e_S}\mathbf{N}$ defined on some subset T of J, of a map $\alpha^c : J\setminus T \to \frac{1}{e_S}\mathbf{N}$. Then, there exists an abstract complex of polyhedral sets $\mathcal{P}_{\sigma_S}$ subdividing σ_S with the same associated cones such that:

$$\textit{for all } P \in \mathcal{P}_{\sigma_S} \textit{ there exists a map } \alpha \textit{ such that } P\cap \mathbf{Z}^r \subseteq P_S(\alpha,\alpha^c). \quad (8)$$

Therefore, if an assignation as before is given for every subset S of $\{1,\dots,r\}$, then there exists an abstract complex of polyhedral sets $\mathcal{P}$ subdividing the fan Σ in Corollary 2.6 with the same associated cones and such that

$$\textit{for all } P \in \mathcal{P} \textit{ there exists a set } S \textit{ and a map } \alpha \textit{ such that } P\cap \mathbf{Z}^r \subseteq P_S(\alpha,\alpha^c).$$

For every subset S of $\{1,\dots,r\}$, let us construct an assignment $\alpha\mapsto\alpha^c$ as in Lemma 2.9 such that the function $\underline{n}\mapsto \overline{D_{\underline{n}}}-\Delta_{\underline{n}}$ has good properties on the sets $P_S(\alpha,\alpha^c)$.

Let us fix S and $T\subseteq J=\{1,\dots,r\}\setminus S$. In a similar way as in Proposition 2.2, given a divisor $D=\sum_{i=1}^r n_iE_i$, among the divisors $D'\geq D$ such that $(D'\cdot E_i)\leq 0$ for all $i\in S\cup T$, there is a minimal one. Let us denote it by $\tilde{D}^{(S\cup T)}$, or $\tilde{D}$ if there is no possible confusion on $S\cup T$. For $\underline{n}\in\sigma_S$, we have

$$\overline{D_{\underline{n}}} \geq \tilde{D}_{\underline{n}} \geq \Delta_{\underline{n}}$$

and $\tilde{D}_{\underline{n}}$ may be computed as follows: Let $\hat{D}_1 := \lceil \Delta_{\underline{n}}\rceil$ where, if $\Delta_{\underline{n}} = \sum_{i=1}^r q_iE_i$, $q_i\in\mathbf{Q}_{\geq 0}$, then $\lceil\Delta_{\underline{n}}\rceil=\sum_{i=1}^r\lceil q_i\rceil E_i$. For $i\geq 1$, let $\tilde{D}:=\hat{D}_i$ if $(\hat{D}_i\cdot E_j)\leq 0$ for all $j\in S\cup T$, or else $\hat{D}_{i+1}:=\hat{D}_i+E_{j_i}$ where $j_i\in S\cup T$ is such that $(\hat{D}_i\cdot E_{j_i})>0$.

Given a map $\alpha: T\to\frac{1}{e_S}\mathbf{N}$, let $P(\alpha,0)$ be the polyhedral set defined by (7) for the map β identically 0. Note that $P(\alpha,0)\supseteq P(\alpha,\beta)$ for any other map β.

Lemma 2.10 *(Lemma 4.7 [7]) For any map $\alpha : T\to\frac{1}{e_S}\mathbf{N}$, if $\underline{n},\underline{m}\in P(\alpha,0)\cap\mathbf{Z}^r$ and $\lceil\Delta_{\underline{n}}\rceil-\Delta_{\underline{n}}=\lceil\Delta_{\underline{m}}\rceil-\Delta_{\underline{m}}$ (for example if $\underline{n}-\underline{m}\in e_S\mathbf{Z}^r$), then*

$$\tilde{D}_{\underline{n}}-\Delta_{\underline{n}}=\tilde{D}_{\underline{m}}-\Delta_{\underline{m}}.$$

Lemma 2.11 *(Lemma 4.8 [7]) For any subset T of J and any map $\alpha : T\to\frac{1}{e_S}\mathbf{N}$, there exists a map $\alpha^c : J\setminus T\to\frac{1}{e_S}\mathbf{N}$ such that*

$$\overline{D_{\underline{n}}}=\tilde{D}^{(S\cup T)} \qquad \textit{for } \underline{n}\in P(\alpha,\alpha^c)\cap\mathbf{Z}^r.$$

Therefore

$$\overline{D_{\underline{n}}} - \Delta_{\underline{n}} = \overline{D_{\underline{m}}} - \Delta_{\underline{m}} \quad \textit{for} \;\; \underline{n}, \underline{m} \in P(\alpha, \alpha^c) \cap \mathbf{Z}^r, \;\; \underline{n} - \underline{m} \in e_S \mathbf{Z}^r.$$

From Lemmas 2.9, 2.10 and 2.11 we conclude

THEOREM 2.12 *(Theorem 4.9 [7]) There exists an abstract complex of polyhedral sets $\mathcal{P}$ subdividing the fan Σ in Corollary 2.6, with the same associated cones such that, for every $P \in \mathcal{P}$, if $P \subseteq \sigma_S$ then*

$$\overline{D_{\underline{n}}} - \Delta_{\underline{n}} = \overline{D_{\underline{m}}} - \Delta_{\underline{m}} \quad \textit{for} \;\; \underline{n}, \underline{m} \in P \cap \mathbf{Z}^r, \;\; \underline{n} - \underline{m} \in e_S \mathbf{Z}^r. \tag{9}$$

3 On finite generation of T_X

In this section we will prove Theorem 1.1. We will find it convenient to write the R-algebra T_X as

$$T_X = \sum_{\underline{n} \in \mathbf{N}^r} \Gamma(X, \mathcal{O}_X(-D_{\underline{n}})) t^{\underline{n}},$$

where $t_1, \ldots, t_r$ are indeterminates, and

$$t^{\underline{n}} = t_1^{n_1} \cdots t_r^{n_r}$$

for $\underline{n} = (n_1, \ldots, n_r) \in \mathbf{N}^r$.

First assume that R is such that T_X is a finitely generated R-algebra for all resolutions $\pi : X \to \mathrm{spec}(R)$.

Suppose that R is not a rational singularity. Then the class group of R is not a torsion group, by Chapter III of [14] and Proposition 17.3 of [11], since k is algebraically closed of characteristic zero. Now by Göhner's conjecture, which is proven in Theorem 4 of [5], there exists a resolution $\pi : X \to \mathrm{spec}(R)$ and an integral exceptional divisor E_1 of π such that

$$A = \bigoplus_{m \geq 0} \Gamma(X, \mathcal{O}_X(-mE_1))$$

is not a finitely generated R-algebra. Since we can regard A as the subalgebra

$$A = \sum_{\{\underline{n} \in \mathbf{N}^r | n_2 = \ldots = n_r = 0\}} \Gamma(X, \mathcal{O}_X(-D_{\underline{n}})) t^{\underline{n}}$$

of T_X, we must have that T_X is not a finitely generated R-algebra, which gives a contradiction to our assumption that R has a rational singularity.

Now assume that R is a rational singularity, and $\pi : X \to \mathrm{spec}(R)$ is a resolution of singularities. Let Σ be the associated fan of Definition 2.4 and Corollary 2.6, and let $\mathcal{P}$ be the abstract complex of polyhedral sets subdividing Σ of Theorem 2.12. We will use the notation introduced in Section 2.

If $P \in \mathcal{P}$, then there exists $\sigma_S \in \Sigma$ such that $P \subset \sigma_S$, and there are α, α^c such that $P \cap \mathbf{Z}^r \subset P(\alpha, \alpha^c) \cap \mathbf{Z}^r$.

Let σ_P be the a cone associated to $P(\alpha, \alpha^c)$. $\sigma_P \cap e_S\mathbf{Z}^r$ is finitely generated as a semi-group (by Gordan's Lemma). Let

$$\underline{m}_1 = \underline{m}_1(P), \ldots, \underline{m}_\lambda = \underline{m}_{\lambda(P)}(P)$$

be generators of $\sigma_P \cap e_S\mathbf{Z}^r$. By Theorem 7.1 [7], $P(\alpha, \alpha^c) \cap \mathbf{Z}^r$ is a finitely generated $\sigma_P \cap \mathbf{Z}^r$ module, and hence is a finitely generated $\sigma_P \cap e_S\mathbf{Z}^r$ module. Let

$$\underline{x}_1 = \underline{x}_1(P), \ldots, \underline{x}_m = \underline{x}_{m(P)}(P)$$

generate $P(\alpha, \alpha^c) \cap \mathbf{Z}^r$ as a $\sigma_P \cap e_S\mathbf{Z}^r$ module.

Set

$$A = R[\Gamma(X, \mathcal{O}_X(-D_{\underline{n}}))t^{\underline{n}} \mid \underline{n} = \underline{m}_i(P) \text{ or } \underline{n} = \underline{x}_j(P) \text{ for some } P \in \mathcal{P}].$$

By construction, A is a finitely generated R-algebra.

Suppose that $\underline{n} \in \mathbf{N}^r$. We will show that $\Gamma(X, \mathcal{O}_X(-D_{\underline{n}}))t^{\underline{n}} \subset A$. It will then follow that $T_X = A$, so that T_X is a finitely generated R-algebra.

There exists $P \in \mathcal{P}$ such that $\underline{n} \in P \cap \mathbf{Z}^r$. With the above notation, we have that

$$\underline{n} = \underline{x}_i + \underline{m}$$

for some $\underline{m} \in (e_S\mathbf{Z}^r) \cap \sigma_P$.

$\sigma_P \subset \sigma_S$ since σ_S is a rational polyhedral cone. We expand

$$\underline{m} = a_1\underline{m}_1 + \cdots + a_\lambda\underline{m}_\lambda$$

with $a_1, \ldots, a_\lambda \in \mathbf{N}$. Thus

$$-\Delta_{\underline{m}} = -\Delta^S_{\underline{m}} = -\overline{D}_{\underline{m}}$$

by Theorem 2.5 and Lemma 2.3, since $\underline{m} \in e_S\mathbf{Z}^r \cap \sigma_S$, and

$$-\overline{D}_{\underline{m}} = a_1(-\Delta_{\underline{m}_1}) + a_2(-\Delta_{\underline{m}_2}) + \cdots + a_\lambda(-\Delta_{\underline{m}_\lambda})$$

as $\Delta_{\underline{m}}$ is linear on σ_S by Theorem 2.5.

By Lemma 2.11 and Theorem 2.5, $\underline{n} - \underline{x}_i = \underline{m} \in e_S\underline{Z}^r$ implies that

$$\begin{aligned} -\overline{D}_{\underline{n}} &= -\overline{D}_{\underline{x}_i} + (-\Delta_{\underline{n}}) + (\Delta_{\underline{x}_i}) \\ &= -\overline{D}_{\underline{x}_i} - (\Delta_{\underline{n}-\underline{x}_i}) \\ &= -\overline{D}_{\underline{x}_i} - \Delta_{\underline{m}}. \end{aligned}$$

Since the $\mathcal{O}_X(-\overline{D}_{\underline{x}_i})$ and $\mathcal{O}_X(-\Delta_{\underline{m}_j})$ are nef ($-\overline{D}_{\underline{x}_i}$, $-\Delta_{\underline{m}_j}$ have non negative intersection products with all of the E_l), and R has a rational singularity, these sheaves are generated by global sections (Theorem 12.1 [11]), and by Lemma 7.3 [11], we have equalitites of products

$$\begin{aligned}&\Gamma(X,\mathcal{O}_X(-D_{\underline{n}}))t^{\underline{n}}\\ &=\Gamma(X,\mathcal{O}_X(-\overline{D}_{\underline{n}}))t^{\underline{n}}\\ &=\Gamma(X,\mathcal{O}_X(-\overline{D}_{\underline{x}_i}))t^{\underline{x}_i}\Gamma(X,\mathcal{O}_X(-\Delta_{\underline{m}}))t^{\underline{m}}\\ &=\Gamma(X,\mathcal{O}_X(-\overline{D}_{\underline{x}_i}))t^{\underline{x}_i}(\Gamma(X,\mathcal{O}_X(-\Delta_{\underline{m}_1}))t^{\underline{m}_1})^{a_1}\cdots(\Gamma(X,\mathcal{O}_X(-\Delta_{\underline{m}_\lambda}))t^{\underline{m}_\lambda})^{a_\lambda}\\ &=\Gamma(X,\mathcal{O}_X(-D_{\underline{x}_i}))t^{\underline{x}_i}(\Gamma(X,\mathcal{O}_X(-D_{\underline{m}_1}))t^{\underline{m}_1})^{a_1}\cdots(\Gamma(X,\mathcal{O}_X(-D_{\underline{m}_\lambda}))t^{\underline{m}_\lambda})^{a_\lambda}.\end{aligned}$$

Thus $\Gamma(X,\mathcal{O}_X(-D_{\underline{n}}))t^{\underline{n}}\subset A$.

4 An example of non finite generation of T_X for all resolutions

In this section, we give an example of a singularity R such that T_X is not a finitely generated R-algebra for any resolution $X\to\mathrm{spec}(R)$.

On pages 72 -73 of [5], an example is given of a surface singularity R such that it has no 1-fibered ideals. That is, the normalization of the blow up any m-primary ideal of R has the property that the reduced exceptional divisor has at least two distinct components. The local ring R of the example is normal and complete of dimension two with residue field $\mathbf{C}$. The minimal resolution $f: X\to\mathrm{spec}(R)$ is constructed for R in the example. $f^{-1}(m)_{red}=C_1+C_2$ where $(C_1\cdot C_2)=1$, $(C_1^2)=-1$ and $(C_2^2)=-2$.

Suppose that T_X is finitely generated. We will derive a contradiction. Under this assumption, the subalgebra

$$T_1=\bigoplus_{n\geq 0}\Gamma(X,\mathcal{O}_X(-nC_1))$$

must also be finitely generated. Now by the Proposition of Section 1 of [5], $X_1=\mathrm{Proj}(T_1)$ is a normal scheme, and the projection $g: X_1\to\mathrm{spec}(R)$ is such that the reduced exceptional divisor of g is irreducible. This is a contradiction to the fact that R has no 1-fibered ideals. We conclude that T_X is not a finitely generated R-algebra.

Now suppose that $g: Y\to\mathrm{spec}(R)$ is any resolution of singularities of R. Then there is a factorization

$$Y\xrightarrow{\Psi}X\xrightarrow{f}\mathrm{spec}(R)$$

of Φ. We can index the irreducible exceptional divisors $E_1,\ldots,E_r$ of Φ so that E_1 is the strict transform of C_1 and E_2 is the strict transform of C_2. There exist natural numbers a_i and b_i such that

$$\Psi^*(C_1)=E_1+\sum_{i=3}^{r}a_iE_i$$

and

$$\Psi^*(C_2) = E_2 + \sum_{i=3}^{r} b_i E_i.$$

There is a natural identification of

$$T_X = \sum_{(m_1,m_2)\in \mathbf{N}^2} \Gamma(X, \mathcal{O}_X(-m_1C_1 - m_2C_2))t_1^{m_1}t_2^{m_2}$$

with the subalgebra

$$\begin{array}{l}\sum_{(m_1,m_2)\in \mathbf{N}^2} \Gamma(Y, \mathcal{O}_Y(-m_1\Psi^*(C_1) - m_2\Psi^*(C_2)))t_1^{m_1}t_2^{m_2}t_3^{a_3m_1+b_3m_2}\cdots t_r^{a_rm_1+b_rm_2}\\ = \sum_{\underline{n}\in\Omega} \Gamma(Y, \mathcal{O}_Y(-D_{\underline{n}}))t^{\underline{n}}\end{array}$$

where Ω is defined by the linear condition

$$\Omega = \{\underline{n} \in \mathbf{N}^r \mid (D_{\underline{n}} \cdot E_i) = 0 \text{ for } 3 \leq i \leq r\}.$$

Thus T_Y cannot be finitely generated.

5 Hilbert functions and Poincaré series

Since $\Gamma(X, \mathcal{O}_X(-D_{\underline{n}}))$ is an m-primary ideal in R, we can define

$$h_X(\underline{n}) = \ell(R/\Gamma(X, \mathcal{O}_X(-D_{\underline{n}}))),$$

where ℓ denotes length as an R-module. h_X is a sort of Hilbert function, so we could hope that it has a nice form, at least for large $\underline{n}$.

The form of the function $h_X(n\underline{m})$ for fixed $\underline{m}$ and $n \gg 0$ is itself very interesting. Even though the ring

$$\oplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(-nD_{\underline{m}}))$$

is in general not a finitely generated R-algebra, the Hilbert function $h_X(n\underline{m})$ is always very nice, as is shown by the following theorem.

Theorem 5.1 *(Theorem 9, [8]) For fixed $\underline{m}$, and $n \gg 0$,*

$$h_X(n\underline{m}) = \ell(R/\Gamma(X, \mathcal{O}_X(-nD_{\underline{m}}))) = an^2 + bn + c(n)$$

where $a, b \in \mathbf{Q}$ and $c(n)$ is a periodic function.

The statement of this theorem with the weaker condition that $c(n)$ is bounded (but not necessarily periodic) was first proven by Morales [13]. This statement (with boundedness of $c(n)$) is in fact a local form of a theorem for linear systems on projective surfaces of Zariski [15]. The statement of periodicity of $c(n)$ was conjectured by Zariski for linear systems on surfaces in [15]. Zariski's conjecture is proven in Theorem 2, [8] for surfaces, and in the local form of Theorem 5.1.

By the local Riemann-Roch Theorem ([11],[9])

$$\begin{aligned} h_X(\underline{n}) &= -\tfrac{1}{2}(K_X \cdot D_{\underline{n}}) + (D_{\underline{n}})^2 + h^1(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X(-D_{\underline{n}})) \\ &= (\text{Quadratic polynomial in } \underline{n}) \ - h^1(X, \mathcal{O}_X(-D_{\underline{n}})). \end{aligned}$$

So we reduce to computing the (finite) length

$$h^1(X, \mathcal{O}_X(-D_{\underline{n}})) = \ell(H^1(X, \mathcal{O}_X(-D_{\underline{n}})).$$

Using the techniques of Zariski decomposition and Laufer decomposition we can subdivide $\mathbf{Q}_+^r$ into rational polyhedral sets (by Theorem 2.12) where $h_X(\underline{n})$ can be better understood.

Theorem 5.2 *(Proposition 6.3 [7]) There exists a subdivision of* $\mathbf{Q}_+^{\mathbf{r}}$ *by rational polyhedral sets* C *such that* $\underline{n} \in C \cap \mathbf{N}^r$ *implies*

$$h_X(\underline{n}) = f_C(\underline{n}) + \sigma_C(\underline{n})$$

where $f_C(\underline{n})$ *is a quadratic polynomial in* $\underline{n}$ *with periodic coefficients and* $\sigma_C(\underline{n})$ *is a bounded function.*

If R *has rational singularities, then* $\sigma_C(\underline{n}) = 0$ *for all* $\underline{n}$*.*

It is natural to consider the following series $P_X(t)$ which we call the Poincaré series of the singularity:

$$P_X(t) = \sum_{\underline{n} \in \mathbf{N}^r} h_X(\underline{n}) t^{\underline{n}}$$

where $t^{\underline{n}} = t_1^{n_1} \cdots t_r^{n_r}$. The series depends on the resolution X in an obvious way, as the number of variables in the series is the number of exceptional components of the resolution.

Related series are considered by Campillo, Delgado, Gusein-Zade [1] [2], Campillo and Galindo [3], for plane curve singularities and for rational surface singularities.

From the series $P_X(t)$ we can recover the topology of X, as is shown in the following theorem.

Theorem 5.3 *(Theorem 3.1 [7]) From* $P_X(t)$ *we can compute*

1. *The intersection matrix* $(E_i \cdot E_j)$*.*
2. *The arithmetic genus* $p_a(E_i)$ *of each* E_i*.*
3. *The arithmetic genus* $h^1(X, \mathcal{O}_X)$ *of* X*.*

Corollary 5.4 *We can determine if R has rational singularities from $P_X(t)$ for any resolution X.*

This follows since $h^1(X, \mathcal{O}_X)$ depends only on R, and $h^1(X, \mathcal{O}_X) = 0$ if and only if R has a rational singularity.

We require the following combinatorial statement.

Theorem 5.5 *(Theorem 7.5 [7]) Suppose that $Q \subset \mathbf{Q}^r_+$ is a rational polyhedral set, $M < \mathbf{Z}^r$ is a subgroup and $\underline{m} \in \mathbf{Z}^r$. Then*

$$\sum_{\underline{n} \in Q \cap (\underline{m}+M)} t^{\underline{n}}$$

is a rational series.

Theorem 5.6 *If R has a rational singularity then $P_X(t)$ is rational for any resolution X.*

Proof: Apply Theorem 5.5 to the conclusions of Theorem 5.2. □

Theorem 5.7 *(The $r \leq 2$ case of Theorem 7.7 [7]) If $r \leq 2$ then $P(t)$ is rational.*

Proof: If $r = 1$ this is trivial, as $\pi^{-1}(m)_{red} = -E_1$ is ample.

If $r = 2$, Theorem 5.2 and the proof of Theorem 5.1 show that h^1 has a nice form on the polyhedral sets C of Theorem 5.2. □

The proof that h^1 has a good form which ensures rationality of $P_X(t)$ does not extend beyond $r = 2$. We consider a condition on the class group $\mathrm{Cl}(R)$ of R which ensures that $P_X(t)$ is rational for all resolutions X of R.

There is an exact sequence of abelian groups

$$0 \to G \to \mathrm{Cl}(R) \to H \to 0$$

where G is a commutative algebraic group and H is a finite group [11]. Furthermore, there is an exact sequence of abelian groups

$$0 \to (k^+)^m \times (k^\times)^n \to G \to A \to 0$$

where A is an abelian variety. G is said to be semi-abelian if $m = 0$. We will say that $\mathrm{Cl}(R)$ is semi-abelian if G is.

It is well known that $G = 0$ if and only if R has a rational singularity [11]. There are many examples of non-rational singularities with semi-abelian class groups. Over any nonsingular curve C of positive genus it is possible to construct cones, with a resolution whose reduced exceptional locus is isomorphic to C, and such that the singularity has semi-abelian class group.

Theorem 5.8 *(Corollary 7.8 [7]) Suppose that Cl(R) is semi-abelian. Then $P_X(t)$ is rational for any resolution X of spec(R).*

If $\mathrm{Cl}(R)$ is semi-abelian, then the functions $\sigma_C(\underline{n})$ of Theorem 5.2 are determined by membership of $\underline{n}$ in translations of subgroups of $\mathbf{Z}^r$. Then we apply Theorem 5.5 to the functions

$$h_X(\underline{n}) = f_C(\underline{n}) + \sigma_C(\underline{n})$$

for $\underline{n} \in C \cap \mathbf{N}^r$.

However, rationality does not hold in general, as is shown in the following theorem.

Theorem 5.9 *(Theorem 9.1 [7]) There exists R with Cl(R) $\cong \mathbf{C}^2$ and a resolution with $r = 3$ such that $P_X(t)$ is not rational.*

We conclude by asking if there is a necessary and sufficient condition on $\mathrm{Cl}(R)$ ensuring that $P_X(t)$ is rational for all resolutions $\pi : X \to \mathrm{spec}(R)$.

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