

ASYMPTOTIC GROWTH OF ALGEBRAS ASSOCIATED TO POWERS OF IDEALS

STEVEN DALE CUTKOSKY, JÜRGEN HERZOG AND HEMA SRINIVASAN

ABSTRACT. We study generalized symbolic powers and form ideals of powers and compare their growth with the growth of ordinary powers, and we discuss the question when the graded rings attached to symbolic powers or to form ideals of powers are finitely generated.

INTRODUCTION

Let (R, \mathfrak{m}) be a local ring (or a graded ring with graded maximal ideal) of dimension d , and $I \subset R$ be an ideal. Let

$$\tilde{I} = I : \mathfrak{m}^\infty = \cup_{i=1}^\infty I : \mathfrak{m}^i$$

denote the saturation of I . In this paper we study the asymptotic behavior of the length $\lambda(\tilde{I}^k/I^k)$ of the R -module \tilde{I}^k/I^k , for large k . This length appears naturally in a number of equivalent formulations, as there are identities

$$(1) \quad \lambda(\tilde{I}^k/I^k) = \lambda(H_{\mathfrak{m}}^1(I^k))$$

if R has depth ≥ 2 , and thus by local duality,

$$(2) \quad \lambda(\tilde{I}^k/I^k) = \lambda(\text{Ext}_R^d(R/I^k, R))$$

if R is Gorenstein. The motivation for this problem come from two directions. One of them is the investigation of regularity of powers of ideals and of powers of ideal sheaves. The other direction is the subject of multiplicities.

To compute regularity, we consider a homogeneous ideal I in the polynomial ring $S = K[x_1, \dots, x_n]$ over a field K , with graded maximal ideal \mathfrak{m} . let $\mathbb{P} = \text{Proj}(R)$ be the projective space, and \mathcal{I} be the sheafification of I .

The regularity (Castelnuovo-Mumford regularity) of the sheaf \mathcal{I} is computed by the formula

$$\text{reg}(\mathcal{I}) = \max\{i \mid H_{\mathfrak{m}}^j(I)_{i-j} \neq 0 \text{ for some } j > 1\}.$$

The regularity of the ideal I (introduced by Eisenbud and Goto [10]) is given by the formula

$$\text{reg}(I) = \max\{i \mid H_{\mathfrak{m}}^j(I)_{i-j} \neq 0 \text{ for some } j\}.$$

The two definitions of regularity are related by the formula $\text{reg}(\tilde{I}) = \text{reg}(\mathcal{I})$. We further have that $\text{reg}(\tilde{I}) \leq \text{reg}(I)$, and their difference is computed from $H_{\mathfrak{m}}^1(I)$.

It was shown by Cutkosky, Herzog and Trung [9], and by Kodiyalam [24], that $\text{reg}(I^k)$ is a linear polynomial for $k \gg 0$. Thus

1991 *Mathematics Subject Classification*. Primary: 13A30 Secondary: 13D45.

The first author was partially supported by NSF.

$$\lim_{k \rightarrow \infty} \frac{\text{reg}(I^k)}{k}$$

exists, and is a rational number.

However, the behavior of $\text{reg}(\tilde{I}^k)$ is not as good. It is shown in [7] that the limit

$$\lim_{k \rightarrow \infty} \frac{\text{reg}(\tilde{I}^k)}{k}$$

does exist, but it is shown in examples in [9], [5] and [7] that the limit could be an irrational number. This limit is the Seshadri constant of the sheaf \mathcal{I} .

A second tie in comes from multiplicity. Kirby [23] and Theodorescu [31] have shown that if I is an \mathfrak{m} -primary ideal in a Cohen Macaulay local ring R of dimension d , then

$$(3) \quad \lambda(\text{Ext}_R^d(R/I^k, R))$$

is a polynomial in k of degree $\leq d$. If R is Gorenstein (of dimension $d \geq 2$), then by local duality, (3) is just the Hilbert function

$$\lambda(\tilde{I}^k/I^k) = \lambda(R/I^k),$$

which is a Hilbert polynomial for large enough k . In particular, in this special case (R is Gorenstein and I is \mathfrak{m} -primary),

$$\lim_{k \rightarrow \infty} \frac{\lambda(\tilde{I}^k/I^k)}{k^d} = \frac{e_I(R)}{d!},$$

where $e_I(R)$ is the multiplicity of R with respect to I .

Cutkosky, Hai, Srinivasan and Theodorescu have shown in Theorem 1.3 of [8] that

$$(4) \quad \lim_{k \rightarrow \infty} \frac{\lambda(\tilde{I}^k/I^k)}{k^d}$$

exists whenever I is a homogeneous ideal in a polynomial ring (over a field of characteristic zero). They also compute the function $\lambda(\tilde{I}^k/I^k)$ for a homogeneous ideal in the example of Theorem 0.2 [8]. The example is far from polynomial in behavior, and the limit (4) is irrational.

In Theorem 1.1 of this paper, we prove that the limit

$$(5) \quad \lim_{k \rightarrow \infty} \frac{\lambda(\tilde{I}^k/I^k)}{k^d},$$

exists, whenever I is an ideal in a regular local ring R of dimension d , which is essentially of finite type over a field of characteristic zero and $\text{spec}(R/I)$ has an isolated singularity. With this assumption, we have that \tilde{I}^k is the ordinary symbolic power $I^{(k)}$ of I , and we have that

$$\lim_{k \rightarrow \infty} \frac{\lambda(I^{(k)}/I^k)}{k^d}$$

exists (as a real number). This includes the case when I is a reduced equidimensional ideal of height $d - 1$ in R ($\text{spec}(R/I)$ is a germ of a singular curve).

As mentioned above, an example in [8] shows that $\lambda(H_{\mathfrak{m}}^0(R/I^k)) = \lambda(\tilde{I}^k/I^k)$ is not a well behaved function in general. However, in [32], Ulrich and Valedashti consider a modified function, which in the case of an ideal I is defined by

$$\Sigma_I(k) = \sum_{i=1}^{k-1} \lambda(H_{\mathfrak{m}}^0(I^i/I^{i+1})).$$

They show that for $k \gg 0$, $\Sigma_I(k)$ is a polynomial of degree $\leq d = \dim(R)$, and is of degree d if and only if $\ell(I) = d$, where $\ell(I)$ is the analytic spread of I (the dimension of $\text{gr}_I(R)/\mathfrak{m} \text{gr}_I(R)$, where $\text{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$). $\Sigma_I(k)$ gives a polynomial approximation to $\lambda(H_{\mathfrak{m}}^0(R/I^k))$, as

$$\lambda(H_{\mathfrak{m}}^0(R/I^k)) \leq \Sigma_I(k)$$

for all k . We deduce that

$$\lim_{k \rightarrow \infty} \frac{\lambda(\tilde{I}^k/I^k)}{k^d} = 0$$

if $\ell(I) < d = \dim(R)$.

When $\ell(I) < d$, we show that for $k \gg 0$, the function $\lambda(\tilde{I}^k/I^k)$ is actually a quasi-polynomial in k of degree $\leq \ell(I) - 1$, with constant $k^{\ell(I)-1}$ term. It follows that

$$(6) \quad \lim_{k \rightarrow \infty} \frac{\lambda(\tilde{I}^k/I^k)}{k^{\ell(I)-1}}$$

is a rational number if $\ell(I) < d$. This is a consequence of Theorem 2.1, which we discuss below.

The limit (6) may be zero. For example, consider $I = (x_1, \dots, x_r) \subset K[[x_1, \dots, x_d]]$ with $r < d$. In this case $\ell(I) = r < d$ and $\tilde{I}^k = I^k$ for all k . So the limit is certainly zero.

We do not know of an example with $\ell(I) = d = \dim(R)$ such that the limit (5) is zero.

An extremely interesting question is to determine the rates of growth possible for the function $\lambda(\tilde{I}^k/I^k)$. A function $f(k)$ of the natural numbers has the growth rate of a function $g(k)$ if there exist positive constants $a < b$ such that $ag(k) \leq f(k) \leq bg(k)$ for all $k \gg 0$.

We see that if $\ell(I) < d$, the only possible growth is the rate k^i with i a natural number with $0 \leq i \leq d - 2$. When $\ell(I) = d$, we only know that $\lambda(\tilde{I}^k/I^k)$ is bounded above by a polynomial of degree d . In the example of Theorem 0.2 [8], where the limit (5) is an irrational number, its growth is of the rate k^d ($d = 4$ in this example).

The reason for the much better behavior of $\lambda(\tilde{I}^k/I^k)$ when $\ell(I) < d = \dim(R)$ than in the case $\ell(I) = d$ is that the algebra $\bigoplus_{k \geq 0} \tilde{I}^k = \bigoplus_{k \geq 0} I^k : \mathfrak{m}^\infty$ is a finitely generated R -algebra when $\ell(I) < d$ (Theorem 2.1), but it may be not finitely generated when $\ell(I) = d$.

In the paper [15], Herzog, Puthenpurakal and Verma consider a generalization of the function $\lambda(\tilde{I}^k/I^k)$. Let $J \subset R$ be an ideal. They replace $\tilde{I}^k = I^k : \mathfrak{m}^\infty$ with the generalized symbolic powers $I_k(J) = I^k : J^\infty = \bigcup_{i=1}^\infty (I^k : J^i)$, and replace length with the multiplicity $e(I_k(J)/I^k)$. If $\text{depth}(R_P) \geq 2$ for all $P \in V(J)$, we have $I_k(J)/I^j \cong H_J^1(I^k)$ for all k .

Generalized symbolic powers occur naturally in many situations. Besides the saturation ($J = \mathfrak{m}$) considered above, another case of interest is obtained when we choose for J the intersection of all asymptotic associated prime ideals of I which are not minimal. In this

case $S_J(I) = \bigoplus_{k \geq 0} I_k(J)$ is the symbolic Rees algebra. This is why we call for any choice of J the ideals $I^k : J^\infty$ *generalized symbolic powers of I* . If I happens to be a prime ideal, then these are the classical symbolic powers. Finite generation of symbolic Rees algebras has been studied in many papers, for instance in [21], [30], [22], and remarkable examples have been found where these algebras are not finitely generated, see [27] and [13]. Symbolic Rees algebras of squarefree monomial ideals can be identified with vertex cover algebras. This class of algebras is always finitely generated, as shown in [14].

It is shown in Theorem 2.5 of [15] that if I and J are monomial ideals, then for $k \gg 0$, $e(I_k(J)/I^k)$ is a quasi-polynomial, with constant leading term. An essential part of the proof is to show that $\bigoplus_{k \geq 0} I_k(J)$ is a finitely generated R -algebra (with the assumption that I and J are monomial ideals).

In Section 2 of this paper, we restrict to the situation where R is an excellent local domain, and I, J are proper ideals of R . We show that $e(I_k(J)/I^k)$ is a quasi-polynomial with constant leading term whenever there exists an integer $r > 0$ such that $\ell((I^r : J^\infty)_P) < R_P$ for all $P \in V(J)$. It follows that (with the assumption that $\ell((I^r : J^\infty)_P) < R_P$) we have that $\lim_{k \rightarrow \infty} e((I^k : J^\infty)/I^k)/k^{\ell(I) + \dim R/J - 1}$ exists and is a rational number. These results are a consequence of finite generation of the R -algebra $S_J(I) = \bigoplus_{k \geq 0} I_k(J)$.

We give very general necessary and sufficient conditions for finite generation of the algebras $S_J(I)$. Suppose that R is an excellent local domain, and I, J are proper ideals of R . We show in Theorem 2.6 that $S_J(I)$ is finitely generated if and only if there exists an integer $r > 0$ such that $\ell((I^r : J^\infty)_P) < \dim R_P$ for all $P \in V(J)$. In Theorem 2.1, we give a short direct proof of the fact that under the above conditions on R and J , $S_J(I)$ is a graded subalgebra of the integral closure of the Rees algebra of I , provided that $\ell(I_P) < \dim R_P$ for all $P \in V(J)$. This fact is also a consequence of Theorem 6.5 [30], and in the case when $J = \mathfrak{m}$, Theorem 4.1 [21]. We further show that $\lim_{k \rightarrow \infty} e((I^k : J^\infty)/I^k)/k^{\ell(I) + \dim R/J - 1}$ exists and is a rational number. In particular it follows from the above results that the saturated power algebra $\bigoplus_{k \geq 0} \tilde{I}^k$ is finitely generated if $\ell(\tilde{I}) < \dim R$, and that in this case $\lim_{k \rightarrow \infty} \lambda(\tilde{I}^k/I^k)/k^{\ell(I) - 1}$ exists and is a rational number.

The results of Section 2 are inspired by a result of McAdam [26] on ideals with small analytic spread. This theory has been extensively developed, giving very general conditions for $S_J(I)$ to be finitely generated, by Huckaba [17], Katz [21], Katz and Ratliff [22], Ratliff [28], Schenzel [30], Verma [34] and others.

Section 3 of this paper deals with the form ideal I^* of I . Let $\text{gr}_{\mathfrak{m}}(R) = \bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}$ denote the associated graded ring of a Noetherian local ring R . For an ideal I of R , the form ideal $I^* \subset \text{gr}_{\mathfrak{m}}(R)$ is the graded ideal in $\text{gr}_{\mathfrak{m}}(R)$ generated by the leading forms of all the elements in I . Alternately, this is the kernel of the natural epimorphism $\text{gr}_{\mathfrak{m}}(R) \rightarrow \text{gr}_{\bar{\mathfrak{m}}}(R/I)$. Here $\bar{\mathfrak{m}}$ is the maximal ideal of R/I . Thus, the Hilbert function $H(R/I^k, j)$ of R/I is given by $H(R/I^k, j) = \sum_{i=0}^j \dim_K(\text{gr}_{\mathfrak{m}}(R)/(I^k)^*)_i$. When R is graded and I is a graded ideal of R , $(I^k)^* = I^k$. So, in this case the algebra $A(I) = \bigoplus_{k \geq 0} (I^k)^*$ is finitely generated over R . However, in general $A(I)$ is not finitely generated, even if I is generated by quasi-homogeneous polynomials. We give such an example in 3.4. There we show that for the ideal $I = (x^2, y^3 - xy) \subset K[[x, y]]$ the algebra $A(I)$ is not finitely generated. Notice that I is quasi-homogeneous if we set $\deg x = 2$ and $\deg y = 1$. This example is also

remarkable, since it is an \mathfrak{m} -primary complete intersection. We also show in Example 3.5 that the finite generation of $A(I)$ may depend on the characteristic of the residue class field.

A motivation for the study of $A(I)$ comes from a result of Hoang and Trung in [18] where they showed that the Hilbert coefficients of the powers I^k of a graded ideal I in the polynomial ring $S = K[x_1, \dots, x_n]$ are polynomial functions in k for $k \gg 0$. In an explicit form this statement is given in [15]. In the same paper the question is raised whether for any ideal in a Noetherian local ring (R, \mathfrak{m}, K) a similar statement is true. We do not have any counterexamples yet. On the other hand, a positive answer is unlikely by the following reason: Hoang and Trung's proof makes essential use of the fact that in the graded case, $A(I)$ is finitely generated over R , indeed is equal to the Rees algebra of I and hence is standard graded over R . Unfortunately, this is not true in general as shown by our example 3.4 in section discussed in the preceding paragraph.

General criteria for the finite generation of $A(I)$ seem to be unavailable. However if I is an ideal in the power series ring $R = K[[x_1, \dots, x_n]]$ the following strategy can be applied: as explained in Lemma 3.6 and Proposition 3.7 there is an ideal $I^\sharp \subset R[[s]]$ attached to I in a natural way with the property that $A(I)$ is finitely generated if and only if $\bigoplus_{k \geq 0} (I^\sharp)^k : s^\infty$ is finitely generated.

We show in Corollary 3.3 that if (R, \mathfrak{m}) is a regular local ring of dimension d , $I \subset R$ is a complete intersection ideal of height d and if either R is 2-dimensional or I^* is a monomial ideal, then the following conditions are equivalent:

- (a) $A(I)$ is standard graded,
- (b) I^* is a complete intersection,
- (c) for infinitely many integers k we have $(I^k)^* = (I^*)^k$.

It would be interesting to know whether these equivalent conditions hold without the extra assumptions on R or I^* .

The authors want to thank Bernd Ulrich for several useful discussion concerning Theorem 3.2.

1. THE EXISTENCE OF A LIMIT OF ASYMPTOTIC GROWTH

Theorem 1.1. *Suppose that (R, \mathfrak{m}) is a regular local ring of dimension d , which is essentially of finite type over a field K of characteristic zero. Suppose that $I \subset R$ is an ideal such that the singular locus of $\text{Spec}(R/I)$ is \mathfrak{m} . Then the limit*

$$\lim_{k \rightarrow \infty} \frac{\lambda(I^{(k)}/I^k)}{k^d} = \lim_{k \rightarrow \infty} \frac{\lambda(\tilde{I}^k/I^k)}{k^d} \in \mathbb{R}$$

exists.

Proof. If I is \mathfrak{m} -primary, then $\tilde{I}^k = R$ for all k , and thus $\lambda(\tilde{I}^k/I^k) = \lambda(R/I^k)$ is a polynomial in k of degree d for $k \gg 0$. Thus the limit exists.

Now assume that I is not \mathfrak{m} -primary. Since R is regular and the singular locus of $\text{Spec}(R/I)$ is \mathfrak{m} , we have that I_P^k is a complete ideal in R_P for all $P \in \text{Spec}(R) - \{\mathfrak{m}\}$ and $k \geq 0$. Thus

$$\tilde{I}^k = I^k : \mathfrak{m}^\infty = \overline{I^k} : \mathfrak{m}^\infty \supset \overline{I^k}$$

for all $k > 0$.

Consider the exact sequence of finite length R -modules

$$0 \rightarrow \overline{I^k}/I^k \rightarrow \widetilde{I^k}/I^k \rightarrow \widetilde{I^k}/\overline{I^k} \rightarrow 0.$$

Since $\bigoplus_{k \geq 0} \overline{I^k}$ is a finitely generated $\bigoplus_{k \geq 0} I^k$ -module, the quotient $\bigoplus_{k \geq 0} \overline{I^k}/I^k$ is a finitely generated $\bigoplus_{k \geq 0} I^k$ module, which is annihilated by \mathfrak{m}^r for some r . Thus $\lambda(\overline{I^k}/I^k)$ is a polynomial of degree $\leq d - 1$ for $k \gg 0$, and we have reduced to showing that

$$\lim_{k \rightarrow \infty} \frac{\lambda(\widetilde{I^k}/\overline{I^k})}{k^d}$$

exists.

The blow up $\text{Proj}(\bigoplus_{k \geq 0} I^k)$ of I is nonsingular away from the fiber over the maximal ideal \mathfrak{m} of R . Let $Y \rightarrow \text{Proj}(\bigoplus_{k \geq 0} I^k)$ be a resolution of singularities which is an isomorphism away from the fiber over the maximal ideal \mathfrak{m} of R . Let $X = \text{Spec}(R)$, and $f : Y \rightarrow X$ be the natural map. Let $\mathcal{L} = I\mathcal{O}_Y$. $\mathcal{L} = \mathcal{O}_Y(-F - E)$ where E is an effective divisor such that $f(E) = \mathfrak{m}$, and F is a reduced effective divisor whose components are the prime divisors corresponding to the P_P -adic valuations in R_P , where P ranges over the minimal primes of I .

For $k \in \mathbb{N}$, we have exact sequences

$$0 \rightarrow \mathcal{O}_Y(-kE) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{kE} \rightarrow 0.$$

Tensoring with $\mathcal{O}_Y(-kF)$, we have exact sequences

$$0 \rightarrow \mathcal{L}^k \rightarrow \mathcal{O}_Y(-kF) \rightarrow \mathcal{O}_{kE}(-kF) \rightarrow 0,$$

where $\mathcal{O}_{kE}(-kF)$ denotes the invertible sheaf $\mathcal{O}_{kE} \otimes \mathcal{O}_Y(-kF)$ on the scheme kE . Taking global sections, we have exact sequences

$$(7) \quad 0 \rightarrow \overline{I^k} \rightarrow \widetilde{I^k} \rightarrow H^0(kE, \mathcal{O}_{kE}(-kF)) \rightarrow H^1(Y, \mathcal{L}^k).$$

Let $N = \bigoplus_{k \geq 0} H^1(Y, \mathcal{L}^k)$. Then N is naturally a $\bigoplus_{k \geq 0} \overline{I^k} \cong \bigoplus_{k \geq 0} H^0(Y, \mathcal{L}^k)$ -module. We will show that N is a finitely generated $\bigoplus_{k \geq 0} \overline{I^k}$ -module.

Let $Z = \text{Proj}(\bigoplus_{k \geq 0} \overline{I^k})$, and $\mathcal{N} = I\mathcal{O}_Z$. Since Y is normal and dominates the blowup of I , the map $f : Y \rightarrow X$ factors as

$$Y \xrightarrow{g} Z \xrightarrow{h} X.$$

From the first terms of the Leray spectral sequence, we have an exact sequence

$$(8) \quad 0 \rightarrow H^1(Z, g_*(\mathcal{L}^k)) \rightarrow H^1(Y, \mathcal{L}^k) \rightarrow H^0(Z, R^1g_*(\mathcal{L}^k)).$$

We have $g_*(\mathcal{L}^k) \cong \mathcal{N}^k$ and $R^1g_*(\mathcal{L}^k) \cong \mathcal{N}^k \otimes R^1g_*\mathcal{O}_Y$ since Z is normal, and by the projection formula. Since \mathcal{N} is ample on Z , there exists k_0 such that $H^1(Z, g_*(\mathcal{L}^k)) = 0$ for $k \geq k_0$. Since h is proper, $H^1(Z, \mathcal{N}^k)$ is a finitely generated R -module for all k . Thus $\bigoplus_{k \geq 0} H^1(Z, g_*(\mathcal{L}^k))$ is a finitely generated $\bigoplus_{k \geq 0} \overline{I^k} = \bigoplus_{k \geq 0} H^0(Z, \mathcal{N}^k)$ -module. Since g is proper, $R^1g_*\mathcal{O}_Y$ is a coherent \mathcal{O}_Z -module. Since \mathcal{N} is ample, there exists $s \in \mathbb{N}$ and $a_i \in \mathbb{Z}$ such that there is a surjection

$$\bigoplus_{i=1}^s \mathcal{N}^{a_i} \rightarrow R^1g_*\mathcal{O}_Y$$

of \mathcal{O}_Z -modules. Let \mathcal{K} be the kernel of this map. We have an exact sequence

$$\bigoplus_{n \geq 0} \left(\bigoplus_{i=1}^s H^0(Z, \mathcal{N}^{n+a_i}) \right) \rightarrow \bigoplus_{n \geq 0} H^0(Z, \mathcal{N}^n \otimes R^1 g_* \mathcal{O}_Y) \rightarrow \bigoplus_{n \geq 0} H^1(Z, \mathcal{K} \otimes \mathcal{N}^n).$$

Since \mathcal{N} is ample, there exists an n_0 such that $H^1(Z, \mathcal{K} \otimes \mathcal{N}^n) = 0$ for $n \geq n_0$. Since R is normal, we have

$$H^0(Z, \mathcal{N}^i) = \begin{cases} R & \text{if } i \leq 0 \\ \bar{I}^i & \text{if } i > 0. \end{cases}$$

Thus $\bigoplus_{n \geq 0} \left(\bigoplus_{i=1}^s H^0(Z, \mathcal{N}^{n+a_i}) \right)$ is a finitely generated $\bigoplus_{n \geq 0} H^0(Z, \mathcal{N}^n)$ -module. Since $H^1(Z, \mathcal{K} \otimes \mathcal{N}^n)$ are finitely generated R -modules for all n , which are zero for $n \geq n_0$, it follows that $\bigoplus_{n \geq 0} H^0(Z, \mathcal{N}^n \otimes R^1 g_* \mathcal{O}_Y)$ is a finitely generated $\bigoplus_{n \geq 0} H^0(Z, \mathcal{N}^n)$ -module. From (8), we see that N is a finitely generated $\bigoplus_{k \geq 0} \bar{I}^k$ -module.

Since N is a finitely generated $\bigoplus_{k \geq 0} \bar{I}^k$ -module, and the support of $H^1(Y, \mathcal{L}^k)$ is contained in $\{\mathfrak{m}\}$ for all k , there exists a positive integer r such that $\mathfrak{m}^r N = 0$. Since

$$\dim \left(\bigoplus_{k \geq 0} \bar{I}^k \right) / \mathfrak{m} \left(\bigoplus_{k \geq 0} \bar{I}^k \right) \leq \dim R = d,$$

there exists a constant c such that $\lambda(H^1(Y, \mathcal{L}^k)) \leq ck^{d-1}$ for all k . From comparison with (7), we have reduced to showing that

$$(9) \quad \lim_{k \rightarrow \infty} \frac{\lambda(H^0(kE, \mathcal{O}_{kE}(-kF)))}{k^d}$$

exists.

If R/\mathfrak{m} is algebraic over K , let $K' = K$. If R/\mathfrak{m} is transcendental over K , let t_1, \dots, t_r be a lift of a transcendence basis of R/\mathfrak{m} over K to R . The rational function field $K(t_1, \dots, t_r)$ is contained in R . Let $K' = K(t_1, \dots, t_r)$. We have that R/\mathfrak{m} is finite algebraic over K' . There exists a nonsingular affine K' -variety U such that R is the local ring of a closed point α of U . Let \bar{X} be a nonsingular projective closure of U , and let \mathcal{I} be an extension of I to an ideal sheaf on \bar{X} . Let $\bar{f}: \bar{Y} \rightarrow \bar{X}$ be a resolution of singularities such that $\bar{Y} \rightarrow \bar{X}$ factors through the blow up of \mathcal{I} , and $\bar{f}^{-1}(X) \cong Y$. We may identify E (and kE for all positive integers k) with a closed subscheme of \bar{Y} . Let \bar{F} be the Zariski closure of F in \bar{Y} . Since the singular locus of $\text{Spec}(R/I)$ is \mathfrak{m} , we may choose U , \bar{X} and \bar{Y} so that the singular locus of the scheme $\text{Spec}(\mathcal{O}_{\bar{X}}/\mathcal{I})$ is the isolated point α , and $\bar{f}_* \mathcal{O}_Y(-\bar{F}) \cong \mathcal{I}_\beta$ for $\beta \in \bar{X} - \{\alpha\}$.

There exists a line bundle \mathcal{M} on \bar{Y} such that $\mathcal{M} \otimes \mathcal{O}_Y \cong \mathcal{L}$ is generated by global sections and is big (the global sections of a high multiple of \mathcal{L} give a birational map on \bar{Y}). We can construct \mathcal{M} by taking any ample line bundle \mathcal{A} on \bar{X} , and taking

$$\mathcal{M} = \bar{f}^*(\mathcal{A}^t) \otimes \mathcal{I} \mathcal{O}_{\bar{Y}}$$

for t sufficiently large. Let $\mathcal{B} = \mathcal{M} \otimes \mathcal{O}_Y(E)$. We have an exact sequence

$$0 \rightarrow \mathcal{O}_{\bar{Y}}(-kE) \rightarrow \mathcal{O}_{\bar{Y}} \rightarrow \mathcal{O}_{kE} \rightarrow 0.$$

Tensoring with \mathcal{B}^k , we have exact sequences

$$0 \rightarrow \mathcal{M}^k \rightarrow \mathcal{B}^k \rightarrow \mathcal{O}_{kE}(-kF) \rightarrow 0.$$

Taking global sections, we have exact sequences

$$(10) \quad 0 \rightarrow H^0(\bar{Y}, \mathcal{M}^k) \rightarrow H^0(\bar{Y}, \mathcal{B}^k) \rightarrow H^0(kE, \mathcal{O}_{kE}(-kF)) \rightarrow H^1(\bar{Y}, \mathcal{M}^k).$$

Since \mathcal{M} is semiample (generated by global sections and big), we have that

$$\lim_{k \rightarrow \infty} \frac{h^1(\bar{Y}, \mathcal{M}^k)}{k^d} = 0$$

(for instance as a special case of [11] or by consideration of the Leray spectral sequence of the mapping from \bar{Y} given by the global sections of a high power of \mathcal{M}). Further, $\bigoplus_{n \geq 0} H^0(\bar{Y}, \mathcal{M}^k)$ is a finitely generated K' -algebra of dimension $d + 1$. Thus

$$\lim_{k \rightarrow \infty} \frac{h^0(\bar{Y}, \mathcal{M}^k)}{k^d} \in \mathbb{Q}$$

exists. Since \mathcal{B} is big, by the corollary given in [25] or [8] to [12], we have that

$$\lim_{k \rightarrow \infty} \frac{h^0(\bar{Y}, \mathcal{B}^k)}{k^d} \in \mathbb{R}$$

exists. From the sequence (10), we see that

$$\lim_{k \rightarrow \infty} \frac{h^0(kE, \mathcal{O}_{kE}(-kF))}{k^d} \in \mathbb{R},$$

and the conclusions of the theorem now follow, by applying the formula

$$h^0(kE, \mathcal{O}_{kE}(-kF)) = \dim_{K'} H^0(kE, \mathcal{O}_{kE}(-kF)) = [R/\mathfrak{m} : K'] \lambda(H^0(kE, \mathcal{O}_{kE}(-kF)))$$

to equation (9). \square

Corollary 1.2. *Suppose that $R = K[[x_1, \dots, x_n]]$ is a power series ring over a field K of characteristic zero, and $I \subset R$ is an equidimensional ideal such that the singular locus of $\text{Spec}(R/I)$ is $\mathfrak{m} = (x_1, \dots, x_n)$. Then the limit*

$$\lim_{k \rightarrow \infty} \frac{\lambda(I^{(k)}/I^k)}{k^d} = \lim_{k \rightarrow \infty} \frac{\lambda(\tilde{I}^k/I^k)}{k^d} \in \mathbb{R}$$

exists.

Proof. By [19, Theorem 1] or [6, Theorem A], there exists an ideal $J \subset K[x_1, \dots, x_n]$ and a K -algebra isomorphism $\varphi : R \rightarrow R$ such that $\varphi(I) = JR$. Thus $\varphi(I^k) = J^k R$ and $\varphi(\tilde{I}^k) = \tilde{J}^k R = \widetilde{J^k} R$ for all k . We have that

$$\lambda(\tilde{I}^k/I^k) = \lambda(\widetilde{J^k} R/J^k R) = \lambda(\tilde{J}^k/J^k)$$

for all $k \in \mathbb{N}$. Now by Theorem 1.1,

$$\lim_{k \rightarrow \infty} \frac{\lambda(\tilde{I}^k/I^k)}{k^d} = \lim_{k \rightarrow \infty} \frac{\lambda(\tilde{J}^k/J^k)}{k^d} \in \mathbb{R}.$$

\square

2. GENERALIZED SYMBOLIC POWERS

Let (R, \mathfrak{m}) be a local ring or a positively graded K -algebra with graded maximal ideal \mathfrak{m} , where K is a field, and let I and J be proper ideals in R that are graded if R is graded. In this section we want to study the algebra $S_J(I) = \bigoplus_{k \geq 0} I^k : J^\infty$ of generalized symbolic powers of I with respect to J . The Rees ring of I will be denoted by $R(I)$ and its integral closure by $\overline{R(I)}$ in case R is a domain. It turns out that the analytic spread $\ell(I)$ of I , which is defined to be the Krull dimension of $R(I)/\mathfrak{m}R(I)$, plays an important role in the study of these algebras. The multiplicity of a finitely generated R -module M will be denoted by $e(M)$.

Conclusion (a) of Theorem 2.1 below is inspired by results of McAdam [26], Ratliff [28], Katz [21], Schenzel [30], Huckaba, [17], Verma [34] and others on the asymptotic associated primes of ideals of small analytic spread. In fact, (a) follows from results of Katz when $J = \mathfrak{m}$ (Theorem 4.1 of [21]) and of Schenzel for arbitrary J (Theorem 6.5 [30]) both of which are valid over general Noetherian rings. Over excellent domains, we get an elegant statement which has a rather simple proof.

Theorem 2.1. *Let (R, \mathfrak{m}) be an excellent domain. Assume that $\ell(I_P) < \dim R_P$ for all $P \in V(J)$. Then*

- (a) $S_J(I) \subset \overline{R(I)}$. In particular, $S_J(I)$ is a finitely generated R -algebra.
- (b) $\lim_{k \rightarrow \infty} e((I^k : J^\infty)/I^k)/k^{\ell(I) + \dim R/J - 1}$ exists and is a rational number.

Proof. (a) Notice that $\overline{R(I)} = \overline{R} \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \dots$. Since $I^k : J^\infty \subset \overline{I^k} :_{\overline{R}} J^\infty \overline{R}$ for all k , it suffices therefore to show that $\overline{I^k} :_{\overline{R}} J^\infty \overline{R} = \overline{I^k}$ for all $k \geq 0$. Recall that (see [2, Proposition 1.2.10])

$$(11) \quad \text{grade}(J\overline{R}, \overline{R}) = \inf\{\text{depth } \overline{R}_P, P \in V(J\overline{R})\}.$$

Now let $P \in V(J\overline{R})$ and $Q = P \cap R$. Then $Q \in V(J)$, and our assumptions imply that $\dim R_Q > \ell(I_Q) > 0$, so that $\dim R_Q \geq 2$. Applying [2, Corollary A.8] to the ring extension \overline{R}/R we obtain that $\dim \overline{R}/P = \dim R/Q$. Hence since R is an excellent domain, we get

$$\dim \overline{R}_P = \dim \overline{R} - \dim \overline{R}/P = \dim R - \dim R/Q = \dim R_Q \geq 2.$$

Being a normal ring, the ring \overline{R} satisfies Serre's condition S_2 . Thus, $\text{depth } \overline{R}_P \geq 2$ for all $P \in \text{Spec}(\overline{R})$ with $\dim \overline{R}_P \geq 2$. Altogether we see that $\text{depth } \overline{R}_P \geq 2$ for all $P \in V(J\overline{R})$ and conclude that $\text{grade}(J\overline{R}, \overline{R}) \geq 2$. This implies that $H_{J\overline{R}}^0(\overline{R}) = H_{J\overline{R}}^1(\overline{R}) = 0$ (see [1, Theorem 6.2.7]), and hence $\overline{I^k} :_{\overline{R}} J^\infty \overline{R}/\overline{I^k} = H_{J\overline{R}}^0(\overline{R}/\overline{I^k}) \cong H_{J\overline{R}}^1(\overline{I^k})$ for all k . Therefore, if $A = \overline{R(I)}$, it remains to be shown that $H_J^1(A) = 0$ which, by [1, Theorem 6.2.7], is equivalent to saying that $\text{grade}(JA, A) \geq 2$.

We apply (11) again, this time to the ideal JA and the ring A , and obtain that

$$\text{grade}(JA, A) = \inf\{\text{depth } A_Q, Q \in V(JA)\}.$$

Again, since A is normal, it satisfies Serre's condition S_2 , and it remains to be shown that $\dim A_Q \geq 2$ for all $Q \in V(JA)$. Let $P = Q \cap R$. Then $P \in V(J)$. Localizing at P we may assume that $P = \mathfrak{m}$. Our assumption on R guarantees that $\dim A_Q = \dim A - \dim A/Q$. Notice that $\dim A = \dim R(I) = \dim R + 1$, since A is a finite $R(I)$ -module, and that

$\dim A/Q \leq \dim A/\mathfrak{m}A = \dim R(I)/\mathfrak{m}R(I) = \ell(I)$. Thus the condition $\ell(I) < \dim(R)$ implies that $\dim A_Q \geq \dim R + 1 - \ell(I) \geq 2$, as desired.

(b) By part (a), the algebra $S_J(I)$ is finitely generated. Thus [15, Theorem 3.2] implies that there exist polynomials P_0, \dots, P_{g-1} , all of same degree and with same leading coefficient, such that $e(I^{mg+i} : J^\infty/J^{mg+i}) = P_i(m)$ for $i = 0, \dots, g-1$ and all $m \gg 0$. For each i , the modules $I^{mg+i} : J^\infty/J^{mg+i}$ have constant dimension for $m \gg 0$, say d_i . Since they are supported in $V(J)$ it follows that $d_i \leq \dim R/J$ for all i . Thus applying [15, Proposition 5.5] we see that $\deg P_i \leq \ell(I) + \dim R/J - 1$ for all i , and thus statement (b) follows. \square

Example 2.2. Let $R = K[[x, y, z]]/(xy, xz)$ and $I = (x, y)R$. Then $\ell(I) = 1 < \dim R = 2$, however $\tilde{I}^k = (x, y^k)R$, and so $A(I)$ is not finitely generated. This examples illustrates that the domain assumption on R in Theorem 2.1 is necessary.

Example 2.3. (Marc Chardin) Let $I = (xw - yz, x^2, z^2) \subset S = K[x, y, z, w]$. Then $\ell(I) = 3 < \dim S$ and $e(\tilde{I}^k/I^k) = \lambda(\tilde{I}^k/I^k) = \binom{k+1}{2}$. It follows that

$$\lim_{k \rightarrow \infty} e(\tilde{I}^k/I^k)/k^{\ell(I)-1} = 1/2.$$

This example shows that the limit would not exist, if we would choose a smaller power of k than $\ell(I) - 1$.

To prove the assertions about the example, we consider a presentation of the Rees ring $R = S[y_1, y_2, y_3] \rightarrow R(I)$ with $y_1 \mapsto (xw - yz)t$, $y_2 \mapsto x^2t$ and $y_3 \mapsto z^2t$. The kernel J of this map is:

$$\begin{aligned} & (z^2y_1 + yzy_3 - xwy_3, \quad yzy_1 + xwy_1 - w^2y_2 + y^2y_3, \\ & xzy_1 - zwy_2 + xy_3, \quad z^2y_2 - x^2y_3, -x^2y_1 - yzy_2 + xwy_2). \end{aligned}$$

Thus $\ell(I) = 3$.

If we set $\deg x = \deg y = \deg z = (1, 0)$ and $\deg y_i = (0, 1)$ then $R(I)$ is a bigraded algebra with

$$R(I)_{(j,k)} = (I^k)_{j+2k} \quad \text{for all } j \text{ and } k.$$

The bigraded R -resolution of $R(I)$ is of the form

$$\mathbb{F} : 0 \rightarrow R(-4, -1) \rightarrow R(-3, -1)^4 \oplus R(-2, -2) \rightarrow R(-2, -1)^5 \rightarrow R \rightarrow R(I) \rightarrow 0$$

where the last map $R(-4, -1) \rightarrow R(-3, -1)^4 \oplus R(-2, -2)$ is given by the 1×5 -matrix $(-x, y, -z, w, 0)^t$.

If we take the graded pieces $\mathbb{F}_{(*,k)} = \bigoplus_j \mathbb{F}_{(j,k)}$ of the resolution we obtain the exact sequences of S -modules

$$\begin{aligned} 0 \rightarrow R(-4, -1)_{(*,j)} \rightarrow R(-3, -1)_{(*,j)}^4 \oplus R(-2, -2)_{(*,k)} & \rightarrow \\ R(-2, -1)_{(*,k)}^5 & \rightarrow R_{(*,k)} \rightarrow I^k(2k) \rightarrow 0. \end{aligned}$$

Since $R(-a, -b)_{(*, k)} = \bigoplus_{a_1+a_2+a_3=k-b} S(-a)y_1^{a_1}y_2^{a_2}y_3^{a_3} \cong S(-a-2k)^{\binom{k-b+2}{2}}$, we obtain for each k the free S -resolution

$$\begin{aligned} 0 \rightarrow S(-4)^{\binom{k+1}{2}} \rightarrow S(-3)^{4\binom{k+1}{2}} \oplus S(-2)^{\binom{k}{2}} &\rightarrow \\ S(-2)^{5\binom{k+1}{2}} \rightarrow S^{\binom{k+2}{2}} \rightarrow I^k(2k) \rightarrow 0, \end{aligned}$$

where the last map in the resolution maps the basis element $y_1^{a_1}y_2^{a_2}y_3^{a_3}$ with $a_1+a_2+a_3=k-1$ to $-xy_1^{a_1}y_2^{a_2}y_3^{a_3} - yy_1^{a_1}y_2^{a_2}y_3^{a_3} - zy_1^{a_1}y_2^{a_2}y_3^{a_3} + wy_1^{a_1}y_2^{a_2}y_3^{a_3}$. Since the cokernel of the transpose of this map is $\text{Ext}_S^3(I^k, S)$ we obtain, applying local duality,

$$(12) \quad \widetilde{I}^k/I^k \cong H_m^0(S/I^k) \cong \text{Ext}_S^4(S/I^k, S)^\vee \cong \text{Ext}_S^3(I^k, S)^\vee \cong K^{\binom{k+1}{2}},$$

as desired. Here N^\vee denotes the dual of N with respect to the injective hull of K .

Let R be the polynomial ring in n variables over a field of characteristic 0 and $I \subset R$ be a graded ideal. In [8, Theorem 0.1] it is shown that $\lim_{k \rightarrow \infty} \ell(\widetilde{I}^k/I^k)/k^n$ exists, but may be an irrational number. Of course, according to Theorem 2.1, this limit can be an irrational number only if $\ell(I) = \dim R$. In the following example this limit is a nonzero rational number.

Example 2.4. Let $I = (xy, xz, yz)$. It is easily seen that $\ell(I) = \dim K[x, y, z] = 3$ and that $\widetilde{I}^2 = (I^2, xyz)$. It is shown in [14, Proposition 5.3 and Example 4.7] that

$$\widetilde{I}^{2k} = (\widetilde{I}^2)^k = (I^2, xyz)^k = (I^{2k}, xyzI^{2(k-1)}, \dots, (xyz)^j I^{2(k-j)}, \dots, (xyz)^k) \text{ for all } k.$$

Let $\text{Gen}(I^k)$ denotes the minimal set of monomial generators of I^k . We claim that the set of the monomials $B = \bigcup_{j=1}^k (xyz)^j G(I^{2(k-j)})$ forms K -basis of $\widetilde{I}^{2k}/I^{2k}$. Indeed, for all $j = 0, \dots, k$ we have

$$\begin{aligned} \{x, y, z\}(xyz)^j \text{Gen}(I^{2(k-j)}) &= (xyz)^{j-1} \{x, y, z\}(xyz) \text{Gen}(I^{2(k-j)}) \\ &\subset (xyz)^{j-1} \text{Gen}(I^{2k}) \text{Gen}(I^{2(k-j)}) = (xyz)^{j-1} \text{Gen}(I^{2(k-j+1)}). \end{aligned}$$

This together with (12) implies that B is system of generators of the K -vector space $\widetilde{I}^{2k}/I^{2k}$. Since for $j = 1, \dots, k$ the degree of the elements in $(xyz)^j \text{Gen}(I^{2(k-j)})$ is $4k - j$, it follows that the elements in B are K -linearly independent modulo I^{2k} .

We have $|\text{Gen}(I^k)| = \binom{k+2}{2}$, since $\ell(I) = 3$. Thus we conclude that

$$\lambda(\widetilde{I}^{2k}/I^{2k}) = \sum_{j=0}^{k-1} |\text{Gen}(I^{2j})| = \sum_{j=0}^{k-1} \binom{2j+2}{2} = \frac{2}{3}k^3 + \frac{1}{2}k^2 - \frac{1}{6}k.$$

Since we know that $\lim_{k \rightarrow \infty} \ell(\widetilde{I}^k/I^k)/k^3$ exists, we see that

$$\lim_{k \rightarrow \infty} \lambda(\widetilde{I}^k/I^k)/k^3 = \lim_{k \rightarrow \infty} \lambda(\widetilde{I}^{2k}/I^{2k})/(2k)^3 = 1/12.$$

Theorem 2.1 has the following surprising consequences:

Corollary 2.5. *Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal generated in degree d .*

- (a) *If $\ell(I) < n$, then all generators of \widetilde{I} are of degree $\geq d$.*
- (b) *Suppose that $\ell(I) < n$ and all powers of I have a linear resolution, and that $\text{depth } S/I^r = 0$ for some r . Then $\ell(I) = n$ and $\lim_{k \rightarrow \infty} \lambda(\widetilde{I}^k/I^k)/n^k \neq 0$.*

Proof. (a) Suppose there exists $g \in \tilde{I}$ with $\deg g = c < d$. Then $g^j(I^{k-j})_{(k-j)d} \subset \tilde{I}^k$ for $j = 1, \dots, k$, and the elements in $g^j(I^{k-j})_{(k-j)d}$ are homogeneous of degree $jc + (k-j)d < kd$. It follows that $\lambda(\tilde{I}^k/I^k) \geq \sum_{j=1}^k \dim_K(I^{k-j})_{(k-j)d}$. Thus we see that $\lambda(\tilde{I}^k/I^k)$ grows like a polynomial of degree $\geq \ell(I)$, contradicting Theorem 2.1(b).

(b) Assuming that I^r has a linear resolution and that $\text{depth } S/I^r = 0$ implies that the $(n-1)$ th syzygy module of I^r has a generator of degree $rd + n - 1$ which in turn implies that there is an element $f \in \tilde{I}^r$ of degree $rd - 1$. It follows from part (a) that $\ell(I) = \ell(I^r) = n$. The proof of part (a) also shows that $\lambda(\tilde{I}^k/I^k) \geq p(k)$, where p is a polynomial of degree $\geq \ell(I)$. Since $\lim_{k \rightarrow \infty} \lambda(\tilde{I}^k/I^k)/n^k$ exists it follows that $\deg p = \ell(I)$ and that $\lim_{k \rightarrow \infty} \lambda(\tilde{I}^k/I^k)/n^k$ is greater than or equal to the leading coefficient of p . \square

Theorem 2.1 can be used to derive the following finiteness criterion. A related result for ordinary symbolic powers was proven by Katz and Ratliff in Theorem A and Corollary 1 of [22].

Theorem 2.6. *Let (R, \mathfrak{m}) be an excellent domain, and let I and J be proper ideals of R . Then the following conditions are equivalent:*

- (a) $S_J(I)$ is finitely generated.
- (b) There exists an integer $r > 0$ such that $\ell((I^r : J^\infty)_P) < \dim R_P$ for all $P \in V(J)$.

Proof. We use the criterion which says that $S_J(I)$ is finitely generated if and only if for some integer $d > 0$ the d th Veronese subalgebra $S_J(I)^{(d)}$ is standard graded, see for example [14, Theorem 2.1]

(a) \Rightarrow (b): We choose an integer r such that $S_J(I)^{(r)}$ is standard graded. Then $(I^r : J^\infty)^k = (I^{rk} : J^\infty)$ for all k . Hence if for a given $P \in V(J)$ we set $L = (I^r : J^\infty)_P$, then it follows that all powers L^k of L are saturated in R_P , and assertion (b) is a consequence of the following claim: let (R, \mathfrak{m}) be a local ring, and $I \subset R$ an ideal with the property that all powers of I are saturated. Then $\ell(I) < \dim R$.

Indeed, let $\text{gr}_I(R)$ be the associated graded ring of I , and assume that $\ell(I) = \dim R$. Then $\dim \text{gr}_I(R) = \dim R = \dim R(I)/\mathfrak{m}R(I) = \dim \text{gr}_I(R)/\mathfrak{m} \text{gr}_I(R)$. Therefore $\mathfrak{m} \text{gr}_I(R)$ is contained in a minimal prime ideal of $\text{gr}_I(R)$, and hence the elements of $\mathfrak{m} \text{gr}_I(R)$ are zerodivisors of $\text{gr}_I(R)$. Since $\mathfrak{m} \text{gr}_I(R)$ is a graded ideal in the graded ring $\text{gr}_I(R)$, there exist a homogeneous element $x = f + I^{k+1}$, $f \in I^k \setminus I^{k+1}$ such that $\mathfrak{m}x = 0$. It follows that $\mathfrak{m}f \subset I^{k+1}$ which shows that I^{k+1} is not saturated.

(b) \Rightarrow (a): Let $r > 0$ be the integer such that $\ell((I^r : J^\infty)_P) < \dim R_P$ for all $P \in V(J)$, and set $L = I^r : J^\infty$. Then by Theorem 2.1(a) we know that $S_J(L)$ is finitely generated. Thus there exists an integer $s > 0$ such that $S_J(L)^{(s)}$ is standard graded. In other words, $(L^s : J^\infty)^k = L^{ks} : J^\infty$ for all k . Since $L = I^r : J^\infty$ this is equivalent to saying that

$$(13) \quad [(I^r : J^\infty)^s : J^\infty]^k = (I^r : J^\infty)^{ks} : J^\infty.$$

Now we claim that for any two integers $i, j > 0$ one has that $(I^i : J^\infty)^j : J^\infty = I^{ij} : J^\infty$. The claim and (13) then implies that $(I^{rs} : J^\infty)^k = I^{rsk} : J^\infty$ for all k . Hence $S_J(I)^{(d)}$ with $d = rs$ is standard graded, and so $S_J(I)$ is finitely generated.

In order to prove the claim first notice that $I^i \subset I^i : J^\infty$, so that $I^{ij} \subset (I^i : J^\infty)^j$ and hence $I^{ij} : J^\infty \subset (I^i : J^\infty)^j : J^\infty$. On the other hand, if $f \in (I^i : J^\infty)^j : J^\infty$, then

$J^r f \in (I^i : J^\infty)^j$ for some $r > 0$. Therefore there exist $g_1, \dots, g_t \in I^i : J^\infty$ and $c_{j_1, \dots, j_t} \in R$ such that $J^r f = \sum c_{j_1, \dots, j_t} g_1^{j_1} \cdots g_t^{j_t}$ where the sum is taken over all sequences (j_1, \dots, j_t) of nonnegative integers with $j_1 + j_2 + \cdots + j_t = j$. For each g_{j_k} there exists an integer $r_k > 0$ such that $J^{r_k} g_{j_k} \in I^i$. Thus for a suitably large positive integer, we get that $J^r f \in I^i$. In other words, $f \in I^i : J^\infty$. \square

Corollary 2.7. *The algebra $\bigoplus_{k \geq 0} \tilde{I}^k$ is finitely generated, if and only if $\ell(\tilde{I}^r) < \dim R$ for some integer $r > 0$.*

3. FORM IDEALS OF POWERS OF COMPLETE INTERSECTIONS

Let (R, \mathfrak{m}) be a Noetherian local ring, $I \subset \mathfrak{m}$ an ideal. For any local ring (S, \mathfrak{n}) we let $\text{gr}_{\mathfrak{n}}(S) = \bigoplus_{k \geq 0} \mathfrak{n}^k / \mathfrak{n}^{k+1}$ denote the associated graded ring of S . The canonical epimorphism $R \rightarrow R/I$ induces an epimorphism $\text{gr}_{\mathfrak{m}}(R) \rightarrow \text{gr}_{\overline{\mathfrak{m}}}(R/I)$ whose kernel we denote by I^* . Here $\overline{\mathfrak{m}}$ denotes the maximal ideal of R/I . The graded ideal I^* is called *the form ideal of I* . If $f \in R$ and d is the maximal number such that $f \in \mathfrak{m}^d$, then we set $f^* = f + \mathfrak{m}^{d+1}$ and call it *the leading form of f* . The leading forms f^* with $f \in I$ generate I^* . Any system of generators f_1, \dots, f_m of I such that f_1^*, \dots, f_m^* generates I^* is called *a standard basis of I* . A standard basis of I is a system of generators of I , but is usually not a minimal system of generators.

The following result, due to Robbiano and Valla [33], is well-known.

Lemma 3.1. *Let (R, \mathfrak{m}) be a local ring such that $\text{gr}_{\mathfrak{m}}(R)$ is domain, and $I \subset R$ an ideal. Let f_1, \dots, f_m be a system of generators of I . Then f_1, \dots, f_m is a standard basis of I if all relations of f_1^*, \dots, f_m^* can be lifted. In other words, whenever there is a homogeneous relation*

$$g_1 f_1^* + g_2 f_2^* + \cdots + g_m f_m^* = 0,$$

with $g_i \in \text{gr}_{\mathfrak{m}}(R)$, then there exist $h_i \in R$ with $g_i = h_i^*$ for $i = 1, \dots, m$ such that

$$h_1 f_1 + h_2 f_2 + \cdots + h_m f_m = 0.$$

Moreover it is sufficient to test the lifting property for a system of homogeneous generators of the relation module of f_1^*, \dots, f_m^* .

We are interested in the algebra $A(I) = \bigoplus_{k \geq 0} (I^k)^*$. If this algebra happens to be finitely generated, then for $k \gg 0$ the coefficients of the Hilbert polynomials $P_{R/I^k}(t)$ are quasi-polynomials as functions of k .

For the formulation of the next result we need the following definition: let A be graded K -algebra and $J \subset A$ a graded ideal. We say that J is *liftable*, if there exists a graded ideal $\tilde{J} \subset A[t_1, \dots, t_r]$ in a polynomial ring extension $A[t_1, \dots, t_r]$ of A with $\deg t_i > 0$ satisfying the following properties: (i) \tilde{J} is generically a complete intersection, (ii) t_1, \dots, t_r is a regular sequence on $A[t_1, \dots, t_r]/\tilde{J}$, and (iii) $A[t_1, \dots, t_r]/\tilde{J}$ modulo (t_1, \dots, t_r) is isomorphic to A/J . If J is liftable to \tilde{J} , then J is called a *specialization of \tilde{J}* .

Theorem 3.2. *Let (R, \mathfrak{m}) be a local ring such that $\text{gr}_{\mathfrak{m}}(R)$ is a domain, and $I \subset R$ an ideal. Then $A(I) = \bigoplus_{k \geq 0} (I^k)^*$ is standard graded, if I^* is a complete intersection.*

Conversely, suppose that $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay and $I \subset R$ is a complete intersection ideal, satisfying:

- (i) $\dim R/I = 0$,
- (ii) $(I^*)^k = (I^k)^*$ for infinitely many $k > 1$ (for example, if $A(I)$ is standard graded),
- (iii) I^* is liftable.

Then I^* is a complete intersection.

Proof. Let g_1, \dots, g_m be the regular sequence generating I^* , and let $f_1, \dots, f_m \in I$ with $f_i^* = g_i$ for $i = 1, \dots, m$. Then f_1, \dots, f_m is a regular sequence generating I , and in particular it is a standard basis of I . Now fix an integer $k > 1$. We claim that the monomials $f^a = f_1^{a_1} f_2^{a_2} \cdots f_m^{a_m}$ in f_1, \dots, f_m of degree k form a standard basis of I^k . This will then imply that $(I^k)^* = (I^*)^k$ for all k , so that $A(I)$ is standard graded.

In order to see that the monomials f^a of degree k form indeed a standard basis of I^k we just need to show that all generating relations of the ideal generated by the leading forms of the elements $(f^a)^*$ with $a \in \mathbb{N}^m$ and $|a| = k$ can be lifted. Observe that $(f^a)^* = g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m}$. Since g_1, \dots, g_m is a regular sequence, the relation module of $(g_1, \dots, g_m)^k$ is generated by relations of the form

$$g_j(g_1^{a_1} g_2^{a_2} \cdots g_i^{a_i+1} \cdots g_m^{a_m}) - g_i(g_1^{a_1} g_2^{a_2} \cdots g_j^{a_j+1} \cdots g_m^{a_m}).$$

These relations can obviously all be lifted.

For the second part of the theorem let $J = I^*$. Since I is a complete intersection ideal with $\dim R/I = 0$, (ii) implies that for infinitely many integers $k > 1$ we have

$$(14) \quad \begin{aligned} \lambda(\mathrm{gr}_{\mathfrak{m}}(R)/J^k) &= \lambda(\mathrm{gr}_{\mathfrak{m}}(R)/(I^k)^*) = \lambda(R/I^k) \\ &= \binom{d+k-1}{k-1} \lambda(R/I) = \binom{d+k-1}{k-1} \lambda(\mathrm{gr}_{\mathfrak{m}}(R)/J), \end{aligned}$$

where $d = \dim \mathrm{gr}_{\mathfrak{m}}(R)$.

Let $\tilde{J} \subset \mathrm{gr}_{\mathfrak{m}}(R)[t_1, \dots, t_r]$ be a lifting of J . Since \tilde{J} is generically a complete intersection, the associativity formula for multiplicities ([2, Corollary 4.7.8]) implies that

$$(15) \quad e(\mathrm{gr}_{\mathfrak{m}}(R)[t_1, \dots, t_r]/\tilde{J}^k) = \binom{d+k-1}{k-1} e(\mathrm{gr}_{\mathfrak{m}}(R)[t_1, \dots, t_r]/\tilde{J}).$$

Since $e(\mathrm{gr}_{\mathfrak{m}}(R)[t_1, \dots, t_r]/\tilde{J}) = \lambda(\mathrm{gr}_{\mathfrak{m}}(R)/J)$, the equations (14) and (15) imply that

$$e(\mathrm{gr}_{\mathfrak{m}}(R)[t_1, \dots, t_r]/\tilde{J}^k) = \lambda(\mathrm{gr}_{\mathfrak{m}}(R)/J^k)$$

for infinitely many $k > 1$ which in turn implies that $\mathrm{gr}_{\mathfrak{m}}(R)[t_1, \dots, t_r]/\tilde{J}^k$ is Cohen–Macaulay, see [2, Corollary 4.7.11]. Thus by a result of Cowsik and Nori [4] and its generalization by Waldi [35, Korollar 1] it follows that \tilde{J} is a complete intersection. Hence $I^* = J$ is a complete intersection as well. \square

Corollary 3.3. *Let (R, \mathfrak{m}) be a regular local ring and $I \subset R$ be a complete intersection ideal with $\dim R/I = 0$. Assume further that either (R, \mathfrak{m}) is 2-dimensional or I^* is a monomial ideal. Then the following conditions are equivalent:*

- (a) $A(I)$ is standard graded;
- (b) I^* is a complete intersection;
- (c) for infinitely many integers k we have $(I^k)^* = (I^*)^k$.

Proof. Under the given assumptions the ideal I^* is liftable. Indeed, if I^* is a monomial ideal, then one applies polarization, see [2, Lemma 4.2.16], and if $\dim R = 2$, then I^* is perfect of codimension 2. Hence if I^* is generated by m elements, the Hilbert–Burch theorem [2, Theorem 1.4.17] implies that I^* is the specialization of the ideal of maximal minors of an $m \times (m + 1)$ -matrix of indeterminates. This ideal is generically a complete intersection.

Now we see that Theorem 3.2 yields the implications (c) \Rightarrow (b) and (b) \Rightarrow (a). The implication (a) \Rightarrow (c) is trivial. \square

It would be interesting to know whether for a complete intersection the conditions (a), (b) and (c) in Corollary 3.3 are equivalent without the assumption that I^* is liftable.

The following simple example shows that even for a complete intersection the algebra $A(I)$ need not to be finitely generated.

Example 3.4. Let K be a field and consider the ideals $I = (x^2, y^3 - xy) \subset S = K[[x, y]]$. We claim that

$$(16) \quad (I^k)^* = ((xy, x^2)^k, \{x^i y^{4k-3i+1}\}_{i=0, \dots, k-1}).$$

The claim implies that y^{4k+1} is a minimal generator of $(I^k)^*$. It follows that for each k , the element $y^{4k+1} \in (I^k)^*$ is a minimal generator of degree k of the form algebra $A = \bigoplus_{k \geq 0} (I^k)^*$ of I . In particular, we see that A is not finitely generated.

We prove (16) by induction of k , and set $f = x^2$ and $g = y^3 - xy$. In order to prove (16) for $k = 1$ we first notice that $y^5 \in I$. Indeed, we have $y^5 = (y^3 + x)g + yf$. It follows that $(x^2, xy, y^5) \subset I^*$. Applying the Buchberger criterion we see immediately that $x^2, xy - y^3, y^5$ is a Groebner basis of I with respect to the lexicographical order. Hence $\text{in}(I) = (x^2, xy, y^5)$ is the initial ideal of I with respect to this monomial order. Therefore $\lambda(S/(x^2, xy, y^5)) = \lambda(S/I)$. On the other hand we have $\lambda(S/I^*) = \lambda(S/I)$. This implies that $(x^2, xy, y^5) = I^*$ and proves the claim for $k = 1$. It also shows that $\lambda(S/I) = 6$.

Now let $k > 1$ and assume that (16) holds for all $j < k$. Then

$$(17) \quad I^*(I^{k-1})^* + (y^{4k+1}) = ((xy, x^2)^k, \{x^i y^{4k-3i+1}\}_{i=0, \dots, k-1}) \subset (I^k)^*.$$

Thus it remains to be shown that

$$(18) \quad (I^k)^* = I^*(I^{k-1})^* + (y^{4k+1}).$$

Since I is generated by a regular sequence it follows that all the modules I^j/I^{j+1} are free S/I -modules of rank $j + 1$. From this we deduce that

$$\lambda(S/(I^k)^*) = \lambda(S/(I^k)) = \lambda(S/I) \binom{k+1}{2} = 6 \binom{k+1}{2}.$$

Now we compute the length of $S/I^*(I^{k-1})^* + (y^{4k+1})$. In view of formula (17) we see that $S/I^*(I^{k-1})^*$ has the following monomial K -basis: $C \cup \bigcup_{i=0}^{k-1} B_i$ where

$$C = \{x^k(x^i y^j)\}_{i+j \leq k-1}, \quad \text{and} \quad B_i = \{x^i y^j\}_{j \leq 4k-3i}.$$

Counting the number of elements of this basis we see that

$$\begin{aligned} \lambda(S/I^*(I^{k-1})^* + (y^{4k+1})) &= |C| + \sum_{i=0}^{k-1} |B_i| \\ &= \binom{k+1}{2} + \sum_{i=0}^{k-1} (4k - 3i + 1) = 6 \binom{k+1}{2}. \end{aligned}$$

Thus

$$(19) \quad \lambda(S/I^*(I^{k-1})^* + (y^{4k+1})) = \lambda(S/(I^k)^*),$$

and hence it suffices to show that $y^{4k+1} \in I^k$. Indeed, we will show that for $i = 0, 1, \dots, 2k$ the monomials $x^{2k-i}y^{2i+1}$ belong to I^k . We proceed by induction on i . For $i = 0$ we have $x^{2k}y = f^k y \in I^k$. Now let $i > 0$ and suppose that $x^{2k-j}y^{2j+1} \in I^k$ for $j < i$. Let the integers a and b be defined by the equations

$$2k - i = 2a + r_1, \quad 0 \leq r_1 \leq 1, \quad \text{and} \quad 2i + 1 = 3b + r_2, \quad 0 \leq r_2 \leq 2.$$

Then $4k + 1 = 4a + 3b + 2r_1 + r_2$ which implies that $4a + 3b \geq 4k - 3$. From this we deduce that $a + b \geq k$. Therefore $x^{r_1}y^{r_2}(x^2)^a(y^3 - xy)^b \in I^k$ and $x^{2k-i}y^{2i+1} - x^{r_1}y^{r_2}(x^2)^a(y^3 - xy)^b$ is a linear combination of monomials of the form $x^{2k-j}y^{2j+1}$ with $j < i$. Since by induction hypothesis these monomials belong to I^k , we conclude that $x^{2k-i}y^{2i+1} \in I^k$.

We can slightly modify Example 3.4 to get finite generation of the algebra $A(I)$ depending on the characteristic of the base field.

Example 3.5. Let $I = (x^2 + y^2, (x + y)y + y^3) \subset K[[x, y]]$. Then the algebra $A(I) = \bigoplus_{k \geq 0} (I^k)^*$ is standard graded if $\text{char } K \neq 2$, and it is not finitely generated if $\text{char } K = 2$. Indeed, if $\text{char } K \neq 2$, then the leading forms $f^* = x^2 + y^2$ and $g^* = (x + y)y$ of $f = x^2 + y^2$ and $g = (x + y)y + y^3$ are prime to each other. Hence Theorem 3.2 implies that $A(I)$ is standard graded. On the other hand, if $\text{char } K = 2$, then $f^* = (x + y)^2$. Applying the linear automorphism $\varphi: K[[x, y]] \rightarrow K[[x, y]]$ with $\varphi(x) = x + y$ and $\varphi(y) = y$ we see that $\varphi(I) = (x^2, xy + y^3)$. It follows that $A(I)$ is not finitely generated since $A(\varphi(I))$ is not finitely generated.

At present we do not know of a complete intersection ideal I for which $A(I)$ is finitely generated but not standard graded.

We now describe the relationship between form ideals of powers and symbolic powers. We fix a field K and consider an ideal $I \subset R = K[[x_1, \dots, x_n]]$. The following lemma establishes the link between the two concepts.

Lemma 3.6. *Let $R[[s]]$ be a power series ring over R , and let $\alpha: R \rightarrow R[[s]]$ be the K -algebra homomorphism with $\alpha(x_i) = x_i s$ for $i = 1, \dots, n$. We denote by $\alpha(I)$ the ideal in $R[[s]]$ generated by the elements $\alpha(f)$ with $f \in I$, and set $I^\sharp = \alpha(I): s^\infty$. Then s is a regular element on $A = R[[s]]/I^\sharp$ and $A/(s) \cong R/I^*R$.*

Proof. We first observe that s is a regular element on $A = R[[s]]/I^\sharp$. Indeed, if $sf \in I^\sharp$ for some $f \in R[[s]]$, then there exists an integer k such that $s^{k+1}f = s^k(sf) \in \alpha(I)$. Then $f \in I^\sharp$.

In order to prove the isomorphism $A/(s) \cong R/I^*R$, we show that I^\sharp is generated by the elements f^\sharp with $f \in I$, where for $f = f_d + f_{d+1} + \cdots \in R$ with each f_i homogeneous of degree i and $f_d \neq 0$, we set $f^\sharp = f_d + sf_{d+1} + \cdots + s^{i-d}f_i + \cdots$. Observe that $f^\sharp = s^{-d}\alpha(f)$, where d is the initial degree of f . This shows that $f^\sharp \in I^\sharp$ for all $f \in I$.

Conversely, let $f \in I^\sharp$. Then there exists an integer k such that $s^k f \in \alpha(I)$. Assigning to s the degree -1 and to each x_i the degree 1 , we see that the generators of $\alpha(I)$ are homogeneous of degree 0 . Here we call a power series $h \in R[[s]]$ homogeneous, if all monomials in the support of h are of same degree. Let $h \in R[[s]]$ be a power series, and $i \in \mathbb{Z}$. We let h_i be the sum of those terms in h whose degree is i . Then h is the formal sum of the h_i and each h_i is homogeneous of degree i . We call h_i the i th homogeneous component of h . The expression $h = \sum_i h_i$ makes sense, because the monomial support of h_i and h_j is disjoint for $i \neq j$. Suppose now that g is homogeneous of degree j . Then $(hg)_i = h_{i-j}g$ for all j . Thus if $J \subset R[[s]]$ is an ideal generated by homogeneous elements g_1, \dots, g_r of degree j_1, \dots, j_r , respectively, then h belongs to J if and only if all its homogeneous components belong to J , as is the case for a positively graded algebra. Indeed, if $h = \sum_{i=1}^r a_i g_i$, then the i th homogeneous component of h is $\sum_{i=1}^r (a_i)_{i-j_i} g_i$, and thus it belongs to J .

Hence, since $\alpha(I)$ is generated by homogeneous elements, we may assume $s^k f$ is homogeneous. In particular f is homogeneous, say $\deg f = d$. Hence there exist $l_1, \dots, l_r \in I$ such that $s^k f = \sum_{i=1}^r g_i \alpha(l_i)$ with each $g_i \in R[[s]]$ homogeneous of degree d_i and such that $d = d_i + k$ for $i = 1, \dots, r$.

Since g_i is homogeneous, it is of the form $\sum_{j \geq 0} g_{ij} s^j$ where for all j , g_{ij} is a homogeneous polynomial in the variables x_1, \dots, x_n of degree $d_i + j$. Let $h_i = \sum_{j \geq 0} g_{ij}$. Then there exist integers k_i such that $g_i = s^{k_i} \alpha(h_i)$ for $i = 1, \dots, r$. We have $k_i = -d_i$ for all i , since $\deg \alpha(h_i) = 0$. It follows that

$$s^k f = \sum_{i=1}^r s^{-d_i} \alpha(b_i) \quad \text{with} \quad b_i = h_i l_i \in I.$$

Write $f = \sum_l f_l s^l$ where each f_l is a homogeneous polynomial of degree $d - l$ in the variables x_1, \dots, x_n , and write each $b_i = \sum_j b_{ij}$, where b_{ij} is a homogenous polynomial of degree j in the variables x_1, \dots, x_n . Then we get

$$s^k f = \sum_l f_l s^{l+k} = \sum_{i=1}^r \left(\sum_j b_{ij} s^{j-d_i} \right) = \sum_{i=1}^r \left(\sum_j b_{ij} s^{j+k-d} \right) = \sum_j \left(\sum_{i=1}^r b_{ij} \right) s^{j+k-d}.$$

Comparing coefficients we see that $f_l = \sum_{i=1}^r b_{i,l+d}$ for all l . This shows that $\sum_l f_l = \sum_{i=1}^r b_i \in I$. Thus if we set $g = \sum_{i=1}^r b_i$, then $g \in I$ and $f = s^m g^\sharp$ for some nonnegative integer m , as desired. \square

We set $J = (s)$. Then we get

Proposition 3.7. *The following conditions are equivalent:*

- (a) $S_J(I^\sharp) = \bigoplus_{k \geq 0} (I^\sharp)^k : s^\infty$ is finitely generated (resp. standard graded).
- (b) $A(I) = \bigoplus_{k \geq 0} (I^k)^*$ is finitely generated (resp. standard graded).

Proof. We first notice that

$$(20) \quad (I^\#)^k : s^\infty = (\alpha(I) : s^\infty)^k : s^\infty = \alpha(I)^k : s^\infty = \alpha(I^k) : s^\infty.$$

An argument as in the proof of Corollary 2.6 shows the second equation in (20).

Set $J_k = (I^\#)^k : s^\infty$ and $I_k = (I^k)^* R$. Then $S_J(I^\#)$ is finitely generated if and only if for some integer $d > 0$ one has $(J_k)^d = J_{dk}$ for all k , and a corresponding statement holds for $A(I)$, see for example [14, Theorem] or [27].

For an $R[[s]]$ -module M , we set $\overline{M} = M/sM$. Then since s is regular on $R[[s]]/J_{dk}$, the exact sequence

$$0 \rightarrow J_{dk}/(J_k)^d \rightarrow R[[s]]/(J_k)^d \rightarrow R[[s]]/J_{dk} \rightarrow 0$$

induces the exact sequence

$$0 \rightarrow \overline{J_{dk}/(J_k)^d} \rightarrow R/(I_k)^d \rightarrow R/I_{dk} \rightarrow 0,$$

see [2, Proposition 1.1.4]. Therefore, $\overline{J_{dk}/(J_k)^d} = I_{dk}/(I_k)^d$, and hence Nakayama's lemma implies that $(J_k)^d = J_{dk}$ if and only if $(I_k)^d = I_{dk}$, which is the case if and only if $((I^k)^*)^d = (I^{dk})^*$. This shows that $S_J(I)$ is finitely generated if and only if $A(I)$ is finitely generated. In the same way one shows that $S_J(I)$ is standard graded if and only if $A(I)$ is standard graded. \square

Example 3.8. Let $J = (x^2, xy - sy^3, y^5) \subset K[[x, y, s]]$. Then $J = I^\#$ for the ideal $I = (x^2, xy - y^3)$ in Example 3.4. The ideal J is a Cohen–Macaulay ideal of codimension 2 and has the relation matrix

$$\begin{pmatrix} y & -x + sy^2 & s^2 \\ 0 & -y^4 & x - sy^2 \end{pmatrix}$$

Since $A(I)$ is not finitely generated, Proposition 3.7 tells us that $\bigoplus_k J^k : s^\infty$ is not finitely generated as well.

Since (J, s) is (x, y, s) -primary it follows that $J^k : s^\infty = \widetilde{J^k}$. Computations with CoCoA suggests that $\lim_{k \rightarrow \infty} \lambda(\widetilde{J^k}/J^k)/k^3$ exists and is equal to $1/3$. The existence of this limit cannot be deduced from the results of Section 1 or from [8].

REFERENCES

- [1] M. Brodmann and R. Y. Sharp, Local Cohomology: An algebraic introduction with geometric applications, Cambridge U. Press, 1998.
- [2] W. Bruns and J. Herzog, “Cohen–Macaulay rings,” Revised Edition, Cambridge University Press, Cambridge, 1996.
- [3] A. Conca, J. Herzog, N. V. Trung and G. Valla, Diagonal subalgebras of bigraded algebras and embeddings of blow-ups of projective spaces, Amer. J. Math. 119 (1997), 859-901
- [4] R. C. Cowsik and M. V. Nori, On the Fibres of Blowing Up, Journal of the Indian Math. Soc. 40 (1976), 217-222.
- [5] D. Cutkosky, Symbolic algebras of monomial primes, Journal für die reine und angewandte Mathematik 416 (1991), 71-89.
- [6] S. D. Cutkosky and H. Srinivasan. An intrinsic criterion for isomorphism of singularities, Amer. J. Math. 115 (1993), 789-821.
- [7] S. D. Cutkosky, L. Ein and R. Lazarsfeld. Positivity and complexity of ideal sheaves, Math. Ann 321 (2001), 213-234.

- [8] S. D. Cutkosky, H. T. Ha, H. Srinivasan and E. Theodorescu, Asymptotic behaviour of the length of local cohomology *Canad. J. Math.* 57 (2005), no. 6, 1178–1192.
- [9] S. D. Cutkosky, J. Herzog and N. V. Trung, Asymptotic behaviour of the Castelnuovo-Mumford regularity, *Compositio Math.* 118 (1999), no. 3, 243–261.
- [10] D. Eisenbud and S. Goto, Linear Free Resolutions and Minimal Multiplicity, *J. Algebra* 88 (1984), 89 – 133.
- [11] T. Fujita, Semipositive line bundles, *J. of the Faculty of Science, The University of Tokyo* 30 (1983), 353–378.
- [12] T. Fujita, Approximating Zariski decomposition of big line bundles, *Kodai Math. J.* 17 (1994), 1–3.
- [13] S. Goto, K. Nishida and K. Watanabe, Non-Cohen-Macaulay symbolic Rees algebras for space monomial curves and counterexamples to Cowsik’s question, *Proc. Amer. Math. Soc.* 120 (1994), 383–392.
- [14] J. Herzog, T. Hibi and N. V. Trung, Symbolic powers of monomial ideals and vertex cover algebras, *Advances in Math.* 210 (2007), 304–322.
- [15] J. Herzog, T. J. Putenpurakal and J. K. Verma, Hilbert polynomials and powers of ideals, to appear in *Mathematical Proc. Cambridge Phil. Soc.*
- [16] H. Hironaka, On the equivalence of singularities, I, *Arithmetical Algebraic Geometry, Proceedings, O.F.G. Schilling, ed., Harper and Row, New York, 1965*, pp. 153–200.
- [17] S. Huckaba, On linear equivalence of the P -adic and P -symbolic topologies, *J. Pure Appl. Alg.* 46 (1987), 179 – 185.
- [18] N. D. Hoang and N. V. Trung, Hilbert polynomials of non-standard bigraded algebras, *Math. Z.* 245 (2003), no. 2, 309–334.
- [19] C. Huneke, On the finite generation of symbolic blow-ups, *Math. Z.*, 179 (1982), 465–472.
- [20] C. Huneke and I. Swanson, *Integral closures of ideals, rings and modules*, London Math. Society lecture Note Series 336, Cambridge University Press, 2006.
- [21] D. Katz, Prime divisors, asymptotic R -sequences and unmixed local rings, *J. Alg.*, 95 (1985), 59–71.
- [22] D. Katz and L.J. Ratliff, On the symbolic Rees ring of a primary ideal, *Comm. Alg.*, 14 no. 5, (1986), 959–970.
- [23] D. Kirby, Hilbert functions and the extension functor, *Math. Proc. Cambridge Philos. Soc.* (3) 105 (1989), 441 - 446.
- [24] V. Kodiyalam, Asymptotic behaviour of Castelnuovo-Mumford regularity, *Proc. Amer. Math. Soc.* 128 no. 2, (2000), 407–411.
- [25] R. Lazarsfeld, *Positivity in Algebraic Geometry*, Springer Verlag, Berlin, 2004.
- [26] S. McAdam, Asymptotic prime divisors and analytic spreads, *proc. Am. mMth. Soc.*, 80 (1980), 555–559.
- [27] L.J. Ratliff, Notes on essentially powers filtrations, *Michigan Math. J.* 26 no. 3, (1979), 313–324.
- [28] L.J. Ratliff, On asymptotic prime divisors, *Pacific J. Math.* 111 (1984), no. 2, 395–413.
- [29] P. C. Roberts, A prime ideal in a polynomial ring whose symbolic blow-up is not Noetherian, *Proc. Amer. Math. Soc.* 94 (1985), no. 4, 589–592.
- [30] P. Schenzel, Finiteness of relative Rees rings and asymptotic prime divisors, *Math. Nachr.* 129 (1986), 123–148.
- [31] E. Theodorescu, Derived functors and Hilbert Polynomials, *math. Proc. Cambridge Philos. Soc.* 132 (2002), 75–88.
- [32] B. Ulrich and J. Validashti, A Criterion for Integral dependence of Modules, *Math. Res. Lett.* 14 (2007) no. 6, 1041 – 1054.
- [33] L. Robbiano, G. Valla, On the equations defining tangent cones, *Math. Proc. Cambridge Philos. Soc.* 88 (1980), no. 2, 281–297. (
- [34] J. Verma, On ideals whose adic and symbolic topologies are linearly equivalent, *J. Pure Appl. Alg.* 47 (1987), 205- - 212.
- [35] R. Waldi, Vollständige Durchschnitte in Cohen-Macaulay-Ringen, *Arch. Math.* 31, no.1, (1978), 439–442.

STEVEN DALE CUTKOSKY, MATHEMATICS DEPARTMENT, 202 MATHEMATICAL SCIENCES BLDG, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211 USA

E-mail address: `dale@math.missouri.edu`

JÜRGEN HERZOG, FACHBEREICH MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, CAMPUS ESSEN, 45117 ESSEN, GERMANY

E-mail address: `juergen.herzog@uni-essen.de`

HEMA SRINIVASAN, MATHEMATICS DEPARTMENT, 202 MATHEMATICAL SCIENCES BLDG, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211 USA

E-mail address: `srinivasan@math.missouri.edu` <`srinivasan@math.missouri.edu`>