

RESOLUTION OF SINGULARITIES FOR 3-FOLDS IN POSITIVE CHARACTERISTIC

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ABSTRACT. A concise, complete proof of resolution of singularities of 3-folds in positive characteristic > 5 is given. Abhyankar first proved this theorem in 1966.

1. INTRODUCTION

The purpose of this paper is to give a complete proof of resolution of singularities of 3-folds in positive characteristic which is concise. The paper is essentially self contained, and includes proofs of resolution of singularities of curves and surfaces in positive characteristic.

Resolution of singularities of algebraic varieties over fields of characteristic zero was solved in all dimensions in a very complete form by Hironaka [H1] in 1964.

Resolution of singularities of varieties over fields of positive characteristic is significantly harder. One explanation for this is the lack of “hypersurfaces of maximal contact” in positive characteristic, which is the main technique used in characteristic zero proofs to reduce to a lower dimension (“Maximal contact and approximate manifolds” of this introduction and Sections 6.2 and 7.4 [C2]).

There are several proofs of resolution of singularities of surfaces in positive characteristic. The first one is by Abhyankar [Ab1] which appeared in 1955. There are later proofs by Hironaka, outlined in his notes [H2], and by Lipman [L2] for excellent surfaces.

Abhyankar proved resolution of singularities of positive characteristic 3-folds in 1966. The proof appears in his book [Ab7] and the papers [Ab5], [Ab6], [Ab8] and [Ab9]. The entire proof is extremely long and difficult. It encompasses 508 pages.

Within the last few years there has been a resurgence of interest in resolution in positive characteristic. Some papers making progress on resolution in positive characteristic are Cossart [Co1], [Co2], [Co3], [Co4], Giraud [G], Hauser [Hau1], de Jong [Jo], Hironaka [H9], Kawanoue [Ka], Kawanoue and Matsuki [KM], Kuhlmann [Ku], Moh [M], Piltant [P], Spivakovsky [S] and Teissier [T]. Several simplifications of Hironaka’s proof of characteristic zero resolution have appeared, making the proof quite accessible now, including Abramovich and de Jong [AJo], Bierstone and Milman [BrM], Bogomolov and Pantev [BP], Bravo, Encinas and Villamayor [BEV], Encinas and Hauser [EH], Encinas and Villamayor [EV], Hauser [Hau2], Kollár [Ko], Villamayor [V], Włodarczyk [W].

Most recently, Cossart and Piltant [CP1], [CP2] have proved resolution of singularities for a 3-fold over an arbitrary field. The proof is extremely long and difficult.

We give a self contained and concise proof of the following theorem in this paper.

Theorem 1.1. *Suppose that V is a projective variety of dimension 3 over an algebraically closed field \mathbf{k} of characteristic $\neq 2, 3$ or 5. Then there exists a nonsingular*

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projective variety W and a birational morphism $\phi : W \rightarrow V$, which is an isomorphism above the nonsingular locus of V .

The restrictions on the field \mathbf{k} in the statement of Theorem 1.1 are those of Abhyankar's original statement.

It is our hope that this paper will motivate study of the fundamental open problems in resolution. Our proof could have been written in 1967, as it only makes use of methods known at that time.

Essential ingredients in the proof are the following theorems on embedded resolution of surface singularities.

Theorem 1.2. (*Embedded resolution of surface singularities*) *Suppose that V is a nonsingular 3-dimensional variety over an algebraically closed field \mathbf{k} , S is a reduced surface (a pure 2-dimensional reduced closed subscheme) in V and E is a simple normal crossings divisor on V . Then there exists a sequence of blow ups*

$$\pi : V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V$$

such that the strict transform S_n of S on V_n is nonsingular, and the divisor $\pi^(S+E)$ is a simple normal crossings divisor on S_n . Further, each $V_i \rightarrow V_{i-1}$ is the blow up of a point or nonsingular curve in the locus in V_{i-1} where the preimage of $S+E$ is not a simple normal crossings divisor.*

An effective divisor is a simple normal crossings divisor if its components are nonsingular and intersect transversely (Section 4). The morphisms $V_i \rightarrow V_{i-1}$ of Theorem 1.2 are required to be permissible (Definition 5.5). In particular, the centers blown up are required to be transversal to the preimage of E .

Theorem 1.3. (*Principalization of ideals*) *Suppose that V is a nonsingular 3 dimensional variety over an algebraically closed field \mathbf{k} and $\mathcal{I} \subset \mathcal{O}_V$ is a nonzero ideal sheaf on V . Then there exists a sequence of blow ups*

$$V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V$$

such that $\mathcal{I}\mathcal{O}_{V_n}$ is locally principal. Further, each $V_i \rightarrow V_{i-1}$ is the blow up of a point or nonsingular curve in the locus in V_{i-1} where $\mathcal{I}\mathcal{O}_{V_{i-1}}$ is not invertible.

Theorems 1.2 and 1.3 were first proven by Abhyankar in the book [Ab7], relying on material proven in [Ab5], [Ab6], [Ab8] and [Ab9]. This is the most difficult part of Abhyankar's proof.

We give self contained proofs of these theorems in Sections 5 - 11. The algorithm that we follow for resolution is essentially that of Levi [Le] to resolve the singularities of a two dimensional hypersurface, embedded in a nonsingular 3-fold. This method was given a completely rigorous proof in characteristic zero by Zariski in [Z4]. The algorithm will be discussed at greater length in Section 2 on resolution of surfaces. The first part of this algorithm extends without much trouble to positive characteristic. Hironaka constructs in [H2] an invariant and outlines a proof that in arbitrary characteristic, the invariant drops under the resolution algorithm of Levi. This involves overcoming new major obstacles in positive characteristic. Our proof requires an extension of Hironaka's proof for a 2 dimensional hypersurface to resolution of ideal sheaves on a nonsingular 3-fold of arbitrary characteristic, with the further condition that the resolution procedure produces transversality with the exceptional divisor. Our main resolution theorem for ideal sheaves on a nonsingular 3-fold is stated in Theorem 5.6. We use the philosophy of Bravo, Encinas and Villamayor in [EV] and [BEV], resolving by taking the weak transform of an ideal rather than the strict transform. Our main resolution invariant is then the order of an ideal, rather than the multiplicity. This leads to vast simplifications in the proofs of resolution theorems.

The most fundamental problem of resolution in positive characteristic (open at the time of this writing) is to construct an embedded resolution of a 3 dimensional hypersurface in a nonsingular 4 dimensional variety.

To circumvent the difficulty of constructing an embedded resolution of a 3-fold, Abhyankar proves and makes use of the following theorem, which is a generalization of a method of Albanese.

Theorem 1.4. *Suppose that K is an algebraic function field of dimension d over an algebraically closed field \mathbf{k} . Then there exists a normal projective variety V such that K is the function field of V , and all points of V have multiplicity $\leq d!$.*

Abhyankar then shows that 3-fold singularities of multiplicity less than p can be resolved essentially using characteristic zero techniques. This is where the restriction $p > 5$ comes in, as $3! = 6$. Abhyankar uses a generalization of Jung's method of local resolution, resolving the branch locus of a finite projection to a nonsingular 3-fold, using Theorem 1.2.

Our proof of Theorem 1.1 can be broken down into 5 steps, which we enumerate.

1. Prove Theorems 1.2 and 1.3 (embedded resolution of surface singularities and principalization of ideals).
2. Prove that a projective variety of dimension n is birationally equivalent to a normal projective variety all of whose points have multiplicity $\leq n!$.
3. Prove that points of multiplicity $< p$ can be resolved locally.
4. Patch local resolutions to produce a nonsingular projective variety, birationally equivalent to V .
5. Modify a resolution of singularities $W \rightarrow V$ to produce a resolution $W_1 \rightarrow V$ which is an isomorphism over the nonsingular locus of V .

Steps 1 - 4 appear in Abhyankar's original proof [Ab7]. They draw on almost all classical approaches to resolution. Steps 1 and 4 already appear in the characteristic zero proof of resolution of algebraic 3-folds of Zariski in [Z4]. Step 5 appears in Cossart's article [Co4].

Step 2 (Theorem 1.4) produces a variety V_0 , birationally equivalent to our given 3-fold V , such that all points of V_0 have multiplicity $\leq 3! = 6$. Thus all points of V_0 have multiplicity less than the characteristic p of the ground field \mathbf{k} (if \mathbf{k} has positive characteristic). Step 3 produces for all $p \in V_0$ local resolutions of singularities $W_p \rightarrow U_p$, where U_p is an affine neighborhood of p in V_0 and $W_p \rightarrow U_p$ is a birational projective morphism such that W_p is nonsingular. We have thus proven local uniformization of the function field of V . The concept of local uniformization is explained in "Valuation rings and local uniformization" in Section 3. Now in Step 4, we show that we can construct from these local resolutions a projective morphism $V_1 \rightarrow V$ such that V_1 is nonsingular. In Step 5, we show that we can construct from V_1 a resolution of singularities $V_2 \rightarrow V$ such that this morphism is an isomorphism away from the singular locus of V . This will complete the proof of Theorem 1.1.

In Section 2 we outline the proof of Step 1, (embedded resolution of surfaces) and in Section 3, we outline the proof of Steps 2-5. The actual proof of Theorem 1.1 starts with Section 4.

There is another, later, proof of Step 3 of Theorem 1.1, making use of contributions of Hironaka, Giraud and Cossart. We outline the proof in "Hypersurfaces of maximal contact for singularities of low multiplicity" in Section 3. In this proof, the essential point is also that 3-fold singularities of multiplicity less than p can be resolved essentially using characteristic zero techniques, but in this case it is accomplished by showing that under this restriction on multiplicity, a hypersurface of maximal contact exists.

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I dedicate this paper to Zariski, Abhyankar, Hironaka and Lipman.

2. ON RESOLUTION OF SURFACES

In this section, we give an overview of the proof of the embedded resolution theorem, Theorem 5.6, and its corollaries, Theorem 1.2 and Theorem 1.3. These theorems are proven in Sections 5 - 11 of this paper. This will complete the proof of the first step of the proof of Theorem 1.1. These theorems are generalizations of the proof of resolution of two dimensional hypersurfaces outlined in Hironaka's lectures [H2], which is itself a generalization of the characteristic zero proof of Levi [Le] and Zariski [Z4] to arbitrary characteristic. Hironaka's invariant is given an elegant geometric interpretation by Hauser in [Hau1], which we will discuss later in this section. All particulars of Hironaka's proof are worked out in Chapter 7, "Resolution of Surfaces in Positive Characteristic", in the book "Resolution of Singularities" [C2]. This method of resolution for a hypersurface on a nonsingular 3-fold extends to resolve an ideal sheaf on a nonsingular 3-fold. To accomplish this, we make a slightly different definition of the τ invariant from the classical one of Hironaka.

Maximal contact and approximate manifolds. Suppose that V is a nonsingular variety over an algebraically closed field \mathbf{k} . We use the definition of order, $\nu_q(\mathcal{I})$, given in Section 4. $\nu_q(\mathcal{I})$ is the largest power of the maximal ideal of $\mathcal{O}_{V,q}$ containing \mathcal{I}_q . Let

$$r = r(\mathcal{I}) = \max\{\nu_q(\mathcal{I}) \mid q \in V\},$$

and let

$$\text{Sing}_r(\mathcal{I}) = \{q \in V \mid \nu_q(\mathcal{I}) = r\}.$$

We summarize a few basic properties of the behavior of the order ν under blow ups. This topic is discussed in depth in [C2], especially in Chapter 4. Some definitions and basic facts about resolution are given in Sections 4 and 5 of this paper.

$\text{Sing}_r(\mathcal{I})$ is a closed subset of V . Suppose that $W \subset \text{Sing}_r(\mathcal{I})$ is a nonsingular subvariety. Let $\pi_1 : V_1 \rightarrow V$ be the blow up of W . The weak transform \mathcal{I}_1 of \mathcal{I} , is defined to be $\mathcal{I}\mathcal{I}_E^{-r}$, where $E = \pi_1^{-1}(W)$ is the exceptional divisor of π_1 , and \mathcal{I}_E is the ideal sheaf of E in V_1 . We have that $r_1 = r(\mathcal{I}_1) \leq r$.

Suppose that $q \in \text{Sing}_r(\mathcal{I})$. Let $R = \mathcal{O}_{V,q}$. A nonsingular subvariety $Y \subset \text{Spec}(R)$ is called a manifold of maximal contact for \mathcal{I} at q if Y contains the germ of $\text{Sing}_r(\mathcal{I})$ at q , and if $W \subset \text{Sing}_r(\mathcal{I})$ is a nonsingular subvariety, and $\pi_1 : V_1 \rightarrow V$ is the blow up of W , then $\pi_1^{-1}(q) \cap \text{Sing}_r(\mathcal{I}_1)$ is contained in the strict transform Y_1 of Y on V_1 . Further, if $q_1 \in \pi_1^{-1}(q) \cap \text{Sing}_r(\mathcal{I}_1)$, then Y_1 is a manifold of maximal contact for \mathcal{I}_1 at q_1 .

The fundamental theorem of resolution of singularities in characteristic zero is that manifolds of maximal contact always exist (for any dimension of V). This allows all theorems on resolution to be reduced by induction to appropriate resolution statements in lower dimension.

We associate a positive integer $\tau = \tau_{\mathcal{I}}(q)$ to $q \in \text{Sing}_r(\mathcal{I})$ such that if a manifold of maximal contact Y exists at q for \mathcal{I} , we have $\dim Y = \tau$. The number τ can be thought of as the minimal number of variables in the tangent cone of \mathcal{I} .

Unfortunately, manifolds of maximal contact do not generally exist in positive characteristic. This can be considered as the major obstacle to resolution. We can

however approximate a manifold of maximal contact to first order, at least over algebraically closed fields of arbitrary characteristic. Such a (nonsingular) subvariety will be called an approximate manifold. The dimension of an approximate manifold is τ .

The method of approximate manifolds, which he formulates slightly differently and calls the directrix, was introduced by Hironaka [H2].

Resolution of an ideal. In this section, we give an overview of our proof of principalization of an ideal sheaf \mathcal{I} on a nonsingular 3-fold V . We adapt the resolution algorithm of Levi [Le] and Zariski [Z4], and Hironaka's resolution invariant and termination proof in [H2] to this situation. The main theorem is stated precisely in Theorem 5.6 of Section 5. We use the philosophy of Bravo, Encinas and Villamayor in [EV] and [BEV], resolving by taking the weak transform of an ideal rather than the strict transform. Our main resolution invariant is then the order of an ideal, rather than the multiplicity. This leads to vast simplifications in the proofs of resolution theorems.

Let $r = r(\mathcal{I})$. We will construct a sequence of blow ups of points and nonsingular curves W_n

$$(1) \quad \cdots \rightarrow V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V$$

such that if \mathcal{I}_n is the weak transform of \mathcal{I}_{n-1} on V_n , then W_n is contained in $\text{Sing}_r(\mathcal{I}_n)$. Further, we will have that two dimensional irreducible components of $\text{Sing}_r(\mathcal{I})$ are isolated, and at a point of such a component, \mathcal{I} is locally principal, and is generated by f^r where $f = 0$ is a local equation of a nonsingular surface S . We may thus assume that 2 dimensional components never appear in $\text{Sing}_r(\mathcal{I}_n)$, as they may be eliminated by blowing up the nonsingular surface S .

Our goal is to construct a sequence which terminates after a finite number of steps with $\text{Sing}_r(\mathcal{I}_n) = \emptyset$.

As shown in Section 8, without too much difficulty, we can reach a V_n where $\text{Sing}_r(\mathcal{I}_n)$ is a finite set of points, so we may as well assume that $\text{Sing}_r(\mathcal{I})$ is a finite set of points. We then apply (in Section 8) the following algorithm to construct (1).

1. If $\text{Sing}_r(\mathcal{I}_n)$ contains a nonsingular curve C , then blow up C .
2. If $\text{Sing}_r(\mathcal{I}_n)$ is a finite set of points, then blow up a point in $\text{Sing}_r(\mathcal{I}_n)$.

We will show that the sequence (1) is finite, and terminates with $\text{Sing}_r(\mathcal{I}_n) = \emptyset$.

The number τ can never go down in such a sequence, so we are reduced to assuming that there is an infinite sequence (1), with points $q_n \in V_n$, such that q_n maps to q_{n-1} for all n , and $\nu_{q_n}(\mathcal{I}_n) = \nu_{q_n}(\mathcal{I}) = r$, $\tau(q_n) = \tau(q)$ for all n .

With our assumptions, in a neighborhood of q_n , $\text{Sing}_r(\mathcal{I}_n)$ is an isolated point, or a nonsingular curve contained in the exceptional divisor. We can thus find approximate manifolds D_{q_n} containing the germ of $\text{Sing}_r(\mathcal{I}_n)$ at q_n . Let our blow up be $\pi_{n+1} : V_{n+1} \rightarrow V_n$. By the theory of approximate manifolds, we have that $\pi_{n+1}^{-1}(q_n) \cap D_{q_{n+1}}$ is the intersection of $\pi_{n+1}^{-1}(q_n)$ and the strict transform of D_{q_n} .

We now distinguish several cases, according to the value of τ . The case $\tau(q) = 3$ is trivial. In this case q is an approximate manifold and the strict transform of q under the blow up of q is the empty set, so $\pi_1^{-1}(q) \cap \text{Sing}_r(\mathcal{I}_1) = \emptyset$.

In a similar way, if $\tau(q) = 2$ and there exists a (nonsingular) curve C through q , then C is an approximate manifold. The blow up π_1 of C then satisfies $\pi_1^{-1}(q) \cap \text{Sing}_r(\mathcal{I}_1) = \emptyset$.

The case of $\tau(q) = 2$ and q_n isolated in $\text{Sing}_r(\mathcal{I}_n)$ for all n is the first nontrivial case. In this case we can actually find a manifold (curve) of maximal contact, by the process of well preparation. This is a generalization of the method used by Newton,

to compute the branches of an algebraic function of two variables. This case is worked out in Section 9. The simpler case of an ideal sheaf on a nonsingular surface is in Section 6.

The most difficult case is when $\tau(q) = 1$. This case is in Section 10. We start with a regular system of parameters x, y, z at q , where $z = 0$ is an approximate manifold of \mathcal{I} at q , and make a formal change of variables, replacing z with $z' = z - \Psi(x, y)$ for some appropriate series $\Psi(x, y)$, to well prepare (Subsection 10.2). We will have that $z' = 0$ is an approximate manifold for \mathcal{I} at q . Even if we start with regular parameters x, y, z in $\mathcal{O}_{V,q}$, our resulting well prepared parameter z' might be formal, existing only in $\hat{\mathcal{O}}_{V,q}$. We may also need to very well prepare (Subsection 10.3), which involves making a change of variables replacing y with an expression $y' = y - \phi(x)$, for some appropriate series $\phi(x)$.

We associate elements $\Omega(q_n) = (\beta_n, \frac{1}{\epsilon_n}, \alpha_n) \in \mathbf{Q}_+^3$ to q_n , in an ordered set, and show that $\Omega(q_{n+1}) < \Omega(q_n)$ for all n , and that $\Omega(q_n)$ cannot decrease indefinitely. This is sufficient to show that (1) has finite length.

Let R_n be the completion of \mathcal{O}_{V_n, q_n} , and $I_n = \mathcal{I}_{n, q_n} R_n$.

We inductively construct a series of (formal) approximate manifolds D_n in $\text{Spec}(R_n)$ by taking a local equation of the strict transform of D_{n-1} , and modifying if necessary by very well preparation. Let x_n, y_n, z_n be the resulting regular parameters in R_n , where $V(z_n) = D_n$. We insist that our parameters x_n, y_n, z_n are such that

$$\text{Sing}(I_n) \subset V(x_n y_n, z_n) = V(x_n, z_n) \cap V(y_n, z_n)$$

for all n . This may prevent very well preparation, if $\text{Sing}(I_n) = V(y_n, z_n)$.

Ω will depend on our choice of regular parameters x_n, y_n, z_n in R_n . We construct a polygon $|\Delta| = |\Delta(I_n, x_n, y_n, z_n)|$ contained in the real upper half plane, from a projection of the nonzero coefficients of monomials in x_n, y_n, z_n in R_n of elements $f \in I_n$.

We choose the parameters x_n, y_n, z_n in R_n to obtain the smallest polygon possible under substitutions $z' = z_n - \Psi(x_n, y_n)$ (to well prepare) or under substitutions $z' = z_n - \Psi(x_n, y_n)$ and $y' = y_n - \phi(x_n)$ (to very well prepare).

(α_n, β_n) is the upper left hand corner of $|\Delta|$, and $-\epsilon_n$ is the slope of the non vertical boundary line segment through (α_n, β_n) .

There are only 4 possible types of maps $R_n \rightarrow R_{n+1}$ which can occur in (1), which are denoted by Tr1 - Tr4. In each case, $\text{Sing}(I_n)$ is a curve or point (Subsection 10.4).

Most of the cases of Tr1 - Tr4 are just monomial substitutions in the parameters, except for the case $\eta \neq 0$ of Tr1. Local equations of Tr1 are recalled two paragraphs below.

The monomial substitutions cause combinatorially simple transformations of the Newton polygon, and do not involve characteristic p phenomena.

The really interesting case, and only non monomial substitution, is when a point is blown up by a Tr1 transformation, with $\eta \neq 0$ (Theorem 10.17). Local equations of the map are then $x_n = x', y_n = x'(y' + \eta), z_n = x'z'$. The blow up is not a monomial substitution in x_n, y_n, z_n in this case, but becomes a monomial substitution after a linear change of variables (a "translation") $y'_n = y_n - \eta x_n$. Then the proof becomes a detailed study of the binomial theorem in positive characteristic.

In characteristic zero the proof is much simpler and reduces to the statement that the term $na^{n-1}b$ is nonzero in the expansion

$$(a + b)^n = a^n + na^{n-1}b + \dots$$

This observation was used by Zariski in his proof [Z1] of resolution of algebraic surface singularities, and in his subsequent proofs of resolution.

Flags. Hauser [Hau1] gives a beautiful formulation of Hironaka’s invariant used in the proof of termination of the resolution algorithm, which we will outline here, as it makes the construction more intuitive. Hauser assumes that \mathcal{I} is the ideal sheaf of a reduced surface.

We define a flag through $q_n \in \text{Sing}_r(\mathcal{I}_n)$ to be the germs at q_n of a nonsingular surface F_2 in V_n and a nonsingular curve F_1 in F_2 . We assume that F_1 is transversal to an approximate manifold D_{q_n} at q_n . If the connected component of $\text{Sing}_r(\mathcal{I}_n)$ is a (nonsingular) curve C , then we require that either F_1 and F_2 are both transversal to C , or F_1 is transversal to C and F_2 contains C .

We say that parameters x_n, y_n, z_n at q_n are subordinate coordinates for the flag if $x_n = 0$ is a local equation of F_2 and $x_n = y_n = 0$ is a local equation of F_1 . All subordinate coordinates for the flag are (essentially) obtained from x_n, y_n, z_n by making changes of variables

$$z'_n = z_n - \Psi(x_n, y_n), \text{ and } y'_n = y_n - \phi(x_n),$$

for appropriate series Ψ and ϕ . These are the type of change of variables which we consider in the local resolution algorithm, to well prepare and very well prepare. Hauser shows that $\Omega(q_n)$ then becomes an actual invariant of the flag. We maximize $(\beta, \frac{1}{\epsilon}, \alpha)$ for subordinate changes of variables for the flag, by well preparing and very well preparing.

Suppose that we have a sequence of points $\{q_n\}$ in (1) with $q_n \in \text{Sing}_r(\mathcal{I}_n)$ and q_n mapping to q_{n-1} for all n .

Hauser shows that a flag at q_{n-1} transforms to a flag at q_n under the sequence (1), that we can always choose subordinate coordinates so that $R_{n-1} \rightarrow R_n$ is monomial in these coordinates, and the invariants computed from these flags satisfy $\Omega(q_n) < \Omega(q_{n-1})$. The descending invariants $\Omega(q_n)$ on the sequence are then computed by taking the transforms of an initial choice of a flag at q .

Embedded resolution.

The concepts of simple normal crossings (SNCs) and of transversal intersection are defined in Section 4. It is discussed in more depth in [C2], especially in Chapter 4 of [C2]. The geometric idea is that all components are nonsingular and intersect transversely. Our notion of simple normal crossings is sometimes called “strict normal crossings” in the literature.

To give the full proof of embedded resolution of singularities of an ideal sheaf \mathcal{I} on a nonsingular 3-fold V (Theorem 5.6) we must start with a SNC divisor E on V as well as our ideal sheaf \mathcal{I} . We then want to construct a sequence (1) so that not only is $\text{Sing}_r(\mathcal{I}_n) = \emptyset$, but the divisor E_n on V_n consisting of the sum of the exceptional divisor of $V_n \rightarrow V$ and the preimage of E on V_n is a SNC divisor. Fortunately, essentially no new characteristic p problems occur in this proof, beyond those we have already encountered.

A little care must be taken however in making sure that all blow ups are of centers which are transversal to the exceptional divisor. The problem is that we cannot deviate from the resolution algorithm, which is to blow up a curve if there is a curve in the singular locus (we can ensure that the curve will always be nonsingular after a little preliminary blowing up before starting the algorithm), and to blow up a point if there is no curve in the singular locus. Hironaka’s invariant which we use to show termination does not behave well if there is deviation from this procedure.

In order for E_n to be a SNC divisor on V_n , we require that all blow ups $V_i \rightarrow V_{i-1}$ (with $i \leq n$) be permissible (Definition 5.5). This means that they are the blow up of a nonsingular subvariety of $\text{Sing}_r(\mathcal{I}_{i-1})$ which is transversal to E_{i-1} .

To accomplish this, we make use of the η invariant. This invariant is used by Abhyankar in his “good point” proof of embedded resolution of (characteristic zero) surface singularities, which is presented in [Ab10], in the lectures [O] of Orbanz, and in Lecture 3 of Lipman’s Arcata notes [L1]. We define η as follows.

On V , set $E^+ = \emptyset$ and $E^- = E$. At a point $q_n \in V_n$ where $\nu_{q_n}(\mathcal{I}_n) = \nu_q(\mathcal{I}) = r$, define E_n^- to be the strict transform of $E^- = E$, and E_n^+ to be the preimage of the exceptional divisor of $V_n \rightarrow V$. If $\nu_{q_n}(\mathcal{I}_n)$ drops to less than r , we define E_n^- to be E_n and E_n^+ to be the empty set.

For $q_n \in \text{Sing}_r(\mathcal{I}_n)$, let $\eta(q_n)$ be the number of irreducible components of E_n^- containing q_n . We have $0 \leq \eta(q_n) \leq \eta(q_{n-1}) \leq 3$ if $\nu_{q_n}(\mathcal{I}_n) = \nu_{q_{n-1}}(\mathcal{I}_{n-1}) = r$.

In Section 8, we reduce by descending induction on η to the case of $\eta = 0$ (E_n^- is disjoint from $\text{Sing}_r(\mathcal{I}_n)$) and $\text{Sing}_r(\mathcal{I}_n) \subset E_n^+$. This is achieved by noticing that our resolution problem reduces to a resolution problem on a nonsingular subvariety of V of dimension $3 - \eta(q) \leq 2$ (an intersection of components of E^-).

Further, we show that the exceptional divisor of $V_n \rightarrow V$ is transversal to an approximate hypersurface of \mathcal{I}_n .

Now any blow up that we make in our algorithm (given after (1)) to reduce the order of \mathcal{I}_n will automatically be permissible. The point is that one dimensional components of $\text{Sing}_r(\mathcal{I}_n) \cap E_n^-$ are not well controlled, while those of $\text{Sing}_r(\mathcal{I}_n) \cap E_n^+$ are projective lines transversal to E_n^+ .

Further generalization of the algorithm. The most interesting generalization would be to make the above reduction argument work for a hypersurface X in a nonsingular variety V of dimension $d \geq 4$. The major problems occur in making the local reduction argument work for small τ . The cases of

$$(2) \quad 0 \leq d - \tau(\mathcal{I}_{X,q}) \leq 2$$

are analogous to the cases of $\tau(\mathcal{I}_q) = 3, 2, 1$ which we analyze in this paper for ideal sheaves on a nonsingular variety of dimension 3. However, the case of $d - \tau(\mathcal{I}_{X,q}) > 2$ appears to be very difficult.

Resolution of excellent surfaces. Hironaka announced in his notes [H2] a proof of resolution of singularities of an arbitrary excellent surface. In the notes he gives a few comments about the realization of the general proof, and gives some parts of the proof in [H3], [H4] and [H5] (see also Cossart’s paper [Co1]).

We outline this approach, when S is a surface which is embedded in a nonsingular variety V of arbitrary dimension d , over an algebraically closed field \mathbf{k} .

Making the proof work over a non perfect base field, or for an arbitrary excellent surface requires an excursion into a further series of complexities and treacherous pitfalls which we will not address here. Some details on this are given in [H4], [H5], [H6] and [Co1]. A complete proof of resolution of excellent surfaces using a different method, is given by Lipman in [L2].

When $d = 3$, the order and multiplicity of \mathcal{I}_S are the same. Also, under permissible blow ups, the weak transform and strict transforms of \mathcal{I}_S are the same. For larger d , however, this is not the case. We now take r to be the largest multiplicity of \mathcal{I}_S , and define $\text{Sing}_r(\mathcal{I})$ to be the locus of points of maximal multiplicity. Under permissible transforms, we take \mathcal{I}_n to be the strict transform of \mathcal{I}_{n-1} . \mathcal{I}_n is the ideal sheaf of the strict transform S_n of S on V_n .

The first part of the proof when $d = 3$ (in “Resolutions of ideals” above) now goes through without trouble for general d . We reduce to the assumption that the locus of maximal multiplicity $\text{Sing}_r(\mathcal{I})$ is a finite set of points, and we have an infinite

sequence of blow ups of points and nonsingular curves (1), constructed by following the algorithm stated after (1). As in the analysis for the case $d = 3$, we reduce to consideration of a sequence of points $q_n \in S_n$ such that q_n maps to q_{n-1} for all n , and the multiplicity of S_n at q_n is r for all n .

Let $R = \hat{\mathcal{O}}_{V,q}$, with maximal ideal m , and let $I = \mathcal{I}_{S,q}R$.

We can obtain any value $\tau(q) = \tau(I) \geq 1$, and as we have commented above in “Further generalization of the algorithm”, our local argument for the reduction of the order of an ideal does not extend to $d - \tau(I) > 2$. We can however get around this by using $\bar{\tau}$ instead of τ , which we now define. We are in fact already forced to use this new concept, since when we are using multiplicity and taking strict transform, instead of using the simpler concepts of order and weak transform. We point out that Hironaka defines τ to be what we call $\bar{\tau}$ in this paper, and does not consider what we call τ , although $\tau = \bar{\tau}$ for locally principal ideals.

Let $\text{in}(I)$ be the ideal of $\text{gr}_m(R) = \bigoplus_{j \geq 0} m^j/m^{j+1}$ generated by initial forms of elements of I . We define $\bar{\tau}(I)$ to be the dimension of the smallest R/m linear subspace T of $\text{gr}_m(R)$ such that the initial forms of all elements of I are contained in the polynomial ring $R/m[T] \subset \text{gr}_m(R)$.

Since $\text{in}(I)$ is contained in the ideal (T) of $\text{gr}_m(R)$ generated by T , we see that

$$2 = \dim R/I = \dim \text{gr}_{mR/I}(R/I) = \dim \text{gr}_m(R)/\text{in}(I) \geq \dim \text{gr}_m(R)/(T).$$

Thus $\bar{\tau}(I) \geq \dim R - 2$.

The problem becomes to extend the reduction argument used in “Resolution of an ideal” in the case when $d = 3$, using $\bar{\tau}$ instead of τ . This is feasible as $\bar{\tau}(I) \geq d - 2$ for the ideal of a surface. The price of this is that we must then use the full range of sophisticated methods developed by Hironaka in [H1] to control the reduction of multiplicity of the strict transform of an ideal under a permissible blow up (instead of just the order of the weak transform of an ideal), and requires a more sophisticated analysis of the polyhedra associated to a singularity which incorporates information from an appropriate standard basis of I . Some of this analysis is done in [H3], in a very general setting.

Abhyankar’s proof of embedded resolution of surfaces. We conclude this section with a quick overview of Abhyankar’s proof in [Ab7] and [Ab5], [Ab6], [Ab8], [Ab9].

The main resolution theorem in Abhyankar’s proof is Theorem 1.1 of [Ab6], which is restated in Theorem 5.1 in [Ab7].

Suppose that \mathbf{k} is an algebraically closed field of characteristic p , and $A = \mathbf{k}[x, y]_{(x,y)}$ is a regular local ring. Suppose that

$$(3) \quad f(Z) = Z^n + f_1 Z^{n-1} + \cdots + f_n$$

is a monic polynomial with coefficients in A , and ω is a rank 1 nondiscrete valuation which dominates A . Let

$$(4) \quad A \rightarrow A_1 \rightarrow \cdots \rightarrow A_t \rightarrow \cdots$$

be the sequence of blow ups of the maximal ideals of the local rings A_i which are dominated by ω . Theorem 1.1 of [Ab6] states that $f(Z)$ has a nice form for all $t \gg 0$. The idea of an approximate manifold is implicit in this argument.

Local uniformization can be easily deduced from Theorem 1.1 of [Ab6]. The essential case is when our germ of a singular surface S has the local equation $f(Z) = 0$, and the f_i have order $\geq i$. Let ν be a valuation which dominates $f(Z) = 0$. Let ω be the restriction of ν to A . The most difficult case where the worst complexities of

characteristic p occur is when ω is nondiscrete of rational rank 1. We may as well assume then that ω satisfies the assumptions of Theorem 1.1 of [Ab6]. We then easily deduce from the theorem that for $t \gg 0$, there exists a substitution in Z which gives a birational extension such that the center of ν has multiplicity less than n on the strict transform of S .

Local uniformization is discussed in ‘‘Valuation rings and local uniformization’’ in Section 3. In his characteristic zero proof of resolution of surface singularities in [Z4], Zariski introduces the notion of the dominant character of a normal sequence. This is the statement that if embedded local uniformization is true by possibly quite different local resolution sequences along each valuation centered on a two dimensional hypersurface, then a reasonable global algorithm of resolution will lead to a global resolution. Zariski proves this in [Z4].

Abhyankar generalizes this notion to resolution of ideals supported on an arbitrary characteristic nonsingular 3-fold, and requires that the resolution procedure generate a resolved object which makes SNCs with the exceptional divisor. This is accomplished in a very difficult proof in Chapter 1 of [Ab7]. In this way, Abhyankar deduces in [Ab7] the embedded resolution theorems for surfaces, stated as Theorems 1.2 and 1.3 in this paper.

The proof of Theorem 1.1 of [Ab6] is realized in the papers [Ab6] and [Ab9], with the aid of a technical result from [Ab8].

The idea is to consider a splitting field L of the polynomial $f(Z)$ over the quotient field K of A . In [Ab6], making ingenious use of the norm and discriminant of a field extension, as well as the observation that the coefficients of a monic polynomial are the elementary symmetric functions in its roots, the proof of Theorem 1.1 of [Ab6] is reduced to the case where $f(Z)$ is irreducible in $K[Z]$, $n = p^m$ for some integer m , and the valuation ω does not split under extension to L . The point is that this isolates the part of the resolution procedure where characteristic p is an obstacle. Now that we have made the reduction, we have obtained nonsplitting type and ramified type for $f(Z)$ (Section 2 of [Ab9]). Lemma 2.6 [Ab6] reduces the proof of Theorem 1.1 of [Ab6] in this case to the technical result of Theorem 5.3 of [Ab9]. The goal is to find t sufficiently large in (4) such that f_n has small positive order, and that this cannot be changed after translation of Z by an element of A_t . Recall that n is a power of p . If we translate by an element r of A_t , replacing Z with $Z + r$, we get

$$Z^n + f_{1,t}Z^{n-1} + \cdots + f_{n,t}$$

where $f_{i,t} \in A_t$, and

$$f_{n,t} = f_n + r^n + \text{contributions from the coefficients of } Z^i \text{ with } 0 < i < n.$$

The non splitting type and ramified type assumptions tell us that the contributions from the coefficients of Z^i with $0 < i < n$ cannot interfere too much in low degree with $f_n + r^n$. We write

$$f_{n,t} = x_t^{a_t} y_t^{b_t} H_t(x_t, y_t)$$

for appropriate regular parameters x_t, y_t in A_t , where x_t, y_t do not divide H_t . Let

$$\text{ord}(H_t(0, y_t)) = c_t.$$

We want to find a t such that c_t is small, and we cannot remove the term $x_t^{a_t} y_t^{b_t+c_t}$ from $f_{n,t}$ by translation of Z . We may reduce to the case where the $x_t^{a_t} y_t^{b_t+c_t}$ term cannot be removed if at least one of a_t and $b_t + c_t$ is not divisible by n . Sections 6 through 9 of [Ab9] give an algorithm for achieving this. The idea is that after translation of Z to remove appropriate monomials in x_t and y_t which are powers of n , if we blow up we have that c can only go up under a very special circumstance

(this is where Lemma 27 of [Ab8] comes in), and after enough blow ups, there must actually be a drop in c_t .

The problem with order going up under resolution also occurs in resolution of vector fields (Seidenberg [Se], Cano [Ca] and Panazzolo [Pa]) and in monomialization of morphisms (Cutkosky [C1], [C3]; and Cutkosky and Piltant [CP]). In the case of dimension two (as in the case of resolution of surfaces) this problem is not overwhelming, but the level of complexity goes up tremendously in dimension three ([Ca], [C1], [C3]).

To compare Abhyankar's proof with Hironaka's invariant used in this paper, we point out that the well preparations of Section 10 of this paper are essentially Abhyankar's translations, and the numbers (α, β) of Section 10 of this paper are analogous to Abhyankar's $(a, b + c)$.

3. ON RESOLUTION OF 3-FOLDS

In this section we give some indication of the proofs of the different steps, discuss some of the history, review some of the background material, and point out a few subtle considerations which come up in the proof. We translate Abhyankar's proof into language which will be more easily accessible to the reader familiar with the language of schemes. We have also simplified some of the proofs, partly by simplifying the statements to the case of an algebraically closed ground field, which is all that we need. We assume throughout this section that \mathbf{k} is an algebraically closed field.

Step 2. In Section 12, we prove the projection theorem, Theorem 1.4. The simple idea of the proof is as follows. Suppose that V_0 is a projective variety of dimension d , with function field $K = \mathbf{k}(V_0)$, and $q \in V_0$. The first observation (in Theorem 12.1) is that the multiplicity $e(\mathcal{O}_{V,q})$ of the local ring $\mathcal{O}_{V,q}$ is less than or equal to the degree $\deg(V_0)$ of V_0 .

The idea of the proof of Theorem 1.4 is to project from points of high multiplicity until we get a rational map from V_0 to a variety V_n such that V_0 and V_n have the same dimension, and if $[V_0 : V_n]$ is the degree of the extension of function fields $\mathbf{k}(V_0)$ over $\mathbf{k}(V_n)$, then every point of V_n has multiplicity $\leq \frac{d!}{[V_0:V_n]}$. Then we take the normalization of V_n in $\mathbf{k}(V_0)$, to get a variety V birational to V_0 , such that $\deg(V) \leq d!$, and thus every point of V has multiplicity $\leq d!$.

Albanese [Alb] used this method in resolution of algebraic surfaces. Abhyankar gives an algebraic proof of this theorem, valid in all dimensions in Section 12 of his book [Ab7]. Lipman also gives a survey of the proof in all dimensions, in geometric language closer to that of Albanese, in Section 5 of Lecture 1 of his Arcata notes [L1].

In Section 12, we essentially present Lipman's argument in his survey [L1], but fill in a few details.

Zariski analyzes Albanese's proof on pages 21 and 22 of his 1935 book [Z6]. Zariski points out that Albanese's proof makes extensive use of intersection properties of curves on a surface, which are well defined for nonsingular surfaces, but for surfaces with singularities, present new difficulties. Zariski cautions us that "with this distinction in mind, Albanese's proof should be closely scrutinized".

The first instance where this issue is apparent is in the necessity to use multiplicity instead of the more intuitive local intersection number $(H^d \cdot V_0)_q$ at a point $q \in V_0$, obtained by computing the length of $\mathcal{O}_{V_0,q}/(x_1, \dots, x_d)$, where $x_i = 0$ are local equations of general hyperplane sections of V_0 through q . This invariant is related to multiplicity by the inequality

$$e(\mathcal{O}_{V_0,q}) \leq (H^d \cdot V_0)_q,$$

with equality if and only if V_0 is Cohen Macaulay at q (Theorem 23, Chapter VIII, Section 10 [ZS]). Intuitively, we might expect that $(H^d \cdot V_0)_q$ is the correct number to use (instead of the multiplicity $e(\mathcal{O}_{V_0, q})$) in the projection formulas of Theorem 12.1. However, we see that this naive approach fails in general.

Another subtle point which comes up is the necessity of using a sophisticated Bézout theorem in the proof to show that we do not eventually reach a situation where the dimension of the image drops (in Theorem 12.3). This issue only appears when the dimension d of V_0 is ≥ 3 . Abhyankar proves the theorem we need in Theorem 12.3.1 of [Ab7]. A slightly more general version of this theorem appears in Fulton's book [F] (Example 12.3.1 of the Refined Bézout Theorem). The only instance where characteristic p appears as an issue in the proof of Theorem 1.4, is in the use of Bertini's theorem for general hyperplane sections, again in the proof that the dimension of the image does not drop (Theorem 12.3).

Valuation rings and local uniformization. Suppose that K is an algebraic function field over \mathbf{k} , and V is a valuation ring of K (containing \mathbf{k}), whose residue field is \mathbf{k} . Suppose that X is a projective (or even proper) variety with function field K . Then there is a unique closed point $p \in X$ such that V dominates (Section 4) the local ring $\mathcal{O}_{X, p}$.

The problem of local uniformization is to show that for every valuation ring V of K , there exists a regular local ring R , which is essentially of finite type over \mathbf{k} and has K as its quotient field, such that V dominates R .

Zariski proved local uniformization over characteristic zero function fields in [Z2]. The most difficult problems in local uniformization occur for valuations whose value groups are not finitely generated.

The Zariski Riemann manifold. Zariski introduced the idea of local uniformization along a valuation to give a local formulation of resolution of singularities.

Suppose that K is an algebraic function field. Zariski constructed a ringed space $\text{ZR}(K)$ which we call the Zariski Riemann manifold of K . A nice survey of this concept is given by Lipman in his appendix to Chapter II of [Z6]. The points of $\text{ZR}(K)$ are the valuation rings of the function field K , which contain \mathbf{k} and whose residue field is \mathbf{k} [Z5]. Such valuations are called zero dimensional. The local ring of the point is the valuation ring. Zariski proved that this space is quasi compact in [Z5] and Section 17, Chapter VI of [ZS].

There is a natural continuous morphism of ringed spaces $\Phi_X : \text{ZR}(K) \rightarrow X$ for any projective variety X whose function field is K . A valuation ring A of K maps to the point of X whose local ring is dominated by A . The point is called the center of A on X . The mappings Φ_X are natural, so that if $\Psi : X_1 \rightarrow X$ is a birational map, then $\Psi \circ \Phi_{X_1} = \Phi_X$.

Because of the quasi compactness of $\text{ZR}(K)$, the construction of a resolution of singularities will follow from patching together a finite number of local resolutions. Zariski accomplished this for (characteristic zero) surfaces [Z3] and 3-folds [Z4].

Step 3. In Section 14, we generalize the method of Jung [J], for locally resolving singularities. This method is discussed (for surfaces in characteristic zero) by Zariski on pages 16 and 17 of [Z6]. This method is also discussed by Lipman in Lecture 2 of [L1], and by Giraud in [G2]. Abhyankar has generalized the method to the case of tame ramification in positive characteristic, and has shown that the situation is extremely complicated when the ramification is wild [Ab2].

Our proof is a simplification of the proof by Abhyankar given in “Uniformization of points of small multiplicity”, in Section 10 of Chapter 3 of [Ab7]. Our final statement is a proof in Theorem 14.4 of local uniformization for the function field of V .

In characteristic zero the method is very simple and elegant. The fundamental result is the following. Suppose that $f : X \rightarrow Y$ is a finite map from a normal variety X (of arbitrary dimension) to a nonsingular variety Y . Then by the purity of the branch locus, the branch locus of f is a divisor D_f on Y . If D_f is a SNC divisor, then X must have abelian quotient singularities, which have a very simple form, and can be easily resolved. In fact, with respect to the toroidal structures determined by D_f and $f^{-1}(D_f)$, $f : X \rightarrow Y$ is a morphism of toroidal varieties. We can thus use one of the combinatorial algorithms (such as in [KKMS] or [BrM2]) to resolve the singularities of Y .

This gives us a way to deduce local uniformization (assuming characteristic zero) in dimension d , assuming embedded resolution in dimension $d - 1$. We take a singular variety V , and a (zero dimensional) valuation ring A of the function field of V , whose center is a (closed) point p on V . We choose a finite map f from an open neighborhood X of p in V to an open subset Y of the affine space \mathbf{A}_k^d . Then we apply embedded resolution of singularities to construct a birational, projective morphism $\pi_1 : Y_1 \rightarrow Y$ so that $\pi_1^{-1}(D_f)$ is a SNC divisor on the nonsingular variety Y_1 . Finally, we take X_1 to be the normalization of the variety $(X \times_Y Y_1)_{\text{red}}$. X_1 is birational to X . Let p_1 be the center of A on X_1 . Then by the above, X_1 has a toric singularity at p_1 , which we can now easily resolve.

In Section 14, we extend this argument to the case when k has characteristic ≥ 0 , and the local ring $\mathcal{O}_{V,p}$ has multiplicity less than p .

Our main local resolution statement is Theorem 14.4, which proves local uniformization for the function field of V .

A slightly different formulation of the theory of ramification from the one we present in Sections 13 and 14, written in the language of etale mappings, is given by Grothendieck and Murre in [GM], especially in Section 2.3, “Tame Ramification and Abhyankar’s Theorem”.

Hypersurfaces of maximal contact for singularities of low multiplicity. Here we outline an alternate proof of Step 3 of Theorem 1.1, which was pointed out to us by one of the reviewers. We start with a variety V' such that all points of V' have multiplicity $< p$. We now make a local study of V' .

Cossart has shown (Theorem II.3 [Co3]) that a hypersurface of maximal contact always exists for singularities at points of multiplicity $< p$. For the sake of motivation, we point out that the existence of a hypersurface of maximal contact is easy to see in an important special case, that of a hypersurface singularity

$$y^r + a_1(x)y^{r-1} + \cdots + a_r(x) = 0$$

where x represents a system of variables, and $r < p$. In this case we may obtain a hypersurface of maximal contact $\bar{y} = 0$, just as in characteristic zero, by making a Tschirnhaus transformation $\bar{y} = y + \frac{a_1(x)}{r}$, to eliminate the \bar{y}^{r-1} coefficient.

Cossart’s proof makes a detailed local analysis, using information obtained from the Hilbert Samuel function of the singularity and a theorem of Giraud [G] to conclude that there is a hypersurface of maximal contact for the singularity. Specifically, Cossart shows that if (f_1, \dots, f_m) is a local standard basis of the singularity, with $\nu_i = \text{ord}_x(f_i)$, for $1 \leq i \leq m$, then $\nu_i < p$, for all i , and there exists a regular system of parameters $(y_1, \dots, y_\tau, u_1, \dots, u_d)$ with $d \leq 3$ such that $W = V(y_1, \dots, y_\tau)$ has maximal contact, by Giraud’s Theorem [G].

Now using techniques of Hironaka on the Hilbert-Samuel function, the resolution of the singularity reduces to the resolution of the idealistic exponent $\mathcal{E} = \cap_{1 \leq i \leq m} ((f_i, \nu_i))$, which is supported by W and is equivalent to an idealistic exponent $((I, b))$ with I an ideal of $\mathcal{O}_{W,x}$, $b \in \mathbf{N}$. \mathcal{E} is solved by principalizing I , which is done by a straight forward generalization of Theorem 5.6 of this paper to idealistic exponents.

Step 4. In Step 3, we proved local uniformization for the function field of V , so we have reduced, by quasi compactness of the Zariski Riemann manifold as we commented in “The Zariski Riemann manifold”, to patching together a finite number of local resolutions. The problem is to take several projective varieties which are birationally equivalent to V , and such that every valuation has a nonsingular center on at least one of the varieties. We must construct a variety V' which birationally dominates all of these varieties and is nonsingular. Zariski’s original characteristic zero proof [Z4] made use of general Bertini theorems which are not valid in positive characteristic. His method was to study a birational morphism through an associated linear system. We eliminate the need for Bertini theorems by using the fact that a birational morphism is the blow up of a sheaf of ideals, and using the universal property of blow ups of ideal sheaves (Proposition II.7.14 [Ha], Theorem 4.2 [C2]). This is essentially the approach of Abhyankar’s proof of this step, although he states and proves it in his own language (Section 8 of Chapter 2 of [Ab7]).

This step is where the theorem on principalization of ideals (Theorem 1.3) is required. It is essential in this application that we blow up only above the locus where the ideal sheaf is not principal. Another essential ingredient of the proof is that a birational morphism of nonsingular 3-folds factors in a canonical way over the general point of a curve. It is just a sequence of blow ups of curves which dominate the original curve. This fails completely in higher dimensions.

The fourth step is proven in Sections 15 and 16.

Step 5. The final fifth step is accomplished in Section 17. The original proof of Abhyankar does not produce a resolution which is an isomorphism over the nonsingular locus. The fact that a resolution can be birationally modified (in dimension 3) to produce a resolution which is an isomorphism over the nonsingular locus is due to Cossart [Co4]. Our proof of this section is based on his extremely simple argument. Again, the main point is that a birational morphism of nonsingular 3-folds factors in a canonical way over the general point of a curve.

4. PRELIMINARIES

In this section we establish notation, and review a few results that we will use in the course of the proof.

If R is a local ring, we will denote the maximal ideal of R by $M(R)$. Suppose that S is another local ring, and $R \rightarrow S$ is an inclusion of rings. We will say that S dominates R if $M(S) \cap R = M(R)$.

Suppose that $\Lambda \subset R$ is a subset. We define the closed subset

$$V(\Lambda) = \{P \in \text{Spec}(R) \mid \Lambda \subset P\}$$

of $\text{Spec}(R)$. We will consider the order $\nu_R(I)$ of an ideal I in R . $\nu_R(I)$ is defined to be the largest integer n such that $I \subset M(R)^n$. If q is a point on a variety W , and \mathcal{J} is an ideal sheaf on W , then we denote $\nu_q(\mathcal{J}) = \nu_{\mathcal{O}_{W,q}}(\mathcal{J}_q)$.

Multiplicity. Suppose that R is a d dimensional Noetherian local ring, and that q is an $M(R)$ primary ideal of R . There exists a degree d polynomial $P_q(n)$ which has the property that the length

$$\ell(R/q^n) = P_q(n)$$

for $n \gg 0$. The multiplicity $e(q)$ is defined to be $d!$ times the leading coefficient of $P_q(n)$. The multiplicity of R is defined to be $e(R) = e(M(R))$. From the definition we infer that if $q' \subset q$ are $M(R)$ -primary, then $e(q) \leq e(q')$. Suppose that R^* is the $M(R)$ -adic completion of R . We see that $e(qR^*) = e(q)$.

The multiplicity has the important property that $e(R) = 1$ if and only if R is regular (Theorem 23, Section 10, Chapter VIII [ZS]).

Theorem 4.1. *Suppose that $R/M(R)$ is infinite. Then there exists an ideal $q' \subset q$ generated by a system of parameters such that $e(q') = e(q)$.*

This is proven in Theorem 22, Section 10, Chapter VIII [ZS].

Theorem 4.2. *Suppose that R is a local domain, and T is an overring of R which is a domain and a finite R module. Then T is a semilocal ring. Let p_1, \dots, p_s be the maximal ideals of T . Let K be the quotient field of R and M be the quotient field of T . Then*

$$[M : K]e(q) = \sum_{i=1}^s [T/p_i : R/M(R)]e(qT_{p_i}).$$

This is proven in Corollary 1 to Theorem 24, Section 10, Chapter VIII [ZS].

Intersection multiplicity and degree. We use the notation of Fulton [F] for intersection theory. The intersection class of a k -cycle α with a Cartier divisor D on a scheme X is denoted by $D \cdot \alpha$. If D_1, \dots, D_k are Cartier divisors on X , the intersection number of D_1, \dots, D_k and α is denoted by $\int_X D_1 \cdot \dots \cdot D_k \cdot \alpha$.

The degree $\deg Y$ of a subscheme Y of \mathbf{P}^m of dimension d is $d!$ times the leading coefficient of the Hilbert Polynomial $P_Y(n)$ of the homogeneous coordinate ring of Y . It can also be computed as $\deg Y = \int_{\mathbf{P}^m} H^d \cdot Y$ where H is a hyperplane of \mathbf{P}^m . Proofs are given in Section 7 of Chapter 1 [Ha] and the Corollary of Section 8.5 [I], or in Examples 2.5.1 and 2.5.2 [F].

Lemma 4.3. *Suppose that R is a local ring of dimension d , and let*

$$Y = \text{Proj}(\oplus_{n \geq 0} M(R)^n / M(R)^{n+1}).$$

Then $\deg Y = e(R)$.

This follows from comparing the Hilbert polynomial of Y , which is equal to

$$\ell(M(R)^n / M(R)^{n+1})$$

for all large n , and the polynomial $P_{M(R)}(n)$.

Completion. Suppose that \mathbf{k} is an algebraically closed field, and R is a normal local ring, which is essentially of finite type over \mathbf{k} (a localization of a ring which is of finite type over \mathbf{k}). The $M(R)$ -adic completion R^* of R is a normal local ring (Theorem 32, Section 13, Chapter VIII [ZS]). Suppose that L is a finite field extension of the quotient field of R^* . Then the integral closure of R^* in L is a local domain (since R^* is Henselian, Corollary 2 to Theorem 18, Section 7, Chapter VIII [ZS]).

We also state here the Zariski subspace theorem. For a proof, see (10.10) [Ab7]. A ring R is essentially of finite type over a ring T if R is a localization of a finitely generated T algebra.

Theorem 4.4. *Suppose that R and S are normal local rings which are locally of finite type over \mathbf{k} . Suppose that S dominates R . Let R^* be the $M(R)$ -adic completion of R and S^* be the $M(S)$ -adic completion of S . Then the natural morphism $R^* \rightarrow S^*$ is an inclusion.*

Birational geometry. In this paper, a variety is an open subset of an integral, closed subscheme of \mathbf{P}^n . A curve, surface or 3-fold is a variety of the corresponding dimension.

The definition of variety in [C2] is a little more general. In [C2], a variety is an open subset of a reduced, equidimensional subscheme of \mathbf{P}^n . The resolution theorems for varieties in [C2] are thus valid for reduced equidimensional subschemes of \mathbf{P}^n .

Suppose that X is a projective variety over an algebraically closed field \mathbf{k} . Let K be the function field of X . Suppose that V is a valuation ring of K which contains \mathbf{k} . Then there exists a unique (not necessarily closed) point $p \in X$ such that V dominates the local ring $\mathcal{O}_{X,p}$. p is called the center of V on X .

Suppose that $\Phi : X \rightarrow Y$ is a birational morphism of normal projective varieties. The exceptional locus of Φ is the closed subscheme of X on which Φ is not an isomorphism. Irreducible components T of the exceptional locus may have any dimension $1 \leq \dim T \leq \dim X - 1$. If Y is nonsingular all components have dimension $\dim X - 1$.

Now suppose that $\Phi : X \rightarrow Y$ is a birational map of normal projective varieties. There is a largest open set U of X such that $\Phi|_U$ is a morphism. The complement, $F = X - U$ is called the fundamental locus of Φ . By Zariski's main theorem (Lemma V.5.1 and Theorem V.5.2 [Ha]), the fundamental locus F has codimension ≥ 2 in X .

Suppose that X is a variety and Y is a subscheme. The ideal sheaf of Y in X will be denoted by \mathcal{I}_Y .

Suppose that X is a nonsingular variety, and Y is a nonsingular subvariety. The blow up of X with center Y is the blow up $X_1 \rightarrow X$ of \mathcal{I}_Y . X_1 is a nonsingular variety.

Suppose that $X_1 \rightarrow X$ is the blow up of the nonsingular center Y , V is a valuation ring of the function field K of X , the center of V on X is p and the center of V on X_1 is p_1 . Then $\mathcal{O}_{X,p} \rightarrow \mathcal{O}_{X_1,p_1}$ will be called a local blow up of $\mathcal{O}_{X,p}$ along V . The induced homomorphism $\hat{\mathcal{O}}_{X,p} \rightarrow \hat{\mathcal{O}}_{X_1,p_1}$ will also be called a local blow up.

We will make use of the fact that any birational morphism $X \rightarrow Y$ of projective varieties is the blow up of an ideal sheaf (Theorem 7.17 [Ha]), and the ‘‘universal property of blow ups’’ (Proposition 7.14 [Ha]). Foundational material on blow ups, strict transforms, and other matters are covered in Chapter 4 of [C2].

Suppose that $X \rightarrow Y$ is a dominant rational map of varieties. If the extension of function fields $\mathbf{k}(Y) \rightarrow \mathbf{k}(X)$ is finite, we denote $[X : Y] = [\mathbf{k}(X) : \mathbf{k}(Y)]$.

Suppose that X is a nonsingular variety. An effective divisor D on X is a simple normal crossings divisor on X if for each $p \in X$ there exist regular parameters x_1, \dots, x_n in $\mathcal{O}_{X,p}$ and $a_1, \dots, a_n \in \mathbf{N}$ such that D is the divisor $x_1^{a_1} \cdots x_n^{a_n} = 0$ in a neighborhood of p . We will abbreviate a simple normal crossings divisor as a SNC divisor. Suppose that X is a nonsingular variety, D is a SNC divisor on X and Y is a subscheme of X . We will say that Y is transversal to D if Y is nonsingular, and for $p \in Y \cap D$, there exist regular parameters x_1, \dots, x_n in $\mathcal{O}_{X,p}$, $r \leq n$ and $a_1, \dots, a_n \in \mathbf{N}$ such that D is the divisor $x_1^{a_1} \cdots x_n^{a_n} = 0$ in a neighborhood of p , and $V(x_1, \dots, x_r) \subset \text{Spec}(\mathcal{O}_{X,p})$ is the germ of Y .

If D_1, D_2 are two Cartier divisors on a variety X , we will write $D_1 \sim D_2$ if D_1 is linearly equivalent to D_2 .

5. THE ν AND τ INVARIANTS

Suppose that V is a nonsingular variety over an algebraically closed field \mathbf{k} of arbitrary characteristic, and \mathcal{I} is a nonzero ideal sheaf of V . Suppose that $q \in V$ is a (not necessarily closed) point. If Z is a subscheme of V with ideal sheaf \mathcal{I}_Z , define $\nu_q(Z) = \nu_q(\mathcal{I}_Z)$ (with notation as in Section 4). Foundational material on singularities can be found in [C2].

For $t \in \mathbf{N}$, let

$$\text{Sing}_t(\mathcal{I}) = \{p \in V \mid \nu_p(\mathcal{I}) \geq t\},$$

which is a Zariski closed subset of V (Theorem A.19 [C2]). Let

$$r = \max\{t \mid \text{Sing}_t(\mathcal{I}) \neq \emptyset\} = \max\{\nu_p(\mathcal{I}) \mid p \in V\}.$$

Suppose that Y is a nonsingular, integral subvariety of V . Let $\pi_1 : V_1 \rightarrow V$ be the blow up of Y with exceptional divisor F . Let $t = \nu_q(\mathcal{I}_q)$, where q is the generic point of Y . Then the weak transform \mathcal{I}_1 of \mathcal{I} on V_1 is defined by (cf page 65 [C2])

$$\mathcal{I}\mathcal{O}_{V_1} = \mathcal{O}_{V_1}(-tF)\mathcal{I}_1.$$

$\nu_q(\mathcal{I}) = t$ implies that \mathcal{I}_1 is an ideal sheaf on V_1 . If Z is the subscheme of V defined by \mathcal{I} , then the weak transform Z_1 of Z is the subscheme of V_1 defined by \mathcal{I}_1 . If Z is a divisor on V , then the weak transform of Z is the strict transform of Z (page 38, page 65 [C2]).

Suppose that $p \in \text{Sing}_r(\mathcal{I})$ is a closed point, x_1, \dots, x_n are regular parameters in $\mathcal{O}_{V,p}$ and $f \in \mathcal{I}_p$. There is an expansion

$$f = \sum_{i_1 + \dots + i_n \geq r} a_{i_1, \dots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

with $a_{i_1, i_2, \dots, i_n} \in \mathbf{k}$ in $\hat{\mathcal{O}}_{V,p} = \mathbf{k}[[x_1, \dots, x_n]]$. The r -leading form L of f (with respect to x_1, \dots, x_n) is defined to be

$$L(x_1, x_2, \dots, x_n) = \sum_{i_1 + \dots + i_n = r} a_{i_1, i_2, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

Define $\tau(p) = \tau_{\mathcal{I}}(p)$ to be the dimension of the smallest linear subspace T of the \mathbf{k} -subspace spanned by x_1, x_2, \dots, x_n in $\mathbf{k}[[x_1, \dots, x_n]]$ such that $L \in \mathbf{k}[T]$ for all $f \in \mathcal{I}_p$. We will call the subvariety $M_p = V(T)$ of $\text{Spec}(\mathcal{O}_{V,p})$ an *approximate manifold* to \mathcal{I} at p . T depends on our choice of regular parameters at p . However, the tangent space to M_p does not depend on T , and the dimension $n - \tau(p)$ of M_p does not depend on T . After making a choice of x_1, \dots, x_n , T is uniquely determined. It follows that if $Y \subset \text{Sing}_r(\mathcal{I})$ is a nonsingular subvariety with $p \in Y$, then there exists an approximate manifold M_p to \mathcal{I} at p such that the germ of Y at p is contained in M_p .

Lemma 5.1. *Suppose that $Y \subset \text{Sing}_r(\mathcal{I})$ is a nonsingular subvariety of V , $\pi_1 : V_1 \rightarrow V$ is the blow up of Y , \mathcal{I}_1 is the weak transform of \mathcal{I} on V_1 , $p \in Y$, M_p is an approximate manifold to \mathcal{I} at p containing the germ of Y at p , and $q \in \pi_1^{-1}(p)$. Then*

1. $\nu_q(\mathcal{I}_1) \leq r$.
2. $\nu_q(\mathcal{I}_1) = r$ implies q is on the strict transform M'_p of M_p and $\tau(p) \leq \tau(q)$.
3. Suppose that $\nu_q(\mathcal{I}_1) = r$ and $\tau(p) = \tau(q)$. Then there exists an approximate manifold M_q to \mathcal{I} at q such that $M_q \cap \pi_1^{-1}(p) = M'_p \cap \pi_1^{-1}(p)$ where M'_p is the strict transform of M_p on V_1 .

Proof. This follows from a local calculation, as in Lemma 6.4 and Lemma 7.5 [C2]. \square

With the hypotheses of Lemma 5.1, we immediately conclude that $\tau(p) \leq n - \dim Y$ and $\tau(p) = n - \dim Y$ implies $\nu_q(\mathcal{I}_1) < r$.

We deduce from Lemma 5.1 the following important corollary.

Lemma 5.2. *Suppose that V is a nonsingular 3-fold, $Y \subset \text{Sing}_r(\mathcal{I})$ is a nonsingular subvariety of V , $\pi_1 : V_1 \rightarrow V$ is the blow up of Y , and \mathcal{I}_1 is the weak transform of \mathcal{I} on V_1 . Let F be the reduced exceptional divisor of π_1 . Suppose that $F \cap \text{Sing}_r(\mathcal{I}_1) \neq \emptyset$. Then*

1. *If Y is a point, then $F \cap \text{Sing}_r(\mathcal{I}_1)$ is either an isolated point, or an irreducible nonsingular curve.*
2. *If Y is a curve, then $F \cap \text{Sing}_r(\mathcal{I}_1)$ is either a finite union of points, each in a distinct fiber over a point of Y , or $\text{Sing}_r(\mathcal{I}_1)$ is an irreducible nonsingular curve which is a section over Y .*

A nonsingular hypersurface $D_p \subset \text{Spec}(\mathcal{O}_{V,p})$ is called an *approximate hypersurface* to \mathcal{I} at p if D_p contains an approximate manifold M_p to \mathcal{I} at p . If D_p is an approximate hypersurface to \mathcal{I} at p , and $Y \subset \text{Sing}_r(\mathcal{I})$ is a nonsingular subvariety containing p , then there exists an approximate hypersurface D'_p to \mathcal{I} at p such that $Y \subset D'_p$ and D_p and D'_p have the same tangent space. We deduce the following lemma from Lemma 5.1.

Lemma 5.3. *Suppose that $Y \subset \text{Sing}_r(\mathcal{I})$ is a nonsingular subvariety of V , $\pi_1 : V_1 \rightarrow V$ is the blow up of Y , \mathcal{I}_1 is the weak transform of \mathcal{I} on V_1 , $p \in Y$, D_p is an approximate hypersurface to \mathcal{I} at p containing the germ of Y at p , and $q \in \pi_1^{-1}(p)$. Then*

1. $\nu_q(\mathcal{I}_1) \leq r$
2. $\nu_q(\mathcal{I}_1) = r$ implies q is on the strict transform D'_p of D_p .
3. *Suppose that $\nu_q(\mathcal{I}_1) = r$. Then there exists an approximate hypersurface D_q to \mathcal{I} at q such that $D_q \cap \pi_1^{-1}(p) = D'_p \cap \pi_1^{-1}(p)$ where D'_p is the strict transform of D_p on V_1 .*

Definition 5.4. *A resolution datum $\mathcal{R} = (E^+, E^-, \mathcal{I}, V)$ is a 4-tuple where E^+ and E^- are effective divisors on a nonsingular variety V over an algebraically closed field \mathbf{k} , such that $E = E^+ + E^-$ is a SNC divisor, E^+ and E^- have no common components, and \mathcal{I} is an ideal sheaf on V .*

For $t \geq 1$, let

$$\text{Sing}_t(\mathcal{R}) = \text{Sing}_t(\mathcal{I}) = \{p \in V \mid \nu_p(\mathcal{I}) \geq t\}.$$

$\text{Sing}_t(\mathcal{R})$ is a proper Zariski closed subset of V for $t \geq 1$.

Let

$$r = \nu(\mathcal{R}) = \nu(\mathcal{I}) = \max\{\nu_p(\mathcal{I}) \mid p \in V\}.$$

For $p \in V$, let

$$\eta(p) = \text{the number of components of } E^- \text{ containing } p.$$

We have $0 \leq \eta(p) \leq \dim V$.

Let $\tau(p) = \tau_{\mathcal{I}}(p)$ for $p \in V$.

Transversality is defined in the discussion of birational geometry in Section 4 on preliminaries.

Definition 5.5. *A permissible transform of \mathcal{R} is the blow up $\pi_1 : V_1 \rightarrow V$ of a nonsingular subvariety $Y \subset \text{Sing}_r(\mathcal{R})$ such that Y is transversal to $E = E^+ + E^-$.*

Let F be the exceptional divisor of a permissible transform $\pi : V_1 \rightarrow V$ of \mathcal{R} . Then $\pi_1^*(E^+ + E^-) + F$ is a SNC divisor. We define the transform \mathcal{R}_1 of \mathcal{R} on V_1 to be $\mathcal{R}_1 = (E_1^+, E_1^-, \mathcal{I}_1, V_1)$ where \mathcal{I}_1 is the weak transform of \mathcal{I} , and

$$E_1^+ = \pi_1^*(E^+) + F, E_1^- = \text{the strict transform of } E^- \text{ if } \nu(\mathcal{I}_1) = \nu(\mathcal{R}),$$

$$E_1^+ = \emptyset, E_1^- = \pi_1^*(E^+ + E^-) + F \text{ if } \nu(\mathcal{I}_1) < \nu(\mathcal{R}).$$

We remark that if $\pi_1(p) = q$ and $\nu_q(\mathcal{I}_1) \leq \nu_p(\mathcal{I}) = r$, then $\eta(q) \leq \eta(p)$.

In the next several sections we prove the following theorem, which is our main resolution theorem for ideals on a nonsingular 3-fold.

Theorem 5.6. *Suppose that V is a nonsingular 3-fold over an algebraically closed field \mathbf{k} , and $\mathcal{R} = (\emptyset, E, \mathcal{I}, V)$ is a resolution datum. Let $r = \nu(\mathcal{R}) \geq 1$. Then there exists a sequence of permissible transforms $\pi : V_1 \rightarrow V$ of \mathcal{R} such that $\text{Sing}_r(\mathcal{R}_1) = \emptyset$, where \mathcal{R}_1 is the transform of \mathcal{R} by π .*

Theorem 5.6 implies the embedded resolution theorems Theorem 1.2 and Theorem 1.3 stated in the introduction. Using classical techniques of resolution, and the resolution algorithm of Levi [Le] and Zariski [Z4], we reduce in Sections 7 and 8 to a local analysis of a sequence of points under which the order of the weak transform of the ideal does not drop. We show that this sequence must be finite in Sections 9 and 10. The essential ingredient in this local analysis is an extension of Hironaka's termination invariant for surfaces [H2]. In Section 11, we deduce Theorem 1.2 and 1.3 from Theorem 5.6.

In the course of the proof of Theorem 5.6, we will construct sequences of permissible transforms

$$V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V.$$

To simplify notation, we will use the convention that $\mathcal{R}_n = (E_n^+, E_n^-, \mathcal{I}_n, V_n)$ where $E_n = E_n^+ + E_n^-$ is the transform of \mathcal{R} on V_n .

We will also construct sequences of permissible transforms involving W_i ,

$$W_m \rightarrow W_{m-1} \rightarrow \cdots \rightarrow W_1 \rightarrow V.$$

We will use the notational convention that $\mathcal{R}'_m = ((E'_m)^+, (E'_m)^-, \mathcal{I}'_m, W_m)$ where $E'_m = (E'_m)^+ + (E'_m)^-$ is the transform of \mathcal{R} on W_m .

6. EMBEDDED RESOLUTION ON A NON SINGULAR SURFACE

In this section, we prove resolution for an ideal sheaf on a nonsingular surface. We define a resolution datum on a nonsingular variety over an arbitrary field in the same way as for algebraically closed fields.

Theorem 6.1. *Suppose that T is a nonsingular surface over a field L , and $\mathcal{R} = (\emptyset, E, \mathcal{I}, T)$ is a resolution datum. Let $r = \nu(\mathcal{R}) \geq 1$. Then there exists a sequence of permissible transforms $\pi : T_1 \rightarrow T$ of \mathcal{R} such that $\text{Sing}_r(\mathcal{R}_1) = \emptyset$, where \mathcal{R}_1 is the transform of \mathcal{R} by π .*

Proof. After a few blow ups of points on the strict transforms of components of E , we construct $T_1 \rightarrow T$ such that $\eta(p) = 0$ for all $p \in \text{Sing}_r(\mathcal{R}_1)$. We may thus assume that this is true on T .

Suppose that $C \subset \text{Sing}_r(\mathcal{I})$ is a curve. Then C is nonsingular, and there exists a neighborhood U of C in T such that $\mathcal{I} \mid \mathcal{O}_U = \mathcal{I}'_C \mid U$. Let $\pi_1 : T_1 \rightarrow T$ be the blow up of C , \mathcal{I}_1 be the weak transform of \mathcal{I} on T_1 . Then $\mathcal{I}_1 \mid \pi_1^{-1}(U) = \mathcal{O}_{\pi_1^{-1}(U)}$. Thus $\text{Sing}_r(\mathcal{I}_1) \cap \pi_1^{-1}(U) = \emptyset$.

We thus reduce to the case where $\text{Sing}_r(\mathcal{I})$ is a finite set of points.

Now the proof is an extension of the proof of Theorem 3.15 [C2]. We outline the proof here, and refer to [C2] for technical details. We construct a sequence of projective morphisms

$$(5) \quad \cdots \rightarrow T_n \xrightarrow{\pi_n} \cdots \rightarrow T_1 \xrightarrow{\pi_1} T_0 = T$$

where each $\pi_{n+1} : T_{n+1} \rightarrow T_n$ is the blow up of all points in $\text{Sing}_r(\mathcal{I}_n)$, where \mathcal{I}_n is the weak transform of \mathcal{I}_{n-1} on T_n . We must show that this sequence is finite.

Suppose that (5) has infinite length. We can then find closed points $p_n \in T_n$ such that $\pi_n(p_n) = p_{n-1}$, and $p_n \in \text{Sing}_r(\mathcal{I}_n)$ for all n . We then have an infinite sequence of local rings

$$R = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_n \rightarrow \cdots$$

where $R_n = \hat{\mathcal{O}}_{T_n, p_n}$. Let L_1 be the residue field of R . By Lemma 3.14 [C2], the residue field of each of these local rings is equal to L_1 .

We will define $\delta_{p_i} \in \frac{1}{r!}\mathbf{N}$ so that $\delta_{p_i} = \delta_{p_{i-1}} - 1$ for all i , leading to a contradiction to the assumed infinite length of (5).

Suppose that x, y are regular parameters in \mathcal{O}_{T_0, p_0} . Identify L_1 with a coefficient field of R_0 . For $f \in I = \mathcal{I}_{p_0}R_0$, we have an expansion

$$f = \sum_{i+j \geq r} a_{ij} x^i y^j$$

with $a_{ij} \in L_1$. We will call x, y good parameters for I if there exists $f \in I$ such that $a_{0r} \neq 0$ ($\text{ord}(f(0, y)) = r$). Suppose that x, y are good parameters for I . For $f \in I$, we define

$$\delta(f; x, y) = \begin{cases} \min \left\{ \frac{i}{r-j} \mid j < r \text{ and } a_{ij} \neq 0 \right\} & \text{if } y^r \nmid f, \\ \infty & \text{if } y^r \mid f \end{cases}$$

and

$$\delta(I; x, y) = \min \{ \delta(f; x, y) \mid f \in I \}.$$

Now define

$$\delta_{p_0} = \sup \{ \delta(I; x, y_0) \mid y_0 = y - \sum_{i=1}^n b_i x^i \text{ with } n \in \mathbf{N} \text{ and } b_i \in L_1 \}.$$

If $\delta_{p_0} = \infty$, then there exists a series $g = y - \sum_{i=1}^{\infty} b_i x^i$ such that $(g^r) = I$, a contradiction to the assumption that p is isolated in $\text{Sing}_r(\mathcal{I})$. Thus $\delta_{p_0} \in \frac{1}{r!}\mathbf{N}$. Let $y_0 \in \mathcal{O}_{T_0, p_0}$ be such that $\delta_{p_0} = \delta(I; x, y_0)$. Then we check that x, y_1 are regular parameters in R_1 , where $y_0 = xy_1$, that x, y_1 are good parameters for $I_1 = \widehat{(\mathcal{I}_1)_{p_1}}$, and $\delta_{p_1} = \delta_{p_0} - 1$. We thus see that $\delta_{p_i} < 0$ for $i \gg 0$, a contradiction to our assumption that (5) has infinite length. \square

Remark 6.2. *Theorem 6.1 and its proof are applicable when $T = \text{Spec}(A)$, where A is an excellent 2 dimensional regular local ring containing a field.*

7. REDUCTION OF EMBEDDED RESOLUTION TO RESOLUTION

In this section, we suppose that V is a nonsingular 3-fold (a 3-dimensional variety over an algebraically closed field \mathbf{k}) and \mathcal{R} is a resolution datum on V with $r = \nu(\mathcal{R}) \geq 1$. We will transform \mathcal{R} into a nice stable form (Theorem 7.2), from which we will obtain our main embedded resolution theorem (Theorem 5.6) in Sections 8 - 10.

Lemma 7.1. *Suppose that $p \in \text{Sing}_r(\mathcal{R})$, and an approximate hypersurface D_p of \mathcal{I} at p is transversal to E^+ and is not a component of E^+ . Suppose that $\pi_1 : V_1 \rightarrow V$ is a permissible transform of a nonsingular subvariety Y of V containing p . Let \mathcal{R}_1 be the transform of \mathcal{R} on V_1 . Suppose that $p_1 \in \pi_1^{-1}(p) \cap \text{Sing}_r(\mathcal{R}_1)$. Let D_{p_1} be an approximate hypersurface of \mathcal{I}_1 at p_1 . Then D_{p_1} is transversal to E_1^+ , and is not a component of E_1^+ .*

Proof. The proof is immediate from 3 of Lemma 5.3. \square

Theorem 7.2. *Let $\mathcal{R} = (\emptyset, E, \mathcal{I}, V)$. Then there exists a sequence of permissible transforms $\pi : V_1 \rightarrow V$ such that if \mathcal{R}_1 is the transform of \mathcal{R} on V_1 , then*

1. $\text{Sing}_r(\mathcal{R}_1) \cap E_1^- = \emptyset$, so that $\eta(p) = 0$ for $p \in \text{Sing}_r(\mathcal{R}_1)$.
2. $\text{Sing}_r(\mathcal{R}_1) \subset E_1^+$ and $\dim \text{Sing}_r(\mathcal{R}_1) \leq 1$.
3. All irreducible curves $C \subset \text{Sing}_r(\mathcal{R}_1)$ are nonsingular.
4. If $p \in \text{Sing}_r(\mathcal{R}_1)$, then there exists an approximate hypersurface D_p of \mathcal{I}_1 at p which is transversal to E_1^+ and is not a component of E_1^+ .

Proof. We first establish 1. We begin by blowing up all triple points of E (points p with $\eta(p) = 3$) in $\text{Sing}_r(\mathcal{R})$ to construct $V_1 \rightarrow V$ such that $\eta(p) \leq 2$ for $p \in \text{Sing}_r(\mathcal{R}_1)$. We then blow up all double curves of E (curves C such that $\eta(p) = 2$ for all $p \in C$) in $\text{Sing}_r(\mathcal{R}_1)$ to construct $V_2 \rightarrow V_1$ such that $\eta(p) \leq 2$ for $p \in \text{Sing}_r(\mathcal{R}_2)$, and $\eta(p) = 2$ at only finitely many points.

Suppose that $p \in \text{Sing}_r(\mathcal{R}_2)$ is an (isolated) point with $\eta(p) = 2$. Then after a finite number of blow ups of points $W_1 \rightarrow V_2$ centered at the point on the strict transform of the double curve of E_2 through p which dominates p , the transform \mathcal{R}'_1 of \mathcal{R} on W_1 satisfies $\eta(p_1) < 2$ at all points $p_1 \in \text{Sing}_r(\mathcal{R}'_1)$ which dominate p . To see this, let x, y, z be regular parameters in $\mathcal{O}_{V_2, p}$ such that $xy = 0$ is a local equation of E_2^- at p . Then $x = y = 0$ is a local equation of the double curve C through p , and after blowing up n times $\psi : W_n \rightarrow V$ along the strict transform of C , the local ring on W_n of the point p_n on the strict transform of C which dominates p has regular parameters $x = x_1 z^n, y = y_1 z^n, z$. Substituting into generators of $(\mathcal{I}_2)_p$, we see that $\nu_{p_n}(\mathcal{I}'_n) < r$ for $n \gg 0$, since $C \not\subset \text{Sing}_r(\mathcal{R}_2)$.

Repeating this for all $p \in \text{Sing}_r(\mathcal{R}_2)$ with $\eta(p) = 2$, we construct $V_3 \rightarrow V_2$ such that $\eta(p) \leq 1$ for $p \in \text{Sing}_r(\mathcal{R}_3)$.

If there is an irreducible component F of E_3^- contained in $\text{Sing}_r(\mathcal{R}_3)$, then we can blow up F by $V_4 \rightarrow V_3$ to eliminate F from $\text{Sing}_r(\mathcal{R}_4)$. In this way we reduce to the assumption that no irreducible components F of E_4^- are contained in $\text{Sing}_r(\mathcal{R}_4)$.

Now suppose that C is a curve in V_4 which is contained in $\text{Sing}_r(\mathcal{R}_4)$, and there is an irreducible component F of E_4^- containing C (so that $\eta(p) = 1$ for all $p \in C$). By blowing up points on the strict transform of C we can construct $W_1 \rightarrow V_4$ such that the strict transform C_1 of C on W_1 is nonsingular and intersects E_1^+ transversely (by Theorem 6.1). Now we construct the sequence of permissible transforms

$$(6) \quad \cdots \rightarrow W_n \rightarrow W_{n-1} \rightarrow \cdots \rightarrow W_2 \rightarrow W_1$$

where each morphism $W_{i+1} \rightarrow W_i$ is the blow up of the section C_i over C_1 on the strict transform F_i of F on W_i (F_i is isomorphic to F_1 by this morphism), and we continue as long as C_i is contained in $\text{Sing}_r(\mathcal{R}'_i)$ (where \mathcal{R}'_i is the transform of \mathcal{R} on W_i).

Let ζ be the generic point of C in V_4 , and let ζ_i be the generic points of C_i in W_i for all i . Base change of (6) by the inclusion of $\text{Spec}(\mathcal{O}_{V_4, \zeta}) \cong \text{Spec}(\mathcal{O}_{W_1, \zeta_1})$ into W_1 induces a sequence of birational projective morphisms of regular, 2 dimensional schemes. For all $i \geq 1$, $\mathcal{I}_{C_i} \otimes_{\mathcal{O}_{V_4}} \mathcal{O}_{V_4, \zeta}$ is the ideal sheaf of a closed point γ_i in

$W_i \times_{V_4} \text{Spec}(\mathcal{O}_{V_4, \zeta})$ which dominates γ_{i-1} , with $\gamma_0 = \text{Spec}(\mathcal{O}_{V_4, \zeta}/\mathcal{I}_{C, \zeta})$. We see that by taking base change of the sequence (6) by localization at ζ , we obtain a sequence of blow ups of closed points above $\text{Spec}(\mathcal{O}_{V_4, \zeta})$. Now by Remark 6.2 to Theorem 6.1, we have that $\gamma_i \notin \text{Sing}_r(W_i \times_{V_4} \text{Spec}(\mathcal{O}_{V_4, \zeta}))$ for all $i \gg 0$. Since $\mathcal{O}_{W_i, \zeta_i} \cong \mathcal{O}_{W_i \times_{V_4} \text{Spec}(\mathcal{O}_{V_4, \zeta}), \gamma_i}$, we see that $C_i \notin \text{Sing}_r(\mathcal{R}'_i)$ for all $i \gg 0$.

Repeating this for all curves C in V_4 which are contained in $\text{Sing}_r(\mathcal{R}_4)$, and such that there is an irreducible component F of E_4^- containing C , we construct $V_5 \rightarrow V_4$ such that $\eta(p) \leq 1$ for all $p \in \text{Sing}_r(\mathcal{R}_5)$, and there are at most finitely many points $p \in \text{Sing}_r(\mathcal{R}_5)$ such that $\eta(p) = 1$.

Suppose that $p \in \text{Sing}_r(\mathcal{R}_5)$ and $\eta(p) = 1$. Let $A = \widehat{\mathcal{O}_{V_5, p}}$, $I = \widehat{(\mathcal{I}_5)_p}$. Let F be the irreducible component of E_5^- containing p . Choose regular parameters x, y, z in $\mathcal{O}_{V_5, p}$ such that $z = 0$ is a local equation of F .

For $f \in I$, we have an expansion

$$f = \sum_{i=0}^{r-1} a_i(x, y) z^i + z^r \Lambda$$

in $A = \mathbf{k}[[x, y, z]]$. Define $C(I)$ to be the ideal in $B = \widehat{\mathcal{O}_{F, p}} = \mathbf{k}[[x, y]]$ generated by

$$\left\{ a_i(x, y) \frac{z^i}{z^{r-i}} \mid f \in I \right\}.$$

We easily verify that for an ideal $H \subset A$, $\nu_A(H) \geq r$ if and only if $\nu_B(C(H)) \geq r!$.

By Remark 6.2 to Theorem 6.1, applied to $\text{Spec}(B)$ and $J = C(I)$, there exists a sequence of permissible transforms

$$(7) \quad T_m \rightarrow T_{m-1} \rightarrow \cdots \rightarrow T_0 = \text{Spec}(B)$$

such that the weak transform J_m of J on T_m satisfies $\nu(J_m) < r!$. Since $\text{Sing}_{r!}(J)$ is the maximal ideal of B (as p is isolated in $F \cap \text{Sing}_r(\mathcal{I})$ by our reduction) every curve blown up in (7) is the strict transform of an exceptional divisor of some $T_{i+1} \rightarrow T_i$. Thus there exists a sequence of permissible transforms

$$(8) \quad W_m \rightarrow W_{m-1} \rightarrow \cdots \rightarrow W_0 = V_5$$

such that each $W_{i+1} \rightarrow W_i$ is either the blow up of a point on the strict transform of F or the blow up of a nonsingular curve on the strict transform of F , and (7) is obtained from (8) by taking the strict transforms of F by the successive morphisms, and taking base change by the inclusion $\text{Spec}(B) \rightarrow F$. Let \mathcal{R}'_i be the transform of \mathcal{R}_4 on W_i .

For p_1 in the strict transform of F by the map $W_1 \rightarrow V_5$ which is also in the fiber above p (where \mathcal{R}'_1 is the transform of \mathcal{R} on W_1), we verify that $C(\widehat{(\mathcal{I}'_1)_{p_1}})$ is the completion of the localization at p_1 of the weak transform \mathcal{J}_1 of J on T_1 , and $p_1 \in \text{Sing}_r(\mathcal{R}'_1)$ if and only if $p_1 \in \text{Sing}_{r!}(\mathcal{J}_1)$. By induction, we see that this property persists throughout the sequences (7) and (8), so that $\text{Sing}_r(\mathcal{R}'_m)$ is disjoint from the strict transform of F above p .

Iterating this construction for all (the finitely many) $p \in \text{Sing}_r(\mathcal{R}_5)$ with $\eta(p) = 1$, we construct $V_6 \rightarrow V_5$ such that the conclusion 1 of the theorem holds on V_6 .

We now establish 2.

Suppose that S is a 2 dimensional irreducible component of $\text{Sing}_r(\mathcal{R}_6)$. Then S is nonsingular and isolated in $\text{Sing}_r(\mathcal{R}_6)$. For $p \in S$, S is an approximate hypersurface D_p for \mathcal{I}_6 at p . E_6^+ is transversal to D_p , by Lemma 7.1. Thus S is transversal to $E_6 = E_6^+ + E_6^-$ at all $p \in S$. The blow up $W_1 \rightarrow V_6$ of S is thus permissible. We have $\text{Sing}_r(\mathcal{R}'_1) \cap \pi_2^{-1}(S) = \emptyset$ (where \mathcal{R}'_1 is the transform of \mathcal{R}_6 on W_1).

Repeating this construction for all 2 dimensional components of $\text{Sing}_r(\mathcal{R}_6)$, we construct $V_7 \rightarrow V_6$ such that $\dim \text{Sing}_r(\mathcal{R}_7) \leq 1$.

Suppose that C_1, \dots, C_n are the Zariski closures in V_7 of the irreducible curves in $\text{Sing}_r(\mathcal{R}_7) - E_7$. There exists (by Corollary 4.4 [C2]) a sequence of blow ups of points $\psi_1 : W_1 \rightarrow V_7$ with transform \mathcal{R}'_1 of \mathcal{R}_7 on W_1 , such that the strict transforms C'_1, \dots, C'_n of C_1, \dots, C_n on W_1 are nonsingular and disjoint. Suppose that $p \in W_1$ is such that E'_1 is not transversal to C'_1 (necessarily $p \notin (E'_1)^-$). There exist regular parameters x, y, z in $\mathcal{O}_{W_1, p}$ such that $x = y = 0$ are local equations of C'_1 at p . Let $f(x, y, z) = 0$ be a local equation of $E'_1 = (E'_1)^+ + (E'_1)^-$ at p in $\hat{\mathcal{O}}_{W_1, p}$. We necessarily have that $f \notin (x, y)\hat{\mathcal{O}}_{W_1, p}$. Let $\psi_2 : W_2 \rightarrow W_1$ be the blow up of p , and suppose that $p_1 \in \pi_2^{-1}(p)$ is on the strict transform of C'_1 . Then there exist regular parameters x_1, y_1, z_1 in \mathcal{O}_{W_2, p_1} such that

$$x = x_1 z_1, y = y_1 z_1, z = z_1.$$

Since $f \notin (x, y)$, we see by substituting

$$x = x_n z_n^n, y = y_n z_n^n, z = z_n$$

into $f(x, y, z)$, that after finitely many blow ups of points $W_{n+1} \rightarrow W_1$, the strict transform of C_1 is disjoint from the strict transform of E'_1 .

Since C'_1 is nonsingular, the strict transform C''_1 of C'_1 on W_{n+1} is transversal to the exceptional divisor of $W_{n+1} \rightarrow W_1$. Thus, C''_1 is transversal to E'_{n+1} . The blow up $W_{n+2} \rightarrow W_{n+1}$ of C''_1 is thus permissible.

We repeat this construction for C_2, \dots, C_n to construct $V_8 \rightarrow V_7$ such that $\text{Sing}_r(\mathcal{R}_8) - E_8$ is a finite set of points. Let $V_9 \rightarrow V_8$ be the blow up of these points. The conclusions 1 and 2 of the theorem hold on V_9 . The conclusions 3 and 4 of the theorem now hold by Lemma 5.2 and Lemma 7.1. \square

Now suppose that S is a reduced effective divisor on V such that S is nonsingular, and suppose that $\mathcal{R} = (E^+, E^-, \mathcal{I}_S, V)$ is a resolution datum on V , such that $E^+ + S$ is a SNC divisor. Let

$$\text{NSNC}(\mathcal{R}) = \{p \in S \mid E^+ + E^- + S \text{ is not a SNC divisor at } p\}.$$

$\text{NSNC}(\mathcal{R})$ is a closed subset of $\text{Sing}_1(\mathcal{R}) = S$.

We define a NC-permissible transform of \mathcal{R} to be the blow up $\pi_1 : V_1 \rightarrow V$ of a nonsingular subvariety Y of $\text{NSNC}(\mathcal{R})$, such that Y is transversal to $E = E^+ + E^-$. We define the transform \mathcal{R}_1 of \mathcal{R} with respect to a NC-permissible transform π_1 as for a permissible transform in Section 5.

Observe that if π_1 is a NC-permissible transform of \mathcal{R} , then $\mathcal{I}_1 = \mathcal{I}_{S_1}$, where S_1 is the strict transform of S by π_1 . We have that S_1 is nonsingular, $E_1^+ + E_1^-$ is a SNC divisor, and $E_1^+ + S_1$ is a SNC divisor.

Lemma 7.3. *Let $\mathcal{R} = (\emptyset, E, \mathcal{I}_S, V)$ be a resolution datum on V , where S is a reduced effective divisor on V such that S is nonsingular. Then there exists a sequence of NC-permissible transforms $\pi_1 : V_1 \rightarrow V$ such that if \mathcal{R}_1 is the transform of \mathcal{R} on V_1 , then $\text{NSNC}(\mathcal{R}_1) = \emptyset$.*

Proof. A simplification of the proof of 1 of Theorem 7.2, with Sing_r replaced with NSNC, shows that there exists a sequence of NC-permissible transforms $\pi : V_1 \rightarrow V$, such that $\eta(p) = 0$ for $p \in \text{NSNC}(\mathcal{R}_1)$. Since $E_1^+ + S_1$ is a SNC divisor, we thus have that $E_1 + S_1$ is a SNC divisor, so that $\text{NSNC}(\mathcal{R}_1) = \emptyset$. \square

8. REDUCTION TO LOCAL RESOLUTION

In this section we reduce the proof of Theorem 5.6 to a local problem, which is proven in Sections 9 and 10.

We may assume that the conclusions of Theorem 7.2 hold (but we now have $\mathcal{R} = (E^+, E^-, \mathcal{I}, V)$).

Suppose that $Y \subset \text{Sing}_r(\mathcal{R})$ is an irreducible component. By the conclusions of Theorem 7.2, Y is nonsingular of dimension ≤ 1 , $Y \cap E^- = \emptyset$, $Y \subset E^+$ and there exists an approximate hypersurface D_p to \mathcal{I} at p which is transversal to E^+ such that $Y \subset D_p$. Thus Y is transversal to E , and the blow up $\pi_1 : V_1 \rightarrow V$ of Y is a permissible transform of \mathcal{R} . Furthermore, it follows from Lemmas 5.2 and 7.1 that the conclusions of Theorem 7.2 hold for the transform \mathcal{R}_1 of \mathcal{R} on V_1 .

We may thus construct a sequence of permissible transforms

$$(9) \quad \cdots \rightarrow V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V$$

by applying the following algorithm:

1. If there exists an irreducible curve $C \subset \text{Sing}_r(\mathcal{R}_n)$ then perform the permissible transform $V_{n+1} \rightarrow V_n$ obtained by blowing up C .
2. If $\text{Sing}_r(\mathcal{R}_n)$ is a finite set of points, then blow up a point in $\text{Sing}_r(\mathcal{R}_n)$.

As remarked in the paragraph above (9), the above blow ups are permissible, and the conclusions of Theorem 7.2 hold for all transforms \mathcal{R}_n of \mathcal{R} along the sequence (9).

We must show that the sequence terminates after a finite number of steps with $\text{Sing}_r(\mathcal{R}_n) = \emptyset$.

We first observe that there exists a V_n in (9) such that $\text{Sing}_r(\mathcal{R}_n)$ is a finite set. Suppose otherwise. Then (9) is an infinite sequence, constructed by only performing Step 1 of the algorithm. By Lemma 5.2, there exists a curve C in $\text{Sing}_r(\mathcal{R})$ such that an infinite number of sections over C are blown up in (9). Let ζ be the generic point of C . As in the argument following (6) in the proof of Theorem 7.2, the sequence of blow ups of closed points over $\text{Spec}(\mathcal{O}_{V, \zeta})$ obtained by making the base change of (9) with the inclusion of $\text{Spec}(\mathcal{O}_{V, \zeta})$ into V , does not reduce the order of the weak transform of \mathcal{I}_n , a contradiction to Remark 6.2 to Theorem 6.1. We may thus assume that $\text{Sing}_r(\mathcal{R})$ is a finite set.

With the assumptions that $\text{Sing}_r(\mathcal{I})$ is a finite set of points, and that the conclusions of Theorem 7.2 hold, we now construct a sequence of permissible transforms

$$(10) \quad \cdots \rightarrow V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V$$

by applying the following algorithm:

1. If there is a curve in $\text{Sing}_r(\mathcal{I}_n)$, then blow up the union of one dimensional components of $\text{Sing}_r(\mathcal{I}_n)$.
2. Otherwise, blow up $\text{Sing}_r(\mathcal{I}_n)$, which is a finite union of points.

By Lemma 5.3, Lemma 5.2, 4 of Theorem 7.2, and since we started with $\text{Sing}_r(\mathcal{I})$ on V being a finite set of points, under this algorithm $\text{Sing}_r(\mathcal{I}_n)$ is always a disjoint union of points and nonsingular curves, which are transversal to E_n . The conclusions of Theorem 7.2 hold on all V_n .

If the sequence (10) is of finite length, then we have achieved the conclusions of Theorem 5.6.

Suppose that (10) has infinite length, and there does not exist a sequence of points $q_n \in \text{Sing}_r(\mathcal{I}_n)$ such that q_{n+1} maps to q_n for all n . We will derive a contradiction.

Let $K = \mathbf{k}(V)$ be the function field of V , and let $\text{ZR}(K)$ be the Zariski Riemann manifold of K (see Section 3). Let $\Phi_n : \text{ZR}(K) \rightarrow V_n$ be the natural projection. Let $U_n = V_n - \text{Sing}_r(\mathcal{I}_n)$, an open subset of V_n .

Suppose that $A \in \text{ZR}(K)$. Let q_n be the center of A on V_n . By assumption, $q_n \notin \text{Sing}_r(\mathcal{I}_n)$ for large n . Thus $A \in \Phi_n^{-1}(U_n)$. It follows that

$$\cup_{n=0}^{\infty} \Phi_n^{-1}(U_n) = \text{ZR}(K).$$

Since $\text{ZR}(K)$ is quasi compact ([Z5] or [ZS]), there exists an n such that $\Phi_n^{-1}(U_n) = \text{ZR}(K)$, so that $\text{Sing}_r(\mathcal{I}_n) = \emptyset$, and (10) has finite length, a contradiction.

Assume that (10) is not finite. We must then have an infinite sequence of points $q_n \in V_n$ such that $q_n \in \text{Sing}_r(\mathcal{I}_n)$, and q_n maps to q_{n-1} for all n .

By Lemma 5.1, $\tau(q_n) \geq \tau(q_{n-1})$ for all n . Thus, by induction on τ , we may assume that $\tau(q_n) = \tau(q)$ for all n .

We will show that this is not possible, by associating to each point q_n an element $\Omega(q_n)$ in an ordered set such that $\Omega(q_{n+1}) < \Omega(q_n)$ for all n . We will further show that Ω cannot decrease indefinitely. It will then follow that there exists an n such that $\text{Sing}_r(V_n) = \emptyset$, proving Theorem 5.6.

We have a sequence

$$(11) \quad R_0 \rightarrow R_1 \rightarrow \dots$$

of infinite length, where $R_i = \hat{\mathcal{O}}_{V_i, q_i}$ is the completion of the local ring of V_i at q_i , and $\nu_{q_i}(\mathcal{I}_i) = r$ for all i . Let $I_i = (\mathcal{I}_i)_{q_i} R_i$.

Since \mathcal{O}_{V_n, q_n} is excellent, we have $\text{Sing}_r(\mathcal{I}_n) \cap \text{Spec}(R_n) = \text{Sing}_r(I_n)$ for all n .

We may extend our definition of τ to an ideal in a power series ring. We have $\tau(q_n) = \tau(I_n)$ for all n .

We will consider separately the cases with different values of τ in the following Sections 9 and 10, to derive a contradiction, showing that (11) must have finite length in all cases.

The case where $\tau(q) = 3$ is immediate from Lemma 5.1.

9. REDUCTION WHEN $\tau(q) = 2$

The proof in this case is an extension of the case of an ideal sheaf on a nonsingular surface, given in Section 6. The proof is based on Hironaka's termination argument in [H2].

Suppose that T is a power series ring in 3 variables over an algebraically closed field \mathbf{k} , and $J \subset T$ is an ideal. Suppose that x, y, z are regular parameters in T , and r is a positive integer.

For $g \in T$, we have an expansion

$$(12) \quad g = \sum b_{ijk} x^i y^j z^k$$

with $b_{ijk} \in \mathbf{k}$. Define

$$\gamma(g; x, y, z) = \min\left\{ \frac{k}{r - (i + j)} \mid b_{ijk} \neq 0 \text{ and } i + j < r \right\} \in \frac{1}{r!} \mathbf{N} \cup \{\infty\}.$$

Define

$$\gamma(J; x, y, z) = \min\{\gamma(g; x, y, z) \mid g \in J\}.$$

We have $\gamma(J; x, y, z) = \infty$ if and only if $J \subset (x, y)^r$. Further,

1. $\nu_T(J) = r$ if and only if $\gamma(J; x, y, z) \geq 1$ and

2. $V(x, y)$ is contained in an approximate manifold of J if and only if

$$\gamma(J; x, y, z) > 1.$$

Let $\gamma = \gamma(J; x, y, z)$. For $g \in J$, define

$$(13) \quad [g]_{xyz} = \sum_{(i+j)\gamma+k=r\gamma} b_{ijk} x^i y^j z^k.$$

There is an expansion

$$(14) \quad g = [g]_{xyz} + \sum_{(i+j)\gamma+k>r\gamma} b_{ijk} x^i y^j z^k.$$

Define

$$A_\gamma = \left\{ (i, j) \mid \frac{k}{r - (i + j)} = \gamma, i + j < r, \text{ and } b_{ijk} \neq 0 \text{ for some } g \in J \right\}.$$

Then for $g \in J$,

$$(15) \quad [g]_{xyz} = L_g(x, y) + \sum_{(i,j) \in A_\gamma} b_{i,j,\gamma(r-i-j)} x^i y^j z^{\gamma(r-i-j)},$$

where $L_g(x, y) = \sum_{i+j=r} b_{ij0} x^i y^j$.

Regular parameters x, y, z in T will be called good parameters for J if $\nu_{\bar{T}}(J\bar{T}) = r$ and $\tau(J\bar{T}) = 2$, where $\bar{T} = T/zT$.

If $\nu_T(J) = r$ and $\tau(J) = 2$, then regular parameters x, y, z in T such that $V(x, y)$ is an approximate manifold of J are good parameters for J .

Definition 9.1. *Suppose that x, y, z are good parameters for J and $\nu_T(J) = r$. Then J is solvable with respect to x, y, z if $\gamma = \gamma(J; x, y, z) \in \mathbf{N}$ and there exist $\alpha, \beta \in \mathbf{k}$ such that*

$$[g]_{xyz} = L_g(x - \alpha z^\gamma, y - \beta z^\gamma).$$

for all $g \in J$ with $\nu_T(g) = r$.

Lemma 9.2. *Suppose that x, y, z are good parameters for J , $\nu_T(J) = r$ and $\text{Sing}_r(J)$ is the maximal ideal of T . Then there exists a change of variables*

$$x_1 = x - \sum_{i=1}^n \alpha_i z^i, y_1 = y - \sum_{i=1}^n \beta_i z^i$$

such that J is not solvable with respect to x_1, y_1, z .

Proof. If there doesn't exist such a change of variables then we construct series

$$x_\infty = x - \sum_{i=1}^{\infty} \alpha_i z^i, y_\infty = y - \sum_{i=1}^{\infty} \beta_i z^i$$

such that $\gamma(J; x_\infty, y_\infty, z) = \infty$, and thus

$$J \subset (x - \sum_{i=1}^{\infty} \alpha_i z^i, y - \sum_{i=1}^{\infty} \beta_i z^i)^r \subset T,$$

a contradiction to our assumption that $\text{Sing}_r(J)$ is the maximal ideal of T . \square

Observe that if $\nu_T(J) = r$, $\tau(J) = 2$, x, y, z are good parameters for J and J is not solvable for x, y, z , then $\gamma(J; x, y, z) > 1$.

Lemma 9.3. *Suppose that $\nu_T(J) = r$, $\tau(J) = 2$, $\text{Sing}_r(J)$ is the maximal ideal of T , x, y, z are good parameters for J and J is not solvable with respect to x, y, z .*

Suppose that T_1 is the completion of a local ring of the blow up of the maximal ideal of T such that the weak transform J_1 of J in T_1 satisfies $\nu_{T_1}(J_1) = r$ and $\tau(J_1) = 2$. Then

1. T_1 has regular parameters x_1, y_1, z_1 defined by

$$(16) \quad x = x_1 z_1, y = y_1 z_1, z = z_1.$$

2. x_1, y_1, z_1 are good parameters for J_1 and $V(x_1, y_1)$ is an approximate manifold of J_1 .
3. $\gamma(J_1; x_1, y_1, z_1) = \gamma(J; x, y, z) - 1$.
4. J_1 is not solvable with respect to x_1, y_1, z_1 .

Proof. Assertion 1 follows from Lemma 5.1, as $V(x, y)$ is an approximate manifold for J . Assertions 3 and 4 follow from substitution of (16) in (12), (13), (14) and (15).

Since we are assuming that $\tau(J_1) = 2$, we have that there exist $\alpha, \beta \in \mathbf{k}$ such that an approximate manifold of J_1 has the form

$$(17) \quad V(x_1 + \alpha z_1, y_1 + \beta z_1),$$

and x_1, y_1, z_1 are good parameters for J_1 .

Since we are assuming that $\nu_{T_1}(J_1) = r$, we have $\gamma(J_1; x_1, y_1, z_1) \geq 1$. If $\gamma(J_1; x_1, y_1, z_1) > 1$, then $V(x_1, y_1)$ is an approximate manifold of I_1 .

Suppose that $\gamma(J_1; x_1, y_1, z_1) = 1$. For $g \in J$ with $\nu_T(g) = r$ and $g_1 = \frac{g}{x_1^r}$, we have $\nu_{T_1}(g_1) = r$, and

$$[g_1]_{x_1 y_1 z_1} = \frac{1}{x_1^r} [g]_{xyz}.$$

Since $\gamma(J_1; x_1, y_1, z_1) = 1$, $[g_1]_{x_1 y_1 z_1}$ is the r -leading form of g_1 with respect to x_1, y_1, z_1 . By (17), there exists a form $\Psi_g(u, v)$, which depends on g , such that

$$[g_1]_{x_1 y_1 z_1} = \Psi_g(x_1 + \alpha z_1, y_1 + \beta z_1) = L_g(x_1, y_1) + z_1 \Omega$$

where $\Omega \in T_1$ and $L_g(x, y)$ is the r -leading form of g with respect to x, y, z . Setting $z_1 = 0$, we have $L_g(x_1, y_1) = \Psi_g(x_1, y_1)$, so that

$$[g_1]_{x_1 y_1 z_1} = L_g(x_1 + \alpha z_1, y_1 + \beta z_1).$$

We conclude that J_1 is solvable with respect to x_1, y_1, z_1 , a contradiction to the assumption that $\gamma(J_1; x_1, y_1, z_1) = 1$. \square

We now prove that (11) cannot have infinite length.

Since $\tau(q_i) = 2$ for all i , the assumption that (11) is infinite and Lemma 5.1 imply that for all i , $\text{Sing}_r(I_i)$ is the maximal ideal of R_i and R_{i+1} is the completion of a local ring of a closed point of the blow up of the maximal ideal of $\text{Spec}(R_i)$.

Moreover, $\tau(q) = 2$ implies that there exist regular parameters (x, y, z) in R_0 such that $V(y, z)$ is an approximate manifold of I_0 , and thus x, y, z are good parameters for I_0 .

By Lemma 9.2, there exists a change of variables in R_0 so that we find good parameters x, y, z for I_0 such that I_0 is not solvable with respect to x, y, z . Set $\Omega(q_0) = \gamma(I_0; x, y, z)$.

By Lemma 9.3, we can inductively define positive rational numbers $\Omega(q_n)$ such that $\Omega(q_{n+1}) = \Omega(q_n) - 1$ for all n , giving a contradiction if (11) has infinite length.

10. REDUCTION WHEN $\tau(\mathbf{q}) = 1$.

This is the most interesting and difficult case. The proof is based on Hironaka's termination argument in [H2].

10.1. Definition of Ω . In this subsection we define Ω for an ideal I in a power series ring T of three variables, over an algebraically closed field \mathbf{k} .

Suppose that x, y, z are regular parameters in T , and r is a positive integer.

For $g \in T$, we have an expansion

$$g = \sum b_{ijk} x^i y^j z^k$$

with $b_{ijk} = b_{ijk}(g) \in \mathbf{k}$. We define

$$\Delta = \Delta(I; x, y, z) = \left\{ \left(\frac{i}{r-k}, \frac{j}{r-k} \right) \in \mathbf{Q}^2 \mid k < r \text{ and } b_{ijk} \neq 0 \text{ for some } g \in I \right\}.$$

Let $|\Delta(I; x, y, z)|$ be the smallest convex set in \mathbf{R}^2 such that $\Delta \subset |\Delta|$, and $(a, b) \in |\Delta|$ implies $(a+c, b+d) \in |\Delta|$ for all $c, d \geq 0$. For $\gamma \in \mathbf{R}$, let $S(\gamma)$ be the line through $(\gamma, 0)$ with slope -1.

Suppose that $|\Delta| \neq \emptyset$. We define $\alpha_{xyz}(I)$ to be the smallest a appearing in any $(a, b) \in |\Delta|$, $\beta_{xyz}(I)$ to be the smallest b such that $(\alpha_{xyz}(I), b) \in |\Delta|$. Let $\gamma_{xyz}(I)$ be the smallest number γ such that $S(\gamma) \cap |\Delta| \neq \emptyset$ and let $\delta_{xyz}(I)$ be such that $(\gamma_{xyz}(I) - \delta_{xyz}(I), \delta_{xyz}(I))$ is the lowest point on $S(\gamma_{xyz}(I)) \cap |\Delta|$. $(\alpha_{xyz}(I), \beta_{xyz}(I))$ and $(\gamma_{xyz}(I) - \delta_{xyz}(I), \delta_{xyz}(I))$ are vertices of $|\Delta|$. Define $\epsilon_{xyz}(I)$ to be the absolute value of the largest slope of a line through $(\alpha_{xyz}(I), \beta_{xyz}(I))$ such that no points of $|\Delta|$ lie below it.

We now define

$$\Omega(I; x, y, z) = (\beta_{xyz}(I), \frac{1}{\epsilon_{xyz}(I)}, \alpha_{xyz}(I)),$$

which is in the ordered set (by the Lex order)

$$\left(\frac{1}{r!} \mathbf{N} \right) \times (\mathbf{Q} \cup \infty) \times \left(\frac{1}{r!} \mathbf{N} \right)$$

We observe that:

1. The vertices of $|\Delta|$ are points of Δ , which lie in the lattice $\frac{1}{r!} \mathbf{Z} \times \frac{1}{r!} \mathbf{Z}$.
2. $\nu_R(I) < r$ holds if and only if $|\Delta|$ contains a point on $S(c)$ with $c < 1$ which holds if and only if there is a vertex (a, b) with $a + b < 1$.
3. $\alpha_{xyz}(I) < 1$ if and only if $I \not\subset (x, z)^r$.
4. A vertex of $|\Delta|$ lies below the line $b = 1$ if and only if $I \not\subset (y, z)^r$.

10.2. Well preparation. (x, y, z) will be called good parameters for I if $\nu_T(I) = r$ and $b_{00r} \neq 0$ for all $g \in I$ such that $\nu_T(g) = r$. Good parameters exist if $\tau(I) = 1$.

Suppose that $\nu_T(I) = r$ and (x, y, z) are good parameters for I . Let $\Delta = \Delta(I; x, y, z)$ and suppose that (a, b) is a vertex of $|\Delta|$. Define

$$S_{(a,b)} = \left\{ k \mid k < r, \text{ and } \left(\frac{i}{r-k}, \frac{j}{r-k} \right) = (a, b) \right\}.$$

For $g \in I$, define

$$\{g\}_{xyz}^{ab} = b_{00r} z^r + \sum_{k \in S_{(a,b)}} b_{a(r-k), b(r-k), k} x^{a(r-k)} y^{b(r-k)} z^k.$$

We will say that the vertex (a, b) is not prepared on $|\Delta|$ if a, b are integers and there exists $\eta \in \mathbf{k}$ such that

$$\{g\}_{xyz}^{ab} = b_{00r}(g)(z - \eta x^a y^b)^r$$

for every $g \in I$ with $\nu_T(g) = r$. We will say that the vertex (a, b) is prepared on $|\Delta|$ if such an η does not exist.

If the vertex (a, b) is not prepared, then we can make an (a, b) preparation, which is the change of parameters

$$z_1 = z - \eta x^a y^b.$$

If all vertices (a, b) of $|\Delta|$ are prepared, then we say that $(I; x, y, z)$ is well prepared.

Lemma 10.1. *Suppose that $\nu_T(I) = r$, $\tau(I) = 1$, and (x, y, z) are good parameters of I . Then $z = 0$ is an approximate manifold of I if and only if the vertices $(0, 1)$ and $(1, 0)$ are prepared on $|\Delta(I; x, y, z)|$.*

Proof. Suppose that $(0, 1)$ and $(1, 0)$ are prepared on $|\Delta(I; x, y, z)|$. Since $\tau(I) = 1$, there is a linear form $ax + by + cz$ with $a, b, c \in \mathbf{k}$ such that $V(ax + by + cz)$ is an approximate manifold of I . For each $g \in I$ with $\nu_T(g) = r$, there exists $0 \neq d_g \in \mathbf{k}$ such that the r -leading form of g is $d_g(ax + by + cz)^r$.

As (x, y, z) are good parameters for I , $b_{00r}(g) \neq 0$, so we must have $c \neq 0$. If $a \neq 0$, then $(1, 0)$ is a vertex of $|\Delta(I; x, y, z)|$, which is not prepared since for $g \in I$ with $\nu_T(g) = r$,

$$\{g\}_{xyz}^{(1,0)} = d_g c^r \left(z + \frac{a}{c}x\right)^r.$$

Thus $a = 0$. If $b \neq 0$, then $(0, 1)$ is a vertex of $|\Delta(I; x, y, z)|$ which is not prepared since for $g \in I$ with $\nu_T(g) = r$,

$$\{g\}_{xyz}^{(0,1)} = d_g c^r \left(z + \frac{b}{c}y\right)^r.$$

Thus $a = b = 0$ and $z = 0$ is an approximate manifold of I .

The proof of the converse also follows from the above arguments. \square

Lemma 10.2. *Consider the terms in the expansion*

$$(18) \quad h = \sum_{\lambda=0}^k \eta^{k-\lambda} \binom{k}{\lambda} x^{i+(k-\lambda)a} y^{j+(k-\lambda)b} z_1^\lambda$$

obtained by substituting $z_1 = z - \eta x^a y^b$ into the monomial $x^i y^j z^k$. Define a projection for $(a, b, c) \in \mathbf{N}^3$ such that $c < r$,

$$\pi(a, b, c) = \left(\frac{a}{r-c}, \frac{b}{r-c}\right).$$

1. Suppose that $k < r$. Then the exponents of monomials in (18) with nonzero coefficients project into the line segment joining (a, b) to $(\frac{i}{r-k}, \frac{j}{r-k})$.
 - a. If $(a, b) = (\frac{i}{r-k}, \frac{j}{r-k})$, then all these monomials project to (a, b) .
 - b. If $(a, b) \neq (\frac{i}{r-k}, \frac{j}{r-k})$, then $x^i y^j z_1^k$ is the unique monomial in (18) which projects onto $(\frac{i}{r-k}, \frac{j}{r-k})$. No monomial in (18) projects to (a, b) .
2. Suppose that $r \leq k$, and $(i, j, k) \neq (0, 0, r)$. Then all exponents in (18) with nonzero coefficients and z_1 exponent less than r project into

$$((a, b) + \mathbf{Q}_{\geq 0}^2) - \{(a, b)\}.$$

3. Suppose that $(i, j, k) = (0, 0, r)$. Then all exponents in (18) with nonzero coefficients and z_1 exponent less than r project to (a, b) .

Lemma 10.2 is proved by a straight forward calculation. We deduce Lemma 10.3 from Lemma 10.2.

Lemma 10.3. *Suppose that the vertex (a, b) is not prepared on $|\Delta|$, and $z_1 = z - \eta x^a y^b$ is an (a, b) preparation. Then*

1. $|\Delta(I; x, y, z_1)| \subset |\Delta(I; x, y, z)| - \{(a, b)\}$.
2. *If (a', b') is another vertex of $|\Delta(I; x, y, z)|$, then (a', b') is a vertex of $|\Delta(I; x, y, z_1)|$, and for $g \in I$ with $\nu_T(g) = r$, $\{g\}_{x, y, z_1}^{a', b'}$ is obtained from $\{g\}_{x, y, z}^{a', b'}$ by substituting z_1 for z .*

Lemma 10.4. *Suppose that $\text{Sing}_r(I)$ has dimension < 2 , $\nu_T(I) = r$, $\tau(I) = 1$ and (x, y, z) are good parameters for I . Then there is a formal series $\Psi(x, y) \in \mathbf{k}[[x, y]]$ such that under the substitution $z = z_1 - \Psi(x, y)$, (x, y, z_1) are good parameters for I and $(I; x, y, z_1)$ is well prepared.*

Lemma 10.4 is proved by successive removal of vertices which are not prepared, using Lemma 10.3.

Remark 10.5. *It is not always possible to well prepare after a finite number of preparations, so that z_1 could be a formal series in z, x, y . An example is given in Example 7.19 [C2].*

10.3. Very Well Preparation. We will also consider change of variables of the form

$$y_1 = y - \eta x^n$$

for $\eta \in \mathbf{k}$, n a positive integer, which we will call translations.

Lemma 10.6. *Consider the expansion*

$$(19) \quad h = \sum_{\lambda=0}^j \eta^{j-\lambda} \binom{j}{\lambda} x^{i+(j-\lambda)n} y_1^\lambda z^k$$

obtained by substituting $y_1 = y - \eta x^n$ into the monomial $x^i y^j z^k$. Consider the projection for $(a, b, c) \in \mathbf{N}^3$ such that $c < r$ defined by

$$\pi(a, b, c) = \left(\frac{a}{r-c}, \frac{b}{r-c} \right).$$

Suppose that $k < r$. Set $(a, b) = \left(\frac{i}{r-k}, \frac{j}{r-k} \right)$. Then $x^i y_1^j z^k$ is the unique monomial in (19) whose coefficients project onto (a, b) . All other monomials in (19) with non-zero coefficient project to points below (a, b) on the line through (a, b) with slope $-\frac{1}{n}$.

Lemma 10.6 is proved by a straight forward calculation. We deduce Lemma 10.7 from Lemma 10.6 and the definition of well preparedness.

Lemma 10.7. *Suppose that $(I; x, y, z)$ is well prepared, $y_1 = y - \eta x^n$ is a translation, and $z_1 = z - \psi(x, y_1)$ is a subsequent well preparation. Then*

$$\alpha_{x, y_1, z_1}(I) = \alpha_{xyz}(I), \beta_{x, y_1, z_1}(I) = \beta_{xyz}(I) \text{ and } \gamma_{x, y_1, z_1}(I) = \gamma_{xyz}(I).$$

Definition 10.8. *Suppose that $I \subset R$ is an ideal such that $\nu_T(I) = r$, $\tau(I) = 1$, $\text{Sing}_r(I)$ has dimension < 2 and (x, y, z) are good parameters for I . Let $\alpha = \alpha_{xyz}(I)$, $\beta = \beta_{xyz}(I)$, $\gamma = \gamma_{xyz}(I)$, $\delta = \delta_{xyz}(I)$, $\epsilon = \epsilon_{xyz}(I)$. $(I; x, y, z)$ will be said to be very well prepared if it is well prepared and one of the following conditions holds.*

1. $(\gamma - \delta, \delta) \neq (\alpha, \beta)$ and if we make a translation $y_1 = y - \eta x$, with subsequent well preparation $z_1 = z - \Psi(x, y_1)$, then

$$\alpha_{xy_1 z_1}(I) = \alpha, \beta_{xy_1 z_1}(I) = \beta, \gamma_{xy_1 z_1}(I) = \gamma$$

and

$$\delta_{xy_1z_1}(I) \leq \delta.$$

2. $(\gamma - \delta, \delta) = (\alpha, \beta)$ and one of the following cases hold:
- $\epsilon = 0$
 - $\epsilon \neq 0$ and $\frac{1}{\epsilon}$ is not an integer
 - $\epsilon \neq 0$ and $n = \frac{1}{\epsilon}$ is a (positive) integer and for any $\eta \in \mathbf{k}$, if $y_1 = y - \eta x^n$ is a translation, with subsequent well preparation $z_1 = z - \Psi(x, y_1)$, then $\epsilon_{xy_1z_1}(I) = \epsilon$. Further, if (c, d) is the lowest point on the line through (α, β) with slope $-\epsilon$ in $|\Delta(I; x, y, z)|$ and (c_1, d_1) is the lowest point on this line in $|\Delta(I; x, y_1, z_1)|$, then $d_1 \leq d$.

Lemma 10.9. *Suppose that $I \subset T$ is an ideal such that $\nu_T(I) = r$, $\tau(I) = 1$, $\text{Sing}_r(I)$ has dimension < 2 and (x, y, z) are good parameters for I . Then there are formal substitutions*

$$z_1 = z - \Psi(x, y), y_1 = y - \phi(x)$$

where $\Psi(x, y), \phi(x)$ are series such that $(I; x, y_1, z_1)$ is very well prepared.

Proof. By Lemmas 10.4 and 10.7, we can find good parameters x, y, z for I such that $(I; x, y, z)$ is well prepared, α, β, γ do not change under translation $y_1 = y - \eta x$ followed by subsequent well preparation, and δ is maximal. If $(\gamma - \delta, \delta) \neq (\alpha, \beta)$, then we have achieved case 1 of Definition 10.8, so that $(I; x, y, z)$ is very well prepared.

We now assume that $(I; x, y, z)$ is well prepared, and $(\alpha, \beta) = (\gamma - \delta, \delta)$. If $\epsilon = 0$ or $\frac{1}{\epsilon}$ is not an integer then $(I; x, y, z)$ is very well prepared.

Suppose that $n = \frac{1}{\epsilon}$ is an integer. We then choose $\eta \in \mathbf{k}$ such that with the translation $y_1 = y - \eta x^n$, and subsequent well preparation, we maximize d for points (c, d) of the line through (α, β) with slope $-\epsilon$ on the boundary of $|\Delta(I; x, y_1, z_1)|$. By Lemma 10.7, α, β and γ are not changed. If we now have that $d \neq \beta$, we are very well prepared.

If the process does not end after a finite number of iterations, then we construct formal series $y' = y - \phi(x)$ and $z' = z - \psi(x, y)$ such that $|\Delta(I; x, y', z')|$ has the single vertex (α, β) , and $(I; x, y', z')$ is thus very well prepared. \square

10.4. Effect of a permissible blow up on Δ .

Definition 10.10. *Suppose that $I \subset T$ is an ideal such that $\nu_T(I) = r$, $\tau(I) = 1$, $\text{Sing}_r(I)$ has dimension < 2 , and (x, y, z) are good parameters for I .*

We consider 4 types of transformations $T \rightarrow T_1$, where T_1 is the completion of the local ring of a blow up of T , and T_1 has regular parameters (x_1, y_1, z_1) related to the regular parameters (x, y, z) of T by one of the following rules. In all cases, if $\nu_{T_1}(I_1) = r$ and $\tau(I_1) = 1$, then (x_1, y_1, z_1) are good parameters for I_1 .

Tr1 $\text{Sing}_r(I) = V(x, y, z)$,

$$x = x_1, y = x_1(y_1 + \eta), z = x_1z_1,$$

with $\eta \in \mathbf{k}$. Then $I_1 = \frac{1}{x_1}IT_1$ is the weak transform of I in T_1 .

Tr2 $\text{Sing}_r(I) = V(x, y, z)$,

$$x = x_1y_1, y = y_1, z = y_1z_1.$$

Then $I_1 = \frac{1}{y_1}IT_1$ is the weak transform of I in T_1 .

Tr3 $\text{Sing}_r(I) = V(x, z)$,

$$x = x_1, y = y_1, z = x_1z_1.$$

Then $I_1 = \frac{1}{x_1}IT_1$ is the weak transform of I in T_1 .

$$\text{Tr4 } \text{Sing}_r(I) = V(y, z),$$

$$x = x_1, y = y_1, z = y_1 z_1.$$

Then $I_1 = \frac{1}{y_1} IT_1$ is the weak transform of I in T_1 .

Lemma 10.11. *Suppose that assumptions are as in Definition 10.10. Further suppose that (x, y, z) and (x_1, y_1, z_1) are related by a transformation of one of the above types Tr1 - Tr4, and $\nu_{T_1}(I_1) = r$, $\tau(I_1) = 1$. Then there is a 1-1 correspondence*

$$\sigma : \Delta(I, x, y, z) \rightarrow \Delta(I_1, x_1, y_1, z_1)$$

defined by

1. $\sigma(a, b) = (a + b - 1, b)$ if the transformation is a Tr1 with $\eta = 0$.
2. $\sigma(a, b) = (a, a + b - 1)$ if the transformation is a Tr2.
3. $\sigma(a, b) = (a - 1, b)$ if the transformation is a Tr3.
4. $\sigma(a, b) = (a, b - 1)$ if the transformation is a Tr4.

The proof of Lemma 10.11 is a straight forward calculation. We deduce Lemma 10.12 from Lemma 10.11.

Lemma 10.12. *In each of the four cases of the preceding lemma, if $\sigma(a, b) = (a_1, b_1)$ is a vertex of $|\Delta(I_1; x_1, y_1, z_1)|$, then (a, b) is a vertex of $|\Delta(I; x, y, z)|$, and if $(I; x, y, z)$ is (a, b) prepared then $(I_1; x_1, y_1, z_1)$ is (a_1, b_1) prepared. In particular, $(I_1; x_1, y_1, z_1)$ is well prepared if $(I; x, y, z)$ is well prepared.*

Lemma 10.13. *Suppose that assumptions are as in Definition 10.10 and (x, y, z) and (x_1, y_1, z_1) are related by a Tr3 transformation. Suppose that $\nu_{T_1}(I_1) = r$ and $\tau(I_1) = 1$. If $(I; x, y, z)$ is very well prepared, then $(I_1; x_1, y_1, z_1)$ is very well prepared,*

$$\beta_{x_1 y_1 z_1}(I_1) = \beta_{xyz}(I), \delta_{x_1 y_1 z_1}(I_1) = \delta_{xyz}(I), \epsilon_{x_1 y_1 z_1}(I_1) = \epsilon_{xyz}(I)$$

and

$$\alpha_{x_1 y_1 z_1}(I_1) = \alpha_{xyz}(I) - 1, \gamma_{x_1 y_1 z_1}(I_1) = \gamma_{xyz}(I) - 1.$$

We further have $\text{Sing}_r(I_1) \subset V(x_1, z_1)$.

Proof. Well preparation is preserved by Lemma 10.12. We deduce from the definition of Tr3 that $(I_1; x_1, y_1, z_1)$ is also very well prepared.

Since $\text{Sing}_r(I) = V(x, z)$, we have $\text{Sing}_r(I_1) \subset V(x_1)$, the exceptional divisor of our Tr3 transformation. $\text{Sing}_r(I_1)$ is either the maximal ideal, or the germ of a nonsingular irreducible curve, by Lemma 5.2.

Suppose that $\text{Sing}_r(I_1)$ is a nonsingular curve C . Then C has formal local equations

$$x_1 = z_1 - \sum_{t=1}^{\infty} a_t y_1^t = 0$$

for some $a_t \in \mathbf{k}$ by Lemma 5.1. Suppose that $g \in I$ with $\nu_T(g) = r$. Expand

$$g = \left(\sum_{i+k=r} \sum_{j=0}^{\infty} b_{ijk} x^i y^j z^k \right) + h$$

with $h \in (x, z)^{r+1}$.

$$\frac{g}{x_1^r} = \left(\sum_{i+k=r} \sum_{j=0}^{\infty} b_{ijk} y_1^j z_1^k \right) + x_1 \frac{h}{x_1^{r+1}}$$

with $\frac{h}{x_1^{r+1}} \in T_1$.

$\nu_T(\frac{g}{x_1^r}) = r$ and $\frac{g}{x_1^r} \in (x_1, z_1 - \sum_{t=1}^{\infty} a_t y_1^t)^r$ implies

$$\sum_{i+k=r} \sum_{j=0}^{\infty} b_{ijk} y_1^j z_1^k = b_{00r} (z_1 - \sum_{t=1}^{\infty} a_t y_1^t)^r.$$

Thus

$$g = b_{00r} (z - \sum_{t=1}^{\infty} a_t x y^t)^r + h.$$

Suppose that $a_t \neq 0$ for some t . Let t_0 be the smallest value of t for which $a_t \neq 0$. Then $(\alpha, \beta) = (1, t_0)$, since $I \subset (x, z)^r$. But $(1, t_0)$ is then a vertex of $|\Delta(I; x, y, z)|$ which is not prepared, a contradiction. Thus $a_t = 0$ for all t and $C = V(x, z_1)$. We conclude that $\text{Sing}_r(I_1) \subset V(x_1, z_1)$

The remaining statements of the lemma follow from Lemma 10.11. \square

Lemma 10.14. *Suppose that assumptions are as in Definition 10.10, (x, y, z) and (x_1, y_1, z_1) are related by a Tr1 transformation with $\eta = 0$, $(I; x, y, z)$ is very well prepared and $\nu_{T_1}(I_1) = r$, $\tau(I_1) = 1$. Let $(I_1; x_1, y', z')$ be a very well preparation of $(I_1; x_1, y_1, z_1)$. Let*

$$\alpha = \alpha_{xyz}(I), \beta = \beta_{xyz}(I), \gamma = \gamma_{xyz}(I), \delta = \delta_{xyz}(I), \epsilon = \epsilon_{xyz}(I),$$

$$\alpha_1 = \alpha_{x_1 y' z'}(I_1), \beta_1 = \beta_{x_1 y' z'}(I_1), \gamma_1 = \gamma_{x_1 y' z'}(I_1), \delta_1 = \delta_{x_1 y' z'}(I_1), \epsilon_1 = \epsilon_{x_1 y' z'}(I_1).$$

Then

$$(\beta_1, \frac{1}{\epsilon_1}, \alpha_1) < (\beta, \frac{1}{\epsilon}, \alpha)$$

in the lexicographical order. If $\beta_1 = \beta$, and $\epsilon \neq 0$, then $\frac{1}{\epsilon_1} = \frac{1}{\epsilon} - 1$.

We further have $\text{Sing}_r(I_1) \subset V(x_1, z')$.

Proof. $(I_1; x_1, y_1, z_1)$ is well prepared by Lemma 10.12.

Let $\sigma : \Delta(I; x, y, z) \rightarrow \Delta(I_1; x_1, y_1, z_1)$ be the 1-1 correspondence of 1 in Lemma 10.11, which is defined by $\sigma(a, b) = (a + b - 1, b)$. σ transforms lines of slope $m \neq -1$ to lines of slope $\frac{m}{m+1}$, and transforms lines of slope -1 to vertical lines.

We deduce the formulas for transformation of $\alpha, \beta, \gamma, \delta, \epsilon$ from consideration of the effect of a Tr1 transformation followed by very well preparation on $(I; x, y, z)$ in the different cases of Definition 10.8. We need the assumption $\text{Sing}_r(I) = V(x, y, z)$ so that $\beta < 1$, to conclude that $\alpha_1 < \alpha$ if $\epsilon = 0$.

By Lemmas 5.1 and 10.1, $\text{Sing}_r(I_1) \subset V(x_1, z_1)$. If $\text{Sing}_r(I_1) = V(x_1, z_1)$, then all vertices of $|\Delta(I_1; x_1, y_1, z_1)|$ have the first coordinate ≥ 1 , and thus $\text{Sing}_r(I_1) = V(x_1, z')$. \square

Lemma 10.15. *Suppose that assumptions are as in Definition 10.10, (x, y, z) and (x_1, y_1, z_1) are related by a Tr2 or a Tr4 transformation and $(I; x, y, z)$ is well prepared. Suppose that $\nu_{T_1}(I_1) = r$ and $\tau(I_1) = 1$. We have that $(I_1; x_1, y_1, z_1)$ is well prepared and $\text{Sing}_r(I_1) \subset V(y_1, z_1)$. Let $(I_1; x_1, y', z')$ be a very well preparation. Then*

$$\beta_{x_1, y_1, z_1}(I_1) = \beta_{x_1, y', z'}(I_1) < \beta_{xyz}(I).$$

Proof. First suppose that $T \rightarrow T_1$ is a Tr2 transformation. The correspondence $\sigma : \Delta(I; x, y, z) \rightarrow \Delta(I_1; x_1, y_1, z_1)$ of 2 in Lemma 10.11 is defined by $\sigma(a, b) = (a, a+b-1)$. σ takes lines of slope m to lines of slope $m+1$. Thus

$$(\alpha_{x_1 y_1 z_1}(I_1), \beta_{x_1 y_1 z_1}(I_1)) = (\alpha_{xyz}(I), \alpha_{xyz}(I) + \beta_{xyz}(I) - 1).$$

Since $\text{Sing}_r(I) = V(x, y, z)$ by assumption, we have $\alpha_{xyz}(I) < 1$, and thus

$$\beta_{x_1 y_1 z_1}(I_1) < \beta_{xyz}(I).$$

$(I_1; x_1, y_1, z_1)$ is well prepared by Lemma 10.12. Thus the vertex $(\alpha_{x_1 y_1 z_1}(I_1), \beta_{x_1 y_1 z_1}(I_1))$ is not affected by very well preparation. We thus have $\beta_{x_1 y'_1 z'_1}(I_1) < \beta_{xyz}(I)$.

Now suppose that $T \rightarrow T_1$ is a Tr4 transformation. The 1-1 correspondence $\sigma : \Delta(I; x, y, z) \rightarrow \Delta(I_1; x_1, y_1, z_1)$ of 4 in Lemma 10.11 is defined by $\sigma(a, b) = (a, b - 1)$. Thus $\beta_{x_1 y_1 z_1}(I_1) < \beta_{xyz}(I)$. $(I_1; x_1, y_1, z_1)$ is well prepared by Lemma 10.12, and the vertex $(\alpha_{x_1 y_1 z_1}(I_1), \beta_{x_1 y_1 z_1}(I_1))$ is not affected by very well preparation. Thus $\beta_{x_1 y'_1 z'_1}(I_1) = \beta_{x_1 y_1 z_1}(I_1) < \beta_{xyz}(I)$.

The proof that $\text{Sing}_r(I_1) \subset V(y_1, z_1)$ is as in the proofs of Lemma 10.13 and Lemma 10.14. \square

Lemma 10.16. *Suppose that p is a prime, s is a nonnegative integer and r_0 is a positive integer such that $p \nmid r_0$. Let $r = r_0 p^s$. Then*

1. $\binom{r}{\lambda} \equiv 0 \pmod{p}$, if $p^s \nmid \lambda$, $0 \leq \lambda \leq r$ is an integer.
2. $\binom{r}{\lambda p^s} \equiv \binom{r_0}{\lambda} \pmod{p}$, if $0 \leq \lambda \leq r_0$ is an integer.

Proof. Compare the expansions over \mathbf{Z}_p of $(x + y)^r = (x^{p^s} + y^{p^s})^{r_0}$. \square

The following theorem is the most delicate part of the proof, where we really see the difference between characteristic 0 and characteristic p .

Theorem 10.17. *Suppose that assumptions are as in Definition 10.10, (x, y, z) and (x_1, y_1, z_1) are related by a Tr1 transformation with $\eta \neq 0$ and $(I; x, y, z)$ is very well prepared. Suppose that $\nu_{T_1}(I_1) = r$, $\tau(I_1) = 1$ and $\beta_{xyz}(I) > 0$. Then there exist good parameters (x_1, y'_1, z'_1) in T_1 such that $(I_1; x_1, y'_1, z'_1)$ is very well prepared, with $\beta_{x_1, y'_1, z'_1}(I_1) < \beta_{x, y, z}(I)$ and $\text{Sing}_r(I_1) \subset V(x_1, z'_1)$.*

Proof. Let $\alpha = \alpha_{xyz}(I)$, $\beta = \beta_{xyz}(I)$, $\gamma = \gamma_{xyz}(I)$, $\delta = \delta_{xyz}(I)$. Lemma 10.1 implies $z = 0$ is an approximate manifold of I . Thus since $\tau(I) = 1$,

$$(20) \quad \alpha + \beta > 1.$$

Apply the translation $y' = y - \eta x$ and well prepare by some substitution $z' = z - \Psi(x, y')$. This does not change α , β or γ . Set $\delta' = \delta_{x, y', z'}(I)$.

First assume that $\delta' < \beta$. We have regular parameters (x_1, y_1, \bar{z}_1) in T_1 such that $x = x_1, y' = x_1 y_1, z' = x_1 \bar{z}_1$, so we have, by 1 in Lemma 10.11, a 1-1 correspondence

$$\sigma : \Delta(I; x, y', z') \rightarrow \Delta(I_1; x_1, y_1, \bar{z}_1)$$

defined by $\sigma(a, b) = (a + b - 1, b)$. $(I_1; x_1, y_1, \bar{z}_1)$ is well prepared by Lemma 10.12. Since $\delta' < \beta$, the line segment in $|\Delta(I; x, y', z')|$ through (α, β) and $(\gamma - \delta', \delta')$ has slope ≤ -1 , so it is transformed by σ to a segment with positive slope or a vertical line in $|\Delta(I_1; x_1, y_1, \bar{z}_1)|$. We have that for $\alpha_1 = \alpha_{x_1 y_1 \bar{z}_1}(I_1)$ and $\beta_1 = \beta_{x_1 y_1 \bar{z}_1}(I_1)$, $(\alpha_1, \beta_1) = \sigma(\alpha, \beta)$ and $\beta_1 = \delta' < \beta$. Since $(I_1; x_1, y_1, \bar{z}_1)$ is well prepared, very well preparation does not effect the vertex (α_1, β_1) , by Lemma 10.7. After very well preparation, we thus have regular parameters x_1, y'_1, z'_1 in T_1 such that $(I_1; x_1, y'_1, z'_1)$ is very well prepared and $\beta_{x_1, y'_1, z'_1}(I_1) < \beta_{xyz}(I)$. $\text{Sing}_r(I_1) \subset V(x_1, z'_1)$ as in the proof of Lemma 10.14.

We have thus reduced the proof to showing that $\delta' < \beta$.

If $(\alpha, \beta) \neq (\gamma - \delta, \delta)$, we have $\delta' \leq \delta < \beta$ by Lemma 10.7, since $(I; x, y, z)$ is very well prepared.

Suppose that $(\alpha, \beta) = (\gamma - \delta, \delta)$. Set

$$W = \left\{ (i, j, k) \in \mathbf{N}^3 \mid k < r \text{ and } \left(\frac{i}{r-k}, \frac{j}{r-k} \right) = (\alpha, \beta) \right\}.$$

For $g = \sum a_{ijk}x^i y^j z^k \in I$, with $\nu_T(g) = r$, set

$$F_g = \sum_{(i,j,k) \in W} a_{ijk}x^i y^j z^k.$$

By assumption, (α, β) is prepared on $|\Delta(I; x, y, z)|$, which implies that if α, β are integers, then there does not exist $\lambda \in \mathbf{k}$ such that

$$a_{00r}z^r + F_g = a_{00r}(z - \lambda x^\alpha y^\beta)^r$$

for all $g \in I$ with $\nu_T(g) = r$. Moreover,

$$(21) \quad F_g(x, y, z) = F_g(x, y' + \eta x, z) = \sum_{(i,j,k) \in W} \sum_{\lambda=0}^j a_{ijk} \eta^\lambda \binom{j}{\lambda} x^{i+\lambda} (y')^{j-\lambda} z^k.$$

By Lemma 10.6, the terms in the expansion of $g(x, y' + \eta x, z)$ contributing to $(\gamma, 0)$ in $|\Delta(I; x, y', z)|$, where $\gamma = \alpha + \beta$, are

$$F_{g,(\gamma,0)} = \sum_{(i,j,k) \in W} a_{ijk} \eta^j x^{i+j} z^k.$$

If $(I; x, y', z)$ is $(\gamma, 0)$ prepared, then $\delta' = 0 < \beta$. Suppose that $(\gamma, 0)$ is not prepared on $|\Delta(I, x, y', z)|$. Then $\gamma \in \mathbf{N}$, and there exists $\psi \in \mathbf{k}$ such that for all $g \in I$ with $\nu_T(g) = r$,

$$(22) \quad a_{00r}(z - \psi x^\gamma)^r = a_{00r}z^r + F_{g,(\gamma,0)},$$

so that, with $\omega = \frac{-\psi}{\eta^\beta} \in \mathbf{k}$, for $0 \leq k < r$, we have for $g \in I$ with $\nu_T(g) = r$:

1. If $\binom{r}{r-k} \neq 0$ (in \mathbf{k}) then $i = \alpha(r-k), j = \beta(r-k) \in \mathbf{N}$, and

$$(23) \quad a_{ijk} = a_{00r} \binom{r}{r-k} \omega^{r-k}.$$

2. If $i = \alpha(r-k), j = \beta(r-k) \in \mathbf{N}$ and $\binom{r}{r-k} = 0$ (in \mathbf{k}), then $a_{ijk} = 0$.

Thus by Lemma 10.16, for $(i, j, k) \in W$, $a_{ijk} = 0$ if $p^s \nmid k$.

If \mathbf{k} has characteristic zero, we obtain a contradiction to our assumption that $(\gamma, 0)$ is not prepared on $|\Delta(I, x, y', z)|$. Thus $0 = \delta' < \beta$ if \mathbf{k} has characteristic zero.

Now we consider the case where \mathbf{k} has characteristic $p > 0$, and $r = p^s r_0$ with $p \nmid r_0$, $r_0 \geq 1$. Then by Lemma 10.16, for $(i, j, k) \in W$, we have $a_{ijk} = 0$ if $p^s \nmid k$, for all $g \in I$ with $\nu_T(g) = r$. By (23), and Lemma 10.16, we have that $i = \alpha p^s, j = \beta p^s \in \mathbf{N}$, and for all $g \in I$ with $\nu_T(g) = r$, we have that $a_{i,j,(r_0-1)p^s} \neq 0$.

We have an expression $\beta p^s = e p^t$ where $p \nmid e$. Suppose that $t \geq s$. Then $\beta \in \mathbf{N}$, which implies that $\alpha = \gamma - \beta \in \mathbf{N}$, so that for all $g \in I$ with $\nu_T(g) = r$,

$$a_{00r}(z + \omega x^\alpha y^\beta)^r = a_{00r}z^r + F_g,$$

a contradiction, since (α, β) is by assumption prepared on $|\Delta(I; x, y, z)|$. Thus $t < s$. Suppose that $e = 1$. Then $\beta = p^{t-s} < 1$ and $\alpha < 1$ (since we must have that $\text{Sing}_r(I) = V(x, y, z)$) which implies (since γ is an integer) that $\gamma = \alpha + \beta = 1$, a contradiction to (20). Thus $e > 1$. Also,

$$(24) \quad \begin{aligned} a_{00r}z^r + F_g &= a_{00r}(z^{p^s} + \omega^{p^s} x^{\alpha p^s} y^{\beta p^s})^{r_0} \\ &= a_{00r}(z^{p^s} + \omega^{p^s} x^{\alpha p^s} (y' + \eta x)^{\beta p^s})^{r_0} \\ &= a_{00r}(z^{p^s} + \omega^{p^s} x^{\alpha p^s} [(y')^{p^t} + \eta^{p^t} x^{p^t}]^e)^{r_0}. \end{aligned}$$

Now make the $(\gamma, 0)$ preparation $z = z' - \eta^\beta \omega x^\gamma$ (from (22)) so that $(I; x, y', z')$ is $(\gamma, 0)$ prepared. For $g \in I$ such that $\nu_T(g) = r$, let $G_g = a_{00r} z^r + F_g$. Then

$$\begin{aligned} G_g &= a_{00r} \left((z')^{p^s} + e\omega^{p^s} \eta^{p^t(e-1)} (y')^{p^t} x^{\alpha p^s + p^t(e-1)} + (y')^{2p^t} \Omega(x, y') \right)^{r_0} \\ &= a_{00r} \left[(z')^{p^s r_0} + r_0 \left[e\omega^{p^s} \eta^{p^t(e-1)} (y')^{p^t} x^{\alpha p^s + p^t(e-1)} + (y')^{2p^t} \Omega \right] (z')^{p^s(r_0-1)} \right] \\ &\quad + \Lambda_2(x, y') (z')^{p^s(r_0-2)} + \cdots + \Lambda_{r_0}(x, y') \end{aligned}$$

for some polynomials (which depend on $g \in I$), $\Omega(x, y')$, $\Lambda_2, \dots, \Lambda_{r_0}$, where $(y')^{ip^t} \mid \Lambda_i$ for all i . All contributions of $S(\gamma) \cap |\Delta(I; x, y', z')|$ must come from these polynomials G_g , for $g \in I$ with $\nu_T(g) = r$ since \mathbf{k} is infinite. Recall that we are assuming $(\alpha, \beta) = (\gamma - \delta, \delta)$. The term of lowest second coordinate on $S(\gamma) \cap |\Delta(I; x, y', z')|$ is

$$(a, b) = \left(\frac{\alpha p^s + p^t(e-1)}{p^s}, \frac{p^t}{p^s} \right),$$

which is not in \mathbf{N}^2 since $t < s$, and is not (α, β) since $e > 1$. (a, b) is thus prepared on $|\Delta(I; x, y', z')|$, and

$$\delta' = \frac{p^t}{p^s} < \frac{ep^t}{p^s} = \beta.$$

□

10.5. Construction of the sequence $\Omega(q_n)$.

Theorem 10.18. *For all $n \in \mathbf{N}$, there are good parameters (x_n, y_n, z_n) for $I_n = (\mathcal{I}_n)_{q_n} R_n$ in R_n such that*

$$\begin{aligned} \Omega(I_{n+1}; x_{n+1}, y_{n+1}, z_{n+1}) &= (\beta_{n+1}, \frac{1}{\epsilon_{n+1}}, \alpha_{n+1}) \\ < \Omega(I_n; x_n, y_n, z_n) &= (\beta_n, \frac{1}{\epsilon_n}, \alpha_n). \end{aligned}$$

Further, if $\beta_{n+1} = \beta_n$, $\epsilon_{n+1} \neq \epsilon_n$ and $\frac{1}{\epsilon_n} \neq \infty$, then

$$\frac{1}{\epsilon_{n+1}} = \frac{1}{\epsilon_n} - 1.$$

Proof. We inductively construct regular parameters (x_n, y_n, z_n) in R_n such that (x_n, y_n, z_n) are good parameters for I_n , $\text{Sing}_r(I_n) \subset V(x_n, z_n)$ or $\text{Sing}_r(I_n) \subset V(y_n, z_n)$ and $(I_n; x_n, y_n, z_n)$ is well prepared. If $\text{Sing}_r(I_n) \subset V(x_n, z_n)$, we will further have that $(I_n; x_n, y_n, z_n)$ is very well prepared. Let $\Omega_n = \Omega(I_n; x_n, y_n, z_n)$.

We first choose possibly formal regular parameters (x, y, z) in R which are good parameters for I , such that $|\Delta(I; x, y, z)|$ is very well prepared (see Remark 10.5). By Lemma 10.1, $z = 0$ is an approximate manifold for I . Let $\alpha = \alpha_{x,y,z}(I)$, $\beta = \beta_{x,y,z}(I)$.

By our reduction in Section 8, q is isolated in $\text{Sing}_r(\mathcal{I})$.

Suppose that regular parameters (x_i, y_i, z_i) in R_i have been defined for $i \leq n$ as specified above.

If $\text{Sing}_r(I_n) = V(y_n, z_n)$, then $R_n \rightarrow R_{n+1}$ must be a Tr4 transformation, by Lemma 5.1, since $V(z_n)$ is an approximate manifold of I_n . R_{n+1} has regular parameters $x'_{n+1}, y'_{n+1}, z'_{n+1}$ defined by

$$x_n = x'_{n+1}, y_n = y'_{n+1}, z_n = y'_{n+1} z'_{n+1}.$$

Thus $(I_{n+1}; x'_{n+1}, y'_{n+1}, z'_{n+1})$ is well prepared by Lemma 10.15. Further $\text{Sing}_r(I_{n+1}) \subset V(y'_{n+1}, z'_{n+1})$ by Lemma 10.15. If $\text{Sing}_r(I_{n+1}) = V(x'_{n+1}, y'_{n+1}, z'_{n+1})$, make a change of variables, subtracting a series in x'_{n+1}, y'_{n+1} from z'_{n+1} , to very well prepare, with resulting variables $x_{n+1}, y_{n+1}, z_{n+1}$. Otherwise, set $x_{n+1} = x'_{n+1}, y_{n+1} = y'_{n+1}, z_{n+1} = z'_{n+1}$. We have $\beta_{n+1} < \beta_n$ by Lemma 10.15, and $\Omega_{n+1} < \Omega_n$.

If $\text{Sing}_r(I_n) = V(x_n, z_n)$ then $(I_n; x_n, y_n, z_n)$ is very well prepared. $R_n \rightarrow R_{n+1}$ must be a Tr3 transformation by Lemma 5.1, since $V(z_n)$ is an approximate manifold of I_n . R_{n+1} has regular parameters $x_{n+1}, y_{n+1}, z_{n+1}$ defined by

$$x_n = x_{n+1}, y_n = y_{n+1}, z_n = x_{n+1}z_{n+1}.$$

Thus $(I_{n+1}; x_{n+1}, y_{n+1}, z_{n+1})$ is also very well prepared, $\alpha_{n+1} < \alpha_n$ and $\Omega_{n+1} < \Omega_n$ by Lemma 10.13. Further $\text{Sing}_r(I_{n+1}) \subset V(x_{n+1}, z_{n+1})$.

Now suppose that $\text{Sing}_r(I_n) = V(x_n, y_n, z_n)$. Then $(I_n; x_n, y_n, z_n)$ is very well prepared, and R_{n+1} must be a Tr1 or Tr2 transformation of R_n by Lemma 5.1, since $V(z_n)$ is an approximate manifold of I_n .

If R_{n+1} is a Tr2 transformation of R_n , then R_{n+1} has regular parameters $x'_{n+1}, y'_{n+1}, z'_{n+1}$ defined by

$$x_n = x_{n+1}y'_{n+1}, y_n = y'_{n+1}, z_n = y_{n+1}z'_{n+1}.$$

$(I_{n+1}; x_{n+1}, y'_{n+1}, z'_{n+1})$ is well prepared by Lemma 10.15. Further, $\text{Sing}_r(I_{n+1}) \subset V(y'_{n+1}, z'_{n+1})$ by Lemma 5.1. If $\text{Sing}_r(I_{n+1}) = V(x_{n+1}, y'_{n+1}, z'_{n+1})$, then make a further change of variables to very well prepare. Otherwise, set $y_{n+1} = y'_{n+1}, z_{n+1} = z'_{n+1}$. $\beta_{n+1} < \beta_n$ by Lemma 10.15, and $\Omega_{n+1} < \Omega_n$.

If R_{n+1} is a Tr1 transformation of R_n , then R_{n+1} has regular parameters $x'_{n+1}, y'_{n+1}, z'_{n+1}$ defined by

$$x_n = x_{n+1}, y_n = x_{n+1}(y'_{n+1} + \eta), z_n = x_{n+1}z'_{n+1}.$$

$\text{Sing}_r(I_{n+1}) \subset V(x_{n+1}, z'_{n+1})$ by Lemma 5.1.

If $\beta_n \neq 0$ and $\eta \neq 0$ in the Tr1 transformation relating R_n and R_{n+1} , we can change variables to $x_{n+1}, y_{n+1}, z_{n+1}$ to very well prepare, with $\beta_{n+1} < \beta_n$, $\Omega_{n+1} < \Omega_n$ and $\text{Sing}_r(I_{n+1}) \subset V(x_{n+1}, z_{n+1})$ by Theorem 10.17.

If $\eta = 0$ in the Tr1 transformation of R_n , and $x_{n+1}, y_{n+1}, z_{n+1}$ are regular parameters of R_{n+1} obtained from $x_{n+1}, y'_{n+1}, z'_{n+1}$ by very well preparation, then by Lemma 10.14, $(I_{n+1}; x_{n+1}, y_{n+1}, z_{n+1})$ is very well prepared and $\Omega_{n+1} < \Omega_n$. If $\beta_{n+1} = \beta_n$, and $\epsilon_n \neq 0$, then

$$\frac{1}{\epsilon_{n+1}} = \frac{1}{\epsilon_n} - 1.$$

Further, $\text{Sing}_r(I_{n+1}) \subset V(x_{n+1}, z_{n+1})$.

If $\beta_n = 0$ we can make the translation $y'_n = y_n - \eta x_n$, and have that (I_n, x_n, y'_n, z_n) is very well prepared, and $\beta_n, \gamma_n, \delta_n, \epsilon_n, \alpha_n$ are unchanged. Thus we are in the case of $\eta = 0$. \square

We now prove that (11) cannot have infinite length. Set $\Omega(q_n) = \Omega(I_n; x_n, y_n, z_n)$ for $n \in \mathbf{N}$. By Theorem 10.18, we have a contradiction if (11) has infinite length, as the sequence $\Omega(q_n)$ cannot decrease indefinitely.

11. EMBEDDED RESOLUTION ON A NONSINGULAR 3-FOLD

Our main resolution theorem for ideals on a nonsingular 3-fold, Theorem 5.6, has now been proven, in Sections 5 - 10.

We now deduce embedded resolution of surface singularities (Theorem 1.2) and principalization of ideals (Theorem 1.3) from Theorem 5.6.

To prove Theorem 1.2, we start with $\mathcal{R} = (\emptyset, E, \mathcal{I}_S, V)$. Let $r = \nu(\mathcal{R})$ and apply Theorem 5.6, to produce a sequence of permissible transforms

$$V_{n_1} \rightarrow V_{n_1-1} \rightarrow \cdots \rightarrow V$$

such that $\text{Sing}_r(\mathcal{R}_{n_1}) = \emptyset$. Since \mathcal{I}_S is locally principal, the weak transform \mathcal{I}_i of \mathcal{I}_S is the ideal sheaf of the strict transform S_i of S on V_i for all i .

Now we redefine \mathcal{R}_{n_1} to be $(\emptyset, E_{n_1}, \mathcal{I}_{n_1}, V_{n_1})$. By descending induction on r , we produce a sequence of permissible transforms

$$V_{n_2} \rightarrow V_{n_2-1} \rightarrow \cdots \rightarrow V_{n_1}$$

such that $\nu(\mathcal{R}_{n_2}) = 1$. Now redefine \mathcal{R}_{n_2} to be $(\emptyset, E_{n_2}, \mathcal{I}_{n_2}, V_{n_2})$. We have that S_{n_2} is nonsingular, since $\nu(\mathcal{R}_{n_2}) = 1$. We now apply Lemma 7.3 to \mathcal{R}_{n_2} , to construct a sequence of permissible transforms

$$V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_2,$$

obtained by blowing up points and curves in the locus where $E_i + S_i$ is not a SNC divisor, to obtain V_n such that the preimage $E_n + S_n$ of $E + S$ on V_n is a SNC divisor.

To prove Theorem 1.3, we observe that we can factor $\mathcal{I} = \mathcal{J}\mathcal{K}$ in \mathcal{O}_V , where \mathcal{J} is an invertible ideal sheaf, and the support of $\mathcal{O}_V/\mathcal{K}$ has codimension ≥ 2 in V . Let $\mathcal{R} = (\emptyset, \emptyset, \mathcal{K}, V)$ and $r = \nu(\mathcal{R})$. By Theorem 5.6, there exists a sequence of permissible transforms

$$V_{n_1} \rightarrow V_{n_1-1} \rightarrow \cdots \rightarrow V,$$

such that $\text{Sing}_r(\mathcal{R}_{n_1}) = \emptyset$. After each permissible transform, we have that the weak transform \mathcal{K}_i of \mathcal{K}_{i-1} is such that the support of $\mathcal{O}_{V_i}/\mathcal{K}_i$ has codimension ≥ 2 in V_i . Now redefine \mathcal{R}_{n_1} to be $(\emptyset, \emptyset, \mathcal{K}_{n_1}, V_{n_1})$. By descending induction on r , we produce a sequence of permissible transforms

$$V_{n_2} \rightarrow V_{n_2-1} \rightarrow \cdots \rightarrow V,$$

such that $\text{Sing}_2(\mathcal{R}_{n_2}) = \emptyset$. Now we have that $\text{Spec}(\mathcal{O}_{V_{n_2}}/\mathcal{K}_{n_2})$ is nonsingular, of codimension ≥ 2 in V_{n_2} . Blowing up the irreducible components of $\text{Spec}(\mathcal{O}_{V_{n_2}}/\mathcal{K}_{n_2})$ is a sequence of permissible transforms above V_{n_2} , after which the conclusions of Theorem 1.3 are achieved.

12. PROJECTION TO POINTS OF SMALL MULTIPLICITY

In this section we prove Theorem 1.4, the projection theorem which allows us to reduce to singularities of small multiplicity. The method is due to Albanese. Our exposition is close to Lipman's survey in [L1]. We fill in a few technical difficulties.

Suppose that \mathbf{k} is an algebraically closed field. Let W be a projective variety over \mathbf{k} of dimension d , with an embedding $W \subset \mathbf{P}^m$. Suppose that $q \in W$. Let $\pi : \mathbf{P}^m \rightarrow \mathbf{P}^{m-1}$ be the rational map which is projection from the point q . Let W_1 be the projective subvariety of \mathbf{P}^{m-1} which is the image of W by π (the closure of $\pi(W - \{q\})$). Let μ be the multiplicity of the local ring $\mathcal{O}_{W,q}$.

Theorem 12.1. *The following are true.*

1. $\mu \leq \text{deg } W$.
2. Suppose that $\mu < \text{deg } W$. Then $\dim W = \dim W_1$ and

$$(25) \quad [W : W_1] \text{deg } W_1 = \text{deg } W - \mu.$$

3. Suppose that $\mu = \text{deg } W$. Then $\dim W > \dim W_1$ and W is a cone over W_1 with vertex q .

Proof. Let $\sigma : Z \rightarrow \mathbf{P}^m$ be the blow up of q , with exceptional divisor E . Let $\lambda : Z \rightarrow \mathbf{P}^{m-1}$ be the morphism induced by $\pi \circ \sigma$ (a resolution of the indeterminacy of π).

Let H_0 be a hyperplane of \mathbf{P}^m and let H_1 be a hyperplane of \mathbf{P}^{m-1} .

We have a linear equivalence of divisors

$$(26) \quad \sigma^*(H_0) - E \sim \lambda^*(H_1).$$

Let \overline{W} be the strict transform of W on Z .

By (26), and since a general hyperplane of \mathbf{P}^m does not contain q , we have equality of intersection numbers

$$(27) \quad \int_Z \lambda^*(H_1)^d \cdot \overline{W} = \int_Z \sigma^*(H_0)^d \cdot \overline{W} + \int_Z (-E)^d \cdot \overline{W}.$$

By the projection formula (Proposition 2.3 [F]),

$$\int_Z \sigma^*(H_0)^d \cdot \overline{W} = \int_{\mathbf{P}^m} H_0^d \cdot W = \deg W.$$

Let $R = \mathcal{O}_{W,q}$, M be the maximal ideal of R . Then the scheme-theoretic intersection of \overline{W} and E is

$$\overline{W} \cap E = \text{Proj}(\oplus_{n \geq 0} M^n / M^{n+1}),$$

and

$$\mathcal{O}_Z(-E) \otimes \mathcal{O}_{\overline{W} \cap E} \cong \mathcal{O}_{\overline{W} \cap E}(1).$$

Thus by Lemma 4.3,

$$\int_Z (-E)^d \cdot \overline{W} = -\int_Z (\overline{W} \cdot E) \cdot (-E)^{d-1} = -\deg \overline{W} \cap E = -e(R) = -\mu.$$

We can now rewrite (27) as

$$(28) \quad \int_Z \lambda^*(H_1)^d \cdot \overline{W} = \deg W - \mu.$$

By the projection formula, we have that

$$\int_Z \lambda^*(H_1)^d \cdot \overline{W} = \begin{cases} [W : W_1] \deg W_1 & \text{if } \dim W = \dim W_1 \\ 0 & \text{if } \dim W > \dim W_1 \end{cases}$$

Substituting into (28), we conclude that $\mu \leq \deg W$, and $\mu = \deg W$ if and only if W_1 has dimension $< d$, which holds if and only if W is a cone with vertex q . If $\mu < \deg W$, we obtain (25). \square

We will say that the induced rational map $\pi : W \rightarrow W_1$ is a permissible projection if $\mu < \deg W$.

Suppose that W is a projective variety. We define $c(W)$ to be the minimum of degrees of irreducible curves on W .

Theorem 12.2. *Suppose that K is an algebraic function field of dimension d over an algebraically closed field \mathbf{k} . Then there exists a projective variety V_0 , whose function field is K , and an embedding of V_0 into a projective space \mathbf{P}^N so that V_0 is not contained in a hyperplane of \mathbf{P}^N , and such that*

$$(29) \quad d! + 1 \geq \frac{\deg V_0}{(N - d - \frac{1}{c} \deg V_0)}$$

where $c = c(V_0)$.

Proof. There exists a projective variety V , whose function field is K . Let H be a very ample divisor on V .

After possibly replacing H with a high multiple of H , we may assume that

$$H^i(V, \mathcal{O}_V(\delta H)) = 0$$

for all $\delta \geq 1$ and $i > 0$. Thus we have

$$h^0(V, \mathcal{O}_V(\delta H)) = \chi(\mathcal{O}_V(\delta H))$$

for all $\delta \geq 1$. $H^0(V, \mathcal{O}_V(\delta H))$ gives an embedding Φ_δ of V into a projective space $\mathbf{P}^{N(\delta)}$, where $N(\delta) = h^0(V, \mathcal{O}_V(\delta H)) - 1$. Let V^δ be the image of V .

Let H_δ be a hyperplane section of V^δ . By our construction, we have that V^δ is not contained in a hyperplane of $\mathbf{P}^{N(\delta)}$, and $\Phi_\delta^*(H_\delta) \sim \delta H$, so that every curve in V^δ has degree $\geq \delta$. Thus $c_\delta = c(V^\delta) \geq \delta$.

We have that $\frac{1}{d!} \deg V^\delta$ is the coefficient of n^d in the polynomial $\chi(\mathcal{O}_{V^\delta}(nH_\delta))$.

Since by our construction, $\chi(\mathcal{O}_{V^\delta}(nH_\delta)) = \chi(\mathcal{O}_V(\delta nH))$, we see that $\deg V^\delta = \delta^d \deg V$.

There exist $a_0, a_1, \dots, a_{d-1} \in \mathbf{Q}$ such that for all $\delta > 0$,

$$N(\delta) + 1 = \chi(\mathcal{O}_V(\delta H)) = \frac{1}{d!} \delta^d \deg V + \sum_{i=0}^{d-1} a_i \delta^i.$$

Thus

$$\frac{\deg V^\delta}{N(\delta) - d - \frac{1}{c_\delta} \deg V^\delta} = \frac{\delta^d \deg V}{\frac{\delta^d}{d!} \deg V + \sum_{i=0}^{d-1} a_i \delta^i - 1 - d - \frac{\delta^d}{c_\delta} \deg V}.$$

Since $c_\delta \geq \delta$ for all δ , the limit as δ goes to infinity of this expression is $d!$, and the conclusions of the theorem follow with $V_0 = V^\delta$ for δ sufficiently large. \square

Theorem 12.3. *Let K be an algebraic function field of dimension d over an algebraically closed field \mathbf{k} . Let $V_0 \subset \mathbf{P}^N$ be as in the conclusions of Theorem 12.2. Then there exists a series of permissible projections*

$$(30) \quad V_0 \xrightarrow{\pi_0} V_1 \xrightarrow{\pi_1} \dots \xrightarrow{\pi_{n-1}} V_n$$

such that every point on V_n has multiplicity $\leq \frac{d!}{[V_0:V_n]}$.

Proof. Suppose that there exists a point $q_0 \in V_0$ such that q_0 has multiplicity μ_0 with $d! < \mu_0 < \deg V_0$. Let π_0 be the projection from q_0 . Let V_1 be the image of V_0 . If there exists a point $q_1 \in V_1$ of multiplicity μ_1 with $\frac{d!}{[V_0:V_1]} < \mu_1 < \deg V_1$, then we perform the projection $\pi_1 : V_1 \rightarrow V_2$ from the point q_1 . After a finite number of steps this process must terminate, as the dimension of the ambient projective space drops by one after each projection. After the final projection, we have a variety V_n such that if $q \in V_n$ and the multiplicity μ of q on V satisfies $\mu > \frac{d!}{[V_0:V_n]}$, then $\mu = \deg V_n$. We see that V_n satisfies the conclusions of the theorem unless V_n is a cone. We thus assume that V_n is a cone with vertex q (in the projective space \mathbf{P}^{N-n}), and will show that this is impossible.

By (25), we have that

$$[V_0 : V_1] \deg V_1 = \deg V_0 - \mu_0 \leq \deg V_0 - d! - 1.$$

We further have that

$$[V_1 : V_2] \deg V_2 \leq \deg V_1 - \frac{d!}{[V_0 : V_1]} - 1,$$

from which we obtain

$$[V_0 : V_2] \deg V_2 \leq [V_0 : V_1] \deg V_1 - d! - [V_0 : V_1] \leq \deg V_0 - 2d! - 2.$$

Continuing in the way, we obtain

$$[V_0 : V_n] \deg V_n \leq \deg V_0 - n(d! + 1).$$

We deduce the inequality

$$(31) \quad n(d! + 1) \leq \deg V_0.$$

There exists a linear subspace L of \mathbf{P}^N of dimension $n-1$ such that $\pi_{n-1} \circ \dots \circ \pi_0$ is induced by the projection morphism $\Psi : \mathbf{P}^N - L \rightarrow \mathbf{P}^{N-n}$ from L . There exist dense open subsets $U_0 \subset V_0$ and $U_n \subset V_n$ such that $\phi = (\pi_{n-1} \circ \dots \circ \pi_0 | U_0) : U_0 \rightarrow U_n$ is a finite morphism, and $U_0 \cap L = \emptyset$ (so that $V_0 \cap \Psi^{-1}(U_n) = U_0$). Let $E = V_n - U_n$.

Since \mathbf{k} is infinite, and by Bertini's theorem for general hyperplane sections (Corollary 3.4.14 and 3.4.10 [FOV]), there exists a hyperplane \overline{H} of \mathbf{P}^{N-n} which does not contain q , intersects each irreducible component of E generically, and such that the scheme theoretic intersection $W = \overline{H} \cap V_n$ is a $d-1$ dimensional variety. V_n is the locus of lines through q and a point of W .

By Bertini's theorem for general hyperplane sections, there exist hyperplanes

$$\overline{H}_1, \dots, \overline{H}_{d-1}$$

of \mathbf{P}^{N-n} such that each \overline{H}_i intersects W and every irreducible component of E generically. In particular, the scheme-theoretic intersection $\overline{H}_1 \cap \dots \cap \overline{H}_{d-1} \cap W$ is a reduced set of points of order $\deg V_n$ which are contained in U_n .

For $1 \leq i \leq d-1$, let H_i be the hyperplane of \mathbf{P}^{N-n} which is spanned by q and the linear space $\overline{H}_i \cap \overline{H}$. By our construction, the scheme theoretic intersection $H_1 \cap \dots \cap H_{d-1} \cap V_n$ is the union of $s = \deg V_n$ distinct lines L_1, \dots, L_s , whose generic points lie in U_n .

Let H_1^*, \dots, H_{d-1}^* be the hyperplanes of \mathbf{P}^N such that $H_i^* \cap (\mathbf{P}^N - L) = \Psi^*(H_i)$. Since $U_0 \rightarrow U_n$ is finite, the irreducible components of

$$H_1^* \cap \dots \cap H_{d-1}^* \cap U_0 = \phi^{-1}(H_1 \cap \dots \cap H_{d-1} \cap U_n)$$

are curves which dominate the irreducible components of $H_1 \cap \dots \cap H_{d-1} \cap U_n$, so there are $\geq \deg V_n$ distinct irreducible components of $H_1^* \cap \dots \cap H_{d-1}^* \cap U_0$. By the weak Bézout theorem (12.3.1) of [Ab7], or Example 12.3.1 of the refined Bézout theorem of [F], we have that

$$(32) \quad \deg V_0 \geq c \deg V_n,$$

where $c = c(V_0)$.

We have that V_n is not contained in a hyperplane of \mathbf{P}^{N-n} , since V_0 is not contained in a hyperplane section of \mathbf{P}^N . Thus we have the classical degree bound (Example 8.4.6 [F]),

$$\deg V_n > N - n - d.$$

From (32), we get that

$$(N - n - d)c < \deg V_0.$$

That is,

$$(33) \quad N - d - \frac{1}{c} \deg V_0 < n.$$

Now from (33) and (31), we have

$$N - d - \frac{1}{c} \deg V_0 < \frac{1}{(d+1)} \deg V_0,$$

which contradicts the assumption of our theorem. Thus V_n is not a cone. \square

We now prove Theorem 1.4.

Let notation be as in the statement of Theorem 12.3. By Theorem 12.3, there exists a dominant rational map $V_0 \rightarrow V_n$ such that every point on V_n has multiplicity $\leq \frac{d!}{[V_0:V_n]}$. Let V be the normalization of V_n in the function field K of V_0 , so that the function field of V is K , and we have a finite morphism $\Psi : V \rightarrow V_n$. By Theorem 4.2, we have that every point on V has multiplicity $\leq d!$.

13. RAMIFICATION

In this section, we collect some basic results on ramification of local rings, which we will need for our proof of local uniformization in the next section. For more details, the book “Ramification Theoretic Methods in Algebraic Geometry” [Ab4] by Abhyankar is an excellent reference.

Suppose that R is a normal local domain with quotient field K , and L is a finite extension of K . Let T be the integral closure of R in L . The finitely many local rings S which are localizations of T and dominate R are called the extensions of R to L .

R is said to be unramified in L if for every extension S of R to L we have that S is residually separable over R and $M(R)S = M(S)$.

Suppose that X and Y are normal integral schemes, and $f : X \rightarrow Y$ is a finite morphism. The branch locus of f is the set of points in Y over which f is ramified. The branch locus is a closed subset of Y (cf Theorems 1.44 and 1.44A [Ab4]). If Y is nonsingular, then the theorem of the purity of the branch locus [N] tells us that if the function field of X is separable over the function field of Y , then the branch locus of f has pure codimension 1 in Y . In particular, if L is separable over K and R is a regular local ring, then R is unramified in L if R_Q is unramified in L for every height one prime Q of R .

Suppose that our local ring R is an equicharacteristic complete regular local ring, with algebraically closed residue field \mathbf{k} . Then R is a power series ring over \mathbf{k} . T is itself a complete local ring. Further, if R is unramified in L , then $R \cong T$ (as follows from Nakayama’s Lemma).

Suppose that $f(Z) \in K[Z]$ is a monic polynomial. Let $\delta(f) \in K$ be the discriminant of f . Suppose that $f(Z)$ factors as $f(Z) = (Z - z_1) \cdots (Z - z_n)$ in a splitting field of f over K . Then $\delta(f) = \prod_{i \neq j} (z_i - z_j)$.

If $f(Z) \in R[Z]$, then since R is integrally closed, $\delta(f) \in R$.

We compute the discriminant in a case which will be important in the sequel. Suppose that $\lambda \in K$. The discriminant of $f(Z) = Z^n - \lambda$ is

$$(34) \quad \delta(f) = \pm n^n \lambda^{n-1}$$

(This can be computed directly from the definition or from the formula of Exercise 12 (c) of Chapter IV [La]).

Lemma 13.1. *Suppose that R is a normal local domain with quotient field K , $L = K(r_1, \dots, r_n)$ is a finite extension field of K , and $f_i(Z) \in R[Z]$ are nonconstant monic polynomials such that $f_i(r_i) = 0$ for $1 \leq i \leq n$. Suppose that $\delta(f_i) \notin M(R)$ for $1 \leq i \leq n$. Then R is unramified in $K(r_1, \dots, r_n)$.*

This is (10.17) [Ab7].

Lemma 13.2. *Let R be a d -dimensional regular local domain with $d > 0$. Let K be the quotient field of R and let (y_1, \dots, y_d) be a basis of $M(R)$. Suppose that $L = K(z_1, \dots, z_d)$ is an overfield of K where $z_i^{n(i)} = y_i$ for $1 \leq i \leq d$, where $n(i)$ are positive integers. Let S be the integral closure of R in L . Then S is a finite R -module, S is a d -dimensional regular local domain, $S = R[z_1, \dots, z_d]$, $M(S) = (z_1, \dots, z_d)S$, $S/M(S) = R/M(R)$, and $[L : K] = n(1) \cdots n(d)$.*

This is (10.20.1) [Ab7].

Suppose that R is one dimensional, and S_i for $1 \leq i \leq n$ are the distinct local rings of T which dominate R . Let w_i be the integers such that $M(R)S_i = M(S)^{w_i}$. Define $q_i = [S_i/M(S_i) : R/M(R)]$. We have an inequality

$$(35) \quad q_1 w_1 + \cdots + q_n w_n \leq [L : K]$$

(by Theorem 21, Section 9, Chapter V [ZS]).

We say that R is tamely ramified in L if w_i is not divisible by the characteristic of $R/M(R)$ and $S_i/M(S_i)$ is separable over $R/M(R)$ for all i .

14. LOCAL UNIFORMIZATION OF POINTS OF SMALL MULTIPLICITY

In this section, we generalize the method of Jung [J], for locally resolving singularities. Our methods are a simplification of those of Abhyankar [Ab7]. We prove local uniformization of the function field of a 3-fold V whose base field is algebraically closed of characteristic > 5 in Theorem 14.4.

Theorem 14.1. *Let R be an equicharacteristic, complete regular local domain of dimension $d \geq 1$, whose residue field is algebraically closed (so that R is a power series ring over an algebraically closed field \mathbf{k}). Let p be the characteristic of \mathbf{k} . Suppose that L is a finite extension of the quotient field K of R . Further suppose that a regular system of parameters y_1, \dots, y_d in R is such that R is unramified in L away from $V(y_1 \cdots y_d)$, and the one dimensional regular local rings $R_{(y_i)}$ are tamely ramified in L for $1 \leq i \leq d$.*

Then there exists a positive integer n such that $p \nmid n$, and an inclusion of fields

$$K \subset L \subset K(z_1, \dots, z_d)$$

such that $z_i^n = y_i$ for $1 \leq i \leq d$.

Proof. Let T be the integral closure of R in L , $H_i = R_{(y_i)}$ for $1 \leq i \leq d$. Let $H_{i,j}$ for $1 \leq j \leq u(i)$ be the distinct local rings of T which dominate H_i for $1 \leq i \leq d$. Let $w_{i,j}$ be the integers such that $M(H_i)H_{i,j} = M(H_{i,j})^{w_{i,j}}$. Define

$$q_{i,j} = [H_{i,j}/M(H_{i,j}) : H_i/M(H_i)].$$

Let n be the least common multiple of the $w_{i,j}$. Since the H_i are tamely ramified in L , we have that $p \nmid n$.

For $1 \leq i \leq d$, choose z_i in an over field of L such that $z_i^n = y_i$. Define polynomials $f_i(Z) = Z^n - y_i \in K[Z]$ for $1 \leq i \leq d$.

Define fields $K^* = K(z_1, \dots, z_d) \subset L^* = L(z_1, \dots, z_d)$. Let R^* be the integral closure of R in K^* , T^* be the integral closure of R in L^* .

Suppose that V^* is a one dimensional localization of R^* , and that W^* is a one dimensional localization of T^* which dominates V^* . We will show that

$$(36) \quad W^* \text{ is residually separable algebraic over } V^* \text{ and } M(V^*)W^* = M(W^*).$$

W^* dominates a one dimensional localization W of T and dominates a one dimensional localization V of R .

First suppose that $V \neq H_i$ for any i . Since $y_1 \dots y_d$ is a unit in V , and thus also in W , we have by (34) that the discriminant $\delta(f_i) \notin M(W)$ for $1 \leq i \leq d$. By Lemma 13.1, W is unramified in L^* . Since V is unramified in L , we then have that (36) holds.

Now suppose that $V = H_i$ for some i . Then $W = H_{i,j}$ for some j . Choose $x \in W$ such that $xW = M(W)$. There exists a unit $x' \in W$ such that $y_i = x'x^{w_{i,j}}$. Let $z = z_i^{\frac{n}{w_{i,j}}}x^{-1} \in L^*$. Define a polynomial $f(Z) = Z^{w_{i,j}} - x' \in L[Z]$. We have that $f(z) = 0$ and $\delta(f) \notin M(W)$ by (34). Further, $\delta(f_j) \notin M(W)$ if $j \neq i$. Lemma 13.1 thus implies that W is unramified in $L' = L(z_1, \dots, z_{i-1}, z, z_{i+1}, \dots, z_d) \subset L^*$.

Let T' be the integral closure of T in L' , and let W' be the local ring of T' such that W^* dominates L' . We have that $M(W') = xW'$ and W' is residually separable algebraic over W . Further, $L^* = L'(z_i)$ and $z_i^{\frac{n}{w_{i,j}}} = zx$. Since z is a unit in W' , $M(W') = (zx)W'$. By Lemma 13.2, we have that $W^*/M(W^*) = W'/M(W')$ and $M(W^*) = z_iW^*$. It follows that (36) holds.

Since we have verified that V^* is unramified in L^* for all one dimensional local rings of R^* , and L^* is separable over K^* , by the purity of the branch locus [N], we have that R^* is unramified in L^* . Since R^* is complete with algebraically closed residue field, $T^* = R^*$. As L^* is the quotient field of T^* , we have that $L^* = K^*$. \square

Corollary 14.2. *Let assumptions be as in Theorem 14.1. Then there exists a subgroup G of $\text{Gal}(K(z_1, \dots, z_d)/K) \cong \mathbf{Z}_n^d$ such that $L = K(z_1, \dots, z_d)^G$. Let A be the integral closure of R in $K(z_1, \dots, z_d)$. Then $A = \mathbf{k}[[z_1, \dots, z_d]]$, and the integral closure T of R in L is $T = A^G$. In particular, there exist elements $s_i \in T$ such that T is generated by s_1, \dots, s_m as an R module, and there exist natural numbers $a(i, j)$ such that*

$$s_j^n = y_1^{a(1,j)} y_2^{a(2,j)} \dots y_d^{a(d,j)}$$

for $1 \leq j \leq m$.

Proof. By its construction, and Lemma 13.2, $K(z_1, \dots, z_d)$ is Galois over K with Abelian Galois group isomorphic to \mathbf{Z}_n^d and $A = \mathbf{k}[[z_1, \dots, z_n]]$. $K(z_1, \dots, z_d)$ is thus Galois over L . Let $H = \text{Gal}(K(z_1, \dots, z_d)/K)$ and $G = \text{Gal}(K(z_1, \dots, z_d)/L) \subset H$.

Elements of H act on $K(z_1, \dots, z_d)$ by multiplying z_i by an n -th root of unity. A^G is integrally closed, is finite over R and has quotient field L . Thus $T = A^G$. The conclusions of the corollary follow. \square

Lemma 14.3. *Let R be a d -dimensional regular local domain with $d \geq 2$, which is essentially of finite type over an algebraically closed field \mathbf{k} with $R/M(R) \cong \mathbf{k}$. Suppose that y_1, \dots, y_d are regular parameters in R and $f \in R$ is such that $f = y_1^{a_1} \dots y_d^{a_d}$ for some natural numbers a_1, \dots, a_d . Further suppose that e is a positive integer. Let \bar{n} denote the remainder modulo e of an integer n . Let V be a valuation ring of the quotient field K of R which contains \mathbf{k} , such that $V/M(V) = \mathbf{k}$, and which dominates R . Then there exists a sequence of local blow ups*

$$R \rightarrow R_1 \rightarrow \dots \rightarrow R_n$$

along V such that there exist regular parameters x_1, \dots, x_d in the regular local ring R_n , a unit $\epsilon \in R_n$ and natural numbers b_1, \dots, b_d such that

$$f = \epsilon x_1^{b_1} \dots x_d^{b_d}$$

with $\bar{b}_1 + \dots + \bar{b}_d < e$.

Proof. The proof is by induction on $\bar{a}_1 + \dots + \bar{a}_d$. Suppose that $\bar{a}_1 + \dots + \bar{a}_d \geq e$, and the assertion of the lemma is true for all values of $\bar{a}_1 + \dots + \bar{a}_d$ smaller than the given value. There exists a unique integer t with $2 \leq t \leq d$ such that for every subsequence $1 \leq u(1) < \dots < u(t-1) \leq d$ we have $\bar{a}_{u(1)} + \dots + \bar{a}_{u(t-1)} < e$, but for some subsequence $1 \leq u(1) < \dots < u(t) \leq d$ we have $\bar{a}_{u(1)} + \dots + \bar{a}_{u(t)} \geq e$. After possibly reindexing y_1, \dots, y_d we may assume that $\bar{a}_1 + \dots + \bar{a}_t \geq e$. Let ν be a valuation whose valuation ring is V . After a further reindexing, we may assume that $\nu(y_1) \leq \nu(y_i)$ for all $i \leq t$. Let

$$R_1 = R\left[\frac{y_2}{y_1}, \frac{y_3}{y_1}, \dots, \frac{y_t}{y_1}\right]_{M_1}$$

where M_1 is the contraction of the maximal ideal of V to $R\left[\frac{y_2}{y_1}, \frac{y_3}{y_1}, \dots, \frac{y_t}{y_1}\right]$. There exist $\lambda_i \in \mathbf{k}$ for $2 \leq i \leq t$ such that $M(R_1) = (y'_1, \dots, y'_d)$, where

$$y_i = \begin{cases} y'_1 & \text{if } i = 1 \\ y'_1(y'_i + \lambda_i) & \text{if } 1 < i \leq t \\ y'_i & \text{if } i > t. \end{cases}$$

We then have that

$$f = \epsilon'(y'_1)^{a'_1}(y'_2)^{a'_2} \cdots (y'_d)^{a'_d}$$

where

$$a'_i = \begin{cases} a_1 + a_2 + \cdots + a_t & \text{if } i = 1 \\ 0 & \text{if } 2 \leq i \leq t \text{ and } \lambda_i \neq 0 \\ a_i & \text{otherwise} \end{cases}$$

and ϵ' is a unit in R_1 . Since $\bar{a}_1 + \cdots + \bar{a}_t - \bar{a}_1 < e$, $\bar{a}_1 < e$ and $\bar{a}_1 + \cdots + \bar{a}_t \geq e$, we see that

$$\bar{a}'_1 = \bar{a}_1 + \cdots + \bar{a}_t - e < \bar{a}_1,$$

and hence

$$\bar{a}'_1 + \cdots + \bar{a}'_d < \bar{a}_1 + \cdots + \bar{a}_d.$$

Therefore we are done by induction. \square

Theorem 14.4. *Let L be a 3 dimensional function field over an algebraically closed field \mathbf{k} of characteristic $p > 5$ (or $p = 0$). Suppose that V is a valuation ring of L containing \mathbf{k} such that $V/M(V) = \mathbf{k}$. Then there exists a regular local ring A with quotient field L which is essentially of finite type over \mathbf{k} and is dominated by V .*

Proof. By Theorem 1.4, there exists a normal local ring S of L which is dominated by V , and such that $e(S) < p$ if \mathbf{k} has positive characteristic p . Suppose that $e(S) > 1$. There exists a system of parameters x_1, x_2, x_3 in S such that $e((x_1, x_2, x_3)S) = e(S)$ by Theorem 4.1. Let $K = \mathbf{k}(x_1, x_2, x_3)$ and $R = \mathbf{k}[x_1, x_2, x_3]_{(x_1, x_2, x_3)}$. R is a 3 dimensional regular local ring which is dominated by S (by Corollary 1 to Theorem 12, Section 9, Chapter VIII [ZS]). There exist r_1, \dots, r_n in the integral closure of R in L such that $L = K(r_1, \dots, r_n)$. For $1 \leq j \leq n$, there exist nonconstant polynomials $f_j(Z) \in R[Z]$ such that $f_j(r_j) = 0$. We further have that $r_j \in S$ for all j . Let $f = \prod_{j=1}^n \delta(f_j) \in R$ (where δ is the discriminant).

By Theorem 1.2, there exists a sequence of local blow ups $R \rightarrow R_0$ along V such that $f = 0$ is a SNC divisor on the spectrum of the regular local ring R_0 . Let S_0 be the normal local ring which is a local ring of the integral closure of R_0 in L , and is dominated by V .

Let K^*, K_0^*, L^*, L_0^* be the respective quotient fields of the respective completions with respect to their maximal ideals R^*, R_0^*, S^*, S_0^* of R, R_0, S, S_0 . By Theorem 4.4, we may identify K^*, K_0^*, L^*, L_0^* with subfields of L_0^* . We have $L^* = K^*(L)$ and $L_0^* = K_0^*(L)$, since S^* is finite over R^* and S_0^* is finite over R_0^* .

We will now show that Theorem 14.1 applies to R_0^* in the field L_0^* . By Theorem 4.2, we have

$$\begin{aligned} e(S_0) &\leq e(M(R_0)S_0) = e(M(R_0^*)S_0^*) = e(R_0^*)[L_0^* : K_0^*] \\ &= [L_0^* : K_0^*] \leq [L^* : K^*] \\ &= e(R^*)[L^* : K^*] = e(M(R^*)S^*) = e(S^*) = e(S). \end{aligned}$$

In summary, we have

$$(37) \quad e(S_0) \leq [L_0^* : K_0^*] \leq [L^* : K^*] = e(S).$$

We have that $L_0^* = K_0^*(r_1, \dots, r_n)$.

Since $f = 0$ is a SNC divisor on the spectrum of R_0 , there exists a system of regular parameters y_1, y_2, y_3 in R_0 , a unit $\epsilon \in R_0$ and natural numbers c_1, c_2, c_3 such that $f = \epsilon y_1^{c_1} y_2^{c_2} y_3^{c_3}$. Suppose that Q is a height one prime of R_0^* which does not contain $y_1 y_2 y_3$. Since $f = \prod_{j=1}^n \delta(f_j)$, we have that $\delta(f_j)$ are units in $(R_0^*)_Q$ for $1 \leq i \leq n$, so that $(R_0^*)_Q$ is unramified in L_0^* by Lemma 13.1. Thus R_0^* is unramified in L_0^* away from $V(y_1 y_2 y_3)$.

By (35) and the fact that $[L_0^* : K_0^*] < p$ by (37) (if \mathbf{k} has positive characteristic p), we see that for $1 \leq i \leq 3$, $(R_0^*)_{(y_i)}$ is tamely ramified in L_0^* . Thus the conclusions of Theorem 14.1 hold for the inclusion $K_0^* \subset L_0^*$. Now by Corollary 14.2, there exists $s \in L_0^*$ and a prime number q which is not equal to p , such that $g(Z) = Z^q - y_1^{a_1} y_2^{a_2} y_3^{a_3} \in K_0^*[Z]$ is the minimal polynomial of s over K_0^* for some natural numbers a_1, a_2, a_3 . Thus, $[K_0^*(s) : K_0^*] = q$. $q[L_0^* : K_0^*(s)] = [L_0^* : K_0^*] \leq e(S)$ implies that

$$(38) \quad [L_0^* : K_0^*(s)] \leq \frac{e(S)}{q}.$$

Now by Lemma 14.3, there exists a sequence of local blowups along V , $R_0 \rightarrow R_1$, such that R_1 has a regular system of parameters z_1, z_2, z_3 such that

$$y_1^{a_1} y_2^{a_2} y_3^{a_3} = \epsilon^* (z_1^{m_1} z_2^{m_2} z_3^{m_3})^q z_1^{b_1} z_2^{b_2} z_3^{b_3}$$

where ϵ^* is a unit in R_1 , $m_1, m_2, m_3, b_1, b_2, b_3$ are natural numbers such that

$$b_1 + b_2 + b_3 < q.$$

Let S_1 be the local ring of the integral closure of R_1 in L which is dominated by V . Let K_1^*, L_1^* be the quotient fields of the respective completions with respect to their maximal ideals R_1^*, S_1^* of R_1 and S_1 . We have

$$[L_1^* : K_1^*(s)] \leq [L_0^* : K_0^*(s)] \leq \frac{e(S)}{q}.$$

Let F be the local ring of the integral closure of R_1^* in $K_1^*(s)$ which is dominated by S_1^* . By Theorem 4.2, we have

$$e(S_1) \leq e(M(F)S_1^*) = e(F)[L_1^* : K_1^*(s)] \leq \left(\frac{e(F)}{q} \right) e(S).$$

We will now show that $e(F) < q$, from which it will follow that $e(S_1) < e(S)$. We will then have achieved a reduction in the multiplicity of S from which the conclusions of the theorem will follow from induction.

Let $x = s(z_1^{m_1} z_2^{m_2} z_3^{m_3})^{-1}$. We have $x^q = (\epsilon^*)^q z_1^{b_1} z_2^{b_2} z_3^{b_3} \in R_1^*$. Thus x^q has order $\nu_{R_1^*}(x^q) = b_1 + b_2 + b_3$, less than q in R_1^* and $K_1^*(s) = K_1^*(x)$.

First assume that x^q is a unit in R_1^* . This is equivalent to $b_1 = b_2 = b_3 = 0$. Thus R_1^* is unramified in $K_1^*(x)$, so that $F \cong R_1^*$ and $e(F) = 1 < q$.

Now assume that x^q has positive order in R_1^* . We have

$$e(F) \leq [K_1^*(x) : K_1^*] \leq q.$$

Assume $[K_1^*(x) : K_1^*] = q$. Then $Z^q - x^q$ is the minimal polynomial of x over K_1^* . Let $F' = R_1^*[x] \subset F$. F' and F have the same quotient field, and F is finite over F' , so $e(F) \leq e(F')$. Since

$$F' \cong \mathbf{k}[[z_1, z_2, z_3, z]] / (z^q - \epsilon^* z_1^{b_1} z_2^{b_2} z_3^{b_3}),$$

we have $e(F') = b_1 + b_2 + b_3 < q$. □

15. PATCHING

In this section we prove a technical result on patching local resolutions, which we will use in the next section to deduce resolution of singularities of a 3-fold from local uniformization.

Theorem 15.1. *Suppose that K is a 3-dimensional algebraic function field over an algebraically closed field \mathbf{k} . Let N be a set of valuation rings of K which contain \mathbf{k} and whose residue fields are \mathbf{k} . Suppose that V_0 and V_1 are normal projective varieties with function field K such that each element of N dominates a regular point on V_0 or V_1 . Then there exists a normal projective variety W , with birational morphisms to V_0 and V_1 , such that each element of N dominates a nonsingular point of W .*

Proof. Let $Z \subset V_0 \times_{\mathbf{k}} V_1$ be the graph of the birational map from V_0 to V_1 . The projection from Z to V_0 is a projective birational morphism. Hence Z is the blow up of an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{V_0}$. Let $U_0 \subset V_0$ be the dense open set of nonsingular points in V_0 . By Theorem 1.3, there exists a sequence of blow ups with nonsingular centers $W_0 \rightarrow U_0$ such that $\mathcal{I}\mathcal{O}_{W_0}$ is invertible. $W_0 \rightarrow U_0$ is the blow up of an ideal sheaf \mathcal{J} on U_0 . Let \mathcal{J}' be an ideal sheaf on V_0 which extends \mathcal{J} , and let $\pi_1 : X_1 \rightarrow V_0$ be the blow up of \mathcal{J}' . $\pi_1^{-1}(U_0) \cong W_0$ and $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{X_1}$ is an ideal sheaf on X_1 which is invertible on W_0 . Let $V_2 \rightarrow X_1$ be the normalization of the blow up of \mathcal{I}_1 , with induced projective morphism $\pi_2 : V_2 \rightarrow V_0$. Let U_2 be the nonsingular locus of V_2 . We have that $W_0 \cong \pi_2^{-1}(U_0) \subset U_2$ is nonsingular, and by the universal property of blowing up, there exists a projective morphism $\alpha : V_2 \rightarrow V_1$. Let U_1 be the nonsingular locus of V_1 , and F_s be the Zariski closure in V_1 of the fundamental locus of $\alpha^{-1} | U_1$.

Since $\alpha : V_2 \rightarrow V_1$ is projective and birational, V_2 is the blow up of an ideal sheaf \mathcal{K} in \mathcal{O}_{V_1} .

Let $W_1 \rightarrow U_1$ be the principalization of $\mathcal{K}\mathcal{O}_{U_1}$ of Theorem 1.3. We have a factorization

$$(39) \quad W_1 = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 = U_1$$

where each map $X_i \rightarrow X_{i-1}$ is the blow up of a point or nonsingular curve in the locus where $\mathcal{K}\mathcal{O}_{X_{i-1}}$ is not invertible.

Since $W_1 \rightarrow U_1$ is a projective birational morphism, there exists an ideal sheaf \mathcal{H} in \mathcal{O}_{U_1} such that W_1 is the blow up of \mathcal{H} . Let \mathcal{H}' be an ideal sheaf in \mathcal{O}_{V_1} which extends \mathcal{H} and such that the support of $\mathcal{O}_{V_1}/\mathcal{H}'$ is the Zariski closure of the support of $\mathcal{O}_{U_1}/\mathcal{H}$, which is equal to F_s . Let $\beta : V_3 \rightarrow V_1$ be the normalization of the blow up of \mathcal{H}' . Let U_3 be the nonsingular locus of V_3 . We have that $W_1 = \beta^{-1}(U_1) \subset U_3$. Let $\gamma : V_3 \rightarrow V_2$ be the induced birational map, and

$$L = \left\{ \begin{array}{l} p_2 \in V_2 \mid p_2 \text{ is the center of some } A \in N \text{ such that} \\ \text{the center of } A \text{ on } V_1 \text{ is a singular point} \end{array} \right\}.$$

We have by assumption that $L \subset U_2$. Suppose that $p_2 \in L$. Let $p_1 = \alpha(p_2)$. Further suppose that there exists a curve Γ_2 in V_2 such that $p_2 \in \Gamma_2$ and Γ_2 is in the fundamental locus of γ^{-1} . Then $\alpha(\Gamma_2) \subset F_s$. Now suppose that $\alpha(\Gamma_2) = \Gamma_1$ is a curve. Let η_1 be the generic point of Γ_1 and η_2 be the generic point of Γ_2 . By [Ab3], the birational local homomorphism $\mathcal{O}_{V_1, \eta_1} \rightarrow \mathcal{O}_{V_2, \eta_2}$ of two dimensional regular local rings factors uniquely as a sequence

$$(40) \quad \mathcal{O}_{V_1, \eta_1} = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_m = \mathcal{O}_{V_2, \eta_2}$$

of blow ups of regular local rings at the maximal ideal, followed by localization at a maximal ideal. Further, by the universal property of \mathcal{K} , we have that $\mathcal{K}_{\eta_1} R_m$ is invertible, but $\mathcal{K}_{\eta_1} R_i$ is not invertible for $i < m$.

By our construction of the sequence (39),

$$V_3 \times_{V_1} \text{Spec}(\mathcal{O}_{V_1, \eta_1}) \rightarrow \text{Spec}(\mathcal{O}_{V_1, \eta_1})$$

can be factored as a product of blow ups of points over η_1 (which are generic points of curves which dominate Γ_1), at which the extension of the ideal sheaf \mathcal{K} is not invertible. Comparing with (40), we see that $V_3 \rightarrow V_2$ is an isomorphism in a neighborhood

of η_2 , a contradiction to our assumption that Γ_2 is in the fundamental locus of γ^{-1} . Thus $\alpha(\Gamma_2) = p_1$, which is a singular point of V_1 in F_s .

It follows that there exists an open subset \tilde{U}_2 of V_2 , which is contained in the nonsingular locus U_2 of V_2 , such that $L \subset \tilde{U}_2$, and the fundamental locus G of the birational map $\tilde{U}_2 \rightarrow V_3$ is a union of curves containing points of L which contract by α to points in the set of points $\alpha(L)$, and possibly of some isolated points in L .

Let $Z_2 \subset \tilde{U}_2 \times_{\mathbf{k}} V_3$ be the graph of the birational map from \tilde{U}_2 to V_3 . There exists an ideal sheaf \mathcal{J} on \tilde{U}_2 such that Z_2 is the blow up of \mathcal{J} , and the support of $\mathcal{O}_{\tilde{U}_2}/\mathcal{J}$ is G . By Theorem 1.3, there exists a sequence of blow ups with non singular centers transforms $W_2 \rightarrow \tilde{U}_2$ such that $\mathcal{J}\mathcal{O}_{W_2}$ is invertible, and $W_2 \rightarrow \tilde{U}_2$ is an isomorphism away from G .

Let \mathcal{J}' be an extension of \mathcal{J} to V_2 such that the support of $\mathcal{O}_{V_2}/\mathcal{J}'$ is the Zariski closure of G in V_2 . Let $\delta : V_4 \rightarrow V_2$ be the normalization of the blow up of \mathcal{J}' . Finally, let V_5 be the normalization of the graph of the birational map $V_4 \rightarrow V_3$. We will show that every element of N has a nonsingular center on V_5 .

Suppose that $A \in N$.

Let p_1, p_2, p_3, p_4, p_5 be the respective centers of A on V_1, V_2, V_3, V_4, V_5 .

Suppose that $p_1 \notin F_s$. Then

$$\mathcal{O}_{V_3, p_3} = \mathcal{O}_{V_1, p_1} \subset \mathcal{O}_{V_2, p_2} = \mathcal{O}_{V_4, p_4}$$

since $\alpha(G) \subset F_s$ implies $p_2 \notin G$. If p_1 is a singular point of V_1 , then $A \in N$ implies p_4 is a nonsingular point on V_4 . If p_1 is a nonsingular point of V_1 , then $\mathcal{O}_{V_1, p_1} = \mathcal{O}_{V_2, p_2}$, so p_4 is a nonsingular point on V_4 . Since $V_4 \rightarrow V_3$ is a morphism at p_4 and p_4 is a nonsingular point of V_4 , we have that $\mathcal{O}_{V_5, p_5} \cong \mathcal{O}_{V_4, p_4}$, and p_5 is a nonsingular point of V_5 .

Suppose that $p_1 \in F_s$ is a nonsingular point of V_1 . Then γ is a morphism at the nonsingular point p_3 , and

$$\mathcal{O}_{V_4, p_4} = \mathcal{O}_{V_2, p_2} \subset \mathcal{O}_{V_3, p_3}.$$

Thus $\mathcal{O}_{V_5, p_5} \cong \mathcal{O}_{V_3, p_3}$ and p_5 is a nonsingular point of V_5 .

Suppose that $p_1 \in F_s$ is a singular point of V_1 . Then $p_2 \in L \subset \tilde{U}_2$. Thus $\mathcal{O}_{V_3, p_3} \subset \mathcal{O}_{V_4, p_4}$ and p_4 is a nonsingular point on V_4 . It follows that $\mathcal{O}_{V_5, p_5} \cong \mathcal{O}_{V_4, p_4}$ and p_5 is a nonsingular point of V_5 .

□

Corollary 15.2. *Suppose that K is a 3-dimensional algebraic function field over an algebraically closed field \mathbf{k} . Let N be the set of all valuation rings of K which contain \mathbf{k} and whose residue field is \mathbf{k} . Suppose that $\{V_1, \dots, V_n\}$ are normal projective varieties with function field K such that each element of N dominates a nonsingular point on some V_i , with $1 \leq i \leq n$. Then there exists a nonsingular projective variety W such that the function field of W is K .*

Proof. The proof is immediate from induction on the statement of Theorem 15.1. □

16. RESOLUTION OF SINGULARITIES OF 3-FOLDS

In this section we deduce resolution of singularities of a 3-fold from local uniformization. We generalize to positive characteristic the patching argument of Zariski [Z4]. Our proof is a little simpler than Abhyankar's argument in [Ab7].

We first prove the existence of nonsingular models.

Theorem 16.1. *Suppose that \mathbf{k} is an algebraically closed field of characteristic $\neq 2, 3$ or 5 , and K is a 3-dimensional algebraic function field over \mathbf{k} . Then there exists a nonsingular projective variety W whose function field is K .*

Proof. Suppose that V is a valuation ring of K which contains \mathbf{k} . We will first show that there exists a normal projective variety W_V whose function field is K , such that the center p of V on W_V is a nonsingular (not necessarily closed) point of W_V .

If the residue field of V is \mathbf{k} , then this follows from Theorem 14.4. Suppose that the residue field of V strictly contains \mathbf{k} , so that it is a transcendental extension. There exists a valuation ring V^* of K which contains \mathbf{k} , whose residue field is \mathbf{k} , and such that V is a localization of V^* at a prime ideal (as follows from the construction of composite valuations on page 57 [Ab4]). By Theorem 14.4, there exists a normal projective variety W_{V^*} , with function field K , such that the center of V^* on W_{V^*} is at a nonsingular point p^* . Thus $\mathcal{O}_{W_{V^*}, p^*}$ is a regular local ring. The center of V on W_{V^*} is thus a point p which is a nonsingular point of W_{V^*} , since the local ring of p is a localization of $\mathcal{O}_{W_{V^*}, p^*}$.

By quasi compactness of the Zariski Riemann manifold (Theorem 40, Section 17, Chapter VI [ZS]), there exists a finite set of normal projective varieties $\{V_1, \dots, V_n\}$ with function field K , such that the center of every valuation ring of K is at a nonsingular point of some V_i . Now the existence of a nonsingular projective variety with function field K follows from Corollary 15.2. \square

We now prove the existence of a resolution of singularities, but do not require it to be an isomorphism above the nonsingular locus of V .

Theorem 16.2. *Suppose that V is a projective variety of dimension 3 over an algebraically closed field \mathbf{k} of characteristic $\neq 2, 3$ or 5 . Then there exists a nonsingular projective variety W and a birational morphism $\phi : W \rightarrow V$.*

Proof. By Theorem 16.1, there exists a nonsingular projective variety W whose function field is K . Let $\Phi : V \rightarrow W$ be the birational map between V and W , induced by the equality of their function fields with K . Let $\Gamma_\Phi \subset V \times_{\mathbf{k}} W$ be the graph of Φ . Projection onto the second factor is a birational projective morphism $\pi_2 : \Gamma_\Phi \rightarrow W$. Thus π_2 is the blow up of a sheaf of ideals \mathcal{I} on W . By Theorem 1.3, there exists a sequence of blow ups with non singular centers $W_1 \rightarrow W$ such that $\mathcal{I}\mathcal{O}_{W_1}$ is invertible. Thus there is a projective birational morphism $W_1 \rightarrow V$. \square

17. A STRONGER RESOLUTION THEOREM

In this section we prove the main resolution theorem, Theorem 1.1. Our argument is based on the proof of Cossart [Co4] of this extension. We require the following lemma.

Lemma 17.1. *Suppose that V is a projective variety over a field \mathbf{k} , and $C \subset V$ is a curve such that the generic point of C is contained in the nonsingular locus of V . Then there exists a sequence of blow ups of closed points*

$$(41) \quad V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V$$

such that the strict transform C_n of C in V_n is contained in the nonsingular locus of V_n .

Proof. By Corollary 4.4 [C2], there exists a sequence of blow ups of points $V_n \rightarrow V$ such that the strict transform C_n of C in V_n is nonsingular. We may thus assume that C is nonsingular from the outset.

Suppose that p is a point in the intersection of C and the singular locus of V . Let

$$(42) \quad \cdots \rightarrow V_n \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V_0 = V$$

be the sequence obtained by first blowing up p , then blowing up the point p_1 on the strict transform C_1 of C above p , and then iterating this construction.

We have a projective embedding of V into a projective space X over \mathbf{k} . The sequence (42) is obtained by constructing the corresponding sequence of blow ups of points

$$\cdots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X$$

and taking the strict transform V_n of V in X_n .

Let $R_n = \mathcal{O}_{X_n, p_n}$ for $n \geq 0$. We have a sequence of homomorphisms of local rings

$$R = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_n \rightarrow \cdots$$

Let

$$I_n = \mathcal{I}_{V_n, p_n}, I_{C_n} = \mathcal{I}_{C_n, p_n},$$

m_n be the maximal ideal of R_n for $n \geq 0$.

There exist $s \leq d$ and regular parameters $x, y_1, \dots, y_s, \dots, y_d$ in R such that

$$y_1 = \cdots = y_d = 0$$

are local equations of C in X , $y_1, \dots, y_s \in I$ and the classes of y_1, \dots, y_s are an R/m basis of $I/m^2 \cap I$.

If y_1, \dots, y_s do not generate I , then there exists $f \in I - (y_1, \dots, y_s)$ such that $f \notin I_C^2$. This can be seen as follows. Since the generic point of C is in the nonsingular locus of V , there exists a basis $y_1, \dots, y_s, h_1, \dots, h_l$ of IR_{I_C} which extends to a regular system of parameters of R_{I_C} . Thus

$$h_1 = \sum_{i=1}^d \lambda_i y_i$$

with $\lambda_i \in R_{I_C}$ and λ_i is a unit in R_{I_C} for some i , with $s+1 \leq i \leq d$. Without loss of generality, $i = d$. For $1 \leq i \leq d$, express

$$\lambda_i = \frac{f_i}{g_i}$$

with $f_i, g_i \in R$ and $g_i \notin I_C$. We have $f_d \notin I_C$.

$$\left(\prod_{j=1}^d g_j \right) h_1 = \sum_{i=1}^d f_i \left(\prod_{j \neq i} g_j \right) y_i \in IR_{I_C} \cap R = I.$$

Thus

$$\sum_{i=1}^d f_i \left(\prod_{j \neq i} g_j \right) y_i \in I - I_C^2.$$

There are regular parameters $x, y_1(n), \dots, y_d(n)$ in R_n defined by $y_i = y_i(n)x^n$ for $1 \leq i \leq d$. We have that $I_{C_n} = (y_1(1), \dots, y_d(n))$. Observe that the residue field of R_n is the residue field of R for all $n \geq 0$.

We have an expansion of f in \hat{R} ,

$$f = \sum_{i=1}^d a_i(x) y_i + \sum_{i_1 + \cdots + i_d \geq 2} a_{i_1, \dots, i_d}(x) y_1^{i_1} \cdots y_d^{i_d}.$$

Let $m_i = \text{ord } a_i(x)$ for $1 \leq i \leq d$, with the convention that $m_i = \infty$ if $a_i = 0$.

After subtracting an appropriate element of (y_1, \dots, y_s) from f , and possibly permuting y_{s+1}, \dots, y_d , we may assume that

$$m_d = \min\{m_1, \dots, m_d\}.$$

For $n > m_d$, we have an expression

$$f = x^{m_d+n} \left[\sum_{i=1}^d x^{m_i-m_d} \bar{a}_i(x) y_i(n) + x\Omega \right]$$

for some $\Omega \in \hat{R}_n$. Let K be the quotient field of R . Then

$$\frac{f}{x^{m_d+n}} \in \hat{R}_n \cap K = R_n$$

is in the prime ideal I_n , as $x = 0$ is a local equation of the exceptional divisor of $X_n \rightarrow X_{n-1}$, and x is thus not in I_n .

$$y_1(n), \dots, y_s(n), \frac{f}{x^{m_d+n}} \in I_n$$

extend to a regular system of parameters in R_n . Iterating this procedure, we achieve that for large n , I_n is generated by part of a regular system of parameters in R_n . Thus p_n is in the nonsingular locus of V_n .

We repeat this procedure for each of the finitely many points of C which are in the singular locus of V , to construct a sequence (41) satisfying the conclusions of the lemma. □

We now prove the main resolution theorem, Theorem 1.1.

By Theorem 16.2, there exists a projective, birational morphism $\Phi : W \rightarrow V$ such that W is nonsingular. Let $U \subset V$ be the open subset consisting of the nonsingular points of V . Let G be the Zariski closure in V of the fundamental locus of the birational map $\Phi^{-1} : U \rightarrow W$. Φ is the blow up of an ideal sheaf \mathcal{I} of V .

By Lemma 17.1, there exists a sequence of blow ups of points, after which we perform a normalization, $\beta : V_1 \rightarrow V$ such that β is an isomorphism above U , and the strict transforms of all curves of G in V_1 are contained in the nonsingular locus of V_1 . β is the blow up of an ideal sheaf \mathcal{J} of V .

By Theorem 1.3, applied to the ideal sheaf $\mathcal{J}\mathcal{O}_W$, there exists a sequence of blow ups of points and nonsingular curves $\gamma : W_1 \rightarrow W$ such that $\mathcal{J}\mathcal{O}_{W_1}$ is locally principal, so that W_1 is nonsingular, $\Psi : W_1 \rightarrow V_1$ is a morphism, and $W_1 \rightarrow W$ is an isomorphism above $\Phi^{-1}(U)$.

Let C_1, \dots, C_r be the curves in V_1 which are in the fundamental locus of the birational map $\Psi^{-1} : V_1 \rightarrow W_1$, are disjoint from $\beta^{-1}(U)$, and intersect the strict transform on V_1 of a curve of G . β maps each C_i into the singular locus of V . Since each C_i intersects the nonsingular locus of V_1 , by the Zariski - Abhyankar factorization theorem [Ab3], $W_1 \rightarrow V_1$ factors as a sequence of blow ups of nonsingular curves above V_1 in a neighborhood of the generic point of each C_i .

We now construct a birational morphism $V_2 \rightarrow V_1$ by blowing up a sequence of points on the intersection points of the strict transform of the C_i and the strict transform of a curve in G , so that the strict transforms of the C_i are disjoint from the strict transforms of the curves in G .

We may now construct a sequence of blow ups of curves (which may possibly be singular) $V_3 \rightarrow V_2$ which dominate the C_i , so that the birational map $W_1 \rightarrow V_3$ is an isomorphism above the generic points of each C_i . We may replace V_3 with its

normalization. Since the C_i contract to the singular locus of V , $\delta : V_3 \rightarrow V$ is an isomorphism over the nonsingular locus of V . $V_3 \rightarrow V_1$ is the blow up of an ideal sheaf \mathcal{K} of V_1 . By Theorem 1.3, applied to the ideal sheaf $\mathcal{K}\mathcal{O}_{W_1}$, we may construct a birational morphism $W_2 \rightarrow W_1$ such that $\mathcal{K}\mathcal{O}_{W_2}$ is locally principal, W_2 is nonsingular, and $W_2 \rightarrow W_1$ is an isomorphism over the complement of a finite number of points of the C_i in V_1 . The induced morphism $\alpha : W_2 \rightarrow V_3$ is the blow up of an ideal sheaf \mathcal{L} of V_3 .

Let G_3 be the fundamental locus of the birational map $\alpha^{-1} : V_3 \rightarrow W_2$, and let H be the union of connected components of G_3 containing the strict transform of G on V_3 . Let $H' = G_3 - H$. By our construction, $H \cap H' = \emptyset$, H is contained in the nonsingular locus of V_3 , and $\delta(H')$ is contained in the singular locus of V . Thus there exists a Zariski open subset U^* of V_3 such that U^* is contained in the nonsingular locus of V_3 , $\delta^{-1}(U) \subset U^*$, δ is an isomorphism above U , and the intersection of U^* and the fundamental locus of α^{-1} is H .

Since V_3 is normal, there exists an effective Weil divisor D on V_3 and an ideal sheaf \mathcal{M} on V_3 such that the support of $\mathcal{O}_{V_3}/\mathcal{M}$ is contained in G_3 , and $\mathcal{L} = \mathcal{O}_{V_3}(-D) \cap \mathcal{M}$. Since $H \cap H' = \emptyset$, $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$ where \mathcal{O}_{V_3} is supported on H' and $\mathcal{O}_{V_3}/\mathcal{M}_2$ is supported on H .

Let $\mathcal{L}_1 = \mathcal{O}_{V_3}(-D) \cap \mathcal{M}_1$. Then $\mathcal{L}_1|_{U^*}$ is a Cartier divisor, and $\mathcal{L}_1|(V_3 - H) = \mathcal{L}|(V_3 - H)$.

Let $V_4 \rightarrow V_3$ be the blow up of \mathcal{L}_1 . By our construction, $V_4 \rightarrow V_3$ is an isomorphism above points of $\delta^{-1}(U)$, so that $V_4 \rightarrow V$ is an isomorphism above the nonsingular locus of V . Since the birational map $V_4 \rightarrow W_3$ is an isomorphism above the complement of H in V_3 , V_4 is nonsingular. Thus $V_4 \rightarrow V$ is a resolution as desired.

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