VANISHING OF DIFFERENTIALS ALONG IDEALS AND NON–ARCHIMEDEAN APPROXIMATION

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Dedicated to E. Kunz on the occasion of his sixty-fifth birthday

One of the major new developments in commutative algebra over the last decade or so was the introduction of the theory of tight closure of an ideal by Hochster and Huneke. It proved to be an extremely useful technique to study ideals, and it also turned out to be closely related to many geometric questions. Fedder [F] used derivations in positive characteristic to obtain characterizations of 2 dimensional graded rational singularities in terms of F-purity and F-injectivity. In an attempt to generalize these techniques and to relate rational singularities with F-rationality, Craig Huneke raised the following problem (c.f [FHH]): Let R be a regular local ring, containing a perfect field k, over which R is essentially of finite type, and let $C(R/k)$ be the subring of derivationally constant elements of R/k (i.e. $C(R/k) = \{x \in R : \delta(x) = 0 \text{ for all } \delta \in \mathrm{Der}_k(R)\}$). Then Huneke asked:

(1) If $I \subseteq R$ is an ideal, does there exist a constant $l = l(R, I) \in \mathbb{N}$ with the following property: If $x \in R$ with $\delta(x) \in I^{n+l}$ then there exists a $c \in C(R/k)$ with $x - c \in I^n$.
(2) If an l as in (1) exists, is it possible to bound it in a way useful for reduction mod p techniques, i.e. if $\mathrm{char}(k) = 0$, does there exist a model $\mathcal{R}/A$, $\mathcal{I} \subseteq \mathcal{R}$ of $R/k, I$ with $A/\mathbb{Z}$ of finite type and a constant $l(\mathcal{I})$ such that $l(\mathcal{R}/\mathfrak{m}\mathcal{R}, \mathcal{I} + \mathfrak{m}/\mathfrak{m}) \le l(\mathcal{I})$ for all $\mathfrak{m} \in \mathrm{Max}(A)$.

A result of the above type has been used successfully by Fedder [F] to relate rationality and F–rationality for two–dimensional graded rings, and a positive answer to the above questions would allow to extend these techniques and results to higher dimension. The

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relation between rational and F–rational rings has been clarified recently by Hara–Watanabe [HW] resp. Mehta–Srinivas [MS], however the interest in the above question has been revived by the work of Huneke and Smith [HS] on the Kodaira vanishing theorem. According to Huneke, a good answer to the above problems would provide an essential part of a proof of Kodaira vanishing via tight closure techniques.

First results in this connection have been obtained in [FHH]. A complete solution for the corresponding characteristic 0 problem was given in [Hü]. Here we show that (1) has a positive answer in positive characteristics as well, and we also give a partial solution to (2). In positive characteristics it turns out to be rather difficult to control the behaviour of the subring $C(R/k)$ of differential constants. Contrary to the characteristic–0–case this subring will change when passing to localizations, completions or discrete valution rings dominating the given ring.

§1 Valuations and the norm associated to an ideal

Let R be an excellent noetherian domain, and let $I \subseteq R$ be a proper ideal. For some $x \in R \setminus \{0\}$ we set $v_I(x) = n$ if $x \in I^n$ but $x \notin I^{n+1}$, and we set $v_I(0) = \infty$. Following Samuel [Sa] we define the I–adic limit order function by

$$\overline{v}_I(x) = \lim_{n \to \infty} \frac{v_I(x^n)}{n}$$

This limit always exists and is finite for all $x \neq 0$. In general however $\overline{v}_I$ is not a valuation. If $\overline{v}_I$ is a valuation, then I is called one–fibered. Such ideals have been studied by J. Sally [Sy], and they are comparatively rare. Rees [Re] however has shown that there exist discrete valuations $v_1, \dots, v_s$ of $K = Q(R)$ such that

$$\overline{v}_I(x) = \inf_{1 \le i \le s} \frac{v_i(x)}{v_i(I)} \quad \text{for } x \in R$$

where $v_i(I) = \inf\{v_i(r) : r \in I\}$. The valuations $v_1, \dots, v_s$ are called the Rees valuations of I, and we set $T(I) = \{v_1, \dots, v_s\}$. This definition of $\overline{v}_I$ extends to all of K (with $\overline{v_I}(0) = \infty$).

Recall that an element $r \in R$ is called integral over I if it satisfies an equation

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

with $a_l \in I^l$. The following results are well known (cf. [McA])

1.1. Remark. i) $\overline{I} := \{r \in R : r \text{ integral over } I\} \subseteq R$ is an ideal with $\overline{\overline{I}} = \overline{I}$.

ii) For $r \in R$ and $n \in \mathbb{N}$ the following are equivalent:

(1) $r \in \overline{I^n}$.
(2) $\overline{v}_I(r) \geq n$.
(3) $v(r) \geq n v(I)$ for all $v \in T(I)$.

1.2. Lemma. There exists a semilocal noetherian Dedekind domain $W \subseteq K$ with $R \subseteq W$ such that $\overline{I^n} = I^n W \cap R$.

Proof. Let $T(I) = \{v_1, \ldots, v_s\}$ and let $(V_i, \mathfrak{m}_{v_i}) \subseteq K$ be the valuation ring associated to v_i. Clearly $R \subseteq V_i$. By Nagata's theorem [Na], (11.11) on the independence of valuations $W := V_1 \cap \cdots \cap V_s$ is a semilocal ring with maximal ideals $\mathfrak{m}_i = \mathfrak{m}_{v_i} \cap W$ $(i = 1, \ldots, s)$ and with $W_{\mathfrak{m}_i} = V_i$. As W is semilocal and locally noetherian, it is noetherian, hence a Dedekind domain. For $r \in R$ we have $r \in I^n W$ if and only if $r \in I^n V_i$ for all $i = 1, \ldots, s$, which again is equivalent to $v_i(r) \geq n v_i(I)$, and the claim follows.

For $x \in K$ we set

$$||x||_I = e^{-\overline{v}_I(x)}$$

(with the convention that $e^{-\infty} = 0$) and call $||x||_I$ the I-adic norm of x.

1.3. Remark. i) Let $T(I) = \{v_1, \ldots, v_s\}$ and set $e_i := v_i(I)$. Then

$$||x||_I = \max\left\{ e^{-\frac{v_i(x)}{e_i}} : i = 1, \ldots, s \right\}$$

ii) For all $x \in K$ $||x||_I \geq 0$ and $||x||_I = 0$ if and only if $x = 0$.
iii) For all $x, y \in K$ we have $||x + y||_I \leq \max\{||x||_I, ||y||_I\}$.
iv) For all $x, y \in K$ we have $||x \cdot y||_I \leq ||x||_I \cdot ||y||_I$ and $||x^n||_I = ||x||_I^n$, and equality holds if I is one-fibered. Conversely if equality holds for all $x, y \in K$ then I is one–fibered. Hence $|| - ||_I$ is an absolute value in the sense of [La], XII if and only if I is one–fibered.

Proof. i) is clear, and ii), iii) and the first part of iv) follow from i). For the second part of iv) suppose that $T(I) = \{v_1, \ldots, v_s\}$ with $s > 1$. By the Chinese remainder theorem there exist $x, y \in K$ with

$v_i(x) \geq 1$ for $i = 1, \ldots, s-1$ and $v_s(x) = 0$.
$v_1(y) = 0$ and $v_i(y) \geq 1$ for $i = 2, \ldots, s$.

Thus $v_i(x \cdot y) \geq 1$ for $i = 1, \ldots, s$ and therefore $||x \cdot y||_I \leq e^{-1}$. On the other hand $||x||_I = 1 = ||y||_I$.

1.4. Proposition. *i) For the Dedekind domain W associated to I as in (1.2) we have*

$$W = \{x \in K : ||x||_I \leq 1\}$$

ii) $\overline{I^n} = \{x \in R : ||x||_I \leq e^{-n}\}$.
iii) For each $0 < r < 1$ *we have*

$$I_r := \{x \in R : ||x||_I \leq r\} \subseteq R$$

is an integrally closed ideal of R.

Proof. i) and ii) follow easily from (1.3)and (1.1), and iii) is a reformulation of [MRS], (2.1)

Definition. A function $|| \; || : K \to \mathbb{R}$ given by $||x|| = \max\left\{ e^{-\frac{v_i(x)}{e_i}} : i = 1, \dots, s \right\}$ for discrete valuations $v_1, \dots, v_s$ of K and positive integers $e_1, \dots, e_s \in \mathbb{N}$ is called a non–archimedean norm on K.

Let K be a field with a non–archimedean norm $|| \; ||_K$ and let V be a K–vectorspace. By a norm on V (compatible with $|| \; ||_K$) we mean a function $|| \; || : V \to \mathbb{R}$ satisfying:

(N1) $||a|| \geq 0$ for all $a \in V$ and $||a|| = 0$ if and only if $a = 0$.
(N2) For $x \in K$ and $v \in V$ we have $||xv|| \leq ||x||_K ||v||$.
(N3) For $v, w \in V$ we have $||v + w|| \leq \max\{||v||, ||w||\}$.

Suppose $|| \; ||_K$ is given by $||x||_K = \max\left\{ e^{-\frac{v_j(x)}{e_j}} : j = 1, \dots, s \right\}$. Let L/K be a finite, purely inseparable extension, let $l_1, \dots, l_m$ be a K–basis of L and let $\rho_1, \dots, \rho_m$ be positive constants. For $y \in L$ write $y = x_1 l_1 + \cdots + x_m l_m$ and set

$$||y|| := \max\{||x_i||_K \cdot \rho_i : i = 1, \dots, m\}$$

As L/K is purely inseparable, each valuation v_i has a unique extension w_i to L, and for $y \in L$ we set

$$||y||' := \max\left\{ e^{-\frac{w_j(y)}{e_j}} : j = 1, \dots, s \right\}$$

1.5. Proposition. *$|| \; ||$ and $|| \; ||'$ are equivalent norms on L, compatible with $|| \; ||_K$, i.e. there exist $c_1, c_2 > 0$ with*

$$|| \; || \leq c_1 || \; ||' \qquad || \; ||' \leq c_2 || \; ||$$

Proof. Clearly both $|| \; ||$ and $|| \; ||'$ are norms on L compatible with $|| \; ||_K$.

Let K_j be the completion of K with respect to the valuation v_j (i.e. if $V_j \subseteq K$ is the valuation ring associated to v_j and if $\widehat{V_j}$ is its completion, then $K_j = Q(\widehat{V_j})$), equipped with the absolute value $||x||_{v_j} = e^{-\frac{v_j(x)}{e_j}}$, then $\widehat{K} = K_1 \times \cdots \times K_s$, equipped with the norm $||(x_1, \dots, x_s)|| = \max\{||x_j||_{v_j} : j = 1, \dots, s\}$ is the completion of the normed ring $(K, || \; ||_K)$. Similarly we define L_j to be the completion of L with respect to w_j, and we set $\widehat{L} = L_1 \times \cdots \times L_s$. Then $\widehat{L} = L \otimes_k \widehat{K}$ and $L_j = L \otimes_K K_j$ as the w_j are the unique extensions of the v_j to L. Thus $l_1, \dots, l_m$ is a basis of $\widehat{L}/\widehat{K}$ resp. L_j/K_j, and we define two norms on $\widehat{L}$ by

$$||y|| = \max\{||x_i|| \cdot \rho_i : i = 1, \dots, m\} \qquad \text{if } y = x_1 l_1 + \cdots + x_m l_m, \quad x_i \in \widehat{K}$$

and by

$$||(y_1, \dots, y_t)||' = \max\{||y_j||_{w_j} : j = 1, \dots, s\}$$

Then $|| \; ||$ and $|| \; ||'$ make $\widehat{L}$ a normed and complete $\widehat{K}$–module, and the canonical inclusions $(L, || \; ||) \to (\widehat{L}, || \; ||)$ and $(L, || \; ||') \to (\widehat{L}, || \; ||')$ are isometric embeddings.

Thus it suffices to show that the two norms $|| \;\; ||$ and $|| \;\; ||'$ are equivalent. For this it suffices to consider each factor L_j/K_j with the induced norms $|| \;\; ||_j$ and $|| \;\; ||'_j$. But then K_j is a complete topological field with a nontrivial absolute value in the sense of [La], XII, §2 and the two norms on L_j are compatible with the absolute value on K_j. Thus by [La], Prop. 3 there exist $c_{1,j}, c_{2,j}$ with

$$|| \;\; ||_j \leq c_{1,j} \cdot || \;\; ||'_j \qquad || \;\; ||'_j \leq c_{2,j} \cdot || \;\; ||_j$$

and form (N3) it follows that $c_\lambda := \max\{c_{\lambda,1}, \ldots, c_{\lambda,s}\}$ $(\lambda = 1, 2)$ will work in the proposition.

1.6. Question. In the situation of (1.5) are any two norms on L, compatible with $|| \;\; ||_K$, equivalent?

§2 Derivations in positive characteristics

In this section we assume that k is a perfect field with $\mathrm{char}(k) = p > 0$. In this situation we will provide a positive answer to question (1) from the introduction.

2.1 Theorem. *Let R be a semi–local regular excellent and irreducible k–algebra such that R/R^p is finite, and let $I \subseteq R$ be an ideal. Then there exists an $l = l(R, I) \in \mathbb{N}$ with the following property: If $x \in R$ with $\delta(x) \in I^{n+l}$ for all $\delta \in Der_k(R)$ then there exists a $c \in C(R/k)$ with $x - c \in I^n$.*

The proof will be divided in several steps. First note that the regularity of R and the finiteness of R/R^p imply that for each $\mathfrak{m} \in \mathrm{Max}(R)$ the Ring $R_\mathfrak{m}$ has a (finite) p–basis. In fact this is clearly true for the completion $\widehat{R_\mathfrak{m}}/\widehat{R_\mathfrak{m}}^p$ by Cohen's structure theorem, and from this it follows for $R_\mathfrak{m}$ by faithfully flat descent. As R is semilocal and irreducible, this implies that R itself has a finite p–basis, i.e. there exist $x_1, \ldots, x_n \in R$ such that $\{x_1^{\mu_1} \cdots x_n^{\mu_n} : 0 \leq \mu_i \leq p - 1\}$ is a basis of R/R^p. In this situation $\Omega^1_{R/k} \cong \Omega^1_{R/R^p}$ is a free R–module with basis $dx_1, \ldots, dx_n$ (cf. [KD], (6.18)). Let $K = Q(R)$ be the field of fractions of R.

2.2 Lemma. Let T be a discrete valuation ring with $R \subseteq T \subseteq K$ and assume that T/R is essentially of finite type. Then T has a p–basis of length m. In particular $\Omega^1_{T/k}$ is a free T–module of rank m and

$$T^p = \ker(d_{T/k} : T \longrightarrow \Omega^1_{T/k})$$

Proof. As T/R is essentially of finite type and as R/R^p is finite, T has a finite p–generating set, and so has $\widehat{T}$, its completion. Thus the universally finite differential module of $\widehat{T}/k$ exists and it is equal to the universal differential module of $\widehat{T}/k$. Now it follows easily from [KD], (14.3) that $\Omega^1_{\widehat{T}/k}$ is free. As $\Omega^1_{\widehat{T}/k} = \Omega^1_{T/k} \otimes_T \widehat{T}$ by the finiteness of $\Omega^1_{T/k}$, also $\Omega^1_{T/k}$ is free, and as $Q(T) = K$ and

$$\mathrm{rk}_T(\Omega^1_{T/k}) = \dim_K(\Omega^1_{T/k} \otimes_T K) = \dim_K(\Omega^1_{K/k}) = \mathrm{rk}_R(\Omega^1_{R/k}) = m$$

it is free of rank m. Thus T has a p–basis of rank m ([KD], (6.18)), and the lemma follows.

2.3 Lemma. Let T be as in (2.2). If $f \in R$ with $\delta(f) \in I^n$ for all $\delta \in \mathrm{Der}_k(R)$, there exists an $x^p \in T^p$ and an $a \in I^nT$ with $f = x^p + a$.

Proof. We may assume that $IT \subseteq \mathfrak{m}_T$. As $\Omega^1_{R/k}$ is free, $\delta(f) \in I^n$ for all $\delta \in \mathrm{Der}_k(R)$ implies that $d_{R/k}f \in I^n\Omega^1_{R/k}$, hence also $d_{T/k}f \in I^n\Omega^1_{T/k}$, and therefore $\delta(f) \in I^nT$ for all $\delta \in \mathrm{Der}_k(T)$.

First case: $p \not| v_T(f)$

Then there exists a $\delta \in \mathrm{Der}_k(R)$ with $v_T(\delta(f)) = v_T(f) - 1$. In fact this is trivially true for $\widehat{T}$, as $\widehat{T} = L[[X]]$ for a field extension L/k (with the derivation $\frac{\partial}{\partial X}$). As

$$\mathrm{Der}_k(\widehat{T}) = \mathrm{Hom}_{\widehat{T}}(\widetilde{\Omega}^1_{\widehat{T}/k}, \widehat{T}) = \mathrm{Hom}_T(\Omega^1_{T/k}, T) \otimes_T \widehat{T}$$

this also holds true over T. In particular we have to have $v_T(f) > v_T(I^n)$ by our assumption on f, and therefore the lemma holds true with $x = 0$.

Second case: $p | v_T(f)$.

Write $f = (x^m)^p \cdot \varepsilon$ for some unit $\varepsilon \in T$ and a regular parameter x of T. If the residue class $\overline{\varepsilon} \notin \overline{T}^p$, where $\overline{T} = T/xT$, then there exists a $\overline{\delta} \in \mathrm{Der}_k(\overline{T})$ such that $\overline{\delta}(\overline{\varepsilon}) \in \overline{T}^*$ is a unit. Therefore there also exists a $\delta \in \mathrm{Der}_k(T)$ such that $\delta(\varepsilon) \in T^*$ is a unit (as $\Omega^1_{T/k}$ is free). Thus for this δ:

$$\delta(f) \cdot T = \delta((x^m)^p \varepsilon) \cdot T = (x^m)^p \delta(\varepsilon) \cdot T = f \cdot T$$

implying again that $f \in I^nT$. Finally, if $\overline{\varepsilon} \in (\overline{T})^p$ then we write $\varepsilon = a^p + b$ for some $b \in xT$, so that $f = (x^m \cdot a)^p + (x^m)^p \cdot b$, and we replace f by $f_1 = f - (x^m \cdot a)^p$. Then $v_T(f_1) > v_T(f)$. If $p \not| v_T(f_1)$, we are done again by case 1, and if $p | v_T(f_1)$, then we proceed by induction till either $p \not| v_T(f_i)$ or $v_T(f_i) \geq v_T(I^n)$.

2.4 Corollary. . *Let W be the Dedekind domain associated to I as in (1.2). If $f \in R$ with $\delta(f) \in I^n$ for all $\delta \in Der_k(R)$ then there exists an $x^p \in W^p$ and an $a \in I^nW$ with $f = x^p + a$.*

Proof. Let $T(I) = \{v_1, \ldots, v_s\}$, and let V_j be the valuation ring of v_j. Then V_j/R is essentially of finite type, and therefore $f = x_j^p + a_j$ with $x_j^p \in V_j^p$ and $a_j \in I^nV_j$ for all $j = 1, \ldots, s$ by (2.3). As we may assume that $I \neq 0$, we have $W/I^nW \cong V_1/I^nV_1 \times \cdots \times V_s/I^nV_s$ by the Chinese remainder theorem, and from this the corollary follows.

Proof of the theorem. Let $T(I) = \{v_1, \ldots, v_s\}$ and set $e_i = v_i(I)$. Furthermore let $|| \ ||_I$ be the I–adic norm on K and let $|| \ ||$ be its restriction to K^p. Then $||x|| = \max\left\{e^{-\frac{w_i(x)}{e_i}} : i = 1, \ldots, s\right\}$, where w_i is the restriction of v_i to K^p, hence $|| \ ||$ is a non–archimedean norm in the sense of §1. Let $x_1, \ldots, x_n$ be a p–basis of R and set

$x^\mu := x_1^{\mu_1} \cdots x_n^{\mu_n}$ for $0 \le \mu_i \le p-1$. Then $\{x^\mu\}$ is a basis of R/R^p and of K/K^p, and we define a norm on K by

$$||y||' := \max\{||a_\mu^p|| \cdot ||x^\mu||_I\} \quad \text{if } y = \sum a_\mu^p x^\mu, \quad a_\mu \in K$$

Then by (1.5) there exists a $c > 0$ with $||\ \ ||' \le c \cdot ||\ \ ||_I$. By the non–archimedean triangle inequality, we have $||\ \ ||_I \le ||\ \ ||'$.

Now choose $l_1 \in \mathbb{N}$ such that $c \le e^{l_1}$ and set $l = l_1 + \dim(R) - 1$. Let $f \in R$ with $\delta(f) \in I^{n+l}$ for all $\delta \in \mathrm{Der}_k(R)$. Then by (2.4)$f = x^p + a$ for some $x \in W$ and $a \in I^{n+l}W$, Thus $||f - x^p||_I \le e^{-(n+l)}$, and therefore $||f - x^p||' \le c \cdot e^{-(n+l)}$. Now write $f = \sum r_\mu^p x^\mu$. Then clearly

$$||f - x^p||' \ge ||f - r_0^p||'$$

Combining all these inequalities we obtain

$$||f - r_0^p||_I \le ||f - r_0^p||' \le c \cdot e^{-(n+l)} \le e^{-(n+(\dim(R)-1))}$$

implying that

$$f - r_0^p \in \overline{\left(I^{n+(\dim(R)-1)}\right)} \subseteq I^n$$

by (1.4) and the Briançon–Skoda theorem (cf. [LS], thm. 1″).

2.5 Remark. The bound l obtained in the proof of (2.1) very much depends on the choice of a p–basis of R. We do not know whether l can be bounded by a constant depending on R only (as is the case if $\mathrm{char}(k) = 0$, cf. [Hü]).

A uniform bound to $l(R, I)$ and a satisfactory solution to the problems from the introduction would be provided by a positive answer to the following question:

2.6 Question. *Let R be as in (2.1), let $I \subseteq R$ be an ideal and let $f \in R$ be an element with $\delta(f) \in I$ for all $\delta \in Der_k(R)$. Does there exist a $c \in C(R/k)$ with $x - c \in \overline{I}$?*

Let K be a field with a non–archimedean norm $||\ \ ||_K$, and let V be a finite K–vector space with a norm $||\ \ ||$, compatible with $||\ \ ||_K$. A basis $x_1, \ldots, x_m$ of V is called orthogonal for $||\ \ ||$ if

$$||r_1 x_1 + \cdots + r_m x_m|| = \max\{||r_i||_K \cdot ||x_i||\}$$

for all $r_1, \ldots, r_m \in K$.

2.7 Question. *In the situation of (2.1) does there exist a basis $1 = x_1, \ldots, x_m$ of R/R^p which is orthogonal for $||\ \ ||_I$ (as a basis of K/K^p)?*

2.8 Remark. i) A positive answer to (2.8) would allow to choose $c = 1$ and $l_1 = 0$ in the proof of (2.1), hence it would imply a positive answer to (2.6).

ii) If the answer to (2.7) is negative, is it possible to characterize those ideals that admit an orthogonal basis in the above sense?

iii) For many problems it would be sufficient to deal with $\mathfrak{m}$–primary ideals in a regular local ring $(R, \mathfrak{m})$, so a positive answer to (2.7) for the class of $\mathfrak{m}$–primary ideals might already be very interesting, as it could lead to a proof of nongraded analogues of Theorem 4.3 [HS] and the vanishing conjecture 3.9 of [HS].

2.9 Remark. There always exists a basis of K/K^p which contains 1 and is orthogonal for $|| \; ||_I$.

Proof. Let $W = V_1 \cap \cdots \cap V_s$ be the Dedekind domain associated to I as in (1.2). By the Chinese remainder theorem there exists a $\pi \in W$ such that $v_i(\pi) = 1$ for all $i \in \{1, \dots, s\}$. Furthermore let $x_2, \dots, x_n \in W$ be elements such that their residue classes $\overline{x_2}, \dots, \overline{x_n} \in W/\mathfrak{m}W$ form a p–basis of $W/\mathfrak{m}W$ for all maximal ideals $\mathfrak{m}$ of W (such elements exist, again by the Chinese remainder theorem), and set $r_\mu = \pi^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$ for $0 \le \mu_i \le p-1$. Then $\{r_\mu\}$ is a basis of K/K^p. Let $y \in K\setminus\{0\}$ and write $y = \sum a_\mu^p r_\mu$, $a_\mu^p \in K^p$. Fix a valuation $v_i \in T(I)$, and for $l \in \{0, \dots, p-1\}$ set $y_l = \sum\limits_{\mu_1 = l} a_\mu^p r_\mu$. Then $y = y_0 + \cdots + y_{p-1}$, and $v_i(y_l) \in l + p\mathbb{Z}$. In particular $v_i(y_l) \ne v_i(y_{l'})$ for $l \ne l'$, and therefore

$$v_i(y) = \inf\{v_i(y_l) : l = 0, \dots, p-1\}$$

Now write $y_l = \pi^l \sum a_\mu^p x_2^{\mu_2} \cdots x_n^{\mu_n}$. Assume $y_l \ne 0$ and set $t(l) = \inf\{v_i(a_\mu) : \mu_1 = l\}$. Then $y_l = \pi^{l+pt(l)} \sum b_\mu^p x_2^{\mu_s} \cdots x_n^{\mu_2}$ with $v_i(b_\mu^p) = 0$ for at least one index μ. Thus

$$\sum b_\mu^p x_2^{\mu_2} \cdots x_n^{\mu_2} \ne 0 \mod \mathfrak{m}_{v_i}$$

by the choice of $x_2, \dots, x_n$, and therefore

$$v_i(y_l) = l + pt(l) = \inf\{v_i(a_\mu^p \pi^l x_2^{\mu_2} \cdots x_n^{\mu_n})\} = \inf\{v_i(a_\mu^p) + v_i(\pi^l x_2^{\mu_2} \cdots x_n^{\mu_n})\}$$

From this we get by an easy calculation

$$||y||_I = \max\{||a_\mu^p|| \cdot ||r_\mu|| : 0 \le \mu_i \le p-1\}$$

2.10 Remark. Let $(R, \mathfrak{m})$ be local, let $I = \mathfrak{m}$, let $x_1, \dots, x_d$ be a regular system of parameters of R and let $x_{d+1}, \dots, x_n$ be elements of R whose residue classes mod $\mathfrak{m}$ form a p–basis of $R/\mathfrak{m}$. Then $r_\mu := x_1^{\mu_1} \cdots x_n^{\mu_n}$ $(0 \le \mu_i \le p-1)$ is an orthogonal basis of R/R^p for $|| \; ||_I$, containing 1.

2.11 Propositon. *In the situation of (2.1) let $(R, \mathfrak{m})$ be local and let $I \subseteq R$ be an ideal defining a strictly normal crossing divisor. Then there exists a basis $1 = r_1, \dots, r_m$ of R/R^p which is orthogonal for $|| \; ||_I$. In particular if $f \in R$ with $\delta(f) \in I^n$ for all $\delta \in Der_k(R)$, then there exists a $c \in C(R/k)$ with $f - c \in I^n$*

Proof. By assumption there exists a regular system of parameters $x_1, \dots, x_d$ of R and positive integers $\nu_1, \dots, \nu_t$ $(t \le d)$ such that $I = x_1^{\nu_1} \cdots x_t^{\nu_t} R$. In this case $\mathfrak{p}_i := x_i R \subseteq R$ is a prime ideal of R, and $R_{\mathfrak{p}_1}, \dots, R_{\mathfrak{p}_t}$ clearly are the Rees–valuations of I (cf. also [MRS], (3.2)). Choosing $x_{d+1}, \dots, x_n \in R$ in such a way that their residue classes $\overline{x_{d+1}}, \dots, \overline{x_n} \in R/\mathfrak{m}$ form a p–basis of $R/\mathfrak{m}$, a calculation similar to that in the proof of (2.10) shows that $\{x_1^{\mu_1} \cdots x_n^{\mu_n} : 0 \le \mu_i \le p-1\}$ is an orthogonal basis of R/R^p for $|| \; ||_I$. As I^n is integrally closed for all $n \in \mathbb{N}$, the proposition follows.

2.12 Remark. Let $(R, \mathfrak{m})$ be a regular local ring, let v be a valuation of $K = Q(R)$ with center on R and with valuation ring V, and let $||x|| = e^{-v(x)}$ be the associated absolute value on K. Then in general there does not exist a basis $1 = r_1, \ldots, r_m$ of R/R^p which is orthogonal for $|| \; ||$ as the following example shows:

Let $R = k[[x, y, z]]$ be a power series ring over a field of characteristic p. Consider the sequence of quadratic transforms (local rings of blowups of points)

$$R \to R_1 \to R_2 \to R_3 \to R_4$$

where the R_i have regular parameters

$$\begin{array}{lll} x = x_1, & y = x_1(y_1+1), & z = x_1 z_1 \\ x_1 = x_2 z_2^{p-1}, & y_1 = y_2 z_2^{p-1}, & z_1 = z_2 \\ x_2 = x_3, & y_2 = x_3(y_3+1), & z_2 = x_3(z_3+1) \\ x_3 = x_4, & y_3 = x_4 y_4, & z_3 = x_4 z_4 \end{array}$$

If we localize R_4 at the prime defining the last exceptional fibre, we get a discrete valuation ring

$$V = R[x_4, y_4, z_4]_{(x_4)}$$

Let v be the valuation defining V. Then we have for the parameters of R:

$$\begin{aligned} x &= x_4^p (x_4 z_4 + 1)^{p-1} \\ y &= x_4^p (x_4 z_4 + 1)^{p-1} + x_4^{2p} (x_4 z_4 + 1)^{2p-2} (x_4 y_4 + 1) \\ z &= x_4^{p+1} (x_4 z_4 + 1)^p \\ (z/x)^p &= x_4^p (x_4 z_4 + 1)^p \end{aligned}$$

and therefore

$$p + 1 = v(y - (z/x)^p) > v(y) = p$$

Since every p^{th} power of a nonunit r in R has valuation $v(r^p) \ge p^2$, there cannot exit an $r \in R$ with the property that $v(y - r^p) \ge p + 1$.

We do not know whether there is an example of this kind in dimension 2 as well.

§3 Reduction mod p

In this section we prove a version of (2.1) as it is necessary for reduction mod p techniques. The result so far requires rather severe restrictions, however we hope that the basic ideas used here may eventually provide a more general result.

We blowup to make the total transform of our ideal a normal crossing divisor on a smooth characteristic 0 scheme. We then reduce the situation mod p for almost all primes p. Proposition 2.11 gives a uniform bound locally on the fiber over p. For p sufficiently large, the obstruction to patching these local solutions lies in the first cohomology group of the reduced exceptional divisor, which we show vanishes.

First let us briefly recall the basic setup: Let $(R,\mathfrak{N})$ be a local regular domain, essentially of finite type over a field K with $\mathrm{char}(K)=0$, where we may assume that $R/\mathfrak{N}$ is finite over K. Furthermore let $I\subseteq R$ be an $\mathfrak{N}$–primary ideal. Then, using the theorem of generic freeness (cf. [Ma], (22.A)) we can construct the following situation (see also [HS] or [MS]): There exists a smooth $\mathbb{Z}$–algebra $A\subseteq K$ of finite type over $\mathbb{Z}$ and a finitely generated flat and smooth A–algebra R_A together with a prime ideal $\mathfrak{q}\subseteq R_A$ such that $R_A/\mathfrak{q}$ is finite and free over A and such that $R=(R_A\otimes_A K)_{\mathfrak{q}R_A\otimes_A K}$. Furthermore we may assume that there exists an ideal $\mathcal{I}_A\subseteq R_A$ with $\mathcal{I}_A R=I$ and such that $R_A/\mathcal{I}_A$ is finite and free as an A–module. For an ideal $\mathfrak{m}\subseteq A$ we denote by $(\mathfrak{m})$ "reduction mod $\mathfrak{m}$". Then for any $\mathfrak{m}\in\mathrm{Max}(A)$ the ring $R_A(\mathfrak{m})/\mathcal{I}_A R_A(\mathfrak{m})$ is semilocal and finite over $k(\mathfrak{m}):=A/\mathfrak{m}$. If $\mathfrak{M}$ is any maximal ideal of $R_A(\mathfrak{m})$ lying over $\mathcal{I}_A R_A(\mathfrak{m})$, then the ring $(R_A(\mathfrak{m})_{\mathfrak{M}},\mathfrak{M}R_A(\mathfrak{m})_{\mathfrak{M}})$ is said to be obtained from R by reduction mod p if $p=\mathrm{char}(k(\mathfrak{m}))$. Using Hironaka's result on the resolution of singularities in characteristic 0 we may assume that there exists a sequence of blow–ups $\pi':X'\longrightarrow\mathrm{Spec}(R_A\otimes_A K)$ of smooth primes such that $\mathcal{I}_A\mathcal{O}_{X'}$ is a strictly normal crossings divisor everywhere, and that after possibly replacing A with A_f for some nonzero $f\in A$, π' is obtained from a projective morphism $\pi:X\longrightarrow\mathrm{Spec}(R_A)$, given by a sequence of blow–ups of smooth primes, which satisfies $R^0\pi_*\mathcal{O}_X=R_A$, $R^l\pi_*\mathcal{O}_X=0$ for $l>0$ and X is flat over A. Note that this implies by base change theory that for each $\mathfrak{m}\in\mathrm{Max}(A)$ also

$$(*)\qquad H^0(X(\mathfrak{m}),\mathcal{O}_{X(\mathfrak{m})})=R_A(\mathfrak{m})\text{ and }H^l(X(\mathfrak{m}),\mathcal{O}_{X(\mathfrak{m})})=0\quad\text{for } l>0$$

We may achieve (after possibly making a further localization of A) that $\mathcal{I}_A\mathcal{O}_X$ is a strictly normal crossing divisor and that $\mathcal{O}_X/\mathcal{I}_A\mathcal{O}_X$ is flat over A. Writing $\mathcal{I}_A\mathcal{O}_X=\mathcal{O}_X(-a_1E_1-\cdots-a_rE_r)$ with positive integers a_i we may finally assume that each E_l is a divisor of X, smooth over A, and that for each $\mathfrak{m}\in\mathrm{Max}(A)$ the ideal $\mathcal{I}_A\mathcal{O}_{X(\mathfrak{m})}$ defines a strictly normal crossing divisor. If $d=\dim(R)=\dim(R_A(\mathfrak{m}))$ we get:

3.1 Theorem. *Let $\mathfrak{m}\in Max(A)$ with $p=char(k(\mathfrak{m}))>n\cdot max\{a_1,\dots,a_r\}$ and let $(R^*,\mathfrak{M}^*)$ be obtained from $(R,\mathfrak{N})$ by reduction mod p. If $f\in R^*$ with $\delta(f)\in\mathcal{I}_A^nR^*$ for every $\delta\in Der_k(R^*)$ then there exists a $c\in C(R^*/k(\mathfrak{m}))$ with $f-c\in\mathcal{I}_A^{n-d}R^*$.*

From now on let us fix the following situation: Let $\mathfrak{m}\in\mathrm{Max}(A)$ and let $\mathfrak{M}\in\mathrm{Max}(R_A(\mathfrak{m}))$ be a maximal ideal, containing $\mathcal{I}_AR_A(\mathfrak{m})$. Then we denote by $R^*:=R_A(\mathfrak{m})_{\mathfrak{M}}$ and $X^*:=X(\mathfrak{m})\times_{R_A(\mathfrak{m})}R^*$.

3.2 Lemma. In the above situation we have for each $n\in\mathbb{N}$:

$$\Gamma(X^*,\mathcal{I}_A^{n+d-1}\mathcal{O}_{X^*})\subseteq\mathcal{I}_A^nR^*$$

Proof. Set $Y = \operatorname{Spec}(R^*)$ and let $f : B := \operatorname{Proj}(\bigoplus \overline{\mathcal{I}_A^n R^*}) \to Y$ be the normalization of the blow–up of $\mathcal{I}_A R^*$. By the universal property of the blow–up and the normalization there exists a $g : X^* \to B$ such that $\pi^* : X^* \to Y$ factors as

$$\pi^* = f \circ g : X^* \xrightarrow{\quad g \quad} B \xrightarrow{\quad f \quad} Y$$

and from this and the projection formula we conclude

$$(\pi^*)_*(\mathcal{I}_A^n \mathcal{O}_{X^*}) = f_* g_*(\mathcal{I}_A^n \mathcal{O}_{X^*}) = f_*(\mathcal{I}_A^n \mathcal{O}_B) = \overline{\mathcal{I}_A^n R^*} \subseteq \mathcal{I}_A^{n-d} R^*$$

where the last inclusion follows from the Briançon–Skoda theorem.

3.3 Lemma. If $f \in R^*$ is an element with $\delta(f) \in \mathcal{I}_A^n R^*$ for all $\delta \in \operatorname{Der}_k(R^*)$ then for all $\mathfrak{P} \in X^*$ there exists a $g_{\mathfrak{P}} \in \mathcal{O}_{X^*,\mathfrak{P}}$ and an $h_{\mathfrak{P}} \in \mathcal{I}_A^n \mathcal{O}_{X^*,\mathfrak{P}}$ with $f = g_{\mathfrak{P}}^p + h_{\mathfrak{P}}$.

Proof. The assumptions on f and the smoothness of $R^*/k(\mathfrak{m})$ imply that we have $d_{R^*/k(\mathfrak{m})}(f) \in \mathcal{I}_A^n \Omega^1_{R^*/k(\mathfrak{m})}$ and from this we conclude that $\delta(f) \in \mathcal{I}_A^n \mathcal{O}_{X^*,\mathfrak{P}}$ for all derivations $\delta \in \operatorname{Der}_k(\mathcal{O}_{X^*,\mathfrak{P}})$, and from this the claim follows by (2.11).

Recall that $\mathcal{I}_A \mathcal{O}_X = \mathcal{O}_X(-a_1 E_1 - \cdots - a_r E_r)$ with smooth divisors $E_i/\operatorname{Spec}(A)$ and positive integers a_i.

3.4 Lemma. $H^1(X^*, \mathcal{O}_{X^*}(-E_1 - \cdots - E_r)) = 0$.

Proof. Let $D = (E_1 + \cdots + E_r)|_{X^*}$. Then we get an exact sequence

$$0 \longrightarrow \mathcal{O}_{X^*}(-D) \longrightarrow \mathcal{O}_{X^*} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

hence a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(X^*, \mathcal{O}_{X^*}(-D)) \longrightarrow & H^0(X^*, \mathcal{O}_{X^*}) \longrightarrow H^0(X^*, \mathcal{O}_D) \\ & \longrightarrow H^1(X^*, \mathcal{O}_{X^*}(-D)) \longrightarrow H^1(X^*, \mathcal{O}_{X^*}) \end{aligned}$$

As $H^0(X^*, \mathcal{O}_{X^*}(-D))$ is the intersection of maximal ideals of discrete valuation rings dominating R^* we conclude that $H^0(X^*, \mathcal{O}_{X^*}(-D)) = \mathfrak{M}$, the maximal ideal of R^*. Let $k = k(\mathfrak{m}) = A/\mathfrak{m}$ and let $\overline{k}$ be an algebraic closure of k. Note that D is projective and strictly normal crossing over k. Setting $\lambda = \dim_k(R^*/\mathfrak{M})$ we conclude that $R^* \otimes_k \overline{k}$ is a reduced semilocal ring with λ distinct maximal ideals.

Let $F = \operatorname{Spec}(\mathcal{O}_D \otimes_k \overline{k}) = (\pi^*)^{-1}(\operatorname{Spec}((R^*/\mathfrak{M}) \otimes_k \overline{k}))_{\text{red}} \subseteq X^* \times_k \overline{k}$. By Zariski's Main Theorem we conclude that F has λ distinct connected components $F_1, \ldots, F_\lambda$ which contract to the distinct maximal ideals of $R^* \otimes_k \overline{k}$. As each $F_i/\overline{k}$ is projective, reduced and connected, we have $H^0(F_i, \mathcal{O}_{F_i}) = \overline{k}$, implying that $H^0(F, \mathcal{O}_F) = H^0(D, \mathcal{O}_D) \otimes_k \overline{k} = \overline{k}^\lambda$, hence $\dim_k(H^0(D, \mathcal{O}_D)) = \lambda$. As $H^0(X^*, \mathcal{O}_{X^*}) = R^*$ and $H^1(X^*, \mathcal{O}_{X^*}) = 0$ by $(*)$ and flat base change, the above long exact sequence now implies that $H^1(X^*, \mathcal{O}_{X^*}(-D)) = 0$

Proof of the theorem. Fix $\mathfrak{m} \in \mathrm{Max}(A)$ with $p = \mathrm{char}(A/\mathfrak{m}) > n \cdot \max\{a_1, \ldots, a_r\}$ and set $k = A/\mathfrak{m}$. Then for all $\mathfrak{P} \in X^*$ there exists an open affine neighbourhood $U_\mathfrak{P}$ of $\mathfrak{P}$ in X^*, a $g_\mathfrak{P} \in \Gamma(U_\mathfrak{P}, \mathcal{O}_{X^*})$ and a $h_\mathfrak{P} \in \Gamma(U_\mathfrak{P}, \mathcal{I}_A^n \mathcal{O}_{X^*})$ with $f = g_\mathfrak{P}^p + h_\mathfrak{P}$ in $\Gamma(U_\mathfrak{P}, \mathcal{O}_{X^*})$ by (3.3). There exist $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$ such that $\{U_{\mathfrak{P}_1}, \ldots, U_{\mathfrak{P}_s}\}$ is an open affine cover of X^*. Set $U_i := U_{\mathfrak{P}_i}$, $g_i := g_{\mathfrak{P}_i}$, and $h_i := h_{\mathfrak{P}_i}$ $(i = 1, \ldots, s)$. Then

$$g_i^p - g_j^p \in \Gamma(U_i \cap U_j, \mathcal{I}_A^n \mathcal{O}_{X^*}) \quad \text{for all } i, j \in \{1, \ldots, s\}$$

As $\mathcal{I}_A^n \mathcal{O}_{X^*} = \mathcal{O}_{X^*}(-na_1E_1 - \cdots - na_rE_r)$ with all $a_i > 0$, and as $E_1 + \cdots + E_r$ is reduced, we conclude that

$$g_i - g_j \in \Gamma(U_i \cap U_j, \mathcal{O}_{X^*}(-E_1 - \cdots - E_r)) \quad \text{for all } i, j \in \{1, \ldots, s\}$$

and clearly $\{g_i - g_j\}_{i,j=1,\ldots,s}$ defines a Čech–1–cocycle. Thus by (3.4) there exist elements $\lambda_i \in \Gamma(U_i, \mathcal{O}_{X^*}(-E_1 - \cdots - E_r))$ such that

$$\lambda_i - \lambda_j = g_i - g_j \quad \text{for all } i, j \in \{1, \ldots, s\}$$

Hence $g_i - \lambda_i = g_j - \lambda_j \in \Gamma(U_i \cap U_j, \mathcal{O}_{X^*})$ for all i, j, and therefore there exists an element $\beta \in \Gamma(X^*, \mathcal{O}_{X^*}) = R^*$ with

$$\beta | U_i = g_i - \lambda_i \quad \text{for all } i \in \{1, \ldots, s\}$$

From this and our assumption on p we get

$$(f - \beta^p) | U_i = f - g_i^p + \lambda_i^p = h_i - \lambda_i^p \in \Gamma(U_i, \mathcal{I}_A^n \mathcal{O}_{X^*}) \quad \text{for all } i \in \{1, \ldots, s\}$$

implying by (3.2) that

$$f - \beta^p \in \Gamma(X^*, \mathcal{I}_A^n \mathcal{O}_{X^*}) \subseteq \mathcal{I}_A^{n-d} R^*$$

and the theorem follows.

3.5 Remark. The proofs of (3.1) and (3.2) actually show that in the situation of (3.1) there exists a $c \in C(R^*/k)$ such that

$$f - c \in \overline{\mathcal{I}_A^n} R^*$$

and therefore (2.6) has a positive answer in this restricted set–up.

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