Poincaré series of line bundles on varieties

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Abstract. We associate a Poincaré series to the dimension of global sections of multiples of r line bundles on a proper variety X. We show that this series is rational on a nonsingular characteristic 0 curve, but can be irrational on a singular curve, and can be irrational in higher dimensions. We also show that the series can be irrational in positive characteristic, even on a nonsingular curve.

Dedicated to Professor Seshadri on the occasion of his 70th birthday

1. Introduction

Suppose that X is a projective variety over an algebraically closed field of characteristic zero, and that $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are line bundles on X. We consider the function

$$h(\underline{n}) = h^0(X, \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r})$$

for $\underline{n} = (n_1, \ldots, n_r) \in \mathbf{N}^r$, and the Poincaré series

$$f(\underline{n}) = \sum_{\underline{n} \in \mathbf{N}^r} h(\underline{n}) t^{\underline{n}}$$

where $t^{\underline{n}} = t_1^{n_1} \cdots t_r^{n_r}$ is a monomial in the variables $t_1, \ldots, t_r$.

If X is a nonsingular curve, then $h(\underline{n})$ is a little complicated. An illustrative example is as follows.

Let C be a nonsingular elliptic curve, over an algebraically closed field k.

Fix a point $p_\infty \in C$ as the 0 in the group law on C. For $p \in C$, let $\overline{p}$ be the divisor $\overline{p} - p_\infty$. We then have a group isomorphism $C \to \mathrm{Pic}^0(C)$ given by $p \mapsto \overline{p}$. Let $p \in C$ be a point such that $\overline{p}$ has infinite order in $\mathrm{Pic}^0(C)$.

Let $\mathcal{L}_1 = \mathcal{O}_C(\overline{p})$, $\mathcal{L}_2 = \mathcal{O}_C(-\overline{p})$. Then

$$h(n_1, n_2) = h^0(C, \mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2}) = \begin{cases} 1 & \text{if } n_1 = n_2 \\ 0 & \text{otherwise} \end{cases}$$

However, this is essentially the most complicated situation that can arise on a nonsingular curve (in characteristic zero), and in fact we have that the Poincaré series

$$f(\underline{n}) = \sum_{\underline{n} \in \mathbf{N}^r} h(\underline{n}) t^{\underline{n}}$$

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on a nonsingular curve is a rational series. We prove this in Theorem 4.1 of Section 4.

We also have that the series $f(\underline{n})$ is always rational on a singular curve, as long as $r \leq 2$. When $r = 1$ this is immediate from the Riemann-Roch formula and Theorem 2.2. The case $r = 2$ follows from the method of proof of Theorem 4.1, using Theorem 15 of [**2**] in place of Proposition 2.1. Theorem 15 of [**2**] is a general statement for cyclic subgroups of algebraic groups.

However, if C is a singular curve, $r \geq 3$ and $\mathrm{Pic}^0(C)$ is not semi-abelian, then the Poincaré series may be irrational. We give in Example 4.2 an example of a singular curve C with $\mathrm{Pic}^0(C) \cong \mathbf{C}^2$, and 3 line bundles such that the associated Poincaré series is irrational.

In Section 5 we consider the corresponding problem on a nonsingular surface X. If X is a nonsingular projective surface over a field of characteristic zero, and $r = 1$, we have that the series $f(n)$ is rational, since $h(n)$ is a polynomial with periodic coefficients for large n (Theorem 4 [**2**]). In Example 5.1 of Section 5, we give an example of a nonsingular surface S and two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ ($r = 2$) such that the associated Poincaré series is not rational. This example is based on the construction of Example 4 [**2**]. Essential use is made of the fact that $\mathcal{L}_1$ and $\mathcal{L}_2^{-1}$ are ample. This leaves open the question of rationality for the more classical case where the line bundles are associated to effective divisors, that is $h^0(X, \mathcal{L}_i) > 0$ for $1 \leq i \leq r$. In Example 6.1 we give an example of 3 line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ on a nonsingular surface X such that $h^0(X, \mathcal{L}_i) > 0$ for $1 \leq i \leq 3$, and the associated Poincaré series is not rational. The underlying idea of this example is the explict calculation of the Abel-Jacobi map of a singular curve C embedded in X, with $\mathrm{Pic}^0(C) \cong \mathbf{C}^2$ of Section 3. This is also the curve used in Example 4.2. We show that the image of C in $\mathrm{Pic}^0(C)$ is just the curve $y = 2x^3$ (Lemma 3.2), which contains infinitely many rational points.

A related example is constructed in [**3**], where the Poincaré series associated to the exceptional divisors of a resolution of a normal surface singularity is considered.

Over a field of characteristic $p > 0$ the Poincaré series can be irrational if X is a nonsingular curve and $r = 2$ (this follows from an analysis of the line bundles on the supersingular curve constructed in Example 3 of [**2**]). An analysis of the Poincaré series of Example 3 of [**2**] shows that if X is a nonsingular surface of characteristic $p > 0$, then the Poincaré series associated to a singular line bundle can be irrational.

2. Preliminaries

Suppose that X is a nonsingular projective surface over a field k. If $\mathcal{L}$ is a line bundle (or a divisor D) on X and C is an integral curve on X, then $\mathcal{L} \cdot C$ (or $D \cdot C$) will denote the line bundle $\mathcal{L} \otimes \mathcal{O}_C$ (or the linear equivalence class of $\mathcal{O}_C \otimes \mathcal{O}_X(D)$). $(\mathcal{L} \cdot C)$ (or $(C \cdot D)$) will denote the degree of $\mathcal{L} \otimes \mathcal{O}_C$ (or of $\mathcal{O}_X(D) \otimes \mathcal{O}_C$).

If $\mathcal{M}$ is a coherent sheaf on X, then $H^i(X, \mathcal{M})$ has finite length as a k-vector space for $0 \leq i$. We will denote

$$h^i(X, \mathcal{M}) = \dim_k H^i(X, \mathcal{M}).$$

The following result is a direct consequence of Lang's conjecture, proven by McQuillan in [**6**]. Recall that a semi-abelian variety is a commutative algebraic group such that there is an exact sequence

$$0 \to \mathbf{G}_m^a \to G \to A \to 0$$

where A is an abelian variety, and $a \in \mathbf{N}$.

PROPOSITION 2.1. *Suppose that G is a semi-abelian variety over an algebraically closed field k of characteristic 0, H is a finitely generated abelian group, and $\pi : H \to G(k)$ is a group homomorphism. Suppose that $Y \subset G$ is a closed integral subvariety such that $Y(k) \cap \pi(H)$ is Zariski dense in Y. Then there exists a subgroup M of H and $n_0 \in H$ such that*

$$\pi(n) \in Y(k) \text{ iff } n \in n_0 + M.$$

PROOF. Let $\Gamma = \pi(H)$, and

$$\overline{\Gamma} = \{x \in G(k) \mid nx \in \Gamma \text{ for some } n \in \mathbf{N}\}.$$

$Y(k) \cap \overline{\Gamma}$ is Zariski dense in X, so by Lang's conjecture (proven in [**6**]) $Y = b + \Delta$ for some $b \in G(k)$ and semi-abelian subvariety Δ of Γ. By assumption, there exists an $n_0 \in H$ such that $\pi(n_0) \in b + \Delta$, so we can assume that $b = \pi(n_0)$. Let

$$M = \{n \in H \mid \pi(n) \in \Delta(k)\}$$

a subgroup of H. We have

$$n_0 + M = \{n \in H \mid \pi(n) \in Y(k)\}.$$

□

By a *rational convex polyhedral set in* $\mathbf{Q}^r$ we mean a set of the form

$$P = \{\underline{n} \in \mathbf{Q}^r \;/\; L_i(\underline{n}) \geq b_i, \quad 1 \leq i \leq m\} \tag{1}$$

where $m \in \mathbf{N}$ and, for $1 \leq i \leq m$, L_i is an integral linear form on $\mathbf{Q}^r$ and $b_i \in \mathbf{Z}$.

THEOREM 2.2. *Let $P \subseteq \mathbf{Q}^r_{\geq 0}$ be a rational convex polyhedral set in $\mathbf{Q}^r$. Let $\underline{m} \in \mathbf{Z}^r$, $M \subset \mathbf{Z}^r$ be a subgroup and suppose that $q(\underline{n})$ is a polynomial in $\mathbf{Q}[t_1, \ldots, t_r]$. Then, there exist $d_i \in \mathbf{N}$, $\underline{a}_i \in \mathbf{N}^r$ and a polynomial $p(t_1, \ldots, t_r) \in \mathbf{Q}[t_1, \ldots, t_r]$, such that*

$$\sum_{\underline{n} \in P \cap (\underline{m} + M)} q(\underline{n}) \underline{t}^{\underline{n}} = \frac{p(t_1, \ldots, t_r)}{\prod_{i=1}^{s} (1 - \underline{t}^{\underline{a}_i})^{d_i}}.$$

This is proven in Section 7 of [**3**].

3. The Abel-Jacobi map of a special quintic curve

The results of this section are Lemma 9.3 and Lemma 9.4 of [**3**], which we reproduce for the reader's convenience.

LEMMA 3.1. *There exists a rational, complex Gorenstein projective curve C which has an isolated singularity $\tilde{p}$, with local ring*

$$\mathcal{O}_{C,\tilde{p}} \cong \mathbf{C}[t^2, t^5]_{(t^2, t^5)},$$

and the following properties:

(1) *C has arithmetic genus $p_a(C) = 2$.*

(2) *If $\mathcal{L}$ is a line bundle on C and deg($\mathcal{L}$) ≥ 8 then $\mathcal{L}$ is generated by global sections.*
(3) *If $\mathcal{L}$ is a line bundle on C and deg($\mathcal{L}$) ≥ 10 then $\mathcal{L}$ is very ample.*
(4) *If $\mathcal{L}$ is a line bundle on C of negative degree, there exists a nonsingular projective surface S and an embedding $C \subset S$ such that $C \cdot C \sim \mathcal{L}$.*

Proof. Let $C_0 = V(x_2^3x_1^2 - x_0^5) \subset X_0 = \mathbf{P}^2_{\mathbf{C}}$. C_0 is a rational curve with two singular points, $q_0 = (0:0:1)$ and $q_1 = (0:1:0)$. We will resolve the singularity at q_1. There are regular parameters $y = \frac{x_0}{x_1}$, $z = \frac{x_2}{x_1}$ at q_1.

$$\mathcal{O}_{C_0,q_1} = (\mathbf{C}[y,z]/z^3 - y^5)_{(y,z)}.$$

Let $\pi_1 : X_1 \to X_0$ be the blowup of q_1. Let C_1 be the strict transform of C_0 on X_1. Let $E_1 = \pi_1^{-1}(q_1)$, $q_2 = E_1 \cap C_1$. Let (y_1, z_1) be the regular parameters at q_2 defined by $y = y_1, z = y_1z_1$. $z_1^3 - y_1^2 = 0$ is a local equation of C_1 at q_2. Let $\pi_2 : X_2 \to X_1$ be the blowup of q_2. Let C_2 be the strict transform of C_1 on X_2, $E_2 = \pi_1^{-1}(q_2)$, $q_3 = E_2 \cap C_2$. Let (y_1, z_1) be the regular parameters at q_2 defined by $y_1 = y_2z_2, z_1 = z_2$. $z_2 - y_2^2 = 0$ is a local equation of C_2 at q_3, and C_2 is thus nonsingular at q_2. Set $\pi = \pi_1 \circ \pi_2$. Identify E_1 with its strict transform on X_2. q_0 is the only singular point on C_2.

$\pi_1^*(C_0) = C_1 + 3E_1$. $\pi^*(C_0) = C_2 + 5E_2 + 3E_1$.

$$C_2 \cdot C_2 \sim \pi^*(C_0) \cdot C_2 - 5E_2 \cdot C_2 - 3E_1 \cdot C_2.$$

$C_2 \cdot E_2$ and $C_2 \cdot E_1$ are supported at q_2, $y_2 = 0$ is a local equation of E_1 at q_2 and $z_2 = 0$ is a local equation of E_2 at q_2. Thus $C_2 \cdot E_1 = q_2$, $C_2 \cdot E_2 = 2q_2$, and

$$C_2 \cdot C_2 \sim \pi^*(C_0) \cdot C_2 - 13q_2.$$

Let V be a general cubic curve on X_0.

$$V \cdot C_0 = p_1 + \cdots + p_{25}$$

where $p_1, \ldots, p_{25} \in C_0$ are distinct nonsingular points.

$$\pi^*(C_0) \cdot C_2 \sim \pi^*(V) \cdot C_2 = p_1 + \cdots + p_{25}.$$

Thus

$$C_2 \cdot C_2 \sim p_1 + \cdots + p_{25} - 13q_2.$$

Set $D = C_2 \cdot C_2$. $\deg(D) = 12$.

$$\hat{\mathcal{O}}_{C_2,q_0} = \mathbf{C}[[\frac{x_0}{x_2}, \frac{x_1}{x_2}]]/(\frac{x_1}{x_2})^2 - (\frac{x_0}{x_2})^5 \cong \mathbf{C}[[t^2, t^5]] \subset \mathbf{C}[[t]]$$

and $\ell(\mathbf{C}[[t]]/\hat{\mathcal{O}}_{C_2,q_0}) = 2$. Let $C = C_2$.

The arithmetic genus of C is thus

$$p_a(C) = p_a(\mathbf{P}^1) + 2 = 2$$

(c.f. Exercise IV 1.8 [**5**]).

Since C is a local complete intersection, the Riemann-Roch Theorem is applicable on C (c.f Exercise IV 1.9 [**5**]). In particular, there is a canonical bundle ω_C on C such that for any line bundle $\mathcal{L}$ on C, $h^1(C, \mathcal{L}) = h^0(C, \omega_C \otimes \mathcal{L}^{-1})$ (Serre duality) and $\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - p_a(C)$. We further have $\deg(\omega_C) = 2p_a(C) - 2 = 2$.

C has an affine cover by opens sets $U_1 = \mathrm{spec}(\mathbf{C}[t^2, t^5])$ and $U_2 = \mathrm{spec}(\mathbf{C}[\frac{1}{t}]$. Let ∞ be the point on C with maximal ideal $(\frac{1}{t})$. ∞ is the point q_2 on C_2. Let $\tilde{p}$ be the point q_1 on C. The maximal ideal of the point $\tilde{p}$ is $m = (t^2, t^5) \subset \mathbf{C}[t^2, t^5])$. Consider the line bundle $\mathcal{L}_1$ defined by

$$H^0(U_1, \mathcal{L}_1) = \mathbf{C}[t^2, t^5]t^2$$
$$H^0(U_2, \mathcal{L}_1) = \mathbf{C}[\tfrac{1}{t}],$$

and the line bundle $\mathcal{L}_2$ defined by

$$H^0(U_1, \mathcal{L}_2) = \mathbf{C}[t^2, t^5]t^5$$
$$H^0(U_2, \mathcal{L}_2) = \mathbf{C}[\tfrac{1}{t}].$$

multiplying $\mathcal{L}_1$ by $\frac{1}{t^2}$ and $\mathcal{L}_2$ by $\frac{1}{t^5}$ we see that $\mathcal{L}_1 \cong \mathcal{O}_C(-2\infty)$ and $\mathcal{L}_2 \cong \mathcal{O}_C(-5\infty)$. We thus have short exact sequences

$$0 \to \mathcal{K}_1 \to \mathcal{O}_C(-2\infty) \oplus \mathcal{O}_C(-5\infty) \to m \to 0 \tag{2}$$

$$0 \to \mathcal{K}_2 \to \mathcal{O}_C(-4\infty) \oplus \mathcal{O}_C(-7\infty) \to m^2 \to 0 \tag{3}$$

of coherent $\mathcal{O}_C$ modules, for some modules $\mathcal{K}_1$ and $\mathcal{K}_2$.

Let $a \in C$ be a closed point, with ideal sheaf m_a. From the exact sequence

$$0 \to \mathcal{L}m_a \to \mathcal{L} \to \mathcal{L}/\mathcal{L}m_a \to 0$$

we see that $\mathcal{L}$ is generated by global sections if $H^1(C, \mathcal{L}m_a) = 0$ for all $a \in C$. By Serre Duality, we have that if $\mathcal{N}$ is a line bundle of degree > 2 then $H^1(C, \mathcal{N}) = 0$. By (3) we see that $\mathcal{L}$ is generated by global sections if $\deg(\mathcal{L}) \geq 8$.

By Proposition II.7.3 [**5**] a line bundle $\mathcal{L}$ on C is very ample if

(1) $\mathcal{L}$ is generated by global sections
(2) $H^0(C, \mathcal{L}m_a) \to H^0(C, \mathcal{L}m_a/\mathcal{L}m_a m_b)$ is surjective for distinct closed point $a, b \in C$
(3) $H^0(C, \mathcal{L}m_a) \to H^0(C, \mathcal{L}m_a/\mathcal{L}m_a^2)$ is surjective for every closed point $a \in C$.

From the exact sequences

$$0 \to \mathcal{L}m_a m_b \to \mathcal{L}m_a \to \mathcal{L}m_a/\mathcal{L}m_a m_b \to 0$$

and

$$0 \to \mathcal{L}m_a^2 \to \mathcal{L}m_a \to \mathcal{L}m_a/\mathcal{L}m_a^2 \to 0,$$

Serre Duality, (2) and (3), we see that $\mathcal{L}$ is very ample if $\deg(\mathcal{L}) \geq 10$.

Suppose that $\mathcal{L} \in \mathrm{Pic}(C_2)$ has degree $-e < 0$. Let $r = e + 12$. $\deg(D - \mathcal{L}) = r > 10$ implies $D - \mathcal{L}$ is very ample, so by Bertini's theorem,

$$D - \mathcal{L} \sim a_1 + \cdots + a_r$$

where $a_1, \ldots, a_r \in C_2$ are distinct nonsingular points in C_2. Let $\lambda : X_3 \to X_2$ be the blowup of $a_1, \ldots, a_r$. Let $F_i = \lambda^{-1}(a_i)$ for $1 \leq i \leq r$. Let $C_3 \cong C$ be the strict transform of C_2. $\lambda^*(C_2) = C_3 + F_1 + \cdots + F_r$.

$$C_3 \cdot C_3 \sim \lambda^*(C_2) \cdot C_3 - a_1 - \cdots - a_r \sim D - a_1 - \cdots - a_r \sim \mathcal{L}.$$

□

Let C be the curve of Lemma 3.1, with singular point $\tilde{p}$. Let $\pi : \mathbf{P}^1 \to C$ be the normalization of C, with function field $\mathbf{C}(\mathbf{P}^1) = \mathbf{C}(t)$ where $t = 0$ is a local equation of $q = \pi^{-1}(\tilde{p})$. Let $\infty \in C$ be the point with local equation $\frac{1}{t} = 0$. Let

$$\mathrm{Div}^0 = \text{ Group of Weil divisors of degree 0 on } C - \tilde{p}.$$

$$\mathrm{Pic}^0(C) \cong \mathrm{Div}^0/\sim$$

where $D_1 \sim D_2$ if $D_1 - D_2 = (f)$ for some $f \in \mathbf{C}(t)$ which is a unit in $\mathcal{O}_{C,\tilde{p}}$ (c.f. II.6 [**5**]).

Suppose that $D \in \mathrm{Div}^0$. There exists $f_D \in \mathbf{C}(t)$ such that $(f_D) = D$ (divisor computed on $\mathbf{P}^1$), and f_D is unique up to multiplication by a nonzero constant in $\mathbf{C}$. Define $\Lambda : \mathrm{Div}^0 \to \mathbf{C}^2$ by

$$\Lambda(D) = \left(\frac{d}{dt}\log(f_D) \mid_{t=0}, \frac{d^3}{dt^3}\log(f_D) \mid_{t=0}\right).$$

$\Lambda(D)$ is a well defined group homomorphism.

For $D \in \mathrm{Div}^0$, we have an expansion

$$f_D = \sum_{i=0}^{\infty} a_i t^i \in \hat{\mathcal{O}}_{\mathbf{P}^1,q} = \mathbf{C}[[t]].$$

$f_D \in \hat{\mathcal{O}}_{C,\tilde{p}}$ if and only if $a_1 = a_3 = 0$. Since $f_D \in \mathbf{C}(t)$ and $\mathcal{O}_{C,\tilde{p}} = \hat{\mathcal{O}}_{C,\tilde{p}} \cap \mathbf{C}(t)$ (c.f. Lemma 2 [**1**]), $f_D \in \mathcal{O}_{C,\tilde{p}}$ if and only if $a_1 = a_3 = 0$. Since $\Lambda(D) = 0$ if and only if $a_1 = a_3 = 0$, $\Lambda(D) = 0$ if and only if $D \sim 0$. By allowing a_1 and a_3 to vary, we see that Λ is onto. Thus Λ is a group isomorphism of $\mathrm{Pic}^0(C)$ with $\mathbf{C}^2$.

We will consider the Abel-Jacobi map

$$AJ : C - \{\tilde{p}\} \to \mathbf{C}^2$$

defined by $AJ(\alpha) = \Lambda(\alpha - \infty)$.

$$\begin{aligned} AJ(\alpha) &= \left(\tfrac{d\log(t-\alpha)}{dt} \mid_{t=0}, \tfrac{d^3\log(t-\alpha)}{dt^3} \mid_{t=0}\right) \\ &= \left(\tfrac{1}{-\alpha}, \tfrac{2}{-\alpha^3}\right). \end{aligned}$$

Define the image of the Abel-Jacobi map to be

$$W = AJ(C - \{\infty\}).$$

W is the subvariety of $\mathbf{C}^2$ defined by $y = 2x^3$.

Lemma 3.2. *Suppose that $D \in Div^0$. Then*

$$h^1(C, \mathcal{O}_C(D + \infty)) = h^0(C, \mathcal{O}_C(D + \infty)) = \begin{cases} 0 & \text{if } \Lambda(D) \notin W \\ 1 & \text{if } \Lambda(D) \in W \end{cases}$$

Proof. There exists $f \in \mathbf{C}(t)$ such that $(f) = D$ (divisor computed on $\mathbf{P}^1$).

$$H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(D + \infty))$$

has $\mathbf{C}$ basis $\frac{1}{f}, \frac{t}{f}$ and

$$H^0(C, \mathcal{O}_C(D + \infty)) = \left\{\lambda \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(D + \infty)) \mid \lambda \in \mathcal{O}_{C,\tilde{p}}\right\}.$$

There is an expansion

$$\frac{t}{f} = a_1t + a_2t^2 + \cdots$$

in $\mathbf{C}[[t]] = \hat{\mathcal{O}}_{\mathbf{P}^1,q}$ where $a_1 \neq 0$, so $\frac{t}{f} \notin \mathcal{O}_{C,\tilde{p}}$. For $0 \neq \alpha \in \mathbf{C}$,

$$\begin{aligned} \tfrac{t-\alpha}{f} \in \mathcal{O}_{C,\tilde{p}} &\Leftrightarrow \Lambda(-D+\alpha-\infty) = 0 \\ &\Leftrightarrow \Lambda(D) = AJ(\alpha) \end{aligned}$$

We further have

$$\frac{1}{f} \in \mathcal{O}_{C,\tilde{p}} \Leftrightarrow \Lambda(-D) = 0 \Leftrightarrow \Lambda(D) = AJ(\infty) = 0.$$

Since $p_a(C) = 2$, the equality of h^0 and h^1 follows from the Riemann-Roch Theorem. □

4. The function $h(\underline{n})$ on a curve

THEOREM 4.1. *Suppose that C is a nonsingular projective integral curve over an algebraically closed field k of characteristic zero, and $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are line bundles on C. Then the Poincaré series*

$$\sum_{n\in\mathbf{N}^r} h(\underline{n})t^{\underline{n}}$$

is a rational series, where

$$h(\underline{n}) = h^0(C, \mathcal{L}_1^{n_1} \otimes \mathcal{L}_r^{n_r}).$$

PROOF. Let $\chi(\mathcal{N})$ be the Euler characteristic of a line bundle $\mathcal{N}$. Let g be the genus of C. Let $d_i = \deg(\mathcal{L}_i)$ for $1 \leq i \leq r$.

$$h(\underline{n}) = \chi(\mathcal{L}_1^{n_1} \otimes \cdots \mathcal{L}_r^{n_r}) \tag{4}$$

if $n_1d_1+\cdots+n_rd_r > 2g-2$. Thus $h(\underline{n})$ is a polynomial for $n_1d_1+\cdots+n_rd_r > 2g-2$, and

$$\sum_{\underline{n}\in\mathbf{N}^r, n_1d_1+\cdots+n_rd_r>2g-2} h(\underline{n})t^{\underline{n}}$$

is a rational series by Theorem 2.2. For $s \in \mathbf{N}$, let

$$H_s = \{\underline{n} \in \mathbf{Z}^r \mid n_1d_1 + \cdots + n_rd_r = s\}.$$

Since $h(\underline{n}) = 0$ if $n_1d_1 + \cdots + n_rd_r < 0$, we are reduced to proving that the $2g-1$ series

$$\sum_{\underline{n}\in H_s\cap\mathbf{N}^r} h(\underline{n})t^{\underline{n}} \tag{5}$$

are rational for $0 \leq s \leq 2g-2$.

For the rest of this proof we will fix s with $0 \leq s \leq 2g-2$, and prove the rationality of (5).

Fix a base point $p_0 \in C$ and set $\overline{\mathcal{L}}_i = \mathcal{L}_i \otimes \mathcal{O}_C(-d_ip_0)$ for $1 \leq i \leq r$.

We have a group homomorphism

$$\pi : \mathbf{Z}^r \to \mathrm{Pic}^0(C)$$

defined by

$$\pi(\underline{n}) = \overline{\mathcal{L}}_1^{n_1} \otimes \cdots \otimes \overline{\mathcal{L}}_r^{n_r}.$$

Let

$$\Omega_{s,i} = \{\mathcal{M} \in \mathrm{Pic}^0(C) \mid h^1(\mathcal{M} \otimes \mathcal{O}_C(sp_0)) \geq i\}.$$

The $\Omega_{s,i}$ are closed sets in $\mathrm{Pic}^0(C)$ (as shown in the proof of Theorem 8 of [**2**]). There exists ω_s such that $\Omega_{s,l} = \emptyset$ for $l > w_s$.

By Proposition 2.1 there exists an index set $\Delta_{s,i}$ with corresponding $m_\delta \in \mathbf{Z}^r$ and subgroup $M_\delta < \mathbf{Z}^r$ to each $\delta \in \Delta_{s,i}$ such that for $\underline{n} \in \mathbf{Z}^r$,

$$\pi(\underline{n}) \in \Omega_{s,i} \Leftrightarrow \underline{n} \in \cup_{\delta \in \Delta_{s,i}} (m_\delta + M_\delta).$$

Define

$$\Psi_i : \mathbf{Z}^r \to \mathbf{N}$$

by

$$\Psi_i(\underline{n}) = \begin{cases} 1 & \text{if } \underline{n} \in \cup_{\delta \in \Delta_{s,i}} (m_\delta + M_\delta) \\ 0 & \text{otherwise} \end{cases}$$

Now define

$$\Psi_s : \mathbf{Z}^r \to \mathbf{N}$$

by

$$\Psi_s(\underline{n}) = \sum_{i=1}^{\omega_s} \Psi_{s,i}(\underline{n}).$$

we have

$$\Psi_s(\underline{n}) = h^1(C, \overline{\mathcal{L}}_1^{n_1} \otimes \cdots \otimes \overline{\mathcal{L}}_r^{n_r} \otimes \mathcal{O}_C(sp_0))$$

for $\underline{n} \in \mathbf{Z}^r$, and for $\underline{n} \in H_s$,

$$h^1(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r}) = \Psi_s(\underline{n}).$$

Thus

$$\sum_{\underline{n} \in H_s \cap \mathbf{N}^r} h^1(\mathcal{L}_1^{n_1} \otimes \cdots \mathcal{L}_r^{n_r}) t^{\underline{n}} = \sum_{i=1}^{\omega_s} (\sum_{\underline{n} \in H_s \cap \mathbf{N}^r} \Psi_{s,i}(\underline{n}) t^{\underline{n}}).$$

Fix i with $1 \leq i \leq \omega_s$. Well order the set $\Delta_{s,i}$. To prove the rationality of (5) we are reduced to proving the rationality of

(6)

$$\sum_{\underline{n} \in H_s \cap \mathbf{N}^r} \Psi_{s,i}(\underline{n}) t^{\underline{n}} = \sum_{j \geq 1} (-1)^{j-1} \left(\sum_{\delta_1 < \cdots < \delta_j} (\sum_{\underline{n} \in H_s \cap \mathbf{N}^r \cap (\underline{n}_{\delta_1} + M_{\delta_1}) \cap \cdots \cap (n_{\delta_j} + M_{\delta_j})} t^{\underline{n}}) \right)$$

where $\delta_1, \ldots, \delta_j \in \Delta_{s,i}$.

If $\underline{n}_0 \in (\underline{n}_{\delta_1} + M_{\delta_1}) \cap \cdots \cap (n_{\underline{\delta}_j} + M_{\delta_j})$, then

$$(\underline{n}_{\delta_1} + M_{\delta_1}) \cap \cdots \cap (n_{\underline{\delta}_j} + M_{\delta_j}) = \underline{n}_0 + M_{\delta_1} \cap \cdots \cap M_{\delta_j}.$$

The rationality of (6) now follows from Theorem 2.2. ☐

If we allow C to be singular, but require C to have semi-abelian Pic^0, then the same proof shows the rationality of the series

$$\sum_{\underline{n} \in \mathbf{N}^r} h(\underline{n}) t^{\underline{n}}.$$

Furthermore, we get rationality in general on a singular curve, as long as $r \leq 2$. If $r = 1$ this is a trivial statment (it is immediate from the Riemann-Roch formula and Theorem 2.2). The case $r = 2$ follows from the method of the proof of Theorem 4.1, using Theorem 15 of [**2**] in place of Proposition 2.1. Theorem 15 of [**2**] is a general statement for cyclic groups on algebraic groups.

However, if $r \geq 3$ and $\mathrm{Pic}^0(C)$ is not semi-abelian, then the Poincaré series may not be rational, as is shown by the following example, where $\mathrm{Pic}^0(C) \cong \mathbf{C}^2$ is not semi-abelian.

EXAMPLE 4.2. *There exists a complex, integral (singular) projective curve C, and line bundles $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ on C such that the Poincaré series*

$$\sum_{\underline{n}\in\mathbf{N}^3} h(\underline{n})t^{\underline{n}}$$

is not rational, where

$$h(\underline{n}) = h^0(C, \mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2} \otimes \mathcal{L}_3^{n_3}).$$

PROOF. Let C be the curve of Lemma 3.1. Let notation be that preceding Lemma 3.2. Set

$$\mathcal{L}_1 = (1,0), \mathcal{L}_2 = (0,1) \in \mathrm{Pic}^0(C),$$
$$\mathcal{L}_3 = \mathcal{O}_C(\infty).$$

Since $p_a(C) = 2$, we have by Serre duality that

$$h(n_1, n_2, n_3) = \chi(\mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2} \otimes \mathcal{L}_3^{n_3})$$

if $n_3 > 2$.

$$h(n_1, n_2, 0) = \begin{cases} 0 & \text{if } n_1 \text{ or } n_2 \neq 0 \\ 1 & \text{if } n_1 = n_2 = 0 \end{cases}$$

By Serre duality, $h(n_1, n_2, 2) = 0$ with one possible exception, that is if n_1, n_2 are such that

$$\omega_C \cong \mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2} \otimes \mathcal{O}_C(2\infty).$$

The interesting case (which follows from Lemma 3.2) is

$$h(n_1, n_2, 1) = \begin{cases} 1 & \text{if } n_2 = 2n_1^3 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\sum_{\underline{n}\in\mathbf{N}^r} h(\underline{n})t^{\underline{n}}$$

is irrational if and only if

$$\sum_{n_1,n_2=0}^{\infty} h(n_1, n_2, 1)t_1^{n_1}t_2^{n_2}$$

is irrational.

Set

$$a_{n_1,n_2} = h(n_1, n_2, 1) = \begin{cases} 1 & \text{if } n_2 = 2n_1^3 \\ 0 & \text{otherwise} \end{cases}$$

We will assume that $f = \sum_{m,n=0}^{\infty} a_{m,n}t_1^m t_2^n$ is rational, and derive a contradiction. Assuming that the series is rational, there exist $r > 0$ and a nonzero polynomial

$$Q = \sum_{i,j=0}^{r} b_{ij}t_1^i t_2^j$$

such that fQ is a polynomial. Thus there exists a constant σ such that

$$\sum_{i,j=0}^{\infty} a_{m-i,n-j}b_{ij} = 0$$

whenever $m+n>\sigma$ and $m,n\ge r$.

We make the following simple observation. Given $n>0$, there exists $m(n)>0$ such that if $x_0>m(n)$ is an integer, and we set $y_0=2x_0^3$, then

$$[x_0-n,x_0+n]\times[y_0-n,y_0+n]\cap \mathbf{Z}^2\cap\{y=2x^3\}=\{(x_0,y_0)\}.$$

Choose (x_0,y_0) as above (with $n=r$) and x_0 so big that $x_0,y_0>r$ and $x_0+y_0>\sigma$.

For fixed a,b with $0\le a\le r$, $0\le b\le r$, set $m=x_0+a$, $n=y_0+b$. Then

$$0=\sum_{i,j=0}^{r} a_{m-i,n-j}b_{ij}=b_{ab}.$$

Thus $Q=0$, a contradiction. □

5. Irrationality on Surfaces I

If X is a normal projective surface over a field k of characteristic 0, and $\mathcal{L}$ is a line bundle on X, then it follows from Theorem 4 of [**2**] that $h^0(X,\mathcal{L}^n)$ is a polynomial with periodic coefficients for $n>>0$. Thus the series

$$\sum_{n=0}^{\infty} h^0(X,\mathcal{L}^n)t^n$$

is a rational series by Theorem 2.2.

We show in the following example that rationality fails for surfaces if $r=2$.

Example 5.1. *There exists a nonsingular (integral) projective surface S and line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on S such that the series*

$$\sum_{i,j=0}^{\infty} h^0(S,\mathcal{L}_1^i\otimes\mathcal{L}_2^j)t_1^it_2^j \tag{7}$$

is an irrational series.

We will use a construction of [**2**]. As in Example 4 of [**2**], Let C be an elliptic curve over an algebraically closed field k, and let $S=C\times C$. Let $\Delta\subset S$ be the diagonal, $p\in C$ be a closed point, $A=\pi_1^{-1}(p)$, $B=\pi_2^{-1}(p)$, where $\pi_i:S\to C$, $i=1,2$ are the projections.

Let $\mathcal{L}_1=\mathcal{O}_S(A+2B+3\Delta)$, $\mathcal{L}_2=\mathcal{O}_S(-A-B-\Delta)$,

$$h(i,j)=h^0(S,\mathcal{L}_1^i\otimes\mathcal{L}_2^j).$$

We will show that the associated Poincaré series

$$f=\sum_{i,j=0}^{\infty} h(i,j)t_1^it_2^j$$

is irrational. Let

$$\tau=2-\frac{\sqrt{3}}{3},$$

an irrational number. For $i, j \geq 0$,

$$h(i,j) = \begin{cases} \chi(\mathcal{L}_1^i \otimes \mathcal{L}^j) = \frac{1}{2}\left((\mathcal{L}_1^i \otimes \mathcal{L}_2^j)^2\right) > 0 & \text{if } j < i\tau \\ 0 & \text{if } j > i\tau \\ 1 & \text{if } i = j = 0 \end{cases}$$

as follows from the calculations of Example 4 [**2**].

Suppose that f is rational. Then there exist $r > 0$ and a nonzero polynomial

$$Q = \sum_{i,j=0}^{r} b_{ij} t_1^i t_2^j$$

such that fQ is a polynomial. Thus there is $\sigma > 0$ such that

$$\sum_{i,j=0}^{r} a_{m-i,n-j} b_{i,j} = 0$$

whenever $m + n \geq \sigma$ and $m, n \geq r$.

Let λ be the largest real number such that there exists a nonzero coefficient b_{ij} of Q such that

$$j - i\tau = \lambda.$$

Since τ is irrational, there exists a unique solution in natural numbers (i_0, j_0) to $j - i\tau = \lambda$. We necessarily have $b_{i_0,j_0} \neq 0$. We further have that $b_{ij} = 0$ if $j - i\tau > \lambda$.

Lemma 5.2. *Suppose that N is a given constant. Then there exists $(\alpha_0, \beta_0) \in \mathbf{N}^2$ such that $\beta_0 < \alpha_0\tau$, $\alpha_0, \beta_0 > N$ and if $(\alpha, \beta) \in \mathbf{N}^2$ is such that*

(1) $\alpha_0 - r \leq \alpha \leq \alpha_0 + r$
(2) $\beta_0 - \tau\alpha_0 \leq \beta - \tau\alpha \leq 0$

Then $(\alpha, \beta) = (\alpha_0, \beta_0)$.

Proof. If x is a real number, $\lceil x \rceil$ will denote the smallest integer $\geq x$. Choose a real number ϵ such that

$$0 < \epsilon < \min\{\lceil i\tau \rceil - i\tau \mid i \in \{-r, -r+1, \ldots, -1, 1, 2, \ldots, r\}\}.$$

As a consequence of Kroneker's theorem (c.f. Theorem 438 [**4**]) there exist natural numbers (α_0, β_0) such that $\alpha_0 > N$ and $0 < \alpha_0\tau - \beta_0 < \epsilon$. Set $\lambda = \alpha_0\tau - \beta_0$. Suppose there exists $(\alpha, \beta) \in \mathbf{N}^2$ such that $\alpha_0 - r \leq \alpha \leq \alpha + r$ and $0 < \alpha\tau - \beta < \lambda$. Then there exists a nonzero integer i such that $-r \leq i \leq r$ and $\alpha = \alpha_0 + i$.

$$(\alpha_0 + i)\tau - \lambda < \beta < (\alpha_0 + i)\tau.$$

We have

$$\alpha_0\tau - \lambda + i\tau < \beta < \alpha_0\tau + i\tau$$

$$\beta_0 + i\tau < \beta < \beta_0 + \lambda + i\tau$$

$$i\tau < \beta - \beta_0 < i\tau + \lambda.$$

There can be no integral solutions in β by our choice of ϵ. □

With the notation of the Lemma 5.2, taking $N=\max\{\sigma, r\}$, set

$$m = i_0 + \alpha_0, n = \beta_0 + j_0.$$

Then $(\alpha_0, \beta_0) = (m - i_0, n - j_0)$.

(8)

$$\begin{array}{ll} 0 & = \sum_{ij=0}^{r} b_{ij} a_{m-i,n-j} \\ & = \sum_{j-i\tau>\lambda} b_{ij} a_{m-i,n-j} + \sum_{j-i\tau<\lambda} b_{ij} a_{m-i,n-j} + \sum_{j-i\tau=\lambda} b_{ij} a_{m-i,n-j} \\ & = \sum_{j-i\tau<\lambda} b_{ij} a_{m-i,n-j} + b_{i_0,j_0} a_{m-i_0,n-j_0} \end{array}.$$

Make a change of variables

$$(\alpha, \beta) = (m - i, n - j)$$

and set $(\alpha_0, \beta_0) = (m - i_0, n - j_0)$, $\psi = \beta_0 - \alpha_0\tau$.

$j_0 - i_0\tau = \lambda$ implies $\psi = \beta_0 - \alpha_0\tau = (n - m\tau) - \lambda$.

If $j - i\tau < \lambda$ then

$$\beta - \alpha\tau = (n - m\tau) + (i\tau - j) > (n - m\tau) - \lambda = \psi.$$

$$\sum_{j-i\tau<\lambda} b_{ij} a_{m-i,n-j} = \sum_{\beta-\alpha\tau>\psi} a_{\alpha,\beta} b_{m-\alpha,n-\beta}. \tag{9}$$

We have $0 \le m - \alpha \le r$ implies

$$\alpha_0 - r \le \alpha_0 + i_0 - r = m - r \le \alpha \le m = \alpha_0 + i_0 \le \alpha_0 + r$$

so $a_{\alpha,\beta} = 0$ in this range. Thus (9) is zero, and substituting into (8), we conclude that $b_{i_0,j_0} = 0$, a contradiction to our assumption that Q is nonzero. Thus the series f is irrational.

6. Irrationality on Surfaces II

In Example 5.1 of Section 5, the fact that $\mathcal{L}_1$ was ample and $\mathcal{L}_2^{-1}$ was ample played a critical role in the construction. This leaves open the question if an example can be constructed on a surface of effective divisors with irrational Poincaré series. We construct such an example in this section.

We construct an example of 3 line bundles which have nonzero global sections (they are the line bundles associated to effective divisors) on a nonsingular complex projective surface such that the associated Poincaré series is not rational.

The construction is similar to that of Theorem 9.1 of [**3**], which shows that the Poincaré series of a resolution of singularities of a surface singularity with 3 exceptional components can be irrational.

Example 6.1. *There exists a nonsingular complex projective surface X of and line bundles $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ on X such that*

$$h^0(X, \mathcal{N}_1) > 0,\ h^0(X, \mathcal{N}_2) > 0,\ h^0(X, \mathcal{N}_3) > 0$$

and the series

$$\sum_{i,j,k=0}^{\infty} h^0(X, \mathcal{N}_1^i \otimes \mathcal{N}_2^j \otimes \mathcal{N}_3^k) t_1^i t_2^j t_3^k \tag{10}$$

is an irrational series.

Let C be the curve of Lemma 3.1. Suppose that $\mathcal{M}$ is a line bundle on C of degree $-d$ with $d \geq 3$, and p_1, p_2 are distinct points on $C - \{\tilde{p}\}$. Let $\overline{\mathcal{L}} = \mathcal{M} \otimes \mathcal{O}_C(p_1 + p_2)$. Lemma 3.1 shows that there exists a nonsingular, complex projective surface S and an embedding $C \subset S$ such that $C \cdot C = \overline{\mathcal{L}}$.

Let $X \to S$ be the blowup of S at the points p_1 and p_2, with exceptional divisors F_1 and F_2 (both isomorphic to $\mathbf{P}^1$). Identify C with its strict transform on X. Then $C \cdot C \sim \mathcal{M}$, $C \cdot F_1 = p_1$, $C \cdot F_2 = p_2$, $(F_1^2) = -1$, $(F_2^2) = -1$.

Recall from the proof of Lemma 3.1 that we have a birational morphism $S \to \mathbf{P}^2$ such that C is the strict transform of a curve of degree 5 in $\mathbf{P}^2$. Let H be the pullback on X of a hyperplane section of $\mathbf{P}^2$. We will now fix $d = 3$, and continue to assume that $\mathcal{M}$ is a line bundle of degree $-d = -3$ on C. Set

$$\begin{aligned} \Delta_1 &= H + 3F_1 + 4F_2 + 4C \\ \Delta_2 &= H + 4F_1 + 3F_2 + 4C \\ \Delta_3 &= H + 4F_1 + 4F_2 + 4C \end{aligned}$$

If we set $F_3 = C$, then we have $(\Delta_i \cdot F_j) = \delta_{ij}$. Let $\overline{H} = H \cdot C - 5\infty$, $\overline{p}_1 = p_1 - \infty$, $\overline{p}_2 = p_2 - \infty$, $\mathcal{L} = \mathcal{M} + 3\infty$. Now choose $\mathcal{L}$ to be

$$\mathcal{L} = (1, \frac{9}{4}) - \frac{1}{4}\overline{H},$$

Set $p_1 = \infty$, and choose p_2 so that $\overline{p}_2 = (-1, -2) \in W$. Then

$$\Delta_1 \cdot C = \overline{H} + 4\overline{p}_2 + 4\mathcal{L} = \overline{H} + (-4, -8) + (4, 9) - \overline{H} = (0, 1)$$

$$\Delta_2 \cdot C = \overline{H} + 3\overline{p}_2 + 4\mathcal{L} = \overline{H} + (-3, -6) + (4, 9) - \overline{H} = (1, 3)$$

LEMMA 6.2. *There exists at most 2 divisors $D = a\Delta_1 + b\Delta_2 + c\Delta_3$ on X such that $D \cdot C \sim \omega_C + (F_1 + F_2 + C) \cdot C$ or $D \cdot C \sim \omega_C$, where ω_C is the canonical bundle of C.*

PROOF. Suppose that $D \cdot C \sim \omega_C + (F_1 + F_2 + C) \cdot C$. Then

$$c = (D \cdot C) = \deg \omega_C - 1 = 1.$$

$$D \cdot C \sim a(0, 1) + b(1, 3) + \Delta_3 \cdot C$$

implies

$$(b, a + 3b) \sim \omega_C + (F_1 + F_2 + C) \cdot C - \Delta_3 \cdot C$$

so there is at most one solution in a and b.

If $D \cdot C \sim \omega_C$, we have $c = 2$ and $(b, a + 3b) \sim \omega_C - 2\Delta_3 \cdot C$ so there is at most one solution in a and b. □

LEMMA 6.3. *There exists $\lambda > 0$ such that $a, b, c \geq 0$ and $a + b + c > \lambda$ implies*

$$h^1(X, \mathcal{O}_X(a\Delta_1 + b\Delta_2 + c\Delta_3)) = \begin{cases} 0 \ \text{if } c \geq 2 & \\ 1 & \text{if } c = 1, a = 2b^3 - 3b - 1 \\ 0 & \text{if } c = 1, a \neq 2b^3 - 3b - 1 \end{cases}$$

PROOF. By Lemma 6.2 we can choose λ so that $a, b, c \geq 0$ and $a+b+c > 0$ implies

$$(a\Delta_1 + b\Delta_2 + c\Delta_3) \cdot C \not\sim \omega_C + (F_1 + F_2 + C) \cdot C$$

and

$$(a\Delta_1+b\Delta_2+c\Delta_3)\cdot C\not\sim\omega_C.$$

Suppose that $\mathcal{N}$ is a line bundle on X. We have exact sequences

(11)

$$0\to\mathcal{O}_{F_1+F_2}(-n(F_1+F_2+C)-C)\otimes\mathcal{N}\to\mathcal{O}_{(n+1)(F_1+F_2+C)}\otimes\mathcal{N}\to\mathcal{O}_{n(F_1+F_2+C)+C}\otimes\mathcal{N}\to 0$$

for $n\ge 0$,

(12)

$$0\to\mathcal{O}_C(-n(F_1+F_2+C))\otimes\mathcal{N}\to\mathcal{O}_{n(F_1+F_2+C)+C}\otimes\mathcal{N}\to\mathcal{O}_{n(F_1+F_2+C)}\otimes\mathcal{N}\to 0$$

for $n\ge 1$,

(13)

$$0\to\mathcal{O}_{F_1}(-n(F_1+F_2+C)-C)\otimes\mathcal{N}\to\mathcal{O}_{n(F_1+F_2+C)+F_1+C}\otimes\mathcal{N}\to\mathcal{O}_{n(F_1+F_2+C)+C}\otimes\mathcal{N}\to 0$$

for $n\ge 0$, and

(14)

$$0\to\mathcal{O}_{F_2}(-n(F_1+F_2+C)-C)\otimes\mathcal{N}\to\mathcal{O}_{n(F_1+F_2+C)+F_2+C}\otimes\mathcal{N}\to\mathcal{O}_{n(F_1+F_2+C)+C}\otimes\mathcal{N}\to 0$$

for $n\ge 0$. Observe that we have

$$\begin{array}{ll} H^1(X,\mathcal{O}_{F_1+F_2}(-n(F_1+F_2+C)-C)\otimes\mathcal{N})=0 & \text{if } (\mathcal{N}\cdot F_i)\ge 0 \text{ for } i=1,2\\ H^1(X,\mathcal{O}_{F_1}(-n(F_1+F_2+C)-C)\otimes\mathcal{N})=0 & \text{if } (\mathcal{N}\cdot F_1)\ge 0\\ H^1(X,\mathcal{O}_{F_2}(-n(F_1+F_2+C)-C)\otimes\mathcal{N})=0 & \text{if } (\mathcal{N}\cdot F_2)\ge 0. \end{array}$$

We also have

$$\begin{array}{l} H^1(X,\mathcal{O}_C(-n(F_1+F_2+C))\otimes\mathcal{N})=0\\ \quad\text{if } n+(\mathcal{N}\cdot C)>2 \text{ or}\\ \quad n+(\mathcal{N}\cdot C)=2 \text{ and } -n(F_1+F_2+C)\cdot C+\mathcal{N}\cdot C\not\sim\omega_C. \end{array}$$

Let $\overline{\Delta}_i=\Delta_i-H$ for $1\le i\le 3$. From the exact sequence

$$0\to\mathcal{O}_X((a+b+c)H)\to\mathcal{O}_X(a\Delta_1+b\Delta_2+c\Delta_3)\to\mathcal{O}_{a\overline{\Delta}_1+b\overline{\Delta}_2+c\overline{\Delta}_3}(a\Delta_1+b\Delta_2+c\Delta_3)\to 0$$

we see that

$$H^1(X,\mathcal{O}_X(a\Delta_1+b\Delta_2+c\Delta_3))\cong H^1(X,\mathcal{O}_{a\overline{\Delta}_1+b\overline{\Delta}_2+c\overline{\Delta}_3}(a\Delta_1+b\Delta_2+c\Delta_3)).$$

For $0\le\alpha\le a$, $0\le\beta\le b$, $0\le\gamma\le c$,

$$a\Delta_1+b\Delta_2+c\Delta_3-(\alpha\overline{\Delta}_1+\beta\overline{\Delta}_2+\gamma\overline{\Delta}_3)=(a-\alpha)\Delta_1+(b-\beta)\Delta_2+(c-\gamma)\Delta_3+(\alpha+\beta+\gamma)H.$$

Consider the exact sequences, with $0\le\alpha\le a$, $0\le\beta\le b$, $0\le\gamma\le c$

(15)

$$\begin{array}{ll} 0\to & \mathcal{O}_{\overline{\Delta}_1}((a-\alpha)\Delta_1+(b-\beta)\Delta_2+(c-\gamma)\Delta_3+(\alpha+\beta+\gamma)H)\to\\ & \mathcal{O}_{(\alpha+1)\overline{\Delta}_1+\beta\overline{\Delta}_2+\gamma\overline{\Delta}_3}(a\Delta_1+b\Delta_2+c\Delta_3)\to\mathcal{O}_{\alpha\overline{\Delta}_1+\beta\overline{\Delta}_2+\gamma\overline{\Delta}_3}(a\Delta_1+b\Delta_2+c\Delta_3)\to 0 \end{array}$$

with $\alpha<a$,

(16)

$$\begin{array}{ll} 0\to & \mathcal{O}_{\overline{\Delta}_2}((a-\alpha)\Delta_1+(b-\beta)\Delta_2+(c-\gamma)\Delta_3+(\alpha+\beta+\gamma)H)\to\\ & \mathcal{O}_{\alpha\overline{\Delta}_1+(\beta+1)\overline{\Delta}_2+\gamma\overline{\Delta}_3}(a\Delta_1+b\Delta_2+c\Delta_3)\to\mathcal{O}_{\alpha\overline{\Delta}_1+\beta\overline{\Delta}_2+\gamma\overline{\Delta}_3}(a\Delta_1+b\Delta_2+c\Delta_3)\to 0 \end{array}$$

with $\beta<b$,

(17)

$$\begin{array}{ll} 0\to & \mathcal{O}_{\overline{\Delta}_3}((a-\alpha)\Delta_1+(b-\beta)\Delta_2+(c-\gamma)\Delta_3+(\alpha+\beta+\gamma)H)\to\\ & \mathcal{O}_{\alpha\overline{\Delta}_1+\beta\overline{\Delta}_2+(\gamma+1)\overline{\Delta}_3}(a\Delta_1+b\Delta_2+c\Delta_3)\to\mathcal{O}_{\alpha\overline{\Delta}_1+\beta\overline{\Delta}_2+\gamma\overline{\Delta}_3}(a\Delta_1+b\Delta_2+c\Delta_3)\to 0 \end{array}$$

with $\gamma<c$.

Set

$$\mathcal{N} = (a-\alpha)\Delta_1 + (b-\beta)\Delta_2 + (c-\gamma)\Delta_3 + (\alpha+\beta+\gamma)H$$

with $a, b \geq 0$ and $c \geq 1$. $(\mathcal{N} \cdot F_1) = a - \alpha$, $(\mathcal{N} \cdot F_2) = b - \beta$, $(\mathcal{N} \cdot C) = (c-\gamma) + 5(\alpha+\beta+\gamma)$. If $a + b + c > 0$, we get the necessary vanishing in (11)-(14) to conclude that

(18) $H^1(X, \mathcal{O}_{\overline{\Delta}_1}((a-\alpha)\Delta_1 + (b-\beta)\Delta_2 + (c-\gamma)\Delta_3 + (\alpha+\beta+\gamma)H)) = 0$

if $0 \leq \alpha < a$, $0 \leq \beta \leq b$, $0 \leq \gamma \leq c$, $\alpha+\beta+\gamma > 0$

(19) $H^1(X, \mathcal{O}_{\overline{\Delta}_2}((a-\alpha)\Delta_1 + (b-\beta)\Delta_2 + (c-\gamma)\Delta_3 + (\alpha+\beta+\gamma)H)) = 0$

if $0 \leq \alpha \leq a$, $0 \leq \beta < b$, $0 \leq \gamma \leq c$ $\alpha+\beta+\gamma > 0$,

(20) $H^1(X, \mathcal{O}_{\overline{\Delta}_3}((a-\alpha)\Delta_1 + (b-\beta)\Delta_2 + (c-\gamma)\Delta_3 + (\alpha+\beta+\gamma)H)) = 0$

if $0 \leq \alpha \leq a$, $0 \leq \beta \leq b$, $0 \leq \gamma < c$, $\alpha+\beta+\gamma > 0$. From (15) - (20) it follows that

$$H^1(X, \mathcal{O}_X(a\Delta_1 + b\Delta_2 + c\Delta_3)) \cong H^1(X, \mathcal{O}_{\overline{\Delta}_3}(a\Delta_1 + b\Delta_2 + c\Delta_3)).$$

It then follows from (11) - (14) and our choice of λ that

$$H^1(X, \mathcal{O}_X(a\Delta_1 + b\Delta_2 + c\Delta_3)) \cong H^1(X, \mathcal{O}_C(a\Delta_1 + b\Delta_2 + c\Delta_3))$$

if $a, b, c \geq 0$, $c \geq 1$ and $a + b + c > \lambda$. The conclusions of Lemma 6.3 now follow from Lemma 3.2 if $c = 1$, since

$$a\Delta_1 + b\Delta_2 + \Delta_3 \sim (a+1)\Delta_1 + b\Delta_2 + F_1$$

and

$$(a+1)\Delta_1 \cdot C + b\Delta_2 \cdot C = (a+1)(0,1) + b(1,3) = (b, a+3b+1).$$

If $c \geq 2$, we have $H^1(C, \mathcal{O}_C(a\Delta_1 + b\Delta_2 + c\Delta_3)) = 0$ by Serre duality and our choice of λ. □

We will now give the proof of Theorem 6.1. Let notation be as above for the surface X. Set $a_{ijk} = h^1(X, \mathcal{O}_X(i\Delta_1 + j\Delta_2 + k\Delta_3))$. We will show that the series

$$f = \sum_{i,j,k=0}^{\infty} a_{ijk} t_1^i t_2^j t_3^k$$

is not rational.

We first observe that, given $n > 0$, there exists $m(n) > 0$ such that if $x_0 > m(n)$ is an integer, and $y_0 = 2x_0^3 - 3x_0 - 1$, then

(21)

$$([x_0 - n, x_0 + n] \times [y_0 - n, y_0 + n]) \cap \mathbf{Z}^2 \cap \{y = 2x^3 - 3x - 1\} = \{(x_0, y_0)\}.$$

Suppose that f is rational. Then there exist $r > 0$ and a nonzero polynomial

$$Q = \sum_{i,j,k=0}^{r} b_{ijk} t_1^i t_2^j t_3^k$$

such that fQ is a polynomial. Thus there is $\sigma > 0$ such that

$$\sum_{i,j,k=0}^{r} a_{l-i,m-j,n-k} b_{i,j,k} = 0$$

whenever $l+m+n\ge\sigma$ and $l,m,n\ge r$, and

$$0=\sum_{i,j=0}^{r}(\sum_{k=0}^{\alpha}a_{l-i,m-j,\alpha-k}b_{ijk})$$

if $l+m+\alpha\ge\sigma$, $l,m\ge r$, $0\le\alpha<r$. Suppose that l,m are such that $l+m\ge\sigma$. Then by lemma 6.3

$$0=\sum_{ijk=0}^{r}a_{l-i,m-j,r+1-k}b_{ijk}=\sum_{i,j=0}^{r}b_{ijr}a_{l-i,m-j,1}.$$

By (21) and lemma 6.3 we can choose l,m to conclude that $b_{ijr}=0$ for $0\le i,j\le r$.

Suppose that α is such that $1\le\alpha\le r$ and $b_{ijk}=0$ for $0\le i,j\le r$ and $\alpha\le k\le r$. Suppose that l,m are such that $l+m\ge\sigma$. Then by lemma 6.3

$$\begin{aligned}0&=\textstyle\sum_{i,j=0}^{r}\sum_{k=0}^{\alpha}b_{ijk}a_{l-i,m-j,\alpha-k}\\&=\textstyle\sum_{i,j=0}^{r}a_{l-i,m-j,0}b_{ij\alpha}+\sum_{i,j=0}^{r}a_{l-i,m-j,1}b_{i,j,\alpha-1}\\&=\textstyle\sum_{i,j=0}^{r}a_{l-i,m-j-1,1}b_{i,j,\alpha-1}.\end{aligned}$$

By (21) and lemma 6.3, we can choose l,m to conclude that $b_{i,j,\alpha-1}=0$ for $0\le i,j\le r$. Thus we conclude be descending induction on α that $Q=0$, a contradiction, so that

$$\sum_{i,j,k=0}^{\infty}h^1(X,\mathcal{O}_X(i\Delta_1+j\Delta_2+k\Delta_3))t_1^it_2^jt_3^k$$

is irrational. Since $\Delta_1,\Delta_2,\Delta_3$ are effective,

$$\sum_{i,j,k=0}^{\infty}h^2(X,\mathcal{O}_X(i\Delta_1+j\Delta_2+k\Delta_3))t_1^it_2^jt_3^k$$

is a polynomial by Serre Duality. Since

$$\sum_{i,j,k=0}^{\infty}\chi(\mathcal{O}_X(i\Delta_1+j\Delta_2+k\Delta_3))t_1^it_2^jt_3^k$$

is rational, (where $\chi(\mathcal{O}_X(i\Delta_1+j\Delta_2+k\Delta_3)$ is the Euler characteristic and is consequentially a polynomial) we must then have that

$$\sum_{i,j,k=0}^{\infty}h^0(X,\mathcal{O}_X(i\Delta_1+j\Delta_2+k\Delta_3))t_1^it_2^jt_3^k$$

is irrational.

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