LECTURES ON RESOLUTION OF SINGULARITIES 2008/2009 WORKSHOP AT IIT, MADRAS

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1. Smoothness, non-singularity and resolution of singularities

We begin by stating some foundational results of Zariski [32]. Proofs may be found in [32], Chapter 2 and the Appendix to [11], or in other sources.

Definition 1.1. *Suppose that X is a scheme. X is non-singular at $P \in X$ if $\mathcal{O}_{X,P}$ is a regular local ring.*

Recall that a local ring R, with maximal ideal m, is regular if the dimension of m/m^2 as a R/m vector space is equal to the Krull dimension of R.

For varieties over a field, there is a related notion of smoothness.

Suppose that $K[x_1,\ldots,x_n]$ is a polynomial ring over a field K, $f_1,\ldots,f_m \in K[x_1,\ldots,x_n]$. We define the Jacobian matrix

$$J(f;x) = J(f_1,\ldots,f_m;x_1,\ldots,x_n) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \in K[x_1,\ldots,x_n]^{mn}.$$

Let $I = (f_1,\ldots,f_m)$ be the ideal generated by $f_1,\ldots,f_m$, $R = K[x_1,\ldots,x_n]/I$. Suppose that $P \in \mathrm{spec}(R)$ Let $K(P)$ be the residue field of $\mathcal{O}_{X,P}$. We will say that $J(f;x)$ has rank l at P if the image of $J(f;x)$ in $K(P)^{mn}$ has rank l.

If A is a square matrix, we will denote the determinant of A by $|A|$.

Definition 1.2. *Suppose that X is a variety of dimension s over a field K, and $P \in X$. Suppose that $U = spec(R)$ is an affine neighborhood of P such that $R \cong K[x_1,\ldots,x_n]/I$ with $I = (f_1,\ldots,f_m)$. Then X is smooth over K if $J(f;x)$ has rank $n-s$ at P.*

This definition depends only on P and X and not on any of the choices of U, x or f.

Theorem 1.3. *Let K be a field. The set of points in a K-variety X which are smooth over K is an open set of X.*

Theorem 1.4. *Suppose that X is a variety over a field K and $P \in X$.*

1. *Suppose that X is smooth over K at P. Then P is a non-singular point of X.*
2. *Suppose that P is a non-singular point of X and $K(P)$ is separably generated over K. Then X is smooth over K at P.*

Corollary 1.5. *Suppose that X is a variety over a perfect field K and $P \in X$. Then X is non-singular at P if and only if X is smooth at P over K.*

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Proof. This is immediate since an algebraic function field over a perfect field K is always separably generated over K (Theorem 13, Section 13, Chapter II, [34]). □

In the case when X is an affine variety over an algebraically closed field K, the notion of smoothness is geometrically intuitive. Suppose that $X = V(I) = V(f_1, \ldots, f_m) \subset \mathbb{A}^n_K$ is an s-dimensional affine variety, where $I = (f_1, \ldots, f_m)$ is a reduced and equidimensional ideal. We interpret the closed points of X as the set of solutions to $f_1 = \cdots = f_m = 0$ in K^n. We identify a closed point $p = (a_1, \ldots, a_n) \in V(I) \subset \mathbb{A}^n_K$ with the maximal ideal $\overline{m} = (x_1 - a_1, \ldots, x_n - a_n)$ of $K[x_1, \ldots, x_n]$. For $1 \le i \le m$,

$$f_i \equiv f_i(p) + L_i \bmod \overline{m}^2$$

where

$$L_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_i}(p)(x_j - a_j).$$

$f_i(p) = 0$ for all i since p is a point of X. The tangent space to X in $\mathbb{A}^n_K$ at the point p is

$$T_p(X) = V(L_1, \ldots, L_m) \subset \mathbb{A}^n_K.$$

We see that $\dim T_p(X) = n - \text{rank}(J(f;x)(p))$. We have that $\dim T_p(X) \ge s$ and X is non-singular at p if and only if $\dim T_p(X) = s$.

Theorem 1.6. *Suppose that X is a variety over a field K. Then the set of non-singular points of X is an open dense set of X.*

Suppose that X is a K-variety, where K is a field.

Definition 1.7. *A resolution of singularities of X is a proper birational morphism $\varphi : Y \to X$ such that Y is a non-singular variety.*

A birational morphism is a morphism $\varphi : Y \to X$ of varieties such that there is a dense open subset U of X such that $\varphi^{-1}(U) \to U$ is an isomorphism. If X and Y are integral and $K(X)$ and $K(Y)$ are the respective function fields of X and Y, φ is birational if and only if $\varphi^* : K(X) \to K(Y)$ is an isomorphism.

A morphism of varieties $\varphi : Y \to X$ is proper if for every valuation ring V with morphism $\alpha : \text{spec}(V) \to X$, there is a unique morphism $\beta : \text{spec}(V) \to Y$ such that $\varphi \circ \beta = \alpha$. If X and Y are integral, and $K(X)$ is the function field of X, then we only need consider valuation rings V such that $K \subset V \subset K(X)$ in the definition of properness.

The geometric idea of properness is that every mapping of a "formal" curve germ into X lifts uniquely to a morphism to Y.

One consequence of properness is that every proper map is surjective. The properness assumption rules out the possibility of "resolving" by taking the birational resolution map to be the inclusion of the open set of non-singular points into the given variety, or the mapping of the non-singular points of a partial resolution to the variety.

We can extend our definition of a resolution of singularities to arbitrary schemes. A reasonable category to consider is excellent (or quasi-excellent) schemes (defined in IV.7.8 [15] and on page 260 of [27]). The definition of excellence is extremely technical, but the idea is to give minimal conditions ensuring that the the singular locus is preserved by natural base extensions such as completion. There are examples of non-excellent schemes which admit a resolution of singularities. Rotthaus [30] gives an example of a regular local ring R of dimension 3 containing a field which is not excellent. In this case, spec(R) is a resolution of singularities of spec(R).

Suppose that (R, m) is a regular local ring containing a field K of characteristic zero. Let K' be the residue field of R. Suppose that $J \subset R$ is an ideal. We define the order of J in R to be

$$\nu_R(J) = \max \{b \mid J \subset m^b\}.$$

If J is a locally principal ideal, then the order of J is its multiplicity. However, order and multiplicity are different in general.

Definition 1.8. *Suppose that q is a point on a variety W and $J \subset \mathcal{O}_W$ is an ideal sheaf. We denote*

$$\nu_q(J) = \nu_{\mathcal{O}_{W,q}}(J\mathcal{O}_{W,q}).$$

If $X \subset W$ is a subvariety, we denote

$$\nu_q(X) = \nu_q(\mathcal{I}_X).$$

Remark 1.9. *If $q \in W$ and $J \subset \mathcal{O}_W$ is an ideal sheaf, then*

$$\nu_{\mathcal{O}_{W,q}}(J_q) = \nu_{\hat{\mathcal{O}}_{W,q}}(J_q\hat{\mathcal{O}}_{W,q}).$$

Suppose that X is a noetherian topological space and I is a totally ordered set. $f : X \to I$ is said to be an upper semi-continuous function if for any $\alpha \in I$,

$$\{q \in X \mid f(q) \geq \alpha\}$$

is a closed subset of X.

Suppose that X is a subvariety of a nonsingular variety W. Define

$$\nu_X : W \to \mathbb{N}$$

by $\nu_X(q) = \nu_q(X)$ for $q \in W$. The order $\nu_q(X)$ is defined in Definition 1.8.

Theorem 1.10. *Suppose that K is a perfect field, W is a nonsingular variety over K and $\mathcal{I}$ is an ideal sheaf on W. Then*

$$\nu_q(\mathcal{I}) : W \to \mathbb{N}$$

is an upper semi-continuous function.

Suppose that A is a local ring with maximal ideal m. A coefficient field of A is a subfield L of A which is mapped onto A/m by the quotient mapping $A \to A/m$.

A basic theorem of Cohen is that an equicharacteristic complete local ring contains a coefficient field (Theorem 27, Section 12, Chapter VIII [34]). This leads to Cohen's Structure Theorem (Corollary, Section 12, Chapter VIII [34]), which shows that an equicharacteristic complete regular local ring A is isomorphic to a formal power series ring over a field. In fact, if L is a coefficient field of A, and if $(x_1, \ldots, x_n)$ is a regular system of parameters of A, then A is the power series ring

$$A = L[[x_1, \ldots, x_n]].$$

We further remark that the completion of a local ring R is a regular local ring if and only if R is regular (c.f. Section 11, Chapter VIII [34]).

Lemma 1.11. *Suppose that S is a non-singular algebraic surface defined over a field K, $p \in S$ is a closed point, $\pi : B = B(p) \to S$ is the blow up of p, and suppose that $q \in \pi^{-1}(p)$ is a closed point such that $K(q)$ is separable over $K(p)$. Let $R_1 = \hat{\mathcal{O}}_{S,p}$ and $R_2 = \hat{\mathcal{O}}_{B,q}$, and suppose that K_1 is a coefficient field of R_1, (x, y) are regular parameters in R_1. Then there exists a coefficient field $K_2 = K_1(\alpha)$ of R_2 and regular parameters (x_1, y_1) of R_2 such that*

$$\hat{\pi}^* : R_1 \to R_2$$

is the map given by

$$\sum_{i,j\geq 0} a_{ij}x^iy^j \to \sum_{i,j\geq 0} a_{ij}x_1^{i+j}(y_1+\alpha)^j$$

where $a_{ij} \in K_1$, or $K_2 = K_1$ and

$$\hat{\pi}^* : R_1 \to R_2$$

is the map given by

$$\sum_{i,j\geq 0} a_{ij}x^iy^j \to \sum_{i,j\geq 0} a_{ij}x_1^iy_1^{i+j}$$

A proof of Lemma 1.11 is given in Lemma 3.5 [11].

Theorem 1.12. *Suppose that R is a reduced affine ring over a field K, and $A = R_p$ where p is a prime ideal of R. Then the completion $A = \hat{R}_p$ of A with respect to its maximal ideal is reduced.*

When K is a perfect field, this is a theorem of Chevalley (Theorem 31, Section 13, Chapter VIII [34]). The general case follows from Scholie IV 7.8.3 (vii) [15].

Theorem 1.13. *(Weierstrass Preparation Theorem) Let K be a field, and suppose that $f \in K[[x_1, \ldots, x_n, y]]$ is such that*

$$0 < r = \nu(f(0, \ldots, 0, y)) = max\{n \mid y^n \text{ divides } f(0, \ldots, 0, y)\} < \infty.$$

Then there exists a unit series u in $K[[x_1, \ldots, x_n, y]]$ and non-unit series $a_i \in K[[x_1, \ldots, x_n]]$ such that

$$f = u(y^r + a_1y^{r-1} + \cdots + a_r).$$

A proof is given in Theorem 5, Section 1, Chapter VII [34].

An important concept in resolution in characteristic zero is the Tschirnhausen transformation, which generalizes the ancient notion of "completion of the square" in the solution of quadratic equations.

Definition 1.14. *Suppose that K is a field of characteristic $p \geq 0$ and $f \in K[[x_1, \ldots, x_n, y]]$ has an expression*

$$f = y^r + a_1y^{r-1} + \cdots + a_r$$

with $a_i \in K[[x_1, \ldots, x_n]]$ and $p = 0$ or $p \not| r$. The Tschirnhausen transformation of f is the change of variables replacing y with

$$y' = y + \frac{a_1}{r}.$$

f then has an expression

$$f = (y')^r + b_2(y')^{r-2} + \cdots + b_r$$

with $b_i \in K[[x_1, \ldots, x_n]]$ for all i.

Definition 1.15. *Suppose that W is a nonsingular variety over a perfect field K and $X \subset W$ is a subvariety.*

For $b \in \mathbb{N}$, define

$$Sing_b(X) = \{q \in W \mid \nu_q(W) \geq b\}.$$

$Sing_b(X)$ is a closed subset of W by Theorem 1.10. Let

$$r = max\{\nu_X(q) \mid q \in W\}.$$

Suppose that $q \in Sing_r(X)$. A subvariety H of an affine neighborhood U of q in W is called a hypersurface of maximal contact for X at q if

1. *$Sing_r(X) \cap U \subset H$ and*
2. *If*

$$W_n \stackrel{\pi_n}{\to} \cdots \to W_1 \stackrel{\pi_1}{\to} W$$

is a sequence of monoidal transforms such that for all i, π_i is centered at a nonsingular subvariety $Y_i \subset Sing_r(X_i)$, where X_i is the strict transform of X on W_i, then the strict transform H_i of H on $U_i = W_i \times_W U$ (which is nonsingular) is such that $Sing_r(X_i) \cap U_i \subset H_i$ for $1 \leq i \leq n$.

With the above assumptions, a nonsingular codimension one subvariety H of $U = spec(\hat{\mathcal{O}}_{W,q})$ is called a formal hypersurface of maximal contact for X at q if 1. and 2. above hold for $U = spec(\hat{\mathcal{O}}_{W,q})$.

2. Blow ups of ideals

Suppose that X is a variety, and $\mathcal{J} \subset \mathcal{O}_X$ is an ideal sheaf. The blow up of $\mathcal{J}$ is

$$\pi : B = B(\mathcal{J}) = \text{proj}(\bigoplus_{n \geq 0} \mathcal{J}^n) \to X.$$

B is a variety and π is proper. If X is projective then B is projective. π is an isomorphism over $X - V(\mathcal{J})$, and $\mathcal{J}\mathcal{O}_B$ is a locally principal ideal sheaf.

If $U \subset X$ is an open affine subset, and $R = \Gamma(U, \mathcal{O}_X)$, $I = \Gamma(U, \mathcal{J}) = (f_1, \ldots, f_m) \subset R$, then

$$\pi^{-1}(U) = B(I) = \text{proj}(\bigoplus_{n \geq 0} I^n).$$

If X is an integral scheme, we have

$$\text{proj}(\bigoplus_{n \geq 0} I^n) = \cup_{i=1}^m \text{spec}(R[\frac{f_1}{f_i}, \cdots, \frac{f_m}{f_i}]).$$

Suppose that $W \subset X$ is a subscheme with ideal sheaf $\mathcal{I}_W$ on X.

The total transform of W, $\pi^*(W)$ is the subscheme of B with the ideal sheaf

$$\mathcal{I}_{\pi^*(W)} = \mathcal{I}_W \mathcal{O}_B.$$

Set theoretically, $\pi^*(W)$ is the preimage of W, $\pi^{-1}(W)$.

Let $U = X - V(\mathcal{J})$. The strict transform $\tilde{W}$ of W is the Zariski closure of $\pi^{-1}(W \cap U)$ in $B(\mathcal{J})$.

The strict transform has the property that

$$\tilde{W} = B(\mathcal{J}\mathcal{O}_W) = \operatorname{proj}(\bigoplus_{n\geq 0}(\mathcal{J}\mathcal{O}_W)^n)$$

This is shown in Corollary II.7.15 [16].

Theorem 2.1. *(Universal property of blowing up) Suppose that $\mathcal{I}$ is an ideal sheaf on a variety V and $f : W \to V$ is a morphism of varieties such that $\mathcal{I}\mathcal{O}_W$ is locally principal. Then there is a unique morphism $g : W \to B(\mathcal{I})$ such that $f = \pi \circ g$.*

This is proven in Proposition II.7.14 [16].

Theorem 2.2. *Suppose that C is an integral curve over a field K. Consider the sequence*

$$\cdots \to C_n \overset{\pi_n}{\to} \cdots \to C_1 \overset{\pi_1}{\to} C \tag{1}$$

where $C_{n+1} \to C_n$ is obtained by blowing up the (finitely many) singular points on C_n. Then this sequence is finite. That is, there exists n such that C_n is non-singular.

Proof. Without loss of generality, $C = \operatorname{spec}(R)$ is affine. Let $\overline{R}$ be the integral closure of R in the function field $K(C)$ of C. $\overline{R}$ is a regular ring. Let $\overline{C} = \operatorname{spec}(\overline{R})$. All ideals in $\overline{R}$ are locally principal (Proposition 9.2, page 94 [4]). By Theorem 2.1, we have a factorization

$$\overline{C} \to C_n \to \cdots \to C_1 \to C$$

for all n. Since $\overline{C} \to C$ is finite, $C_{i+1} \to C_i$ is finite for all i, and there exist affine rings R_i such that $C_i = \operatorname{spec}(R_i)$. For all n we have sequences of inclusions

$$R \to R_1 \to \cdots \to R_n \to \overline{R}.$$

$R_i \neq R_{i+1}$ for all i since a maximal ideal m in R_i is locally principal if and only if $(R_i)_m$ is a regular local ring. Since $\overline{R}$ is finite over R, we have that (1) is finite. □

Corollary 2.3. *Suppose that X is a variety over a field K and C is an integral curve on X. Consider the sequence*

$$\cdots \to X_n \overset{\pi_n}{\to} X_{n-1} \to \cdots \to X_1 \overset{\pi_1}{\to} X$$

where π_{i+1} is the blow up of all points of X which are singular on the strict transform C_i of C on X_i. Then this sequence is finite. That is, there exists an n such that the strict transform C_n of C is non-singular.

Proof. The induced sequence

$$\cdots \to C_n \to \cdots \to C_1 \to C$$

of blow ups of points on the strict transform C_i of C is finite by Theorem 2.2. □

Theorem 2.4. *Suppose that V and W are varieties and $V \to W$ is a projective birational morphism. Then $V \cong B(\mathcal{I})$ for some ideal sheaf $\mathcal{I} \subset \mathcal{O}_W$.*

This is proved in Proposition II.7.17 [16].

Suppose that Y is a non-singular subvariety of a variety X. The monoidal transform of X with center Y is $\pi : B(\mathcal{I}_Y) \to X$. We define the exceptional divisor of the monoidal transform π (with center Y) to be the reduced divisor with the same support as $\pi^{-1}(Y)$.

In general, we will define the exceptional locus of a proper birational morpism $\varphi : V \to W$ to be the (reduced) closed subset of V on which φ is not an isomorphism.

3. Exercise Set 1.

1. Suppose that K is an algebraically closed field, and $f \in K[x_1, \ldots, x_n]$ is irreducible. Let $X = V(f) \subset \mathbb{A}^n_K$. Show that $q \in X$ is a nonsingular point if and only if $\nu_q(X) = 1$. Generalize this result to the case of an irreducible hypersurface X embedded in a regular scheme.
2. Let K be an algebraically closed field, and X be the affine surface
$$X = \mathrm{spec}(K[x, y, z]/(xy - z^2)).$$
Let $Y = V(x, z) \subset X$, and let $\pi : B \to X$ be the monoidal transform centered at the non-singular curve Y. Show that π is a resolution of singularities. Compute the exceptional locus of π and compute the exceptional divisor of the monoidal transform π of Y.
3. Let K be an algebraically closed field, and X be the affine 3-fold
$$X = \mathrm{spec}(K[x, y, z, w]/(xy - zw)).$$
Show that the monodial transform $\pi : B \to X$ with center Y is a resolution of singularities in each of these cases, and describe the exceptional locus and exceptional divisor (of the monoidal transform): Y=V((x,y,z,w)), Y=V((x,z)), Y=V((y,z)). Show that the monoidal transforms of the second and third case factor the monoidal transform of the first case.
4. Consider the curve $y^2 - x^3 = 0$ in $\mathbb{A}^2_K$, over an algebraically closed field K of characteristic 0 or $p > 3$.
   a. Show that at the point $p_1 = (1, 1)$, the tangent space $T_{p_1}(X)$ is the line
   $$-3(x - 1) + 2(y - 1) = 0.$$
   b. Show that at the point $p_2 = (0, 0)$, the tangent space $T_{p_2}(X)$ is the entire plane $\mathbb{A}^2_K$.
   c. Show that the curve is singular only at the origin p_2.
5. Let $K = \mathbb{Z}_p(t)$ where p is a prime, t is an indeterminate. Let $R = K[x, y]/(x^p + y^p - t)$, $X = \mathrm{spec}(R)$. Prove that X is non-singular, but there are no points of X which are smooth over K.
6. Let $K = \mathbb{Z}_p(t)$ where $p > 2$ is a prime, t is an indeterminate. Let $R = K[x, y]/(x^2 + y^p - t)$, $X = \mathrm{spec}(R)$. Prove that X is non-singular, and X is smooth over K at every point except at the prime $(y^p - t, x)$.
7. Show that the property of being a domain is not preserved under completion. Let$f = y^2 - x^2 + x^3$. Show that f is irreducible in $\mathbb{C}[x, y]$, but is reducible in the completion $\mathbb{C}[[x, y]]$. Thus $R = \mathbb{C}[x, y]/(f)$ is a domain, but its completion $\hat{R}$ is not.

4. Resolution of curves embedded in a non-singular surface

In this section we consider curves embedded in a non-singular surface, defined over an arbitrary field K. A more general version of this proof, valid in arbitrary two dimensional regular local rings, can be found in [3] or [29]. The proof of this section is from Section 3.5 of [11].

Lemma 4.1. *Suppose that K is a field, S is a non-singular surface over K, C is a curve on S and $p \in C$ is a closed point. Suppose that $\pi : B = B(p) \to S$ is the blow up of p, $\tilde{C}$ is the strict transform of C on B and $q \in \pi^{-1}(p) \cap \tilde{C}$. Then*

$$\nu_q(\tilde{C}) \leq \nu_p(C)$$

and if

$$\nu_q(\tilde{C}) = \nu_p(C)$$

then $K(p) = K(q)$.

A proof of this lemma is given in Lemma 3.14 [11].

Theorem 4.2. *Suppose that C is a curve which is a subvariety of a non-singular surface X over a field K. Then there exists a sequence of blow ups of points $\lambda : Y \to X$ such that the strict transform $\tilde{C}$ of C on Y is non-singular.*

Proof. Let $r = \max(\nu_p(C) \mid p \in C\}$. If $r = 1$, C is non-singular, so we may assume that $r > 1$. The set $\{p \in C \mid \nu_p(C) = r\}$ is a subset of the singular local of C which is a proper closed subset of the 1-dimensional variety C, so it is a finite set.

The proof is by induction on r.

We can construct a sequence of projective morphisms

$$\cdots \to X_n \stackrel{\pi_n}{\to} \cdots \to X_1 \stackrel{\pi_1}{\to} X_0 = X \tag{2}$$

where each $\pi_{n+1} : X_{n+1} \to X_n$ is the blow up of all points on the strict transform C_n of C which have multiplicity r on C_n. If this sequence is finite, then there is an integer n such that all points on the strict transform C_n of C have multiplicity $\leq r - 1$. By induction on r, we can then repeat this process to construct the desired morphism $Y \to X$ which induces a resolution of C.

We will assume that the sequence (2) is infinite, and derive a contradiction. If it is infinite, then for all $n \in \mathbb{N}$ there are closed points $p_n \in C_n$ which have multiplicity r on C_n and such that $\pi_{n+1}(p_{n+1}) = p_n$. Let $R_n = \hat{\mathcal{O}}_{X_n, p_n}$ for all n. We then have an infinite sequence of completions of quadratic transforms (blow ups of maximal ideals) of local rings

$$R = R_0 \to R_1 \to \cdots \to R_n \to \cdots$$

We will define

$$\delta_{p_i} \in \frac{1}{r!}\mathbb{N}$$

such that

$$\delta_{p_i} = \delta_{p_{i-1}} - 1 \tag{3}$$

for all $i \geq 1$. We can thus conclude that p_i has multiplicity $< r$ on the strict transform C_i of C for some natural number $i \leq \delta_p + 1$. From this contradiction it will follow that (2) is a sequence of finite length.

Suppose that $f \in R_0 = \hat{\mathcal{O}}_{X,p}$ is such that $f = 0$ is a local equation C and (x, y) are regular parameters in $R = \hat{\mathcal{O}}_{X,p}$ such that $r = \text{mult}(f(0, y))$. We will call such (x, y) good parameters for f. Let K' be a coefficient field of R. There is an expansion

$$f = \sum_{i+j \geq r} a_{ij} x^i y^j$$

with $a_{ij} \in K'$ for all i, j and $a_{0r} \neq 0$. Define

$$\delta(f; x, y) = \min \left\{ \frac{i}{r-j} \mid j < r \text{ and } a_{ij} \neq 0 \right\}. \tag{4}$$

$\delta(f; x, y) \geq 1$ since (x, y) are good parameters We thus have an expression (with $\delta = \delta(f; x, y)$)

$$f = \sum_{i+j\delta \geq r\delta} a_{ij} x^i y^j = L_\delta + \sum_{i+j\delta > r\delta} a_{ij} x^i y^j$$

where

$$L_\delta = \sum_{i+j\delta = r\delta} a_{ij} x^i y^j = a_{0r} y^r + \Lambda \tag{5}$$

is such that $a_{0r} \neq 0$ and Λ is not zero.

Suppose that (x, y) are fixed good parameters of f. Define

$$\delta_p = \sup\{\delta(f; x, y_1) \mid y = y_1 + \sum_{i=1}^{n} b_i x^i \text{ with } n \in \mathbb{N} \text{ and } b_i \in K'\} \in \frac{1}{r!}\mathbb{N} \cup \{\infty\}. \tag{6}$$

We cannot have $\delta_p = \infty$, since then there would exist a series

$$y = y_1 + \sum_{i=1}^{\infty} b_i x^i$$

such that $\delta(f; x, y_1) = \infty$, and thus there would be a unit series γ in R such that

$$f = \gamma y_1^r.$$

But then $r = 1$ since f is reduced in R, a contradiction. We see then that $\delta_p \in \frac{1}{r!}\mathbb{N}$.

After possibly making a substitution

$$y = y_1 + \sum_{i=1}^{n} b_i x^i$$

with $b_i \in K'$, we may assume that $\delta_p = \delta(f; x, y)$.

Let $\delta = \delta_p$.

Since $\nu_{p_1}(C_1) = r$, by Lemma 4.1, we have an expression $R_1 = K'[[x_1, y_1]]$, where $R \to R_1$ is the natural K'-algebra homomorphism such that either

$$x = x_1, y = x_1(y_1 + \alpha)$$

for some $\alpha \in K'$, or

$$x = x_1 y_1, y = y_1.$$

We first consider the case where $x = x_1y_1, y = y_1$. A local equation of C_1 (in spec(R_1)) is $f_1 = 0$, where

$$f_1 = \frac{f}{y_1^r} = \sum_{i+j=r} a_{ij}x_1^i + y_1\Omega = a_{0r} + x_1g + y_1h$$

for some series $\Omega, g, h \in R_1$. Thus f_1 is a unit in R_1, a contradiction.

Now consider the case where $x = x_1, y = x_1(y_1 + \alpha)$ with $0 \neq \alpha \in K'$. A local equation of C_1 is

$$f_1 = \frac{f}{x_1^r} = \sum_{i+j=r} a_{ij}(y_1 + \alpha)^j + x_1\Omega$$

for some $\Omega \in R_1$. Since $\nu_{p_1}(C_1) = r$,

$$\sum_{i+j=r} a_{ij}(y_1 + \alpha)^j = a_{0r}y_1^r.$$

Substituting $t = y_1 + \alpha$ we have

$$\sum_{i+j=r} a_{ij}t^j = a_{0r}(t - \alpha)^r.$$

If we now substitute $t = \frac{y}{x}$ and multiply the series by x^r we obtain the leading form L of f,

$$L = \sum_{i+j=r} a_{ij}x^iy^j = a_{0r}(y - \alpha x)^r = a_{0r}y^r + \cdots + (-1)^r a_{0r}\alpha^r x^r.$$

Comparing with (5), we see that $r = r\delta$ and $\delta = 1$, so that $L_\delta = L$. But we can replace y with $y - \alpha x$ to increase δ, a contradiction to the maximality of δ. Thus $\nu_{p_1}(C_1) < r$, a contradiction.

Finally, consider the case $x = x_1, y = x_1y_1$. Set $\delta' = \delta - 1$. A local equation of C_1 is

$$f_1 = \frac{f}{x_1^r} = \sum_{i+j\delta \geq r\delta} a_{ij}x_1^{i+j-r}y_1^j = \sum_{\bar{i}+j\delta'=r\delta'} a_{\bar{i}-j+r,j}x_1^{\bar{i}}y_1^j + \sum_{\bar{i}+j\delta'>r\delta'} a_{\bar{i}-j+r,j}x_1^{\bar{i}}y_1^j$$

where $\bar{i} = i + j - r$. Since

$$L_{\delta'}(x_1, y_1) = \frac{1}{x_1^r}L_\delta(x_1, x_1y_1)$$

has at least two non-zero terms, we see that $\delta(f_1; x_1, y_1) = \delta' = \delta - 1$.

Finally, we will show that $\delta_{p_1} = \delta(f_1; x_1, y_1)$. Suppose not. Then we can make a substitution

$$y_1 = y_1' - \sum b_ix_1^i = y_1' - bx_1^d + \text{ higher order terms in } x_1$$

with $0 \neq b \in K'$ such that

$$\delta(f_1; x_1, y_1') > \delta(f_1; x_1, y_1).$$

Then we have an expression

$$\sum_{\bar{i}+j\delta'=r\delta'} a_{\bar{i}-j+r,j}x_1^{\bar{i}}(y_1' - bx_1^d)^j = a_{0r}(y_1')^r + \sum_{i+j\delta'>r\delta'} b_{ij}x_1^i(y_1')^j$$

so that $\delta' = d \in \mathbb{N}$, and

$$\sum_{\bar{i}+j\delta'=r\delta'} a_{\bar{i}-j+r,j}x_1^{\bar{i}}(y_1' - bx_1^d)^j = a_{0r}(y_1')^r.$$

Thus

$$\sum_{i+j\delta=r\delta} a_{ij}x_1^{i+j-r}y_1^j = \sum_{\bar{i}+j\delta'=r\delta'} a_{\bar{i}-j+r,j}x_1^{\bar{i}}y_1^j = a_{0r}(y_1+bx_1^{\delta'})^r.$$

If we now multiply these series by x_1^r we obtain

$$L_\delta = \sum_{i+j\delta=r\delta} a_{ij}x^iy^j = a_{0r}(y+bx^\delta)^r.$$

But we can now make the substitution $y' = y - bx^\delta$ and see that

$$\delta(f;x,y') > \delta(f;x,y) = \delta_p,$$

a contradiction, from which we conclude that $\delta_{p_1} = \delta' = \delta_p - 1$. We can then inductively define δ_{p_i} for $i \geq 0$ by this procedure so that (3) holds. The conclusions of the theorem now follow. □

5. Exercise Set 2.

1. Resolve the singularities by blowing up.
   a. $x^2 = x^4 + y^4$
   b. $xy = x^6 + y^6$
   c. $x^3 = y^2 + x^4 + y^4$
   d. $x^2y + xy^3 = x^4 + y^4$
   e. $y^2 = x^n$
   f. $y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7$.
2. Suppose that D is an effective divisor on a non-singular surface S. That is, there exist irreducible curves C_i on S, $r_i \in \mathbb{N}$ such that $\mathcal{I}_D = \mathcal{I}_{C_1}^{r_1} \cdots \mathcal{I}_{C_n}^{r_n}$. D has simple normal crossings (SNCs) on S if for every $p \in S$ there exist regular parameters (x,y) in $\mathcal{O}_{S,p}$ such that if $f = 0$ is a local equation for D at p, then $f = \text{unit } x^ay^b$ in $\mathcal{O}_{S,p}$.

   Find a sequence of blow ups of points making the total transform of $y^2 - x^3 = 0$ a SNCs divisor.
3. Suppose that D is an effective divisor on a non-singular surface S. Show that there exists a sequence of blow ups of points

   $$S_n \to \cdots \to S$$

   such that the total transform $\pi^*(D)$ is a SNCs divisor.
4. Suppose that $T = T_0$ is a nonsingular surface over a field L, and $\mathcal{I} = \mathcal{I}_0$ is an ideal sheaf of T. Show that there exists a sequence of blow ups of points

   $$T_n \overset{\varphi_n}{\to} T_{n-1} \to \cdots \to T_1 \overset{\pi_1}{\to} T_0$$

   such that $\mathcal{I}\mathcal{O}_{T_n}$ is everywhere locally a principal ideal whose generator defines a SNC divisor, and the blow ups π_{i+1} are centered at a point where $\mathcal{I}\mathcal{O}_{T_i}$ is not locally a principal ideal whose generator defines a SNC divisor.

5. Prove that the δ_p defined in formula (6) is equal to

$$\delta_p = \sup\{\delta(f; x, y) \mid (x, y) \text{ are good parameters for } f\}.$$

Thus δ_p does not depend on the initial choice of good parameters, and δ_p is an invariant of p.
6. Suppose that $\nu_p(C) > 1$. Let $\pi : B(p) \to X$ be the blow up of p, $\tilde{C}$ be the strict transform of C. Show that there is at most one point $q \in \pi^{-1}(p)$ such that $\nu_q(\tilde{C}) = \nu_p(C)$.
7. The Newton Polygon $N(f; x, y)$ is defined as follows. Let I be the ideal in $R = K[[x, y]]$ generated by the monomials $x^\alpha y^\beta$ such that the coefficient $a_{\alpha\beta}$ of $x^\alpha y^\beta$ in $f(x, y) = \sum a_{\alpha\beta} x^\alpha y^\beta$ is not zero. Set

$$P(f; x, y) = \{(\alpha, \beta) \in \mathbb{Z}^2 | x^\alpha y^\beta \in I\}.$$

Now define $N(f; x, y)$ to be the smallest convex subset of $\mathbb{R}^2$ such that $N(f; x, y)$ contains $P(f; x, y)$ and if $(\alpha, \beta) \in N(f; x, y)$, $(s, t) \in \mathbb{R}^2_+$, then $(\alpha + s, \beta + t) \in N(f; x, y)$. Now suppose that (x, y) are good parameters for f (so that $\nu(f(0, y)) = \nu(f) = r$). Then $(0, r) \in N(f; x, y)$. Let the slope of the steepest segment of $N(f; x, y)$ be $s(f; x, y)$. This is the slope of the nonvertical line on the boundary of $N(f; x, y)$ which contains the point $(0, r)$. We have $0 \geq s(f; x, y) \geq -1$, since $a_{\alpha\beta} = 0$ if $\alpha + \beta < r$. Show that

$$\delta(f; x, y) = -\frac{1}{s(f; x, y)}.$$

8. Show that y_1 giving δ_p in (6) is a hypersurface of maximal contact (Definition 1.15).
9. Prove Lemma 4.1.

6. Some Reduction Theorems for Resolution

Resolution of singularities. Suppose that V is a variety. A resolution of singularities of V is a proper birational morphism $\varphi : W \to V$ such that W is non-singular.

Principalization of ideals. Suppose that V is a non-singular variety, $\mathcal{I} \subset \mathcal{O}_V$ is an ideal sheaf. A principalization of $\mathcal{I}$ is a proper birational morphism $\varphi : W \to V$ such that W is non-singular and $\mathcal{I}\mathcal{O}_W$ is locally principal.

Resolution of indeterminacy. Suppose that $\varphi : W \to V$ is a rational map of proper K-varieties such that W is non-singular. A resolution of indeterminacy of φ is a proper non-singular K-variety X with a birational morphism $\psi : X \to W$ and a morphism $\lambda : X \to V$ such that $\lambda = \varphi \circ \psi$.

Lemma 6.1. *Suppose that resolution of singularities is true for K-varieties of dimension n. Then resolution of indeterminacy is true for rational maps from K-varieties of dimension n.*

Proof. Let $\varphi : W \to V$ be a rational map of proper K-varieties where W is non-singular. Let U be a dense open set of W on which φ is a morphism. Let Γ_φ be the Zariski closure of the image in $W \times V$ of the map $U \to W \times V$ defined by $p \mapsto (p, \varphi(p))$. By resolution of singularities, there is a proper birational morphism $X \to \Gamma_\varphi$ such that X is non-singular. □

Theorem 6.2. *Suppose that K is a perfect field, resolution of singularities is true for projective hypersurfaces over K of dimension n and principalization of ideals is true for non-singular varieties of dimension n over K. Then resolution of singularities is true for projective K-varieties of dimension n.*

Proof. Suppose that V is an n dimensional projective K-variety. If $V_1, \dots, V_r$ are the irreducible components of V, we have a projective birational morphism from the disjoint union of the V_i to V. Thus it suffices to assume that V is irreducible. The function field $K(V)$ of V is a finite separable extension of a rational function field $K(x_1, \dots, x_n)$ (Chapter II, Theorem 30, page 104 [34]). By the theorem of the primitive element,

$$K(V) \cong K(x_1, \dots, x_n)[x_{n+1}]/(f).$$

We can clear the denominator of f so that

$$f = \sum a_{i_1 \dots i_{n+1}} x_1^{i_1} \cdots x_{n+1}^{i_{n+1}}$$

is irreducible in $K[x_1, \dots, x_{n+1}]$. Let

$$d = \max\{i_1 + \cdots i_{n+1} \mid a_{i_1 \dots i_{n+1}} \neq 0\}.$$

Set

$$F = \sum a_{i_1 \dots i_{n+1}} X_0^{d-(i_1+\cdots+i_{n+1})} X_1^{i_1} \cdots X_{n+1}^{i_{n+1}},$$

the homogenization of f. Let $\overline{V}$ be the variety defined by $F = 0$ in $\mathbb{P}^{n+1}$. $K(\overline{V}) \cong K(V)$ implies there is a birational rational map from V to $\overline{V}$. That is, there is a birational morphism $\varphi : \tilde{V} \to \overline{V}$ where $\tilde{V}$ is a dense open subset of V. Let Γ_φ be the Zariski closure of $\{(a, \varphi(a)) \mid a \in \tilde{V}\}$ in $V \times \overline{V}$. We have birational projection morphisms $\pi_1 : \Gamma_\varphi \to V$ and $\pi_2 : \Gamma_\varphi \to \overline{V}$. Γ_φ is the blow up of an ideal sheaf $\mathcal{J}$ on $\overline{V}$ (Theorem 2.4). By resolution of singularities for n dimensional hypersurfaces, we have a resolution of singularities $f : W' \to \overline{V}$. By principalization of ideals in non-singular varieties of dimension n, we have a principalization $g : W \to W'$ for $\mathcal{J}\mathcal{O}_{W'}$. By the universal property of blowing up (Theorem 2.1), we have a morphism $h : W \to \Gamma_\varphi$. Hence $\pi_1 \circ h : W \to V$ is a resolution of singularities. □

Corollary 6.3. *Suppose that C is a projective curve over a perfect field K. Then C has a resolution of singularities.*

Proof. By Theorem 4.2 resolution of singularities is true for projective plane curves over K. All ideal sheaves on a non-singular curve are locally principal since the local rings of points are Dedekind local rings. □

7. Resolution of surface singularities in characteristic zero

In this section, we give a simple proof of resolution of surface singularities in characteristic zero. The proof follows the algorithm of Beppo Levi [25], [33].

Theorem 7.1. *Suppose that S is a projective surface over an algebraically closed field K of characteristic 0. Then there exists a resolution of singularities*

$$\Lambda : T \rightarrow S.$$

Theorem 7.1 is a consequence of the following Theorem 7.2, Theorem 6.2 and Problem 4 of Exercise Set 2.

Theorem 7.2. *Suppose that S is a hypersurface of dimension 2 in a non-singular projective variety V of dimension 3, over an algebraically closed field K of characteristic 0. Then there exists a sequence of blow ups of points and non-singular curves contained in the strict transform S_i of S*

$$V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V$$

such that the the strict transform S_n of S on V_n is non-singular.

The remainder of this section will be devoted to the proof of Theorem 7.2. Suppose that V is a non-singular three dimensional variety over an algebraically closed field K of characteristic 0, and $S \subset V$ is a surface.

For a natural number t, define (Definitions 1.8 1.15)

$$\mathrm{Sing}_t(S) = \{p \in V \mid \nu_p(S) \geq t\}.$$

By Theorem 1.10, $\mathrm{Sing}_t(S)$ is Zariski closed in V.

Let

$$r = \max\{t \mid \mathrm{Sing}_t(S) \neq \emptyset\}$$

be the maximal multiplicity of points of S. There are two types of blow ups of non-singular subvarieties on a non-singular three dimensional variety, a blow up of a point, and a blow up of a non-singular curve.

We will first consider the blow up of a closed point $p \in V$, $\pi : B(p) \rightarrow V$. Suppose that $U = \mathrm{spec}(R) \subset V$ is an affine open neighborhood of p, and p has ideal $m_p = (x, y, z) \subset R$.

$$\pi^{-1}(U) = \mathrm{proj}(\bigoplus m_p^n) = \mathrm{spec}(R[\frac{y}{x}, \frac{z}{x}]) \cup \mathrm{spec}(R[\frac{x}{y}, \frac{z}{y}]) \cup \mathrm{spec}(R[\frac{x}{z}, \frac{y}{z}]).$$

The exceptional divisor is $E = \pi^{-1}(p) \cong \mathbb{P}^2$.

At each closed point $q \in \pi^{-1}(p)$, we have regular parameters (x_1, y_1, z_1) of the following forms:

$$x = x_1, y = x_1(y_1 + \alpha), z = x_1(z_1 + \beta),$$

with $\alpha, \beta \in K$, $x_1 = 0$ a local equation of E, or

$$x = x_1 y_1, y = y_1, z = y_1(z_1 + \alpha)$$

with $\alpha \in K$, $y_1 = 0$ a local equation of E, or

$$x = x_1 z_1, y = y_1 z_1, z = z_1,$$

$z_1 = 0$ a local equation of E.

We will now consider the blow up $\pi : B(C) \to V$ of a non-singular curve $C \subset V$. If $p \in V$ and $U = \mathrm{spec}(R) \subset V$ is an open affine neighborhood of p in V such that $m_p = (x, y, z)$ and the ideal of C is $I = (x, y)$ in R then

$$\pi^{-1}(U) = \mathrm{proj}(\bigoplus I^n) = \mathrm{spec}(R[\frac{x}{y}]) \cup \mathrm{spec}(R[\frac{y}{x}]).$$

$\pi^{-1}(p) \cong \mathbb{P}^1$, $\pi^{-1}(C \cap U) \cong (C \cap U) \times \mathbb{P}^1$. Let $E = \pi^{-1}(C)$ be the exceptional divisor. E is a projective bundle over C. At each point $q \in \pi^{-1}(p)$, we have regular parameters (x_1, y_1, z_1) such that:

$$x = x_1, y = x_1(y_1 + \alpha), z = z_1$$

where $\alpha \in K$, $x_1 = 0$ is a local equation of E, or

$$x = x_1 y_1, y = y_1, z = z_1$$

where $y_1 = 0$ is a local equation of E.

In this section, we will analyze the blow up

$$\pi : B(W) = B(\mathcal{I}_W) \to V$$

of a non-singular subvariety W of V above a closed point $p \in V$, by passing to a formal neighborhood $\mathrm{spec}(\hat{\mathcal{O}}_{V,p})$ of p and analyzing the map $\overline{\pi} : B(\hat{\mathcal{I}}_{W,p}) \to \mathrm{spec}(\hat{\mathcal{O}}_{V,p})$. We have a natural isomorphism $B(\hat{\mathcal{I}}_{W,p}) \cong B(\mathcal{I}_W) \times_V \mathrm{spec}(\hat{\mathcal{O}}_{V,p})$. Observe that we have a natural identification of $\pi^{-1}(p)$ with $\overline{\pi}^{-1}(p)$.

Lemma 7.3. *Suppose that V is a non-singular three dimensional variety, $S \subset V$ is a surface, $C \subset \mathrm{Sing}_r(S)$ is a non-singular curve, $\pi : B(C) \to V$ is the blow up of C, and $\tilde{S}$ is the strict transform of S in $B(C)$. Suppose that $p \in C$. Then $\nu_q(\tilde{S}) \leq r$ for all $q \in \pi^{-1}(p)$, and there exists at most one point $q \in \pi^{-1}(p)$ such that $\nu_q(\tilde{S}) = r$. In particular, if $E = \pi^{-1}(C)$, then either $\mathrm{Sing}_r(\tilde{S}) \cap E$ is a non-singular curve which maps isomorphically onto C or $\mathrm{Sing}_r(\tilde{S}) \cap E$ is a finite union of points.*

Proof. By the Weierstrass preparation theorem and after a Tschirnhausen transformation (Definition 1.14), a local equation $f = 0$ of S in $\hat{\mathcal{O}}_{V,p} = K[[x, y, z]]$ has the form

$$f = z^r + a_2(x, y) z^{r-2} + \cdots + a_r(x, y). \tag{7}$$

$f \in \hat{\mathcal{I}}^r_{C,p}$ implies $\frac{\partial f}{\partial z} \in \hat{\mathcal{I}}^{r-1}_{C,p}$, and $r! z = \frac{\partial^{r-1} f}{\partial z^{r-1}} \in \hat{\mathcal{I}}_{C,p}$. Thus $z \in \hat{\mathcal{I}}_{C,p}$. After a change of variables in x and y, we may assume that $\hat{\mathcal{I}}_{C,p} = (x, z)$. $f \in \hat{\mathcal{I}}^r_{C,p}$ implies $x^i \mid a_i$ for all i. If $q \in \pi^{-1}(p)$, then $\hat{\mathcal{O}}_{B(C),q}$ has regular parameters (x_1, y, z_1) such that

$$x = x_1 z_1, z = z_1$$

or

$$x = x_1, z = x_1(z_1 + \alpha)$$

for some $\alpha \in K$.

In the first case, a local equation of the strict transform of S is a unit. In the second case, the strict transform of S has a local equation

$$f_1 = (z_1 + \alpha)^r + \frac{a_2}{x_1^2}(z_1 + \alpha)^{r-2} + \cdots + \frac{a_r}{x_1^r}.$$

$\nu(f_1) \leq r$, and $\nu(f_1) < r$ if $\alpha \neq 0$. ∎

Lemma 7.4. *Suppose that $p \in Sing_r(S)$ is a point, $\pi : B(p) \to V$ is the blow up of p, $\tilde{S}$ is the strict transform of S in $B(p)$, and $E = \pi^{-1}(p)$. Then $\nu_q(\tilde{S}) \le r$ for all $q \in \pi^{-1}(p)$, and either $Sing_r(\tilde{S}) \cap E$ is a non-singular curve or $Sing_r(\tilde{S}) \cap E$ is a finite union of points.*

Proof. By the Weierstrass preparation theorem and after a Tschirnhausen transformation, a local equation $f = 0$ of S in $\hat{\mathcal{O}}_{V,p} = K[[x, y, z]]$ has the form

$$f = z^r + a_2(x, y)z^{r-2} + \cdots + a_r(x, y). \tag{8}$$

If $q \in \pi^{-1}(p)$, and $\nu_q(\tilde{S}) \ge r$, then $\hat{\mathcal{O}}_{B(C),q}$ has regular parameters (x_1, y, z_1) such that

$$x = x_1y_1, y = y_1, z = y_1z_1$$

or

$$x = x_1, y = x_1(y_1 + \alpha), z = x_1z_1$$

for some $\alpha \in K$, and $\nu_1(\tilde{S}) = r$. Thus $\mathrm{Sing}_r(\tilde{S}) \cap E$ is contained in the line which is the intersection of E with the strict transform of $z = 0$ in $B(C) \times_V \mathrm{spec}(\hat{\mathcal{O}}_{V,p})$. ☐

Theorem 7.5. *There exists a sequence of blow ups of points in $Sing_r(S_i)$*

$$V_n \to V_{n-1} \to \cdots \to V,$$

where S_i is the strict transform of S on V_i, so that all curves in $Sing_r(S_n)$ are non-singular.

Proof. This is possible by Corollary 2.3 and since (by Lemma 7.4) the blow up of a point in $\mathrm{Sing}_r(S_i)$ can introduce at most one new curve into $\mathrm{Sing}_r(S_{i+1})$, which must be non-singular. ☐

Theorem 7.6. *Suppose that all curves in $Sing_r(S)$ are non-singular, and*

$$\cdots \to V_n \to V_{n-1} \to \cdots \to V$$

is a sequence of blow ups of non-singular curves in $Sing_r(S_i)$ where S_i is the strict transform of S on V_i. Then $Sing_r(S_i)$ is a union of non-singular curves and a finite number of points for all i. Further, this sequence in finite. That is, there exists an n such that $Sing_r(S_n)$ is a finite set.

Proof. Suppose that C is a non-singular curve in $\mathrm{Sing}_r(S)$. Let $\pi_1 : V_1 = B(C) \to V$ be the blow up of C, S_1 be the strict transform of S. If $\mathrm{Sing}_r(S_1) \ne \emptyset$, we can choose another non-singular curve C_1 in $\mathrm{Sing}_r(S_1)$ and blow up by $\pi_2 : V_2 = B(C_1) \to V_1$. Let S_2 be the strict transform of S_1. We either reach a surface S_n such that $\mathrm{Sing}_r(S_n) = \emptyset$, or we obtain an infinite sequence of blow ups

$$\cdots \to V_n \to V_{n-1} \to \cdots \to V$$

such that each $V_{i+1} \to V_i$ is the blow up of a curve C_i in $\mathrm{Sing}_r(S_i)$, where S_i is the strict transform of S on V_i. Each curve C_i which is blown up must map onto a curve in S by Lemma 7.4. Thus there exists a curve $\gamma \subset S$ such that there are infinitely many blow ups of curves mapping onto γ in the above sequence. Let $R = \mathcal{O}_{V,\gamma}$, a two dimensional regular local ring. $\mathcal{I}_{S,\gamma}$ is a height 1 prime ideal in this ring, and $P = \mathcal{I}_{\gamma,\gamma}$ is the maximal ideal of R.

$$\dim R + \mathrm{trdeg}_K R/P = 3$$

by the dimension formula (Theorem 15.6 [27]). Thus R/P has transcendence degree 1 over K. Let $t \in R$ be the lift of a transcendence basis of R/P over K. $K[t] \cap P = (0)$, so the field $K(t) \subset R$. We can write $R = A_Q$ where A is a finitely generated $K(t)$ algebra (which is a domain) and Q is a maximal ideal in A. Thus R is the local ring of a non-singular point q on the $K(t)$ surface spec(A). q is a point of multiplicity r on the curve in spec(A) with ideal sheaf $\mathcal{I}_{S,\gamma}$ in R.

The sequence

$$\cdots \to V_n \times_V \operatorname{spec}(R) \to V_{n-1} \times_V \operatorname{spec}(R) \to \cdots \to \operatorname{spec}(R)$$

consists of infinitely many blow ups of points on a $K(t)$ surface, which are of multiplicity r on the strict transform of the curve spec$(R/\mathcal{I}_{S,\gamma})$. But this is impossible by Theorem 4.2. ☐

We can now state the main resolution theorem.

Theorem 7.7. *Suppose that $Sing_r(S)$ is a finite set. Consider a sequence of blow ups*

(9) $$\cdots \to V_n \to \cdots \to V_2 \to V_1 \to V$$

where S_i is the strict transform of S on V_i and $V_{i+1} \to V_i$ is the blow up of a curve in $Sing_r(S_i)$ if such a curve exists, and $V_{i+1} \to V_i$ is the blow up of a point in $Sing_r(S_i)$ otherwise. Then $Sing_r(S_i)$ is a disjoint union of non-singular curves and a finite number of points for all i. Further, this sequence is finite. That is, there exists V_n such that $Sing_r(S_n) = \emptyset$.

Proof. The fact that $\operatorname{Sing}_r(S_i)$ is a disjoint union of non-singular curves and a finite number of points for all i follows from the algorithm, and Lemmas 7.3 and 7.4.

We now prove that (9) is finite. Suppose there is an infinite sequence of the form of (9). Then there is an infinite sequence of points $p_n \in \operatorname{Sing}_r(S_n)$ such that $\pi_n(p_n) = p_{n-1}$ for all n. We then have an infinite sequence of homomorphisms of local rings

(10) $$R_0 = \hat{\mathcal{O}}_{V,p} \to R_1 = \hat{\mathcal{O}}_{V_1,p_1} \to \cdots \to R_n = \hat{\mathcal{O}}_{V_n,p_n} \to \cdots$$

Without loss of generality, we may asume that no $R_i \to R_{i+1}$ is an isomorphism.

By the Weierstrass preparation theorem and after a Tshirnhausen transformation (Definition 1.14) there exist regular parameters (x, y, z) in R_0 such that there is a local equation $f = 0$ for S in R_0 of the form

$$f = z^r + a_2(x, y)z^{r-2} + \cdots + a_r(x, y)$$

with $\nu(a_i(x, y)) \geq i$ for all i. Since $\nu_{p_i}(S_i) = r$ for all i, by Lemmas 7.3 and 7.4, we have regular parameters (x_i, y_i, z_i) in R_i for all i and a local equation $f_i = 0$ for S_i such that in the case when $R_{i-1} \to R_i$ is the blow up of the maximal ideal,

$$x_{i-1} = x_i, y_{i-1} = x_i(y_i + \alpha_i), z_{i-1} = x_i z_i$$

with $\alpha_i \in K$ or

$$x_{i-1} = x_i y_i, y_{i-1} = y_i, z_{i-1} = y_i z_i.$$

We set

$$\tilde{x}_{i-1} = x_{i-1}, \tilde{y}_{i-1} = y_{i-1}.$$

In the case when $R_{i-1} \to R_i$ is the blow up of the completion $\hat{\mathcal{I}}_{C,p_{i-1}}$ of the ideal sheaf of a nonsingular curve $C \subset V_{i-1}$, we must have a change of variables

$$\tilde{x}_{i-1}(x_{i-1},y_{i-1}), \tilde{y}_{i-1}(x_{i-1},y_{i-1}), z_{i-1}$$

in R_{i-1} such that $\hat{\mathcal{I}}_{C,p_{i-1}} = (\tilde{x}_{i-1}, z_{i-1})$ or $(\tilde{y}_{i-1}, z_{i-1})$, and

$$\tilde{x}_{i-1} = x_i, \tilde{y}_{i-1} = y_i, z_{i-1} = x_i z_i$$

or

$$\tilde{x}_{i-1} = x_i, \tilde{y}_{i-1} = y_i, z_{i-1} = y_i z_i.$$

Furthermore,

$$f_i = z_i^r + a_{2,i}(x_i,y_i) z_i^{r-2} + \cdots + a_{r,i}(x_i,y_i)$$

where

$$a_{ji}(x_i,y_i) = \begin{cases} \frac{a_{j,i-1}}{x_i^j} \text{ or} \\ \frac{a_{j,i-1}}{y_i^j} \end{cases}$$

for all j. Thus the sequence

(11) $$K[[x,y]] \to K[[x_1,y_1]] \to \cdots$$

is a sequence of blow ups of the maximal ideal, or the trivial extension of blowing up a height one prime, followed by completion.

Let

$$I_i = (a_{2,i}(x_i,y_i)^{\frac{r!}{2}}, a_{3,i}(x_i,y_i)^{\frac{r!}{3}}, \ldots, a_{r,i}(x_i,y_i)^{\frac{r!}{r}}) \subset K[[x_i,y_i]].$$

We have $\nu_{p_i}(S_i) = r$ if and only if $\nu(I_i) \ge r!$. Further,

$$I_{i+1} = \frac{1}{x_i^{r!}} I_i \text{ or } I_{i+1} = \frac{1}{y_i^{r!}} I_i.$$

We now see that infinitely many points must be blown up in the sequence (10), since any element of I_i can have only finitely many irreducible factors, and by our assumption that (10) is infinite. Thus by Problem 4 of Exercise Set 2 (on principalization of ideals in two variables) there exists m_0 such that I_i is principal for $i \ge m_0$, and for suitable choice of x_i, y_i, I_i is generated by a "monomial" $x_i^{a_i} y_i^{b_i}$.

We now consider the change of (a_i, b_i) in the sequence (11). If we blow up (x_i, z_i), then we must have $a_i \ge r!$. In R_{i+1}, we have regular parameters $x_{i+1}, y_{i+1}, z_{i+1}$ defined by

$$x_i = x_{i+1}, y_i = y_{i+1}, z_i = x_{i+1} z_{i+1}.$$

Thus $I_{i+1} = (x_{i+1}^{a_i - r!} y_{i+1}^{b_i})$. We have a similar conclusion if we blow up (y_i, z_i). We then have $b_i \ge r!$ and $I_{i+1} = (x_{i+1}^{a_i} y_{i+1}^{b_i - r!})$.

If we blow up (x_i, y_i, z_i), then since p_i is isolated in $\mathrm{Sing}_r(S_i)$, we must have $a_i < r!$, $b_i < r!$ and $a_i + b_i \ge r!$. In R_{i+1}, we have regular parameters $x_{i+1}, y_{i+1}, z_{i+1}$ defined by

(12) $$x_i = x_{i+1}, y_i = x_{i+1}(y_{i+1} + \alpha_{i+1}), z_i = x_{i+1} z_{i+1}$$

or

(13) $$x_i = x_{i+1} y_{i+1}, y_i = y_{i+1}, z_i = y_{i+1} z_{i+1}$$

In the case when $\alpha_{i+1} = 0$ in (12), we have

$$I_{i+1} = (x_{i+1}^{a_i+b_i-r!} y_{i+1}^{b_i}).$$

If $\alpha_{i+1} \neq 0$ in (12), then

$$I_{i+1} = (x_{i+1}^{a_i+b_i-r!}).$$

If (13) holds, then

$$I_{i+1} = (x_{i+1}^{a_i} y_{i+1}^{a_i+b_i-r!}).$$

After a finite number of blow ups, we see that we must have $\nu(I_{i+1}) < r!$, so that $\nu_{p_{i+1}}(S_{i+1}) < r$, a contradiction. □

8. Resolution of Surface Singularities in all Characteristics

8.1. **Resolution and some invariants.** In this section we prove resolution of singularities for surfaces over an algebraically closed field of characteristic $p \geq 0$. The characteristic 0 proof in Section 7 depended on the existence of hypersurfaces of maximal contact, which is false in positive characteristic (see Exercise Sets 3 and 4). The first proof of resolution of singularities of surfaces in characteristic p is by Abhyankar [1]. The most general proof of resolution in dimension two is Lipman's proof of resolution of excellent two dimensional schemes in [26]. Proofs of embedded resolution of surfaces in positive characteristic are given by Abhyankar in [2] and Hironaka in [20]. Abhyankar uses embedded resolution of surface singularities to prove resolution of 3-folds in characteristic greater than 5 in [2]. A simplified proof of this theorem is given by Cutkosky in [10]. Recently, Cossart and Piltant have proven resolution of 3-folds in all characteristics in [8] and [9].

The algorithm of this chapter is sketched by Hironaka in his notes [20]. A more detailed proof is given in [11] and an extension to ideals in [10]. In this section we prove the following theorem:

Theorem 8.1. *Suppose that S is a surface over an algebraically closed field K of characteristic $p \geq 0$. Then there exists a resolution of singularities*

$$\Lambda : T \to S.$$

Theorem 8.1 in the case when S is a hypersurface follows from induction on $r = \max\ \{t \mid \mathrm{Sing}_t(S) \neq 0\}$ in Theorems 8.6, 8.7 and 8.8. We then obtain Theorem 8.1 in general from Theorem 6.2 and Problem 4 of Exercise Set 2.

For the remainder of this chapter we will assume that K is an algebraically closed field of characteristic $p \geq 0$, V is a non-singular 3 dimensional variety over K, and S is a surface in V.

It follows from Theorem 1.10 that for $t \in \mathbb{N}$,

$$\mathrm{Sing}_t(S) = \{p \in S \mid \nu_p(S) \geq t\}$$

is Zariski closed.

Let

$$r = \max\{t \mid \mathrm{Sing}_t(S) \neq \emptyset\}.$$

Suppose that $p \in \mathrm{Sing}_r(S)$ is a closed point, $f = 0$ is a local equation of S at p, and (x, y, z) are regular parameters at p in V (or in $\hat{\mathcal{O}}_{V,p}$). There is an expansion

$$f = \sum_{i+j+k \geq r} a_{ijk} x^i y^j z^k$$

with $a_{ijk} \in K$ in $\hat{\mathcal{O}}_{V,p} = K[[x, y, z]]$. The leading form of f is defined to be

$$L(x, y, z) = \sum_{i+j+k=r} a_{ijk} x^i y^j z^k.$$

We define a new invariant, $\tau(p)$, to be the dimension of the smallest linear subspace T of the K-subspace spanned by x, y and z in $K[x, y, z]$ such that $L \in k[T]$. This subspace is uniquely determined. This dimension is in fact independent of choice of regular parameters (x, y, z) at p (or in $\hat{\mathcal{O}}_{V,p}$). If x, y, z are regular parameters in $\mathcal{O}_{V,p}$, we will call the subvariety $M = V(T)$ of $\mathrm{spec}(\mathcal{O}_{V,p})$ an *approximate manifold* to S at p. If (x, y, z) are regular parameters in $\hat{\mathcal{O}}_{V,p}$, we call $M = V(T) \subset \mathrm{spec}(\hat{\mathcal{O}}_{V,p})$ a (formal) approximate manifold to S at p. M is dependent of our choice of regular parameters at p. Observe that

$$1 \leq \tau(q) \leq 3.$$

However, the tangent space to N does not depend on our choice of M and τ does not depend on our choice of M. If there is a non-singular curve $C \subset \mathrm{Sing}_r(S)$ such that $p \in C$, then $\tau(p) \leq 2$, and there exists an approximate manifold M such that M contains the germ of C at p.

Example 8.2. *Let $f = x^2 + y^2 + z^2 + x^5$. If char$(K) \neq 2$, then $\tau = 3$ and $x = y = z = 0$ are local equations of the approximate manifold at the origin.*

However, if char$(K) = 2$, then $\tau = 1$ and $x + y + z = 0$ is a local equation of an approximate manifold at the origin.

Example 8.3. *Let $f = y^2 + 2xy + x^2 + z^2 + z^5$, char$(K) > 2$. $L = (y + x)^2 + z^2$ so that $\tau = 2$ and $x + y = z = 0$ are local equations of an approximate manifold at the origin.*

Lemma 8.4. *Suppose that $Y \subset$ Sing$_r(S)$ is a nonsingular subvariety of V (a point or a curve), $\pi_1 : V_1 \to V$ is the blow up of Y, S_1 is the strict transform of S on V_1, $p \in Y$, M_p is an approximate manifold to S at p containing the germ of Y at p, and $q \in \pi_1^{-1}(p)$. Then*

1. *$\nu_q(S_1) \leq r$.*
2. *$\nu_q(S_1) = r$ implies q is on the strict transform M_p' of M_p and $\tau(p) \leq \tau(q)$.*
3. *Suppose that $\nu_q(S_1) = r$ and $\tau(p) = \tau(q)$. Then there exists an approximate manifold M_q to S_1 at q such that $M_q \cap \pi_1^{-1}(p) = M_p' \cap \pi_1^{-1}(p)$ where M_p' is the strict transform of M_p on V_1.*

Proof. (in the case when $Y = p$) Let $f = 0$ be a local equation of S at q, (x, y, z) be regular parameters at p such that

1. $x = y = z = 0$ are local equations of M if $\tau(q) = 3$,
2. $y = z = 0$ are local equations of M if $\tau(q) = 2$,
3. $z = 0$ is a local equation of M if $\tau(q) = 1$.

In $\hat{\mathcal{O}}_{V,p} = K[[x,y,z]]$, write

$$f = \sum a_{ijk} x^i y^j z^k = L + G$$

where L is the leading form of degree r, and G has order $> r$. q_1 has regular parameters (x_1, y_1, z_1) such that one of the following cases hold:

1. $x = x_1, y = x_1(y_1 + \alpha), z = x_1(z_1 + \beta)$ with $\alpha, \beta \in K$
2. $x = x_1 y_1, y = y_1, z = y_1(z_1 + \beta)$ with $\beta \in K$
3. $x = x_1 z_1, y = y_1 z_1, z = z_1$

Suppose that the first case holds. S_1 has a local equation $f_1 = 0$ at q_1 such that

$$f_1 = L(1, y_1 + \alpha, z_1 + \beta) + x_1 \Omega.$$

Thus $\nu_{q_1}(f_1) \le r$, and $\nu_{q_1}(f_1) = r$ implies

$$L(1, y_1 + \alpha, z_1 + \beta) = \sum_{i+j+k=r} a_{ijk}(y_1 + \alpha)^j (z_1 + \beta)^k = \sum_{j+k=r} b_{jk} y_1^j z_1^k$$

for some $b_{jk} \in k$. Thus

$$L(1, \frac{y}{x}, \frac{z}{x}) = \sum_{j+k=r} (\frac{y}{x} - \alpha)^j (\frac{z}{x} - \beta)^k$$

implies $V(y - \alpha x, z - \beta x) \subset M$. We conclude that $\tau(q) \le 2$ if $\nu_{q_1}(S_1) = \nu_q(S)$.

Suppose that $\tau(q) = 2$ and $\nu_{q_1}(S_1) = \nu_q(S)$. Then we must have that $\alpha = \beta = 0$, so that $q_1 \in N_1$. We further have

$$f_1 = L(y_1, z_1) + x_1 \Omega$$

so that $\tau(q_1) \ge \tau(q)$.

If $\tau(q) = 1$, then $L = cz^r$ for some (non-zero) $c \in K$, and we must have $\beta = 0$, so that $q_1 \in N_1$. We further have

$$f_1 = L(z_1) + x_1 \Omega$$

so that $\tau(q_1) \ge \tau(q)$.

There is a similar analysis in cases 2 and 3. $\square$

With the hypotheses of Lemma 8.4, we immediately conclude that $\tau(p) \le 3 - \dim Y$ and $\tau(p) = 3 - \dim Y$ implies $\nu_q(\mathcal{I}_1) < r$.

We deduce from Lemma 8.4 the following important corollary.

Lemma 8.5. *Suppose that V is a nonsingular 3-fold, $Y \subset \mathrm{Sing}_r(S)$ is a nonsingular subvariety of V, $\pi_1 : V_1 \to V$ is the blow up of Y, and S_1 is the strict transform of S on V_1. Let F be the reduced exceptional divisor of π_1. Suppose that $F \cap \mathrm{Sing}_r(S_1) \ne \emptyset$. Then*

1. *If Y is a point, then $F \cap \mathrm{Sing}_r(S_1)$ is either an isolated point, or an irreducible nonsingular curve.*
2. *If Y is a curve, then $F \cap \mathrm{Sing}_r(S_1)$ is either a finite union of points, each in a distinct fiber over a point of Y, or $\mathrm{Sing}_r(S_1)$ is an irreducible nonsingular curve which is a section over Y.*

Theorem 8.6. *There exists a sequence of blow ups of points in $Sing_r(S_i)$*

$$V_n \to V_{n-1} \to \cdots \to V,$$

where S_i is the strict transform of S on V_i, so that all curves in $Sing_r(S_n)$ are non-singular.

Proof. This is possible by Corollary 2.3 and since (by Lemma 8.5) the blow up of a point in $\mathrm{Sing}_r(S_i)$ can introduce at most one new curve into $\mathrm{Sing}_r(S_{i+1})$, which must be non-singular. □

Theorem 8.7. *Suppose that all curves in $Sing_r(S)$ are non-singular, and*

$$\cdots \to V_n \to V_{n-1} \to \cdots \to V$$

is a sequence of blow ups of non-singular curves in $Sing_r(S_i)$ where S_i is the strict transform of S on V_i. Then $Sing_r(S_i)$ is a union of non-singular curves and a finite number of points for all i. Further, this sequence in finite. That is, there exists an n such that $Sing_r(S_n)$ is a finite set.

The proof of Theorem 8.7 is exactly the same as the characteristic zero proof of Theorem 7.6, with the reference to Lemma 7.4 replaced with a reference to Lemma 8.5.

We can now state the main resolution theorem.

Theorem 8.8. *Suppose that $Sing_r(S)$ is a finite set. Consider a sequence of blow ups*

$$\cdots \to V_n \to \cdots \to V_2 \to V_1 \to V \tag{14}$$

where $V_{i+1} \to V_i$ is the blow up of a curve in $Sing_r(S_i)$ if such a curve exists, and $V_{i+1} \to V_i$ is the blow up of a point in $Sing_r(S_i)$ otherwise. Let S_i be the strict transform of S on V_i. Then $Sing_r(S_i)$ is a disjoint union of non-singular curves and a finite number of points for all i. Further, this sequence is finite. That is, there exists V_n such that $Sing_r(S_n) = \emptyset$.

We will prove Theorem 8.8 by contradiction. Assume that (14) is not finite. We must then have an infinite sequence of points $q_n \in V_n$ such that $q_n \in \mathrm{Sing}_r(S_n)$, and q_n maps to q_{n-1} for all n.

By Lemma 8.4, $\tau(q_n) \geq \tau(q_{n-1})$ for all n. Thus, by induction on τ, we may assume that $\tau(q_n) = \tau(q)$ for all n.

We will show that this is not possible, by associating to each point q_n an element $\Omega(q_n)$ in an ordered set such that $\Omega(q_{n+1}) < \Omega(q_n)$ for all n. We will further show that Ω cannot decrease indefinitely. It will then follow that there exists an n such that $\mathrm{Sing}_r(V_n) = \emptyset$, proving Theorem 8.8.

We have a sequence

$$R_0 \to R_1 \to \cdots \tag{15}$$

of infinite length, where $R_i = \hat{\mathcal{O}}_{V_i,q_i}$ is the completion of the local ring of V_i at q_i, and $\nu_{q_i}(S_i) = r$ for all i. Let $I_i = (\mathcal{I}_{S_i,q_i})R_i$.

Since $\mathcal{O}_{V_n,q_n}$ is excellent, we have $\mathrm{Sing}_r(S_n) \cap \mathrm{Spec}(R_n) = \mathrm{Sing}_r(I_n)$ for all n.

We may extend our definition of τ to an ideal in a power series ring. We have $\tau(q_n) = \tau(I_n)$ for all n.

We will consider separately the cases with different values of τ in the following Sections 9 and 10, to derive a contradiction, showing that (14) must have finite length in all cases.

The case where $\tau(q) = 3$ is immediate from Lemma 8.4.

9. Reduction When $\tau(q) = 2$

The proof in this case is an extension of the case of resolution of a plane curve, given in Section 4. The proof is based on Hironaka's termination argument in [20].

Suppose that T is a power series ring in 3 variables over an algebraically closed field K. Suppose that x, y, z are regular parameters in T, and r is a positive integer.

For $g \in T$, we have an expansion

$$g = \sum b_{ijk} x^i y^j z^k \tag{16}$$

with $b_{ijk} \in K$. Define

$$\gamma(g; x, y, z) = \min\{\frac{k}{r - (i+j)} \mid b_{ijk} \neq 0 \text{ and } i + j < r\} \in \frac{1}{r!}\mathbf{N} \cup \{\infty\}.$$

We have $\gamma(g; x, y, z) = \infty$ if and only if $g \in (x, y)^r$. Further,

1. $\nu_T(g) = r$ if and only if $\gamma(g; x, y, z) \geq 1$ and
2. $V(x, y)$ is contained in an approximate manifold of g if and only if

$$\gamma(g; x, y, z) > 1.$$

Let $\gamma = \gamma(g; x, y, z)$. Define

$$[g]_{xyz} = \sum_{(i+j)\gamma + k = r\gamma} b_{ijk} x^i y^j z^k. \tag{17}$$

There is an expansion

$$g = [g]_{xyz} + \sum_{(i+j)\gamma + k > r\gamma} b_{ijk} x^i y^j z^k. \tag{18}$$

Define

$$A_\gamma = \left\{(i, j) \mid \frac{k}{r - (i+j)} = \gamma, i + j < r, \text{ and } b_{ijk} \neq 0\right\}.$$

Then

$$[g]_{xyz} = L_g(x, y) + \sum_{(i,j) \in A_\gamma} b_{i,j,\gamma(r-i-j)} x^i y^j z^{\gamma(r-i-j)}, \tag{19}$$

where $L_g(x, y) = \sum_{i+j=r} b_{ij0} x^i y^j$.

Regular parameters x, y, z in T will be called good parameters for J if $\nu_{\overline{T}}(\overline{g}) = r$ and $\tau(\overline{g}) = 2$, where $\overline{T} = T/zT$ and $\overline{g} = g + zT$..

If $\nu_T(g) = r$ and $\tau(g) = 2$, then regular parameters x, y, z in T such that $V(x, y)$ is an approximate manifold of g are good parameters for g.

Definition 9.1. *Suppose that x, y, z are good parameters for g and $\nu_T(g) = r$. Then g is solvable with respect to x, y, z if $\gamma = \gamma(g; x, y, z) \in \mathbf{N}$ and there exist $\alpha, \beta \in K$ such that*

$$[g]_{xyz} = L_g(x - \alpha z^\gamma, y - \beta z^\gamma).$$

Lemma 9.2. *Suppose that x, y, z are good parameters for g, $\nu_T(g) = r$ and $Sing_r(g)$ is the maximal ideal of T. Then there exists a change of variables*

$$x_1 = x - \sum_{i=1}^{n} \alpha_i z^i, y_1 = y - \sum_{i=1}^{n} \beta_i z^i$$

such that g is not solvable with respect to x_1, y_1, z.

Proof. If there doesn't exist such a change of variables then we construct series

$$x_\infty = x - \sum_{i=1}^{\infty} \alpha_i z^i, y_\infty = y - \sum_{i=1}^{\infty} \beta_i z^i$$

such that $\gamma(g; x_\infty, y_\infty, z) = \infty$, and thus

$$g \in (x - \sum_{i=1}^{\infty} \alpha_i z^i, y - \sum_{i-1}^{\infty} \beta_i z^i)^r \subset T,$$

a contradiction to our assumption that $\text{Sing}_r(g)$ is the maximal ideal of T. □

Observe that if $\nu_T(g) = r$, $\tau(g) = 2$, x, y, z are good parameters for g and g is not solvable for x, y, z, then $\gamma(g; x, y, z) > 1$.

Lemma 9.3. *Suppose that $\nu_T(g) = r$, $\tau(g) = 2$, $Sing_r(g)$ is the maximal ideal of T, x, y, z are good parameters for g and g is not solvable with respect to x, y, z.*

Suppose that T_1 is the completion of a local ring of the blow up of the maximal ideal of T such that the strict transform g_1 of g in T_1 satisfies $\nu_{T_1}(g_1) = r$ and $\tau(g_1) = 2$. Then

1. *T_1 has regular parameters x_1, y_1, z_1 defined by*

$$x = x_1 z_1, y = y_1 z_1, z = z_1. \tag{20}$$

2. *x_1, y_1, z_1 are good parameters for g_1 and $V(x_1, y_1)$ is an approximate manifold of g_1.*
3. *$\gamma(g_1; x_1, y_1, z_1) = \gamma(g; x, y, z) - 1$.*
4. *g_1 is not solvable with respect to x_1, y_1, z_1.*

Proof. Assertion 1 follows from Lemma 8.4, as $V(x, y)$ is an approximate manifold for g. We have

$$g_1 = \frac{g}{z_1^r}.$$

Assertions 3 and 4 follow from substitution of (20) in (16), (17), (18) and (19).

Since we are assuming that $\tau(J_1) = 2$, we have that there exist $\alpha, \beta \in \mathbf{k}$ such that an approximate manifold of J_1 has the form

$$V(x_1 + \alpha z_1, y_1 + \beta z_1), \tag{21}$$

and x_1, y_1, z_1 are good parameters for J_1.

Since we are assuming that $\nu_{T_1}(g_1) = r$, we have $\gamma(g_1; x_1, y_1, z_1) \geq 1$. If $\gamma(g_1; x_1, y_1, z_1) > 1$, then $V(x_1, y_1)$ is an approximate manifold of g_1.

Suppose that $\gamma(g_1; x_1, y_1, z_1) = 1$. Then

$$[g_1]_{x_1 y_1 z_1} = \frac{1}{z_1^r}[g]_{xyz}.$$

Since $\gamma(g_1;x_1,y_1,z_1)=1$, $[g_1]_{x_1y_1z_1}$ is the r-leading form of g_1 with respect to x_1,y_1,z_1. By (21), there exists a form $\Psi_g(u,v)$, which depends on g, such that

$$[g_1]_{x_1y_1z_1}=\Psi_g(x_1+\alpha z_1,y_1+\beta z_1)=L_g(x_1,y_1)+z_1\Omega$$

where $\Omega\in T_1$ and $L_g(x,y)$ is the r-leading form of g with respect to x,y,z. Setting $z_1=0$, we have $L_g(x_1,y_1)=\Psi_g(x_1,y_1)$, so that

$$[g_1]_{x_1y_1z_1}=L_g(x_1+\alpha z_1,y_1+\beta z_1).$$

We conclude that g_1 is solvable with respect to x_1,y_1,z_1, a contradiction to the assumption that $\gamma(g_1;x_1,y_1,z_1)=1$. □

We now prove that (15) cannot have infinite length.

Since $\tau(q_i)=2$ for all i, the assumption that (15) is infinite and Lemma 8.4 imply that for all i, $\mathrm{Sing}_r(I_i)$ is the maximal ideal of R_i and R_{i+1} is the completion of a local ring of a closed point of the blow up of the maximal ideal of $\mathrm{Spec}(R_i)$.

Moreover, $\tau(q)=2$ implies that there exist regular parameters (x,y,z) in R_0 such that $V(y,z)$ is an approximate manifold of $I_0=(g_0)$, and thus x,y,z are good parameters for I_0.

By Lemma 9.2, there exists a change of variables in R_0 so that we find good parameters x,y,z for g_0 such that g_0 is not solvable with respect to x,y,z. Set $\Omega(q_0)=\gamma(g_0;x,y,z)$.

By Lemma 9.3, we can inductively define positive rational numbers $\Omega(q_n)$ such that $\Omega(q_{n+1})=\Omega(q_n)-1$ for all n, giving a contradiction if (15) has infinite length.

10. Reduction When $\tau(\mathbf{q})=\mathbf{1}$.

This is the most interesting and difficult case. The proof is based on Hironaka's termination argument in [20].

10.1. **Definition of Ω.** In this subsection we define Ω for an element g in a power series ring T of three variables, over an algebraically closed field K.

Suppose that x,y,z are regular parameters in T, and r is a positive integer.

For $g\in T$, we have an expansion

$$g=\sum b_{ijk}x^iy^jz^k$$

with $b_{ijk}=b_{ijk}(g)\in K$. We define

$$\Delta=\Delta(g;x,y,z)=\left\{\left(\frac{i}{r-k},\frac{j}{r-k}\right)\in\mathbf{Q}^2\mid k<r\text{ and }b_{ijk}\neq 0\right\}.$$

Let $|\Delta(g;x,y,z)|$ be the smallest convex set in $\mathbf{R}^2$ such that $\Delta\subset|\Delta|$, and $(a,b)\in|\Delta|$ implies $(a+c,b+d)\in|\Delta|$ for all $c,d\geq 0$. For $\gamma\in\mathbf{R}$, let $S(\gamma)$ be the line through $(\gamma,0)$ with slope -1.

Suppose that $|\Delta|\neq\emptyset$. We define $\alpha_{xyz}(g)$ to be the smallest a appearing in any $(a,b)\in|\Delta|$, $\beta_{xyz}(g)$ to be the smallest b such that $(\alpha_{xyz}(g),b)\in|\Delta|$. Let $\gamma_{xyz}(g)$ be the smallest number γ such that $S(\gamma)\cap|\Delta|\neq\emptyset$ and let $\delta_{xyz}(g)$ be such that $(\gamma_{xyz}(g)-\delta_{xyz}(g),\delta_{xyz}(g))$ is the lowest point on $S(\gamma_{xyz}(g))\cap|\Delta|$. $(\alpha_{xyz}(I),\beta_{xyz}(I))$ and $(\gamma_{xyz}(g)-\delta_{xyz}(g),\delta_{xyz}(g))$ are vertices of $|\Delta|$. Define $\varepsilon_{xyz}(g)$ to be the absolute value of the largest slope of a line through $(\alpha_{xyz}(g),\beta_{xyz}(g))$ such that no points of $|\Delta|$ lie below it.

We now define

$$\Omega(g;x,y,z) = (\beta_{xyz}(g), \frac{1}{\varepsilon_{xyz}(g)}, \alpha_{xyz}(g)),$$

which is in the ordered set (by the Lex order)

$$\left(\frac{1}{r!}\mathbf{N}\right) \times (\mathbf{Q} \cup \infty) \times \left(\frac{1}{r!}\mathbf{N}\right)$$

We observe that:

1. The vertices of $|\Delta|$ are points of Δ, which lie in the lattice $\frac{1}{r!}\mathbf{Z} \times \frac{1}{r!}\mathbf{Z}$.
2. $\nu_R(g) < r$ holds if and only if $|\Delta|$ contains a point on $S(c)$ with $c < 1$ which holds if and only if there is a vertex (a,b) with $a+b<1$.
3. $\alpha_{xyz}(g) < 1$ if and only if $g \notin (x,z)^r$.
4. A vertex of $|\Delta|$ lies below the line $b=1$ if and only if $g \notin (y,z)^r$.

10.2. **Well preparation.** (x,y,z) will be called good parameters for g if $\nu_T(g) = r$ and $b_{00r} \neq 0$. Good parameters exist if $\tau(g) = 1$.

Suppose that $\nu_T(g) = r$ and and (x,y,z) are good parameters for g. Let $\Delta = \Delta(g;x,y,z)$ and suppose that (a,b) is a vertex of $|\Delta|$. Define

$$S_{(a,b)} = \left\{ k \mid k < r, \text{ and } (\frac{i}{r-k}, \frac{j}{r-k}) = (a,b) \right\},$$

and

$$\{g\}^{ab}_{xyz} = b_{00r}z^r + \sum_{k \in S_{(a,b)}} b_{a(r-k),b(r-k),k}x^{a(r-k)}y^{b(r-k)}z^k.$$

We will say that the vertex (a,b) is not prepared on $|\Delta|$ if a,b are integers and there exists $\eta \in K$ such that

$$\{g\}^{ab}_{xyz} = b_{00r}(g)(z - \eta x^a y^b)^r.$$

We will say that the vertex (a,b) is prepared on $|\Delta|$ if such an η does not exist.

If the vertex (a,b) is not prepared, then we can make an (a,b) preparation, which is the change of parameters

$$z_1 = z - \eta x^a y^b.$$

If all vertices (a,b) of $|\Delta|$ are prepared, then we say that $(g;x,y,z)$ is well prepared.

Lemma 10.1. *Suppose that $\nu_T(g) = r$, $\tau(g) = 1$, and (x,y,z) are good parameters of g. Then $z = 0$ is an approximate manifold of g if and only if the vertices $(0,1)$ and $(1,0)$ are prepared on $|\Delta(g;x,y,z)|$.*

Proof. Suppose that $(0,1)$ and $(1,0)$ are prepared on $|\Delta(g;x,y,z)|$. Since $\tau(g) = 1$, there is a linear form $ax+by+cz$ with $a,b,c \in K$ such that $V(ax+by+cz)$ is an approximate manifold of g. There exists $0 \neq d_g \in K$ such that the r-leading form of g is $d_g(ax+by+cz)^r$.

As (x,y,z) are good parameters for g, $b_{00r}(g) \neq 0$, so we must have $c \neq 0$. If $a \neq 0$, then $(1,0)$ is a vertex of $|\Delta(g;x,y,z)|$, which is not prepared since

$$\{g\}^{(1,0)}_{xyz} = d_g c^r (z + \frac{a}{c}x)^r.$$

Thus $a=0$. If $b\neq 0$, then $(0,1)$ is a vertex of $|\Delta(g;x,y,z)|$ which is not prepared since

$$\{g\}_{xyz}^{(0,1)}=d_gc^r(z+\frac{b}{c}y)^r.$$

Thus $a=b=0$ and $z=0$ is an approximate manifold of g.

The proof of the converse also follows from the above arguments. □

Lemma 10.2. *Consider the terms in the expansion*

$$h=\sum_{\lambda=0}^{k}\eta^{k-\lambda}\binom{k}{\lambda}x^{i+(k-\lambda)a}y^{j+(k-\lambda)b}z_1^{\lambda} \tag{22}$$

obtained by substituting $z_1=z-\eta x^ay^b$ into the monomial $x^iy^jz^k$. Define a projection for $(a,b,c)\in\mathbf{N}^3$ such that $c<r$,

$$\pi(a,b,c)=(\frac{a}{r-c},\frac{b}{r-c}).$$

1. *Suppose that $k<r$. Then the exponents of monomials in (22) with nonzero coefficients project into the line segment joining (a,b) to $(\frac{i}{r-k},\frac{j}{r-k})$.*
   a. *If $(a,b)=(\frac{i}{r-k},\frac{j}{r-k})$, then all these monomials project to (a,b).*
   b. *If $(a,b)\neq(\frac{i}{r-k},\frac{j}{r-k})$, then $x^iy^jz_1^k$ is the unique monomial in (22) which projects onto $(\frac{i}{r-k},\frac{j}{r-k})$. No monomial in (22) projects to (a,b).*
2. *Suppose that $r\leq k$, and $(i,j,k)\neq(0,0,r)$. Then all exponents in (22) with nonzero coefficients and z_1 exponent less than r project into*

$$\left((a,b)+\mathbf{Q}^2_{\geq 0}\right)-\{(a,b)\}.$$

3. *Suppose that $(i,j,k)=(0,0,r)$. Then all exponents in (22) with nonzero coefficients and z_1 exponent less than r project to (a,b).*

Lemma 10.2 is proved by a straight forward calculation. We deduce Lemma 10.3 from Lemma 10.2. Detailed proofs are given in Lemma 7.17 and Lemma 7.18 of [11].

Lemma 10.3. *Suppose that the vertex (a,b) is not perpared on $|\Delta|$, and $z_1=z-\eta x^ay^b$ is an (a,b) preparation. Then*

1. *$|\Delta(g;x,y,z_1)|\subset|\Delta(g;x,y,z)|-\{(a,b)\}$.*
2. *If (a',b') is another vertex of $|\Delta(g;x,y,z)|$, then (a',b') is a vertex of $|\Delta(g;x,y,z_1)|$, and $\{g\}_{x,y,z_1}^{a',b'}$ is obtained from $\{g\}_{x,y,z}^{a',b'}$ by substituting z_1 for z.*

Lemma 10.4. *Suppose that $Sing_r(g)$ has dimension <2, $\nu_T(g)=r$, $\tau(g)=1$ and (x,y,z) are good parameters for g. Then there is a formal series $\Psi(x,y)\in K[[x,y]]$ such that under the substitution $z=z_1-\Psi(x,y)$, (x,y,z_1) are good parameters for g and $(g;x,y,z_1)$ is well prepared.*

Lemma 10.4 is proved by successive removal of vertices which are not prepared, using Lemma 10.3.

Remark 10.5. *It is not always possible to well prepare after a finite number of preparations, so that z_1 could be a formal series in z,x,y. An example is given in Example 7.19* [11].

10.3. **Very Well Preparation.** We will also consider change of variables of the form

$$y_1 = y - \eta x^n$$

for $\eta \in K$, n a positive integer, which we will call translations.

Lemma 10.6. *Consider the expansion*

$$h = \sum_{\lambda=0}^{j} \eta^{j-\lambda} \binom{j}{\lambda} x^{i+(j-\lambda)n} y_1^{\lambda} z^k \tag{23}$$

obtained by substituting $y_1 = y - \eta x^n$ into the monomial $x^i y^j z^k$. Consider the projection for $(a,b,c) \in \mathbf{N}^3$ such that $c < r$ defined by

$$\pi(a,b,c) = (\frac{a}{r-c}, \frac{b}{r-c}).$$

Suppose that $k < r$. Set $(a,b) = (\frac{i}{r-k}, \frac{j}{r-k})$. Then $x^i y_1^j z^k$ is the unique monomial in (23) whose coefficients project onto (a,b). All other monomials in (23) with non-zero coefficient project to points below (a,b) on the line through (a,b) with slope $-\frac{1}{n}$.

Lemma 10.6 is proved by a straight forward calculation. We deduce Lemma 10.7 from Lemma 10.6 and the definition of well preparedness. A detailed proof is given in Lemma 7.21 [11].

Lemma 10.7. *Suppose that $(g;x,y,z)$ is well prepared, $y_1 = y - \eta x^n$ is a translation, and $z_1 = z - \psi(x, y_1)$ is a subsequent well preparation. Then*

$$\alpha_{x,y_1,z_1}(g) = \alpha_{xyz}(g), \beta_{x,y_1,z_1}(g) = \beta_{xyz}(g) \text{ and } \gamma_{x,y_1,z_1}(g) = \gamma_{xyz}(g).$$

Definition 10.8. *Suppose that $I \subset R$ is an ideal such that $\nu_T(g) = r$, $\tau(g) = 1$, $Sing_r(g)$ has dimension < 2 and (x,y,z) are good parameters for g. Let $\alpha = \alpha_{xyz}(g)$, $\beta = \beta_{xyz}(g)$, $\gamma = \gamma_{xyz}(g)$, $\delta = \delta_{xyz}(g)$, $\varepsilon = \varepsilon_{xyz}(g)$. $(g;x,y,z)$ will be said to be very well prepared if it is well prepared and one of the following conditions holds.*

1. *$(\gamma - \delta, \delta) \neq (\alpha, \beta)$ and if we make a translation $y_1 = y - \eta x$, with subsequent well preparation $z_1 = z - \Psi(x, y_1)$, then*

   $$\alpha_{xy_1z_1}(g) = \alpha, \beta_{xy_1z_1}(g) = \beta, \gamma_{xy_1z_1}(g) = \gamma$$

   and

   $$\delta_{xy_1z_1}(g) \leq \delta.$$

2. *$(\gamma - \delta, \delta) = (\alpha, \beta)$ and one of the following cases hold:*
   a. $\varepsilon = 0$
   b. *$\varepsilon \neq 0$ and $\frac{1}{\varepsilon}$ is not an integer*
   c. *$\varepsilon \neq 0$ and $n = \frac{1}{\varepsilon}$ is a (positive) integer and for any $\eta \in K$, if $y_1 = y - \eta x^n$ is a translation, with subsequent well preparation $z_1 = z - \Psi(x, y_1)$, then $\varepsilon_{xy_1z_1}(I) = \varepsilon$. Further, if (c,d) is the lowest point on the line through (α, β) with slope $-\varepsilon$ in $|\Delta(g;x,y,z)|$ and (c_1, d_1) is the lowest point on this line in $|\Delta(g;x,y_1,z_1)|$, then $d_1 \leq d$.*

Lemma 10.9. *Suppose that $g \in T$ is such that $\nu_T(g) = r$, $\tau(g) = 1$, $Sing_r(g)$ has dimension < 2 and (x, y, z) are good parameters for g. Then there are formal substitutions*

$$z_1 = z - \Psi(x, y), y_1 = y - \varphi(x)$$

where $\Psi(x, y), \varphi(x)$ are series such that $(g; x, y_1, z_1)$ is very well prepared.

Proof. By Lemmas 10.4 and 10.7, we can find good parameters x, y, z for g such that $(g; x, y, z)$ is well prepared, α, β, γ do not change under translation $y_1 = y - \eta x$ followed by subsequent well preparation, and δ is maximal. If $(\gamma - \delta, \delta) \neq (\alpha, \beta)$, then we have achieved case 1 of Definition 10.8, so that $(g; x, y, z)$ is very well prepared.

We now assume that $(g; x, y, z)$ is well prepared, and $(\alpha, \beta) = (\gamma - \delta, \delta)$. If $\varepsilon = 0$ or $\frac{1}{\varepsilon}$ is not an integer then $(g; x, y, z)$ is very well prepared.

Suppose that $n = \frac{1}{\varepsilon}$ is an integer. We then choose $\eta \in K$ such that with the translation $y_1 = y - \eta x^n$, and subsequent well preparation, we maximize d for points (c, d) of the line through (α, β) with slope $-\varepsilon$ on the boundary of $|\Delta(g; x, y_1, z_1)|$. By Lemma 10.7, α, β and γ are not changed. If we now have that $d \neq \beta$, we are very well prepared.

If the process does not end after a finite number of iterations, then we construct formal series $y' = y - \varphi(x)$ and $z' = z - \psi(x, y)$ such that $|\Delta(g; x, y', z')|$ has the single vertex (α, β), and $(g; x, y', z')$ is thus very well prepared. ☐

10.4. **Effect of a permissible blow up on Δ.**

Definition 10.10. *Suppose that $g \in T$ is such that $\nu_T(g) = r$, $\tau(g) = 1$, $Sing_r(g)$ has dimension < 2, and (x, y, z) are good parameters for g.*

We consider 4 types of transformations $T \to T_1$, where T_1 is the completion of the local ring of a blow up of T, and T_1 has regular parameters (x_1, y_1, z_1) related to the regular parameters (x, y, z) of T by one of the following rules. In all cases, if $\nu_{T_1}(g_1) = r$ and $\tau(g_1) = 1$, then (x_1, y_1, z_1) are good parameters for g_1.

Tr1 *$Sing_r(g) = V(x, y, z)$,*

$$x = x_1, y = x_1(y_1 + \eta), z = x_1 z_1,$$

with $\eta \in K$. Then $g_1 = \frac{g}{x_1^r}$ is the strict transform of g in T_1.

Tr2 *$Sing_r(g) = V(x, y, z)$,*

$$x = x_1 y_1, y = y_1, z = y_1 z_1.$$

Then $g_1 = \frac{g_1}{y_1^r}$ is the strict transform of g in T_1.

Tr3 *$Sing_r(g) = V(x, z)$,*

$$x = x_1, y = y_1, z = x_1 z_1.$$

Then $g_1 = \frac{g_1}{x_1^r}$ is the strict transform of g in T_1.

Tr4 *$Sing_r(g) = V(y, z)$,*

$$x = x_1, y = y_1, z = y_1 z_1.$$

Then $g_1 = \frac{g_1}{y_1^r}$ is the strict transform of g in T_1.

Lemma 10.11. *Suppose that assumptions are as in Definition 10.10. Further suppose that (x,y,z) and (x_1,y_1,z_1) are related by a transformation of one of the above types Tr1 - Tr4, and $\nu_{T_1}(g_1)=r$, $\tau(g_1)=1$. Then there is a 1-1 correspondence*

$$\sigma:\Delta(g,x,y,z)\rightarrow\Delta(g_1,x_1,y_1,z_1)$$

defined by

1. $\sigma(a,b)=(a+b-1,b)$ *if the transformation is a Tr1 with* $\eta=0$.
2. $\sigma(a,b)=(a,a+b-1)$ *if the transformation is a Tr2.*
3. $\sigma(a,b)=(a-1,b)$ *if the transformation is a Tr3.*
4. $\sigma(a,b)=(a,b-1)$ *if the transformation is a Tr4.*

The proof of Lemma 10.11 is a straight forward calculation. We deduce Lemma 10.12 from Lemma 10.11. Details are given in Lemma 7.26 [11].

Lemma 10.12. *In each of the four cases of the preceeding lemma, if $\sigma(a,b)=(a_1,b_1)$ is a vertex of $|\Delta(g_1;x_1,y_1,z_1)|$, then (a,b) is a vertex of $|\Delta(g;x,y,z)|$, and if $(g;x,y,z)$ is (a,b) prepared then $(g_1;x_1,y_1,z_1)$ is (a_1,b_1) prepared. In particular, $(g_1;x_1,y_1,z_1)$ is well prepared if $(g;x,y,z)$ is well prepared.*

Lemma 10.13. *Suppose that assumptions are as in Definition 10.10 and (x,y,z) and (x_1,y_1,z_1) are related by a Tr3 transformation. Suppose that $\nu_{T_1}(g_1)=r$ and $\tau(g_1)=1$. If $(g;x,y,z)$ is very well prepared, then $(g_1;x_1,y_1,z_1)$ is very well prepared,*

$$\beta_{x_1y_1z_1}(g_1)=\beta_{xyz}(g),\delta_{x_1y_1z_1}(g_1)=\delta_{xyz}(g),\varepsilon_{x_1y_1z_1}(g_1)=\varepsilon_{xyz}(g)$$

and

$$\alpha_{x_1y_1z_1}(g_1)=\alpha_{xyz}(g)-1,\gamma_{x_1y_1z_1}(g_1)=\gamma_{xyz}(g)-1.$$

We further have $\mathrm{Sing}_r(g_1)\subset V(x_1,z_1)$.

Proof. Well preparation is preserved by Lemma 10.12. We deduce from the definition of Tr3 that $(g_1;x_1,y_1,z_1)$ is also very well prepared.

Since $\mathrm{Sing}_r(g)=V(x,z)$, we have $\mathrm{Sing}_r(g_1)\subset V(x_1)$, the exceptional divisor of our Tr3 transformation. $\mathrm{Sing}_r(g_1)$ is either the maximal ideal, or the germ of a nonsingular irreducible curve, by Lemma 8.5.

Suppose that $\mathrm{Sing}_r(g_1)$ is a nonsingular curve C. Then C has formal local equations

$$x_1=z_1-\sum_{t=1}^{\infty}a_ty_1^t=0$$

for some $a_t\in K$ by Lemma 8.4. Expand

$$g=(\sum_{i+k=r}\sum_{j=0}^{\infty}b_{ijk}x^iy^jz^k)+h$$

with $h\in(x,z)^{r+1}$.

$$\frac{g}{x_1^r}=\left(\sum_{i+k=r}\sum_{j=0}^{\infty}b_{ijk}y_1^jz_1^k)\right)+x_1\frac{h}{x_1^{r+1}}$$

with $\frac{h}{x_1^{r+1}}\in T_1$.

$\nu_T(\frac{g}{x_1^r}) = r$ and $\frac{g}{x_1^r} \in (x_1, z_1 - \sum_{t=1}^{\infty} a_t y_1^t)^r$ implies

$$\sum_{i+k=r} \sum_{j=0}^{\infty} b_{ijk} y_1^j z_1^k = b_{00r}(z_1 - \sum_{t=1}^{\infty} a_t y_1^t)^r.$$

Thus

$$g = b_{00r}(z - \sum_{t=1}^{\infty} a_t x y^t)^r + h.$$

Suppose that $a_t \neq 0$ for some t. Let t_0 be the smallest value of t for which $a_t \neq 0$. Then $(\alpha, \beta) = (1, t_0)$, since $g \in (x, z)^r$. But $(1, t_0)$ is then a vertex of $|\Delta(g; x, y, z)|$ which is not prepared, a contradiction. Thus $a_t = 0$ for all t and $C = V(x, z_1)$. We conclude that $\mathrm{Sing}_r(g_1) \subset V(x_1, z_1)$

The remaining statements of the lemma follow from Lemma 10.11.

Lemma 10.14. *Suppose that assumptions are as in Definition 10.10, (x, y, z) and (x_1, y_1, z_1) are related by a Tr1 transformation with $\eta = 0$, $(g; x, y, z)$ is very well prepared and $\nu_{T_1}(g_1) = r$, $\tau(g_1) = 1$. Let $(g_1; x_1, y', z')$ be a very well preparation of $(g_1; x_1, y_1, z_1)$. Let*

$$\alpha = \alpha_{xyz}(g), \beta = \beta_{xyz}(g), \gamma = \gamma_{xyz}(g), \delta = \delta_{xyz}(g), \varepsilon = \varepsilon_{xyz}(g),$$

$$\alpha_1 = \alpha_{x_1 y' z'}(g_1), \beta_1 = \beta_{x_1 y' z'}(g_1), \gamma_1 = \gamma_{x_1 y' z'}(g_1), \delta_1 = \delta_{x_1 y' z'}(g_1), \varepsilon_1 = \varepsilon_{x_1 y' z'}(g_1).$$

Then

$$(\beta_1, \frac{1}{\varepsilon_1}, \alpha_1) < (\beta, \frac{1}{\varepsilon}, \alpha)$$

in the lexicographical order. If $\beta_1 = \beta$, and $\varepsilon \neq 0$, then $\frac{1}{\varepsilon_1} = \frac{1}{\varepsilon} - 1$.

We further have $\mathrm{Sing}_r(g_1) \subset V(x_1, z')$.

Proof. $(g_1; x_1, y_1, z_1)$ is well prepared by Lemma 10.12.

Let $\sigma : \Delta(g; x, y, z) \to \Delta(g_1; x_1, y_1, z_1)$ be the 1-1 correspondence of 1 in Lemma 10.11, which is defined by $\sigma(a, b) = (a + b - 1, b)$. σ transforms lines of slope $m \neq -1$ to lines of slope $\frac{m}{m+1}$, and transforms lines of slope -1 to vertical lines.

We deduce the formulas for transformation of $\alpha, \beta, \gamma, \delta, \varepsilon$ from consideration of the effect of a Tr1 transformation followed by very well preparation on $(g; x, y, z)$ in the different cases of Definition 10.8. We need the assumption $\mathrm{Sing}_r(g) = V(x, y, z)$ so that $\beta < 1$, to conclude that $\alpha_1 < \alpha$ if $\varepsilon = 0$.

By Lemmas 8.4 and 10.1, $\mathrm{Sing}_r(g_1) \subset V(x_1, z_1)$. If $\mathrm{Sing}_r(g_1) = V(x_1, z_1)$, then all vertices of $|\Delta(g_1; x_1, y_1, z_1)|$ have the first coordinate ≥ 1, and thus $\mathrm{Sing}_r(g_1) = V(x_1, z')$.

Lemma 10.15. *Suppose that assumptions are as in Definition 10.10, (x, y, z) and (x_1, y_1, z_1) are related by a Tr2 or a Tr4 transformation and $(g; x, y, z)$ is well prepared. Suppose that $\nu_{T_1}(g_1) = r$ and $\tau(g_1) = 1$. We have that $(g_1; x_1, y_1, z_1)$ is well prepared and $\mathrm{Sing}_r(g_1) \subset V(y_1, z_1)$. Let $(g_1; x_1, y', z')$ be a very well preparation. Then*

$$\beta_{x_1, y_1, z_1}(g_1) = \beta_{x_1, y', z'}(g_1) < \beta_{xyz}(g).$$

Proof. First suppose that $T \to T_1$ is a Tr2 transformation. The correspondence $\sigma : \Delta(g; x, y, z) \to \Delta(g_1; x_1, y_1, z_1)$ of 2 in Lemma 10.11 is defined by $\sigma(a, b) = (a, a + b - 1)$. σ takes lines of slope m to lines of slope $m + 1$ Thus

$$(\alpha_{x_1 y_1 z_1}(g_1), \beta_{x_1 y_1 z_1}(g_1) = (\alpha_{xyz}(g), \alpha_{xyz}(g) + \beta_{xyz}(g) - 1).$$

Since $\mathrm{Sing}_r(g) = V(x,y,z)$ by assumption, we have $\alpha_{xyz}(g) < 1$, and thus

$$\beta_{x_1y_1z_1}(g_1) < \beta_{xyz}(g).$$

$(g_1; x_1, y_1, z_1)$ is well prepared by Lemma 10.12. Thus the vertex $(\alpha_{x_1y_1z_1}(g_1), \beta_{x_1y_1z_1}(g_1))$ is not affected by very well preparation. We thus have $\beta_{x_1y'z'}(g_1) < \beta_{xyz}(g)$.

Now suppose that $T \to T_1$ is a Tr4 transformation. The 1-1 correspondence $\sigma : \Delta(g; x, y, z) \to \Delta(g_1; x_1, y_1, z_1)$ of 4 in Lemma 10.11 is defined by $\sigma(a,b) = (a, b-1)$. Thus $\beta_{x_1y_1z_1}(g_1) < \beta_{xyz}(g)$. $(g_1; x_1, y_1, z_1)$ is well prepared by Lemma 10.12, and the vertex $(\alpha_{x_1y_1z_1}(g_1), \beta_{x_1y_1z_1}(g_1))$ is not affected by very well preparation. Thus $\beta_{x_1y'z'}(g_1) = \beta_{x_1y_1z_1}(g_1) < \beta_{xyz}(g)$.

The proof that $\mathrm{Sing}_r(g_1) \subset V(y_1, z_1)$ is as in the proofs of Lemma 10.13 and Lemma 10.14. □

Lemma 10.16. *Suppose that p is a prime, s is a nonnegative integer and r_0 is a positive integer such that $p \not| r_0$. Let $r = r_0p^s$. Then*

1. $\binom{r}{\lambda} \equiv 0$ *mod p, if $p^s \not| \lambda$, $0 \le \lambda \le r$ is an integer.*
2. $\binom{r}{\lambda p^s} \equiv \binom{r_0}{\lambda}$ *mod p, if $0 \le \lambda \le r_0$ is an integer.*

Proof. Compare the expansions over $\mathbf{Z}_p$ of $(x+y)^r = (x^{p^s} + y^{p^s})^{r_0}$. □

The following theorem is the most delicate part of the proof, where we really see the difference between characteristic 0 and characteristic p.

Theorem 10.17. *Suppose that assumptions are as in Definition 10.10, (x, y, z) and (x_1, y_1, z_1) are related by a Tr1 transformation with $\eta \ne 0$ and $(g; x, y, z)$ is very well prepared. Suppose that $\nu_{T_1}(g_1) = r$, $\tau(g_1) = 1$ and $\beta_{xyz}(g) > 0$. Then there exist good parameters (x_1, y_1', z_1') in T_1 such that $(g_1; x_1, y_1', z_1')$ is very well prepared, with $\beta_{x_1,y_1',z_1'}(g_1) < \beta_{x,y,z}(g)$ and $\mathrm{Sing}_r(g_1) \subset V(x_1, z_1')$.*

Proof. Let $\alpha = \alpha_{xyz}(g)$, $\beta = \beta_{xyz}(g)$, $\gamma = \gamma_{xyz}(g)$, $\delta = \delta_{xyz}(g)$. Lemma 10.1 implies $z = 0$ is an approximate manifold of g. Thus since $\tau(g) = 1$,

$$\alpha + \beta > 1. \tag{24}$$

Apply the translation $y' = y - \eta x$ and well prepare by some substitution $z' = z - \Psi(x, y')$. This does not change α, β or γ. Set $\delta' = \delta_{x,y',z'}(g)$.

First assume that $\delta' < \beta$. We have regular parameters $(x_1, y_1, \overline{z}_1)$ in T_1 such that $x = x_1, y' = x_1y_1, z' = x_1\overline{z}_1$, so we have, by 1 in Lemma 10.11, a 1-1 correspondence

$$\sigma : \Delta(g; x, y', z') \to \Delta(g_1; x_1, y_1, \overline{z}_1)$$

defined by $\sigma(a,b) = (a+b-1, b)$. $(g_1; x_1, y_1, \overline{z}_1)$ is well prepared by Lemma 10.12. Since $\delta' < \beta$, the line segment in $|\Delta(g; x, y', z')|$ through (α, β) and $(\gamma - \delta', \delta')$ has slope ≤ -1, so it is transformed by σ to a segment with positive slope or a vertical line in $|\Delta(g_1; x_1, y_1, \overline{z}_1)|$. We have that for $\alpha_1 = \alpha_{x_1y_1\overline{z}_1}(g_1)$ and $\beta_1 = \beta_{x_1y_1\overline{z}_1}(g_1)$, $(\alpha_1, \beta_1) = \sigma(\gamma - \delta', \delta')$ and $\beta_1 = \delta' < \beta$. Since $(I_1; x_1, y_1, \overline{z}_1)$ is well prepared, very well preparation does not effect the vertex (α_1, β_1), by Lemma 10.7. After very well preparation, we thus have regular parameters x_1, y_1', z_1' in T_1 such that $(g_1; x_1, y_1', z_1')$ is very well prepared and $\beta_{x_1,y_1',z_1'}(g_1) < \beta_{xyz}(g)$. $\mathrm{Sing}_r(g_1) \subset V(x_1, z_1')$ as in the proof of Lemma 10.14.

We have thus reduced the proof to showing that $\delta' < \beta$.

If $(\alpha,\beta) \neq (\gamma-\delta,\delta)$, we have $\delta' \leq \delta < \beta$ by Lemma 10.7, since $(g;x,y,z)$ is very well prepared.

Suppose that $(\alpha,\beta) = (\gamma-\delta,\delta)$. Set

$$W = \left\{(i,j,k) \in \mathbf{N}^3 \mid k < r \text{ and } \left(\frac{i}{r-k}, \frac{j}{r-k}\right) = (\alpha,\beta)\right\}.$$

Expand $g = \sum a_{ijk}x^iy^jz^k$, and set

$$F_g = \sum_{(i,j,k)\in W} a_{ijk}x^iy^jz^k.$$

By assumption, (α,β) is prepared on $|\Delta(g;x,y,z)|$, which implies that if α,β are integers, then there does not exist $\lambda \in K$ such that

$$a_{00r}z^r + F_g = a_{00r}(z-\lambda x^\alpha y^\beta)^r.$$

Moreover,

$$F_g(x,y,z) = F_g(x,y'+\eta x,z) = \sum_{(i,j,k)\in W}\sum_{\lambda=0}^{j} a_{ijk}\eta^\lambda \binom{j}{\lambda} x^{i+\lambda}(y')^{j-\lambda}z^k. \tag{25}$$

By Lemma 10.6, the terms in the expansion of $g(x,y'+\eta x,z)$ contributing to $(\gamma,0)$ in $|\Delta(g;x,y',z)|$, where $\gamma = \alpha+\beta$, are

$$F_{g,(\gamma,0)} = \sum_{(i,j,k)\in W} a_{ijk}\eta^j x^{i+j}z^k.$$

If $(g;x,y',z)$ is $(\gamma,0)$ prepared, then $\delta' = 0 < \beta$. Suppose that $(\gamma,0)$ is not prepared on $|\Delta(g,x,y',z)|$. Then $\gamma \in \mathbf{N}$, and there exists $\psi \in K$ such that

$$a_{00r}(z-\psi x^\gamma)^r = a_{00r}z^r + F_{g,(\gamma,0)}, \tag{26}$$

so that, with $\omega = \frac{-\psi}{\eta^\beta} \in K$, for $0 \leq k < r$, we have

1. If $\binom{r}{r-k} \neq 0$ (in K) then $i = \alpha(r-k), j = \beta(r-k) \in \mathbf{N}$, and

$$a_{ijk} = a_{00r}\binom{r}{r-k}\omega^{r-k}. \tag{27}$$

2. If $i = \alpha(r-k), j = \beta(r-k) \in \mathbf{N}$ and $\binom{r}{r-k} = 0$ (in K), then $a_{ijk} = 0$.

Thus by Lemma 10.16, for $(i,j,k) \in W$, $a_{ijk} = 0$ if $p^s \not| k$.

If K has characteristic zero, we obtain a contradiction to our assumption that $(\gamma,0)$ in not prepared on $|\Delta(g,x,y',z)|$. Thus $0 = \delta' < \beta$ if K has characteristic zero.

Now we consider the case where K has characteristic $p > 0$, and $r = p^sr_0$ with $p \not| r_0$, $r_0 \geq 1$. Then by Lemma 10.16, for $(i,j,k) \in W$, we have $a_{ijk} = 0$ if $p^s \not| k$. By (27), and Lemma 10.16, we have that $i = \alpha p^s, j = \beta p^s \in \mathbf{N}$, and f $a_{i,j,(r_0-1)p^s} \neq 0$.

We have an expression $\beta p^s = ep^t$ where $p \not| e$. Suppose that $t \geq s$. Then $\beta \in \mathbf{N}$, which implies that $\alpha = \gamma - \beta \in \mathbf{N}$, so that

$$a_{00r}(z+\omega x^\alpha y^\beta)^r = a_{00r}z^r + F_g,$$

a contradiction, since (α,β) is by assumption prepared on $|\Delta(g;x,y,z)|$. Thus $t<s$. Suppose that $e=1$. Then $\beta=p^{t-s}<1$ and $\alpha<1$ (since we must have that $\mathrm{Sing}_r(g)=V(x,y,z)$) which implies (since γ is an integer) that $\gamma=\alpha+\beta=1$, a contradiction to (24). Thus $e>1$. Also,

$$
\begin{aligned}
a_{00r}z^r+F_g &= a_{00r}(z^{p^s}+\omega^{p^s}x^{\alpha p^s}y^{\beta p^s})^{r_0}\\
&= a_{00r}(z^{p^s}+\omega^{p^s}x^{\alpha p^s}(y'+\eta x)^{\beta p^s})^{r_0}\\
&= a_{00r}(z^{p^s}+\omega^{p^s}x^{\alpha p^s}[(y')^{p^t}+\eta^{p^t}x^{p^t}]^e)^{r_0}.
\end{aligned}
\tag{28}
$$

Now make the $(\gamma,0)$ preparation $z=z'-\eta^\beta\omega x^\gamma$ (from (26)) so that $(g;x,y',z')$ is $(\gamma,0)$ prepared. Let $G_g=a_{00r}z^r+F_g$. Then

$$
\begin{aligned}
G_g &= a_{00r}\left((z')^{p^s}+e\omega^{p^s}\eta^{p^t(e-1)}(y')^{p^t}x^{\alpha p^s+p^t(e-1)}+(y')^{2p^t}\Omega(x,y')\right)^{r_0}\\
&= a_{00r}\left[(z')^{p^sr_0}+r_0\left[e\omega^{p^s}\eta^{p^t(e-1)}(y')^{p^t}x^{\alpha p^s+p^t(e-1)}+(y')^{2p^t}\Omega\right](z')^{p^s(r_0-1)}\right]\\
&\quad +\Lambda_2(x,y')(z')^{p^s(r_0-2)}+\cdots+\Lambda_{r_0}(x,y')
\end{aligned}
$$

for some polynomials $\Omega(x,y'),\Lambda_2,\ldots,\Lambda_{r_0}$, where $(y')^{ip^t}\mid\Lambda_i$ for all i. All contributions of $S(\gamma)\cap|\Delta(g;x,y',z')|$ must come from these polynomials G_g. Recall that we are assuming $(\alpha,\beta)=(\gamma-\delta,\delta)$. The term of lowest second coordinate on $S(\gamma)\cap|\Delta(g;x,y',z')|$ is

$$
(a,b)=\left(\frac{\alpha p^s+p^t(e-1)}{p^s},\frac{p^t}{p^s}\right),
$$

which is not in $\mathbf{N}^2$ since $t<s$, and is not (α,β) since $e>1$. (a,b) is thus prepared on $|\Delta(g;x,y',z')|$, and

$$
\delta'=\frac{p^t}{p^s}<\frac{ep^t}{p^s}=\beta.
$$

Example 10.18. *With the notation of Theorem 10.17, it is possible to have $(\gamma,0)$ solvable in $\mid\Delta(g;x,y',z)\mid$ if K has positive characteristic, as is shown by the following simple example. Suppose that the characteristic p of K is greater than 5.*

Let $g=z^p+xy^{p^2-1}+x^{p^2-1}y^2$.

The origin is isolated in $\mathrm{Sing}_p(g=0)$, and $(g;x,y,z)$ is very well prepared. The vertices of $\mid\Delta(g;x,y,z)\mid$ are $(\alpha,\beta)=(\gamma-\delta,\delta)=(\frac{1}{p},p-\frac{1}{p})$ and $\left(p-\frac{1}{p},\frac{2}{p}\right)$. Set $y=y'+\eta x$ with $0\neq\eta\in K$.

$$
\begin{aligned}
g &= z^p+x(y'+\eta x)^{p^2-1}+x^{p^2-1}(y'+\eta x)^2\\
&= z^p+\left(\textstyle\sum_{i=0}^{p^2-1}\binom{p^2-1}{i}\eta^{p^2-1-i}x^{p^2-i}(y')^i\right)\\
&\quad +\eta^2x^{p^2+1}+2\eta y'x^{p^2}+x^{p^2-1}(y')^2
\end{aligned}
$$

The vertices of $\mid\Delta(g;x,y',z)\mid$ are $(\alpha,\beta)=\left(\frac{1}{p},p-\frac{1}{p}\right)$ and $(\gamma,0)=(p,0)$. However, $(\gamma,0)$ is solvable. Set $z'=z+\eta^{p-\frac{1}{p}}x^p$.

$$
g=(z')^p+\left(\sum_{i=1}^{p^2-1}\binom{p^2-1}{i}\eta^{p^2-1-i}x^{p^2-i}(y')^i\right)+\eta^2x^{p^2+1}+2\eta y'x^{p^2}+x^{p^2-1}(y')^2
$$

Since

$$
\binom{p^2-1}{1}\equiv -1 \ \mathit{mod}\ p,
$$

$\delta_1 = \delta_{xy'z'}(g) = \frac{1}{p}$ *and the vertices of* $\mid \Delta(g; x, y', z') \mid$ *are* $(\alpha, \beta) = (\frac{1}{p}, p - \frac{1}{p})$, $(\gamma_1 - \delta_1, \delta_1) = (p - \frac{1}{p}, \frac{1}{p})$ *and* $(p + \frac{1}{p}, 0)$.

10.5. **Construction of the sequence** $\Omega(q_n)$**.**

Theorem 10.19. *For all* $n \in \mathbf{N}$*, there are good parameters* (x_n, y_n, z_n) *in* R_n *for* $g_n \in R_n$*, where* $I_n = (\mathcal{I}_{S_n,q_n})R_n = g_nR_n$ *such that*

$$\begin{array}{rlcl} & \Omega(g_{n+1}; x_{n+1}, y_{n+1}, z_{n+1}) & = & (\beta_{n+1}, \frac{1}{\varepsilon_{n+1}}, \alpha_{n+1}) \\ < & \Omega(g_n; x_n, y_n, z_n) & = & (\beta_n, \frac{1}{\varepsilon_n}, \alpha_n). \end{array}$$

Further, if $\beta_{n+1} = \beta_n$*,* $\varepsilon_{n+1} \neq \varepsilon_n$ *and* $\frac{1}{\varepsilon_n} \neq \infty$*, then*

$$\frac{1}{\varepsilon_{n+1}} = \frac{1}{\varepsilon_n} - 1.$$

Proof. We inductively construct regular parameters (x_n, y_n, z_n) in R_n such that (x_n, y_n, z_n) are good parameters for g_n, $\mathrm{Sing}_r(g_n) \subset V(x_n, z_n)$ or $\mathrm{Sing}_r(g_n) \subset V(y_n, z_n)$ and $(g_n; x_n, y_n, z_n)$ is well prepared. If $\mathrm{Sing}_r(g_n) \subset V(x_n, z_n)$, we will further have that $(g_n; x_n, y_n, z_n)$ is very well prepared. Let $\Omega_n = \Omega(g_n; x_n, y_n, z_n)$.

We first choose possibly formal regular parameters (x, y, z) in R which are good parameters for $g = g_0$, such that $|\Delta(g; x, y, z)|$ is very well prepared (see Remark 10.5). By Lemma 10.1, $z = 0$ is an approximate manifold for g. Let $\alpha = \alpha_{x,y,z}(g)$, $\beta = \beta_{x,y,z}(g)$.

By assumption, q is isolated in $\mathrm{Sing}_r(S)$.

Suppose that regular parameters (x_i, y_i, z_i) in R_i have been defined for $i \leq n$ as specified above.

If $\mathrm{Sing}_r(g_n) = V(y_n, z_n)$, then $R_n \to R_{n+1}$ must be a Tr4 transformation, by Lemma 8.4, since $V(z_n)$ is an approximate manifold of g_n. R_{n+1} has regular parameters $x'_{n+1}, y'_{n+1}, z'_{n+1}$ defined by

$$x_n = x'_{n+1}, y_n = y'_{n+1}, z_n = y'_{n+1}z'_{n+1}.$$

Thus $(g_{n+1}; x'_{n+1}, y'_{n+1}, z'_{n+1})$ is well prepared by Lemma 10.15. Further $\mathrm{Sing}_r(g_{n+1}) \subset V(y'_{n+1}, z'_{n+1})$ by Lemma 10.15. If $\mathrm{Sing}_r(g_{n+1}) = V(x'_{n+1}, y'_{n+1}, z'_{n+1})$, make a change of variables, subtracting a series in x'_{n+1}, y'_{n+1} from z'_{n+1}, to very well prepare, with resulting variables $x_{n+1}, y_{n+1}, z_{n+1}$. Otherwise, set $x_{n+1} = x'_{n+1}, y_{n+1} = y'_{n+1}, z_{n+1} = z'_{n+1}$. We have $\beta_{n+1} < \beta_n$ by Lemma 10.15, and $\Omega_{n+1} < \Omega_n$.

If $\mathrm{Sing}_r(g_n) = V(x_n, z_n)$ then $(g_n; x_n, y_n, z_n)$ is very well prepared. $R_n \to R_{n+1}$ must be a Tr3 transformation by Lemma 8.4, since $V(z_n)$ is an approximate manifold of g_n. R_{n+1} has regular parameters $x_{n+1}, y_{n+1}, z_{n+1}$ defined by

$$x_n = x_{n+1}, y_n = y_{n+1}, z_n = x_{n+1}z_{n+1}.$$

Thus $(g_{n+1}; x_{n+1}, y_{n+1}, z_{n+1})$ is also very well prepared, $\alpha_{n+1} < \alpha_n$ and $\Omega_{n+1} < \Omega_n$ by Lemma 10.13. Further $\mathrm{Sing}_r(g_{n+1}) \subset V(x_{n+1}, z_{n+1})$.

Now suppose that $\mathrm{Sing}_r(g_n) = V(x_n, y_n, z_n)$. Then $(g_n; x_n, y_n, z_n)$ is very well prepared, and R_{n+1} must be a Tr1 or Tr2 transformation of R_n by Lemma 8.4, since $V(z_n)$ is an approximate manifold of g_n.

If R_{n+1} is a Tr2 transformation of R_n, then R_{n+1} has regular parameters $x'_{n+1}, y'_{n+1}, z'_{n+1}$ defined by

$$x_n = x_{n+1}y'_{n+1}, y_n = y'_{n+1}, z_n = y_{n+1}z'_{n+1}.$$

$(g_{n+1}; x_{n+1}, y'_{n+1}, z'_{n+1})$ is well prepared by Lemma 10.15. Further, $\mathrm{Sing}_r(g_{n+1}) \subset V(y'_{n+1}, z'_{n+1})$ by Lemma 8.4. If $\mathrm{Sing}_r(g_{n+1}) = V(x_{n+1}, y'_{n+1}, z'_{n+1})$, then make a further change of variables to very well prepare. Otherwise, set $y_{n+1} = y'_{n+1}, z_{n+1} = z'_{n+1}$. $\beta_{n+1} < \beta_n$ by Lemma 10.15, and $\Omega_{n+1} < \Omega_n$.

If R_{n+1} is a Tr1 transformation of R_n, then R_{n+1} has regular parameters $x'_{n+1}, y'_{n+1}, z'_{n+1}$ defined by

$$x_n = x_{n+1}, y_n = x_{n+1}(y'_{n+1} + \eta), z_n = x_{n+1} z'_{n+1}.$$

$\mathrm{Sing}_r(g_{n+1}) \subset V(x_{n+1}, z'_{n+1})$ by Lemma 8.4.

If $\beta_n \neq 0$ and $\eta \neq 0$ in the Tr1 transformation relating R_n and R_{n+1}, we can change variables to $x_{n+1}, y_{n+1}, z_{n+1}$ to very well prepare, with $\beta_{n+1} < \beta_n$, $\Omega_{n+1} < \Omega_n$ and $\mathrm{Sing}_r(g_{n+1}) \subset V(x_{n+1}, z_{n+1})$ by Theorem 10.17.

If $\eta = 0$ in the Tr1 transformation of R_n, and $x_{n+1}, y_{n+1}, z_{n+1}$ are regular parameters of R_{n+1} obtained from $x_{n+1}, y'_{n+1}, z'_{n+1}$ by very well preparation, then by Lemma 10.14, $(g_{n+1}; x_{n+1}, y_{n+1}, z_{n+1})$ is very well prepared and $\Omega_{n+1} < \Omega_n$. If $\beta_{n+1} = \beta_n$, and $\varepsilon_n \neq 0$, then

$$\frac{1}{\varepsilon_{n+1}} = \frac{1}{\varepsilon_n} - 1.$$

Further, $\mathrm{Sing}_r(g_{n+1}) \subset V(x_{n+1}, z_{n+1})$.

If $\beta_n = 0$ we can make the translation $y'_n = y_n - \eta x_n$, and have that (g_n, x_n, y'_n, z_n) is very well prepared, and $\beta_n, \gamma_n, \delta_n, \varepsilon_n, \alpha_n$ are unchanged. Thus we are in the case of $\eta = 0$. □

We now prove that (15) cannot have infinite length. Set $\Omega(q_n) = \Omega(g_n; x_n, y_n, z_n)$ for $n \in \mathbf{N}$. By Theorem 10.19, we have a contradiction if (15) has infinite length, as the sequence $\Omega(q_n)$ cannot decrease indefinitely.

11. Exercise Set 3.

1. Show that the Tschirnhausen transformation gives a formal hypersurface of maximal contact (for a characteristic zero surface embedded in a nonsingular 3-fold).
2. (Narasimhan [28]) Let K be an algebraically closed field of characterisitc 2, and let X be the 3-fold in $\mathbb{A}^4_K$ with equation

$$f = w^2 + xy^3 + yz^3 + zx^7 = 0.$$

   a. Show that the maximal multiplicity of a singular point on X is 2, and the locus of singular points on X is the monomial curve C with local equations

$$y^3 + zx^6 = 0,\ \ xy^2 + z^3 = 0,\ \ yz^2 + x^7 = 0, w^2 + zx^7 = 0$$

   which has the parameterization $t \to (t^7, t^{19}, t^{15}, t^{32})$. Thus C has embedding dimension 4 at the origin, so there cannot exist a hypersurface (or formal hypersurface) of maximal contact for X at the origin.

   b. Resolve the singularities of X.
3. (Hauser [17]) Let K be an algebraically closed field of characteristic 2, and let S be the surface in $\mathbb{A}^2_K$ with equation $f = x^2 + y^4 z + y^2 z^4 + z^7 = 0$.
   a. Show that the maximal multiplicity of points on S is 2, and the singular locus of S is defined by the singular curve $y^2 + z^3 = 0$, $x + yz^2 = 0$.

b. Suppose that X is a hypersurface on a non-singular variety W, and $p \in X$ is a point in the locus of maximal multiplicity r of X. A hypersurface H through p is said to have permanent contact with X if under any sequence of blow ups $\pi : W_1 \to W$ of W, with non-singular centers, contained in the locus of points of multiplicity r on the strict transform of X, the strict transform of H contains all points of the intersection of the strict transform of X with multiplicity r which are in the fiber $\pi^{-1}(p)$. Show that there does not exist a non-singular hypersurface of permanent contact at the origin for the above surface S. Conclude that a hypersurface of maximal contact does not exist for S.

c. Resolve the singularities of S.

4. (Hauser [18]) Let K be a field, and let $f = z^2 + (x^7 + xy^4) \in K[x, y, z]$. Analyze a resolution of singularities of $f = 0$ when K has characteristic $\neq 2$, and when K has characteristic 2. At each step of the resolution process (where the multiplicity is 2), we have regular parameters $\overline{x}, \overline{y}, \overline{z}$ such that the strict transform of $f = 0$ has a local equation $\overline{z}^2 + Mg(\overline{x}, \overline{y}) = 0$ for some polynomial g, and monomial g in local equations of the exceptional locus, such that M and g have no common divisors. After well preparation (in the variable $\overline{z}$), how does the multiplicity of g change under blow ups in different characteristics?

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