THE ALGEBRAIC FUNDAMENTAL GROUP OF A CURVE SINGULARITY

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Dedicated to Professor David Buchsbaum

Introduction

A classical theorem of Brauner [3], (also Kähler [10], Zariski [15]), gives a formula for the generators and relations of the topological fundamental group of the knot determined by the germ of an analytically irreducible singular curve in $\mathbf{C^2}$. These elegant formulas depend only on the characteristic pairs of a Puiseux series expansion of the curve. Zariski states this Theorem as part of his discussion on Puiseux series in the chapter on resolution of singularities in his book "Algebraic Surfaces" [16].

Let γ be the germ of an analyically irreducible plane curve singularity at the origin. Let D_ε be an epsilon ball centered at the origin in $\mathbf{C^2}$, and let S_ε be the boundary of D_ε. For ε sufficiently small, the pair $(D_\varepsilon, \gamma\cap D_\varepsilon)$ is homeomorphic to the pair consisting of the cone over S_ε and the cone over $\gamma\cap S_\varepsilon$ (c.f. Theorem 2.10 [11]). Thus $S_\varepsilon-\gamma\cap S_\varepsilon$ is a strong deformation retract of $B_\varepsilon-\gamma\cap B_\varepsilon$, and the topological fundamental group of the knot is isomorphic to $\pi_1^{top}(B_\varepsilon-\gamma\cap B_\varepsilon)$.

The arithmetic analogue of the topological fundamental group of the knot determined by the germ of an analytically irreducible singular plane curve is thus the algebraic fundamental group $\pi_1(\mathrm{Spec}(R)-V(f))$, where $R=k[[x,y]]$ is a power series ring over an algebraically closed field k, and $f\in R$ is irreducible.

The basic theory of the algebraic fundamental group is classical, being understood for Riemann surfaces in the 19th century. The algebraic fundamental group is constructed from the finite topological covers, which are algebraic. Abhyankar extended the algebraic fundamental group to arbitrary characteristic [2] and Grothendieck [8] defined the fundamental group in general.

In positive characteristic, Puiseux series expansions do not always exist (c.f section 2.1 of [5]). However, in characteristic zero, the characteristic pairs of the Puiseux series are determined by the resolution graph of a resolution of singularites of the curve germ. As such, the characteristic pairs can be defined in any characteristic from the resolution graph (c.f [5]).

In this paper we prove an arithmetic analogue of Brauner's theorem, valid in arbitrary characteristic. The generators and relations in our Theorem (Theorem 0.1

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below) for the prime to p part of the algebraic fundamental group, coincide with those of Brauner's Theorem ([3], [10], [15]).

Theorem 0.1. *(Main Theorem) Let $R = k[[x,y]]$ be a power series ring over an algebraically closed field k of characteristic $p \geq 0$. Suppose that $f \in R$ is irreducible. Let $U = \mathrm{Spec}(R_f) = \mathrm{Spec}(R) - V(f)$. Let (m_i, n_i), $1 \leq i \leq g$ be the characteristic pairs of f. Then $\pi_1^{(p)}(U)$ is isomorphic to the prime to p part of the pro-finite completion of the free group on the symbols $Q_0, \ldots, Q_g, P_1, \ldots, P_g$ with the relations $Q_i^{m_i} = P_i^{\overline{n}_i} Q_{i-1}^{m_{i-1}m_i}$, $1 \leq i \leq g$, $Q_0 = 1$, $P_{i+1} P_i^{y_i} Q_{i-1}^{m_{i-1}x_i} = Q_i^{x_i}$, $1 \leq i \leq g-1$. x_i and y_i are integers such that $x_i \overline{n}_i = y_i m_i + 1$, $\overline{n}_i = n_i - n_{i-1} m_i$.*

The prime to p part of a fundamental group π_1 is the quotient $\pi_1^{(p)}$ of π_1 by the closed normal subgroup generated by its p-Sylow subgroups.

Let $C = V(f)$, m be the maximal ideal of R. The basic strategy of the proof is to construct an embedded resolution of singularities $\tau : X \to \mathrm{Spec}(R)$ of C such that the preimage of C is a simple normal crossings divisor. Then Abhyankar's Lemma [2] shows that $\pi_1^{(p)}(\mathrm{Spec}(\hat{\mathcal{O}}_{X,x}) - \tau^{-1}(C))$ has an extremely simple form for any $x \in \tau^{-1}(m)$. We further compute the prime to p part of the algebraic fundamental group of the complement of $\tau^{-1}(C)$ in the formal completion of X along each of the exceptional curves of $X \to \mathrm{Spec}(R)$. Then we make use of an arithmetic analogue of Van Kampen's Theorem by Grothendieck and Murre [9] to give $N+1$ generators (where N is the number of exceptional curves of $X \to \mathrm{Spec}(R)$), and certain relations. This is accomplished in Theorem 6.2 of Section 3. We make essential use of the theory of the fundamental group developed by Grothendieck in [8] and by Grothendieck and Murre in [9]. A related result to Theorem 6.2 has been proven by the authors in [6], where the prime to p part of the local algebraic fundamental group of a quasi-rational surface singularity is computed.

In Section 7 we prove the main theorem, Theorem 0.1. In this section we make an analysis of the combinatorics of the resolution graph of C, as is developed in sections 2 and 4.

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1. Continued fractions

Let n, m be positive integers with $(n, m) = 1$. By the Euclidean algorithm, We can expand $\frac{n}{m}$ in a continued fraction in nonnegative integers t_i,

$$\frac{n}{m} = t_1 + \frac{1}{t_2 + \ddots + \frac{1}{t_\nu}}$$

We represent this by

$$\frac{n}{m} = [t_1, \ldots, t_\nu]. \tag{1}$$

Conversely, given a μ-tuple of nonnegative integers $[t_1, \dots, t_\nu]$ with $t_i > 0$ for $i > 1$, there is a unique factorization $\frac{n}{m}$ with n, m relatively prime positive integers such that

$$\frac{n}{m} = [t_1, \dots, t_\nu].$$

Definition 1.1. *Let $\sigma = (t_1, \dots, t_\nu)$ be a ν-tuple of nonnegative integers with $t_i > 0$ for $i > 1$.*

$$\{p_k = p_k(\sigma), -1 \le k \le \nu\}$$

is the sequence given by

$$p_{-1} = 0, p_0 = 1, p_1 = t_1, p_k = t_k p_{k-1} + p_{k-2},\ k \le \nu$$

Definition 1.2. *Let $\sigma = (t_1, \dots, t_\nu)$ be a ν-tuple as in definition 1.1. Set*

$$q_k = q_k(\sigma) = p_{k-1}((t_2, \dots, t_\nu)).$$

We have the identity

$$q_k = t_k q_{k-1} + q_{k-2}.$$

Lemma 1.3.

$$p_\nu q_{\nu-1} - q_\nu p_{\nu-1} = (-1)^\nu$$

Proof. Let $S = p_\nu q_{\nu-1} - q_\nu p_{\nu-1}$. Now

$$\begin{aligned} p_k q_{k-1} - q_k p_{k-1} &= (t_k p_{k-1} + p_{k-2}) q_{k-1} - (t_k q_{k-1} + q_{k-2}) p_{k-1} \\ &= p_{k-2} q_{k-1} - q_{k-2} p_{k-1} \end{aligned}$$

for any k. Hence by repeated application of the above calculation, we get

$$S = \begin{cases} p_1 q_0 - q_1 p_0 & \text{if } \nu \text{ is odd} \\ p_0 q_1 - q_0 p_1 & \text{if } \nu \text{ is even.} \end{cases}$$

But $p_1 q_0 - q_1 p_0 = 0 - 1 = -1$, so $S = (-1)^\nu$ as desired. □

Lemma 1.4. *Let $\sigma = (t_1, \dots, t_\nu)$ be a ν-tuple of nonnegative integers such that $t_i > 0$ for $i > 1$. Let $\frac{n}{m}$, $(n, m) = 1$ be the unique factorization such that $\frac{n}{m} = [t_1, \dots, t_\nu]$. Let $p_k = p_k(\sigma)$ and $q_k = q_k(\sigma)$ be as in definition 1.1 and 1.2. Then $p_\nu = n$ and $q_\nu = m$. Furthermore,*

$$\frac{p_k}{q_k} = [t_1, \dots, t_k].$$

2. Resolution graphs

Let $R = k[[x, y]]$ be a power series ring over an algebraically closed field k (of arbitrary characteristic). Suppose that $f \in R$ is irreducible (and singular). Let $C = \mathrm{V}(f)$.

There is a unique sequence of blowups of points

$$X_N \to X_{N-1} \to \cdots \to X_1 \to X_0 = \mathrm{Spec}(R)$$

such that if $h : X_N \to X_0$ is the composed morphism, $h^{-1}(C)$ is a divisor with simple normal crossings, and N is the smallest number of blowups such that the preimage of C has this property.

We can associate to C a graph $\Gamma(f)$ called the resolution graph of C. $\Gamma(f)$ is the intersection graph of the exceptional curves of g and the strict transform of C. The vertices of $\Gamma(f)$ are a vertex E_0, corresponding to the strict transform of C on X_N, and vertices E_i, for $1 \le i \le N$, which correspond to the (strict transform in X_N of the) exceptional curve of $X_i \to X_{i-1}$. two vertices are connected by an edge if the corresponding curves intersect. The weight of a vertex is -1 times the self intersection number of the corresponding curve in X_N.

The resolution graph $\Gamma(f)$ can be constructed inductively, by elementary transformations on the intersection graphs Γ_i of the i-th blow up X_i.

$$\Gamma_1 \overset{\sigma_2}{\to} \Gamma_2 \overset{\sigma_3}{\to} \cdots \overset{\sigma_{\tau(g,0,0)}}{\to} \Gamma_{\tau(g,0,0)}. \tag{2}$$

Γ_1 is the single vertex E_1 with $\mathrm{wt}(E_1) = 1$. An elementary transformation of type one of a weighted graph Γ, $\sigma(X,Y)$, adds a new vertex X with weight 1, an edge from X to Y, and increases the weight of Y by 1. An elementary transformation of type two of a weighted graph Γ, $\sigma(X,Y,Z)$, adds a new vertex X with weight 1, edges from X to Y and from X to Z, increases the weights of Y and Z by 1, and removes the edge from Y to Z.

There are unique natural numbers

$$L = \{g; \nu_i, 1 \le i \le g; t_{ij}, 1 \le i \le g, 1 \le j \le \nu_i\} \tag{3}$$

such that g, ν_i are positive, t_{ij} is positive if $j > 1$ or if $i = 1$, and $[t_{i,1}, \ldots, t_{i,\nu_i}] \notin \mathbf{Z}$ for $1 \le i \le g$, such that the data L uniquely determines the construction of $\Gamma(f)$ by the following rules.

Define

$$\tau(i,j,k) = \sum_{a=1}^{i}\sum_{b=1}^{\nu_a} t_{ab} + \sum_{c=1}^{j} t_{i+1,c} + k. \tag{4}$$

$$\sigma_{\tau(i,0,1)} = \sigma(E_{\tau(i,0,1)}, E_{\tau(i,0,0)}) \quad 1 \le i \le g-1$$

If $t_{i+1,1} = 0$, $\nu_{i+1} = 2$, $1 \le i \le g-1$,

$$\sigma_{\tau(i,0,k)} = \sigma(E_{\tau(i,0,k)}, E_{\tau(i,0,k-1)}, E_{\tau(i,0,0)}) \quad 2 \le k \le t_{i+1,2}$$

If $t_{i+1,1} = 0$, $\nu_{i+1} > 2$, $1 \le i \le g-1$,

$$\begin{array}{ll}
\sigma_{\tau(i,0,k)} = \sigma(E_{\tau(i,0,k)}, E_{\tau(i,0,k-1)}, E_{\tau(i,0,0)}) & 2 \le k \le t_{i+1,2}+1 \\
\sigma_{\tau(i,j,k)} = \sigma(E_{\tau(i,j,k)}, E_{\tau(i,j,k-1)}, E_{\tau(i,j,0)}) & 2 \le j \le \nu_{i+1}-2 \\
 & 2 \le k \le t_{i+1,j+1}+1 \\
\sigma_{\tau(i,\nu_{i+1}-1,k)} = \sigma(E_{\tau(i,\nu_{i+1}-1,k)}, E_{\tau(i,\nu_{i+1}-1,k-1)}, E_{\tau(i,\nu_{i+1}-1,0)}) & 2 \le k \le t_{i+1,\nu_{i+1}}
\end{array}$$

If $t_{i+1,1} \ge 1$ and $0 \le i \le g-1$,

$$\begin{array}{ll}
\sigma_{\tau(i,0,k)} = \sigma(E_{\tau(i,0,k)}, E_{\tau(i,0,k-1)}) & 2 \le k \le t_{i+1,1}+1 \\
\sigma_{\tau(i,j,k)} = \sigma(E_{\tau(i,j,k)}, E_{\tau(i,j,k-1)}, E_{\tau(i,j,0)}) & 1 \le j \le \nu_{i+1}-2 \\
 & 2 \le k \le t_{i+1,j+1}+1 \\
\sigma_{\tau(i,\nu_{i+1}-1,k)} = \sigma(E_{\tau(i,\nu_{i+1}-1,k)}, E_{\tau(i,\nu_{i+1}-1,k-1)}, E_{\tau(i,\nu_{i+1}-1,0)}) & 2 \le k \le t_{i+1,\nu_{i+1}}.
\end{array}$$

Conversely, suppose we are given data $L = (g, \nu_i, t_{ij})$ as in (3). Then we can associate a graph $\Gamma(L)$ to L by the following rules.

The vertices of $\Gamma(L)$ are E_i, with $0 \le i \le \tau(g,0,0)$. The edges of Γ are

$$\begin{array}{ll} e(0,\tau(g,0,0)) & \\ e(\tau(i,0,0),\tau(i,0,1)) & 1 \le i \le g-1 \end{array}$$

If $1 \le i \le g-1$, $t_{i+1,1} = 0$ and $\nu_{i+1} = 2$, Γ contains the edges

$$\begin{array}{ll} e(\tau(i,0,k),\tau(i,0,k-1)) & 2 \le k \le t_{i+1,2} \\ e(\tau(i+1,0,0),\tau(i,0,0)). & \end{array}$$

If $1 \le i \le g-1$, $t_{i+1,1} = 0$ and $\nu_{i+1} > 2$, Γ contains the edges

$$\begin{array}{ll} e(\tau(i,0,k),\tau(i,0,k-1)) & 2 \le k \le t_{i+1,2}+1 \\ e(\tau(i,2,1),\tau(i,0,0)) & \\ e(\tau(i,j,k),\tau(i,j,k-1)) & 2 \le j \le \nu_{i+1}-2 \\ & 2 \le k \le t_{i+1,j+1}+1 \\ e(\tau(i,j+1,1),\tau(i,j,0)) & 2 \le j \le \nu_{i+1}-2 \\ e(\tau(i,\nu_{i+1}-1,k),\tau(i,\nu_{i+1}-1,k-1)) & 2 \le k \le t_{i+1,\nu_{i+1}} \\ e(\tau(i,\nu_{i+1},0),\tau(i,\nu_{i+1}-1,0)) & \end{array}$$

If $0 \le i \le g-1$ and $t_{i+1,1} \ge 1$, then Γ contains the edges

$$\begin{array}{ll} e(\tau(i,0,k),\tau(i,0,k-1)) & 2 \le k \le t_{i+1,1}+1 \\ e(\tau(i,j,k),\tau(i,j,k-1)) & 1 \le j \le \nu_{i+1}-2 \\ & 2 \le k \le t_{i+1,j+1}+1 \\ e(\tau(i,j,0),\tau(i,j+1,1)) & 1 \le j \le \nu_{i+1}-2 \\ e(\tau(i,\nu_{i+1}-1,k),\tau(i,\nu_{i+1}-1,k-1)) & 2 \le k \le t_{i+1,\nu_{i+1}} \\ e(\tau(i,\nu_{i+1}-1,0),\tau(i,\nu_{i+1},0)). & \end{array}$$

The weights of the vertices are

$$\mathrm{wt}(\tau(i-1,0,k)) = \begin{cases} 2 & k \ne \sum_{n=1}^{j} t_{i,n} \quad 1 \le j \le \nu_i \\ t_{i,j+1}+2 & k = \sum_{n=1}^{j} t_{i,n} \quad 1 \le j < \nu_i - 1 \\ t_{i,\nu_i}+1 & k = \sum_{n=1}^{\nu_i-1} t_{i,n} \\ t_{i+1,2}+1 & k = \sum_{n=1}^{\nu_i} t_{i,n} \quad i < g, t_{i+1,1} = 0, \nu_{i+1} = 2 \\ t_{i+1,2}+2 & k = \sum_{n=1}^{\nu_i} t_{i,n} \quad i < g, t_{i+1,1} = 0, \nu_{i+1} > 2 \\ 2 & k = \sum_{n=1}^{\nu_i} t_{i,n} \quad i < g, t_{i+1,1} \ne 0 \\ 1 & k = \sum_{n=1}^{\nu_i} t_{i,n} \quad i = g \\ 0 & k = 0 \quad i = 1 \end{cases}$$

From the algorithm following (3), we can associate to L a sequence of blowups of points $h : X_{\tau(g,0,0)} \to X$ of (2), such that $\Gamma(L)$ is the intersection graph of the exceptional curves of g, augmented by a vertex E_0 of weight 0, and an edge from E_0 to $E_{\tau(g,0,0)}$.

Let $\overline{C}$ be a nonsingular curve on $X_{\tau(g,0,0)}$ which is disjoint from E_j for $j < \tau(g,0,0)$ and intersects $E_{\tau(g,0,0)}$ transversely at a point. This is possible since X is Henselian. Let $f \in R$ be a local equation for $C = h(\overline{C})$. Then $\Gamma(L)$ is the resolution graph $\Gamma(f)$ of f.

Consider the sets of characteristic pairs

$$N = \{(m_1,n_1),\ldots,(m_g,n_g)\}$$

where g is a positive integer, (m_i, n_i) are relatively prime positive integers, $n_1 > m_1$ and $m_i > 1$ for all i.

There is a 1-1 correspondence between the sets of characteristic pairs N and the set of resolution graphs Γ which arise from resolution of irreducible (singular) $f \in R$.

Suppose that $N = \{(m_1, n_1), \dots, (m_g, n_g)\}$ is a set of characteristic pairs. We will associate a resolution graph to N.

Define $\overline{n}_1 = n_1$, $\overline{n}_i = n_i - n_{i-1}m_i$ for $2 \le i \le g$. From the continued fraction representations

$$\frac{\overline{n}_i}{m_i} = [t_{i,1}, \dots, t_{i,\nu_i}] \tag{5}$$

of (1), we obtain data $L = (g, \nu_i, t_{ij})$. we have shown above that $\Gamma(L)$ is a resolution graph.

We can associate characteristic pairs $\{(m_1, n_1), \dots, (m_g, n_g)\}$ to a resolution graph $\Gamma(f)$ by formula (3) and the formula

$$\frac{\overline{n}_i}{m_i} = \frac{n_i + n_{i-1}m_i}{m_i} = [t_{i,1}, \dots, t_{i,\nu_i}], 1 \le i \le g. \tag{6}$$

$n_1 > m_1$ since $t_{11} > 0$.

3. Puiseux Series

In characteristic zero, the characteristic pairs of f play a remarkable role in the Puiseux series expansion of f (c.f chapter 1 [16]). In positive characteristic, Puiseux series expansions do not always exist (c.f section 2.1 of [5]).

In this section we will suppose that k has characteristic zero. Given a nonunit $f \in R$, Newton developed an algorithm to construct a fractional power series expansion (c.f. [4])

$$\begin{aligned} y = \; & \textstyle\sum_{i=1}^{k_1} a_{1,i}x^i + b_1 x^{\frac{\tilde{n}_1}{\tilde{m}_1}} + \sum_{i=1}^{k_2} a_{2,i} x^{\frac{\tilde{n}_1+i}{\tilde{m}_1}} + b_2 x^{\frac{\tilde{n}_2}{\tilde{m}_1\tilde{m}_2}} + \cdots \\ & + \textstyle\sum_{i=1}^{k_g} a_{g,i} x^{\frac{\tilde{n}_{g-1}+i}{\tilde{m}_1\tilde{m}_2\cdots\tilde{m}_{g-1}}} + b_g x^{\frac{\tilde{n}_g}{\tilde{m}_1\tilde{m}_2\cdots\tilde{m}_g}} + \sum_{i=1}^{\infty} c_i x^{\frac{\tilde{n}_g+i}{\tilde{m}_1\tilde{m}_2\cdots\tilde{m}_g}}. \end{aligned} \tag{7}$$

We can write the above fractional powerseries as a power series $y(x^{\frac{1}{\nu}})$ in $x^{\frac{1}{\nu}}$, where $\nu = \tilde{m}_1\tilde{m}_2\cdots\tilde{m}_g$. This series is a solution to

$$f(x, y(x^{\frac{1}{\nu}})) = 0.$$

After suitable choice of regular parameters x, y in R, we can normalize (7) by the conditions

$$1 < \frac{\tilde{n}_1}{\tilde{m}_1} < \frac{\tilde{n}_2}{\tilde{m}_1\tilde{m}_2} < \cdots < \frac{\tilde{n}_g}{\tilde{m}_1\cdots\tilde{m}_g}$$

$\tilde{m}_j > 1$ for $1 \le j \le g$, $(\tilde{m}_j, \tilde{n}_j) = 1$ for $1 \le j \le g$ and $b_j \ne 0$ for $1 \le i \le g$.

If f is irreducible, then the expansion (7) is a Puiseux series expansion of f. Let ω be a primitive ν^{th} root of unity. Then

$$f = \gamma \prod_{i=0}^{\nu-1}(y - y(\omega^i x^{\frac{1}{\nu}}))$$

for some unit $\gamma \in R$.

Classically (c.f. [4], [7], [16]) the characteristic pairs are defined as the

$$\{(\tilde{m}_1, \tilde{n}_1), \dots, (\tilde{m}_g, \tilde{n}_g)\} \tag{8}$$

obtained from the Puiseux series expansion (7) of f. We will call (8) the classical characteristic pairs of f.

Enriques and Chisini [7] construct the resolution graph of f from the classical characteristic pairs of f. Neuerburg, in his thesis [13], gives a modern proof. He shows that the algorithm following (3) applied to the data L constructed from (8) by the formula (6) constructs the resolution graph of f. The case $t_{i,1} = 0$ requires special treatment. This point is not clear in the literature.

Thus (assuming that k has characteristic zero), the characteristic pairs are precisely the classical characteristic pairs.

4. A reordering of the resolution graph

We will now reindex the nodes of the resolution graph $\Gamma(f)$ to give a combinatorial description of $\Gamma(f)$ which will be more natural for our analysis of the fundamental group.

We can describe $\Gamma(f)$ as follows. Let (m_i, n_i), $1 \le i \le g$ be the characteristic pairs of f, $\overline{n}_i = n_i - m_i n_{i-1}$. Let

$$\frac{\overline{n}_i}{m_i} = [t_{i,1}, \dots, t_{i,\nu_i}]$$

be the continued fraction representation as in (5). Let

$$\begin{array}{lcl} b_{i,2k-1} & = & \sum_{j=1}^{k} t_{i,2j-1} \\ b_{i,2k} & = & \sum_{j=1}^{k} t_{i,2j} \\ b_{i,k} & = & 0, k \le 0. \end{array}$$

The graph $\Gamma(f)$ has $g+1$ horizontal branches with each branch connected to the next by a single edge. The $(g+1)^{st}$ horizontal branch has one node O_{g+1} with weight 0. For $i \le g$, the i^{th} horizontal branch has

$$\sum_{j=1}^{\nu_i} t_{i,j}$$

nodes. There are

$$\sum_{j=1}^{\left[\frac{\nu_i+1}{2}\right]} t_{i,2j-1}$$

nodes

$$L_i(1), L_i(2), \dots, L_i\Big(\sum_{j=1}^{\left[\frac{\nu_i+1}{2}\right]} t_{i,2j-1}\Big)$$

numbered from the left, and

$$\sum_{j=1}^{\left[\frac{\nu_i}{2}\right]} t_{i,2j}$$

nodes

$$R_i(1), R_i(2), \dots, R_i(\sum_{j=1}^{[\frac{\nu_i}{2}]} t_{i,2j})$$

numbered from the right. R_i denotes the right end of the i^{th} branch where as the left extreme L_i of the i^{th} branch is connected by an edge to the node O_{i-1} of the $(i-1)^{st}$ branch. When $t_{i,1}=0$, and $\nu_i = 2$, O_i coincides with L_i. Further,

$$O_i = \begin{cases} L_i(b_{i,\nu_i}) & \text{if } \nu_i \text{ is odd} \\ R_i(b_{i,\nu_i}) & \text{if } \nu_i \text{ is even.} \end{cases}$$

O_{g+1} is connected by an edge to the node O_g.

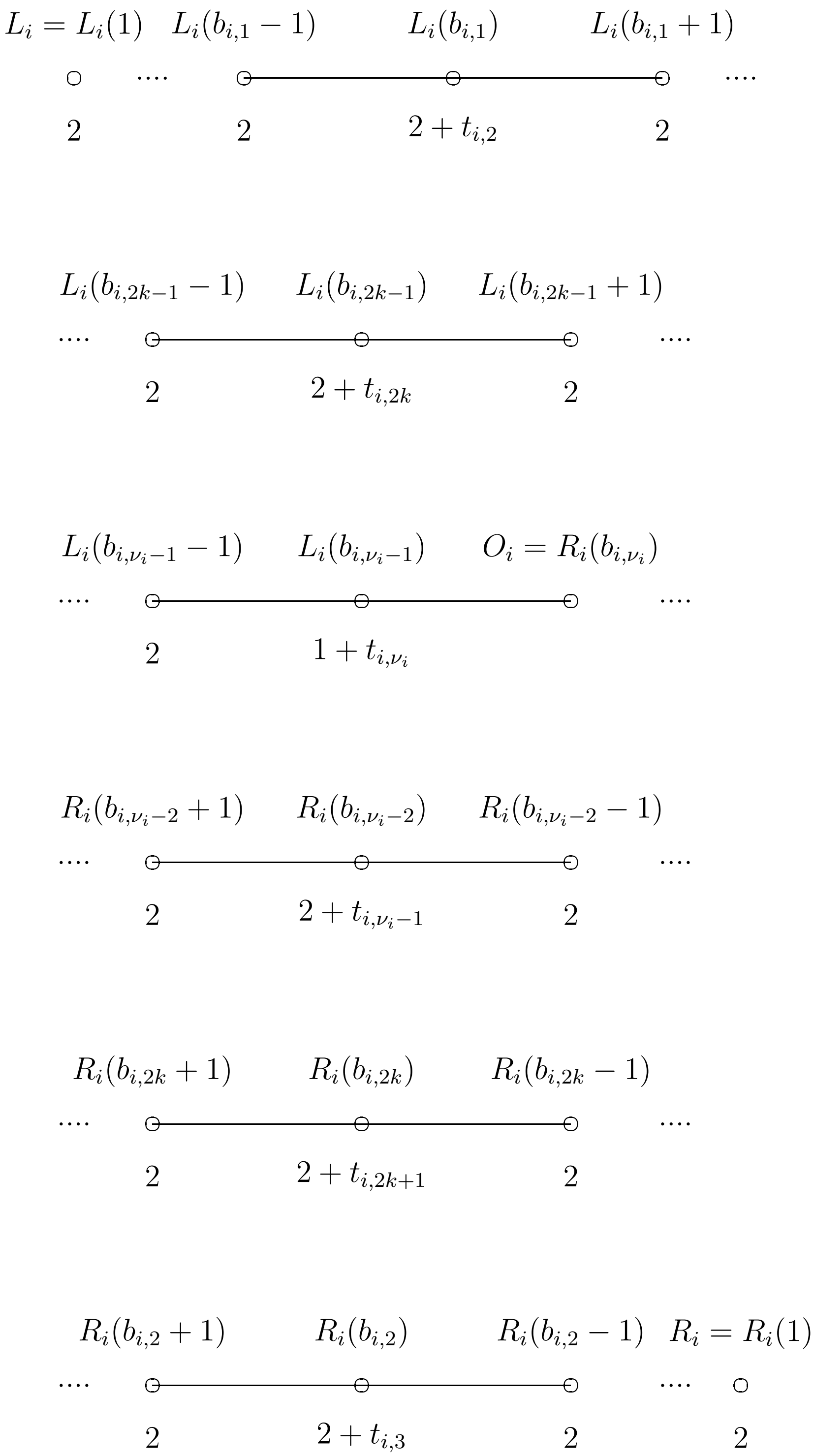
All nodes except O_i, $1 \le i \le g$ and $L_i(b_{i,2k-1})$, $R_i(b_{i,2k})$ have weight 2. ($L_i(b_{i,1})$ does not exist if $t_{i,1}=0$).

$$\mathrm{wt}(O_i) = \begin{cases} 2 & \text{if } t_{i+1,1} \neq 0 \\ 1 + t_{i+1,2} & \text{if } t_{i+1,1} = 0, \nu_{i+1} = 2 \\ 2 + t_{i+1,2} & \text{if } t_{i+1,1} = 0, \nu_{i+1} \neq 2 \\ 1 & \text{if } i = g \end{cases}$$

$$\mathrm{wt}(L_i(b_{i,2k-1})) = \begin{cases} 2 + t_{i,2k} & 2k-1 < \nu_i - 1 \\ 1 + t_{i,2k} & 2k-1 = \nu_i - 1 \end{cases}$$

$$\mathrm{wt}(R_i(b_{i,2k})) = \begin{cases} 2 + t_{i,2k+1} & 2k < \nu_i - 1 \\ 1 + t_{i,2k+1} & 2k = \nu_i - 1 \end{cases}$$

ν_i even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$L_i(b_{i,2k-1}-1)$ $L_i(b_{i,2k-1})$ $L_i(b_{i,2k-1}+1)$
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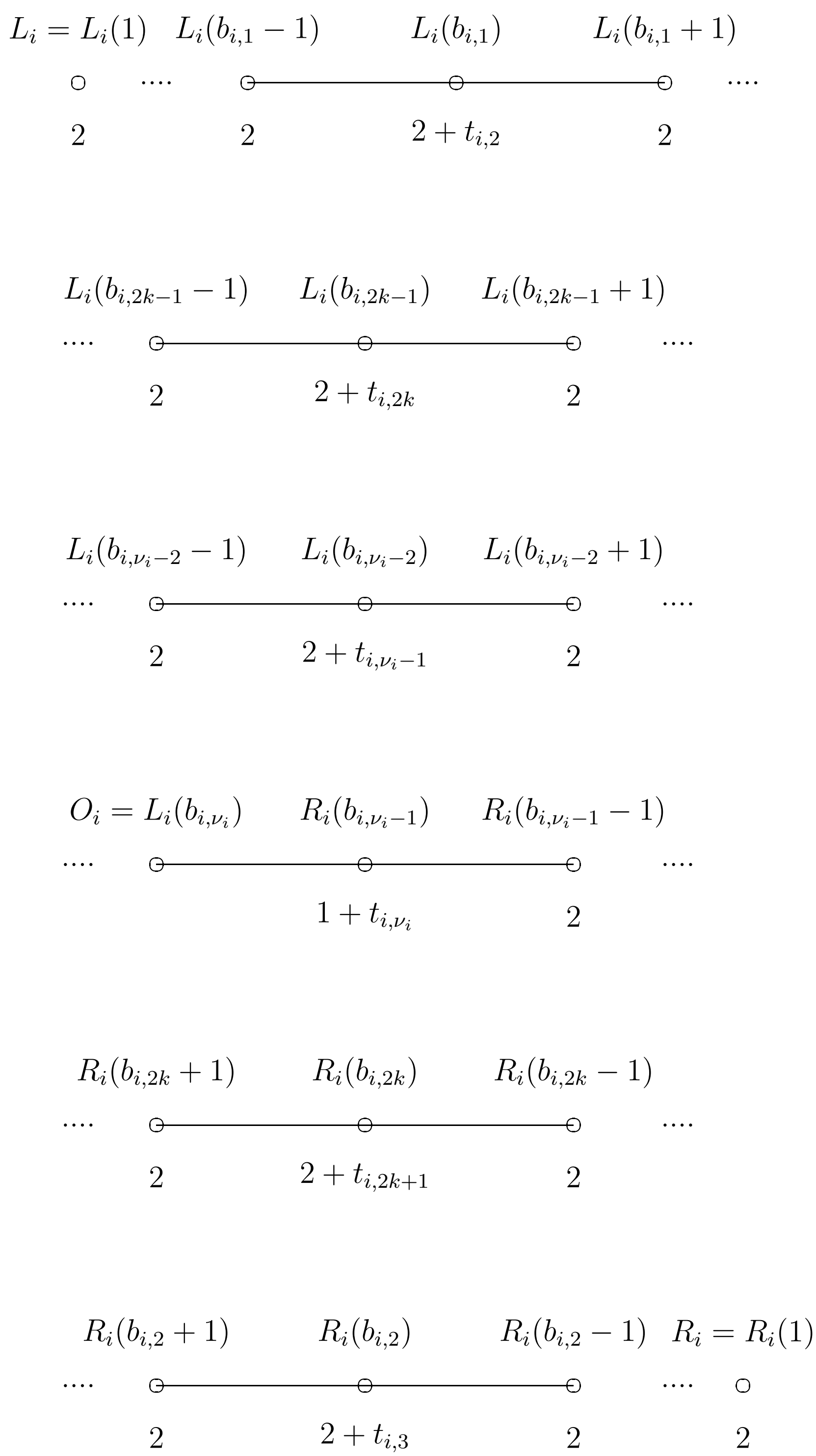
ν_i odd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$L_i(b_{i,1})$
$L_i(b_{i,1} + 1)$
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2
$L_i(b_{i,2k-1} - 1)$
$L_i(b_{i,2k-1})$
$L_i(b_{i,2k-1} + 1)$
2
$2 + t_{i,2k}$
2
$L_i(b_{i,\nu_i-2} - 1)$
$L_i(b_{i,\nu_i-2})$
$L_i(b_{i,\nu_i-2} + 1)$
2
$2 + t_{i,\nu_i-1}$
2
$O_i = L_i(b_{i,\nu_i})$
$R_i(b_{i,\nu_i-1})$
$R_i(b_{i,\nu_i-1} - 1)$
$1 + t_{i,\nu_i}$
2
$R_i(b_{i,2k} + 1)$
$R_i(b_{i,2k})$
$R_i(b_{i,2k} - 1)$
2
$2 + t_{i,2k+1}$
2
$R_i(b_{i,2} + 1)$
$R_i(b_{i,2})$
$R_i(b_{i,2} - 1)$
$R_i = R_i(1)$
2
$2 + t_{i,3}$
2
2

Remark 4.1. *When compared with the resolution graph in section 2, where the j-th node E_j corresponds to the exceptional divisor of the j-th blowup, we see that for $1 \leq k \leq t_{i,j+1}$, $E_{\tau(i-1,j,k)}$ is the $(b_{i,j-1}+k)^{th}$ node from the left on the i^{th} branch if j is even, and is the $(b_{i,j-1}+k)^{th}$ node from the right on the i^{th} branch if j is odd.*

5. Algebraic fundamental groups

Suppose that X is a normal irreducible scheme. Let η be the generic point of X, $K = \mathcal{O}_{\mathcal{K},\eta}$, the "function field" of X. Let Ω be an algebraic closure of K.

Suppose that L is a finite extension of K which is Galois over K. we can form the normalization of Y of X in L. The set of all such Y which are unramified over X form a projective system Y_m. If $L_n \subset L_m$ we have natural morphisms $\varphi_{n,m} : Y_m \rightarrow Y_n$. We have natural maps $\tau_m : \mathrm{Spec}(\Omega) \rightarrow Y_m$ which are compatible with the $\varphi_{n,m}$. Let M be the union of the fields L_m. M is a subfield of Ω which is Galois over K.

$$\mathrm{Aut}_K(M) = \varprojlim \mathrm{Aut}_K(L_m) \cong \varprojlim(\mathrm{Aut}_X(Y_m))^{op}.$$

$\varprojlim(\mathrm{Aut}_X(Y_m))^{op}$ is the algebraic fundamental group $\pi_1(X)$.

Let $\mathcal{C}$ be the category of etale covers of X. Consider the functor $F : \mathcal{C} \rightarrow$ finite sets defined by

$$F(Z) = \mathrm{Hom}_X(\mathrm{Spec}(\Omega), Z).$$

$\mathcal{C}$ is a Galois category, and F is a fundamental functor for $\mathcal{C}$ (Exposé V [8], Chapter IV [12]).

Suppose that $Z \in \mathcal{C}$. For each Y_m we have a natural map $\mathrm{Hom}_X(Y_m, Z) \rightarrow F(Z)$, given by $h \mapsto h \circ \tau_m$.

$$\varinjlim(\mathrm{Hom}_X(Y_m, Z) \rightarrow F(Z)$$

is an isomorphism. Thus the system $\{Y_m\}$ pro-represents F. F is an equivalence between $\mathcal{C}$ and the category $\mathcal{C}(\pi)$ of finite sets on which $\pi = \pi_1(X)$ acts continuously (Theorem V 4.1 [8], Theorem 4.4 [12]).

Suppose that D is a divisor on X. This method can be extended to construct a fundamental group of $X - D$. In this case we consider normal Galois Y_m such that $Y_m \mid (X - D)$ is unramified. We can further consider the category $\mathcal{C}^{\mathcal{D}}(\mathcal{X})$ of normal covers Y of X which are etale over $X - D$ and such that p does not divide $|\mathrm{Aut}_X(Z) \mid$ whenever Z is a Galois closure of an irreducible component of Y. In this case the Galois Y_m which are unramified over $X - D$ and such that p does not divide $|\mathrm{Aut}_X(Y_m) \mid$ pro-represent the fundamental functor $F \mid \mathcal{C}^{\mathcal{D}}(\mathcal{X})$. The prime to p part $\pi_1^{(p)}(X - D)$ of the fundamental group $\pi_1(X - D)$ is a fundamental group for $\mathcal{C}^{\mathcal{D}}(\mathcal{X})$. The prime to p part of a fundamental group π_1 is the quotient $\pi_1^{(p)}$ of π_1 by the closed normal subgroup generated by its p-Sylow subgroups.

Let $\mathcal{S}$ be a normal, connected formal scheme with a simple normal crossings divisor D on $\mathcal{S}$. Let $\mathcal{C}^{\mathcal{D}}(\mathcal{S})$ be the category of formal $\mathcal{S}$ schemes Y which are normal, etale over $\mathcal{S} - \mathcal{D}$, and such that p does not divide $|\mathrm{Aut}_{\mathcal{X}}(Z) \mid$ whenever Z is a Galois closure of an irreducible component of Y. $\mathcal{C}^{\mathcal{D}}(\mathcal{S})$ is a Galois category by Proposition 4.2.2 of [9], and thus has a fundamental group by Exposé V [8]. A fundamental functor for $\mathcal{C}^{\mathcal{D}}(\mathcal{S})$ can be constructed as follows (4.2.3 [9]). Let $s \in \mathcal{S}$, and let $V = \mathrm{spf}(A)$ be a neighborhood of s in $\mathcal{S}$. Let $V^* = \mathrm{Spec}(A)$, and choose $s^* \in V^*$

such that s^* is not on the divisor on V^* corresponding to D, and s is in the closure of $\{s^*\}$. Let Ω be an algebraically closed field containing $k(s^*)$. We can define a fundamental functor $F : \mathcal{C}^{\mathcal{D}}(\mathcal{S}) \rightarrow$ finite sets by

$$F(\mathcal{T}) = \mathrm{Hom}_{\mathcal{V}^*}(\mathrm{Spec}(\Omega), \mathcal{T}^*)$$

where $T^* = \mathrm{Spec}(B)$ if $T \mid V^* = \mathrm{spf}(B)$.

Suppose that E is an integral divisor on a normal formal scheme $\mathcal{X}$, $\mathcal{Y} \rightarrow \mathcal{X}$ is finite, and the integral divisor E' maps to E. The isotropy group $I(E'/E)$ is defined as follows. Let $R = \mathcal{O}_{\mathcal{X},\mathcal{E}}$, $S = \mathcal{O}_{\mathcal{Y},\mathcal{E}'}$. Let K be the quotient field of R, L be the quotient field of S. Let a be the maximal ideal of R, b the maximal ideal of S. Define

$$I(E'/E) = \{g \in \mathrm{Aut}_{\mathcal{X}}(\mathcal{Y}) \mid \}(\S) \equiv \S(\bmod \dashv) \text{ for all } \S \in \mathcal{S}\}.$$

By the ramification theory of Dedekind domains (c.f Chapter V, Section 10 [16]), $I(E'/E)$ is canonically isomorphic to a subgroup of the multiplicative group $(R/a)^*$.

6. Preliminary analysis

We will now follow the notation of the statement of Theorem 0.1. Set $N = \tau(g, 0, 0)$. Set $E = \sum_{i=0}^{N} E_i$, a simple normal crossings divisor on X. Let $\mathcal{X}$ be the formal completion of X along $E' = \sum_{i=1}^{N} E_i$. Let $p_{ij} = E_i \cap E_j$ whenever E_i and E_j intersect properly. Let $\mathcal{X}_{\rangle}$ be the formal completion of X along E_i, and let $\mathcal{X}_{\rangle|}$ be the formal completion of X along p_{ij}.

Let π be a fundamental group for $\mathcal{C}^{\mathcal{E}}(\mathcal{X})$. By Corollary 9.9 [9] we have

$$\pi_1^{(p)}(\mathrm{Spec}(R_f)) \cong \pi.$$

Let π_i be a fundamental group for $\mathcal{C}^{\mathcal{E}}(\mathcal{X}_{\rangle})$, and let π_{ij} be a fundamental group for $\mathcal{C}^{\mathcal{E}}(\mathcal{X}_{\rangle|})$.

Let μ_r be the r-th roots of unity of k. Set $\mu^t = \mathrm{Lim}_{p \not| r} \mu_r$.

We first analyze π_{ij}. Let $T = \mathcal{O}_{\mathcal{X},\sqrt{\rangle|}} \cong \|[[\S, \dagger]]$. Whenever p does not divide m and p does not divide n, set $T_{mn} = k[[s,t]]/(s^m - x, t^n - y)$, $Y_{mn} = \mathrm{spf}(T_{mn})$. By Abhyankar's Lemma ([2], XIII 5.3 [8]), $\pi_{ij} \cong \varprojlim \mathrm{Aut}_{\mathcal{X}_{\rangle|}}(\mathcal{Y}_{\Updownarrow \setminus}) \cong \mu^{\sqcup} \oplus \mu^{\sqcup}$, where the summands are the direct limits of inertia groups of prime divisors ramified over $E_i \cap \mathcal{X}_{\rangle|}$ and $E_j \cap \mathcal{X}_{\rangle|}$ respectively. We thus have

$$\pi_{ij} \cong (F(\alpha_i, \alpha_j)/[\alpha_i, \alpha_j])^{(p)}.$$

Furthermore,

$$\pi_0 \cong (F(\alpha_0, \alpha_N)/[\alpha_0, \alpha_N])^{(p)}.$$

We now analyze π_i for $i > 0$, and define a continuous homomorphism (a path)

$$\pi_{ij} \rightarrow \pi_i. \tag{9}$$

This construction will be used in the proof of Lemma 6.1. Let $\{T_b, \psi_{ab}\}$ pro-represent a fundamental functor for $\mathcal{C}^{\mathcal{E}}(\mathcal{X}_{\rangle})$, where the T_b are Galois. Thus $\pi_i \cong \varprojlim \mathrm{Aut}_{\mathcal{X}_{\rangle}}(T_b)^{op}$. Let Ω be an algebraically closed field containing the quotient field of A, where $\mathcal{X}_{\rangle|} = \mathrm{spf}(\mathcal{A})$. $F(Z) = \mathrm{Hom}(\mathrm{Spec}(\Omega), Z)$ is a fundamental functor for

$\mathcal{C}^{\mathcal{E}}(\mathcal{X}_{\rangle|})$. $G(Z) = \mathrm{Hom}(\mathrm{Spec}(\Omega), Z \mid \mathcal{X}_{\rangle|})$ is a fundamental functor for $\mathcal{C}^{\mathcal{E}}(\mathcal{X}_{\rangle})$. Let $t_b \in G(T_b)$ be a coherent system of points.

Suppose that $\{S_b, \varphi_{ab}\}$ pro-represent F, where the S_b are Galois, and $\tau_b \in F(S_b)$ are a coherent system of points. $\pi_{ij} \cong \varprojlim \mathrm{Aut}_{\mathcal{X}_{\rangle|}}(S_b)^{op}$.

For given T_b, we will define a homomorphism $u_b : \pi_{ij} \to \mathrm{Aut}_{\mathcal{X}_{\rangle}}(T_b)^{op}$. For $a >> 0$, there exists $h \in \mathrm{Hom}_{\mathcal{X}_{\rangle|}}(S_a, T_b \mid \mathcal{X}_{\rangle|})$ such that $t_b = h \circ \tau_a$, since $\varinjlim \mathrm{Hom}_{\mathcal{X}_{\rangle|}}(S_a, T_b \mid \mathcal{X}_{\rangle|}) \cong \mathcal{G}(\mathcal{T}_{\lfloor})$. Suppose that $\sigma \in \pi_{ij}$. $h \circ \sigma \circ \tau_a \in G(T_b)$, and $G(T_b)$ is a principal homogeneous space under the action of $\mathrm{Aut}_{\mathcal{X}_{\rangle}}(T_b)^{op}$ on the right, since T_b is Galois. Thus there exists a unique $u_b(\sigma) \in \mathrm{Aut}_{\mathcal{X}_{\rangle}}(T_b)^{op}$ such that $u_b(\sigma) \circ t_b = h \circ \sigma \circ \tau_a$. Since $\psi_{ba} \circ u_b = u_a$, we can construct a continuous homomorphism $\pi_{ij} \to \pi_i$.

Lemma 6.1. *Suppose that $i \neq 0$, for some l, a path $\lambda_i^{ij_l} : \pi_{ij_l} \to \pi$ is given, and that τ is a permutation of $\{1, \ldots, m(i)\}$. Then there exist paths $\lambda_i^{ij_k} : \pi_{ij_k} \to \pi_i$ such that*

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})/\alpha_{j_{\tau(1)}}\alpha_{j_{\tau(2)}}\cdots\alpha_{j_{\tau(\tau(m(i))}}\alpha_i^{e_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{j_{m(i)}}] = 1)^{(p)},$$

where $e_i = (E_i^2)$.

Proof. Suppose that $\{Y_m\}$ pro-represent $\mathcal{C}^{\mathcal{E}}(\mathcal{X}_{\rangle})$, where each Y_m is normal, connected and Galois over $\mathcal{X}_{\rangle}$. Let $\varphi_m : Y_m \to \mathcal{X}_{\rangle}$. By Abhyankar's Lemma ([1], XIII 5.3 [8]), and since $\mathcal{X}_{\rangle}$ is complete along E_i, $E_i^m = (\varphi_m^{-1}(E_i))_{\mathrm{red}}$ is irreducible. Hence the inertia group $H_m = I(E_i^m/E_i)$ is a normal subgroup of $\mathrm{Aut}_{\mathcal{X}_{\rangle}}(Y_m)$. The quotient $Y_m/H_m \in \mathcal{C}^{\sum_{|\neq\rangle}\mathcal{E}_|}(\mathcal{X}_{\rangle})$. Let Π be a fundamental group for $\mathcal{C}^{\sum_{|\neq\rangle}\mathcal{E}_|}(\mathcal{X}_{\rangle})$. We have an onto group homomorphism $\Psi : \pi_i \to \Pi$, whose kernel is $N = \varprojlim I(E_i^m/E_i)$. We further have $\Pi \cong \pi_1^{(p)}(E_i - \sum_j p_{ij})$ since $\mathcal{X}_{\rangle}$ is complete along E_i (Lemma 9.6 [9]). We thus have the exact sequence of Corollary 5.1.11 [9].

$$\mu^t \to \pi_i \to \pi_1^{(p)}(E_i - \sum_j p_{ij}) \to 0. \tag{10}$$

Suppose that $\{Z_m\}$ pro-represents $\mathcal{C}^{\sum_{|\neq\rangle}\mathcal{E}_|}(\mathcal{X}_{\rangle})$. The path $\lambda_i^{ij_l}$ determines a coherent system of irreducible divisors $E_{j_l}^m$ in Z_m which map to to E_{j_l} in $\mathcal{X}_{\rangle}$. There exist irreducible divisors $E_{j_k}^m$ in Z_m which map to E_{j_k} in $\mathcal{X}_{\rangle}$ for $1 \leq k \leq m(i)$, and $k \neq j_l$, such that if α_j are generators of $\varprojlim I(E_m^j/E) \cong \mu^t$,

$$\Pi \cong (F(\alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})/\alpha_{j_{\tau(1)}}\cdots\alpha_{j_{\tau(m(i))}} = 1)^{(p)}.$$

This can be deduced from section 7 of [1], Theorem 12.1 [14], and the isomorphism $\Pi \cong \pi_1^{(p)}(E_i - \sum_j p_{ij})$.

Let α_i be a generator of $N = \varprojlim I(E_i^m/E_i)$. $\pi_1^{(p)}(\mathcal{X}_{\rangle} - \sum \mathcal{E}_|)$ is a quotient of $F(\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})$. By (9) we can choose paths $\lambda_i^{ij_k}$ such that $\lambda_i^{ij_k}(\alpha_{j_k}) = \alpha_{j_k}$. $\lambda_i^{ij_k}(\alpha_i) = \alpha_i$ since $\varphi_m^{-1}(E_i)$ is irreducible.

Let $d_i = (E_i^2)$. Let s be an integer between 1 and $m(i)$. Let r be an integer such that $(r, p) = 1$, $(r, d_i) = 1$ and $r > -d_i$. Since E_i can be contracted inside $\mathcal{X}_{\rangle}$ to a rational singularity, there exists $g \in \Gamma(\mathcal{X}_{\rangle}, \mathcal{O}_{\mathcal{X}_{\rangle}})$ such that $(g) = -d_i E_s + E_i$. Let $\varphi : \mathcal{W}_{\nabla} \to \mathcal{X}_{\rangle} \in \mathcal{C}^{\mathcal{E}}(\mathcal{X}_{\rangle})$ be defined so that $\varphi_*(\mathcal{O}_{\mathcal{W}_{\nabla}})$ is the normalization of

$\mathcal{O}_{\mathcal{X}_\rangle}[\sqcup]/(\sqcup^\nabla - \})$. We can choose a surjection $\Lambda : \pi_j \to \mathrm{Aut}_{\mathcal{X}_\rangle}(\mathcal{W}_\nabla)$. φ is unramified over E_{j_k} if $k \neq s$. Hence $\Lambda(\alpha_{j_k}) = 1$ if $k \neq s$. Consideration of the induced map $\pi_{ij} \to \mathrm{Aut}_{\mathcal{X}_\rangle}(\mathcal{W}_\nabla)$ shows that

$$\mathrm{Aut}_{\mathcal{X}_\rangle}(\mathcal{W}_\nabla) = (\mathcal{F}(\alpha_\rangle, \alpha_{|_\int})/\alpha_\rangle^\nabla = \alpha_{|_\int}^\nabla = [\alpha_\rangle, \alpha_{|_\int}] = \alpha_{|_\int}\alpha_\rangle^{\lceil_\rangle} = \infty). \tag{11}$$

By taking r arbitrarily large, we see from (11) that (10) is left exact, and

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_{m(j)}})/\alpha_{j_{\tau(1)}}\alpha_{j_{\tau(2)}} \cdots \alpha_{j_{\tau(\tau(m(i))}}\alpha_i^{e_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{j_{m(i)}}] = 1)^{(p)}$$

for some integer e_i. Now (11) shows that $e_i = d_i$. □

The following theorem is related to a similar result of the authors proved in Theorem 3 of [6]. This is one of the main ingredients in the proof of our main theorem.

Theorem 6.2. *Suppose that $E_{j_1}, \ldots, E_{j_{m(i)}}$ are the curves which intersect E_i properly, and τ_i are permutations of $\{j_1, \ldots, j_{m(i)}\}$. Then $\pi_1^{(p)}(\mathrm{Spec}(R_f))$ is isomorphic to the prime to p part of the pro-finite completion of the free group on the symbols $\{\alpha_0, \alpha_1, \ldots, \alpha_N\}$, with the relations*

$$\begin{array}{ll} \alpha_{j_{\tau_i(1)}}\alpha_{j_{\tau_i(2)}} \cdots \alpha_{j_{\tau_i(m(i))}}\alpha_i^{e_i} = 1 & 1 \le i \le N \\ [\alpha_i, \alpha_{j_1}] = \cdots = \left[\alpha_i, \alpha_{j_{m(i)}}\right] = 1 & 1 \le i \le N \end{array}$$

where $e_i = (E_i^2)$.

Proof. Since the resolution graph Γ is a tree, it follows from Lemma 6.1 and induction that it is possible to choose paths $\lambda_i^{ij} : \pi_{ij} \to \pi_i$ and $\varphi_i : \pi_i \to \pi$ such that

$$\begin{array}{cccc} & \pi_{ij} & \stackrel{\lambda_i^{ij}}{\rightarrow} & \pi_i \\ \lambda_i^{ij} & \downarrow & & \downarrow & \varphi_i \\ & \pi_i & \stackrel{\varphi_j}{\rightarrow} & \pi \end{array} \tag{12}$$

commutes and

$$\pi_i = (F(\alpha_i, \alpha_{j_1}, \ldots, \alpha_{j_{m(i)}})/\alpha_{j_{\tau_i(1)}}\alpha_{j_{\tau_i(2)}} \cdots \alpha_{j_{\tau_i(m(i))}}\alpha_i^{e_i} = [\alpha_i, \alpha_{j_1}] = \cdots = [\alpha_i, \alpha_{j_{m(i)}}] = 1)^{(p)},$$

where $E_{j_1}, \ldots, E_{j_{m(i)}}$ are the curves which intersect E_i properly, and τ_i are given permutations of $\{j_1, \ldots, j_{m(i)}\}$. We can then identify α_i with $\varphi_i(\alpha_i) = \varphi_j(\alpha_i)$ in π. An application of the arithmetic analogue of Van Kampen's Theorem proved in Corollary 8.3.6 [9] to (12) finishes the proof of Theorem 6.2. □

7. The group associated to a tree

Suppose that Γ is a tree with weighted nodes $f_1, \ldots, f_t$ and weights $e_1, \ldots, e_t$ respectively, such that one of the nodes f_s is a distinguished node. The group $G(\Gamma)$ associated to Γ is the free group generated by the nodes $f_1, \ldots, f_t$ modulo the relations

$$f_{i1} \cdots f_{i_r} f_i^{-e_i} = 1, i \neq s \tag{13}$$

$$f_i f_{i_j} = f_{i_j} f_i,\ 1 \le j \le r \tag{14}$$

where $f_{i_1}, \dots, f_{i_r}$ are the nodes adjacent to f_i. A node with only one adjacent node is called an end.

In the relations (13), we have made a choice of an ordering of the nodes $f_{i_1}, f_{i_2}, \dots, f_{i_r}$ adjacent to f_i. A change in this ordering results in gorups which are isomorphic but not equal.

By Theorem 6.2, the prime to p part of the algebraic fundamental group of the complement of f is the pro-finite completion with respect to normal subgroups of finite index prime to p of the group $G(f) = G(\Gamma)$ associated to the resolution graph $\Gamma = \Gamma(f)$ of f, where our distinguished node is O_{g+1}.

We will not specify the ordering of the nodes in the relations (13) at this time. The ordering is only significant at the nodes which are adjacent to three nodes. We will make this choice as is appropriate in the course of the proof of Theorem 7.3.

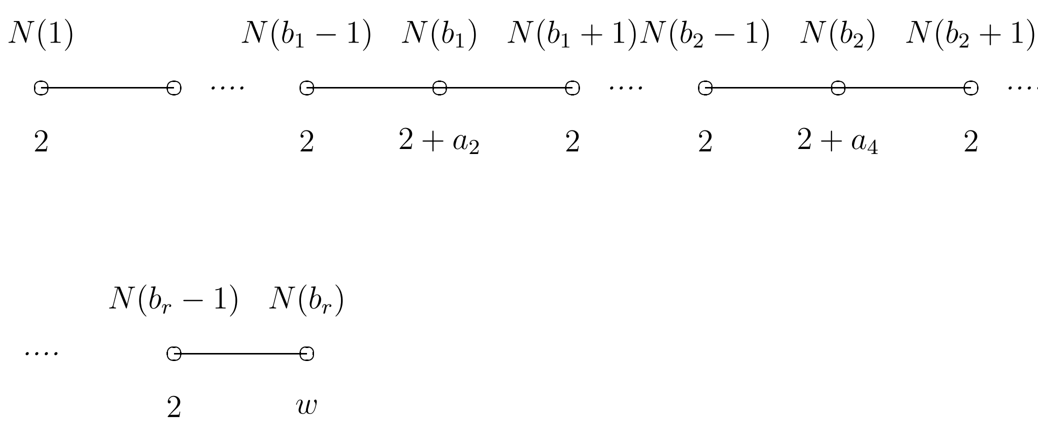
Lemma 7.1. *Let $\sigma = (a_1, \dots, a_{2r-1})$ be an $(2r-1)$ tuple of natural numbers, such that a_i is positive for $i > 1$, w be a natural number. Let*

$$\frac{n}{m} = [a_1, \dots, a_{2r-1}]$$

where n, m are relatively prime positive integers. Let $p_k = p_k(\sigma)$, $q_k = q_k(\sigma)$ be as in definition 1. Let

$$b_i = \begin{cases} \sum_{j=1}^{i} a_{2j-1}, & 1 \le i \le r \\ 0, & i = 0 \end{cases}$$

Let $\Delta(\sigma, w)$ be the weighted graph



$\Delta(\sigma, w)$ has b_r nodes $N(1), N(2), \dots, N(b_r)$. $N(b_r)$ is the distinguished node. The weight of $N(t)$ is 2 unless $t = b_i$ for some i. The weight of $N(b_i)$ is $2 + a_{2i}$ for $1 \le i < r$, the weight of $N(b_r)$ is w.

Then, the group $G(\Delta(\sigma, w))$ is freely generated by $N(1)$. Furthermore,

$$N(b_i + k) = N(1)^{p_{2i-1}+kp_{2i}} \quad 0 \le k \le a_{2i+1}, 1 \le b_i + k \le b_r \tag{15}$$

In particular,

$$N(b_r) = N(1)^n \tag{16}$$

and

$$N(b_r - 1) = N(1)^{p_{2r-1}-p_{2r-2}} \tag{17}$$

Remark 7.2. *The right arm of the i^{th} horizontal branch of $\Gamma(f)$ is $\Delta(\sigma, w)$ with $\sigma = (t_{i,2}, t_{i,3}, \dots, t_{i,2r})$, $r = \left[\frac{\nu_i}{2}\right]$,*

$$w = \begin{cases} wt(O_i) & \nu_i \text{ even} \\ 1 + t_{i,\nu_i} & \nu_i \text{ odd.} \end{cases}$$

We have a natural isomorphism of $G(\Delta(\sigma, w))$ with the subgroup of $G(\Gamma)$ generated by $R_i(k)$, $1 \le k \le b_{i,2\left[\frac{\nu_i}{2}\right]}$. Let $\sigma_i = (t_{i,1}, t_{i,2}, \dots, t_{i,\nu_i})$. By Definition 1.2

$$\begin{aligned} p_j(\sigma) &= q_{j+1}(t_{i,1}, t_{i,2}, \dots, t_{i,2\left[\frac{\nu_i}{2}\right]}) \\ &= q_{j+1}(\sigma_i) \end{aligned}$$

for $j \le 2\left[\frac{\nu_i}{2}\right]$. We have

$$R_i(b_{i,2j} + k) = R_i^{q_{2j}(\sigma_i)+kq_{2j+1}(\sigma_i)}$$

for $1 \le b_{i,2j} + k \le b_{i,2\left[\frac{\nu_i}{2}\right]}$,$0 \le k \le t_{i,2j+2}$.

If ν_i is even,

$$R_i(b_{i,2\left[\frac{\nu_i}{2}\right]}) = R_i(b_{i,\nu_i}) = O_i$$

and

$$O_i = R_i^{q_{\nu_i}(\sigma_i)} = R_i^{m_i}.$$

If ν_i is odd, $R_i(b_{i,2\left[\frac{\nu_i}{2}\right]}) = R_i(b_{i,\nu_i-1})$. In this case $O_i = L_i(b_{i,\nu_i})$, and we have the relation

$$L_i(b_{i,\nu_i})R_i(b_{i,\nu_i-1} - 1)R_i(b_{i,\nu_i-1})^{-(1+t_{i,\nu_i})} = 1.$$

$$\begin{aligned} O_i = L_i(b_{i,\nu_i}) &= R_i(b_{i,\nu_i-1})^{(1+t_{i,\nu_i})}R_i(b_{i,\nu_i-1} - 1)^{-1} \\ &= R_i^{(1+t_{i,\nu_i})q_{\nu_i-1}(\sigma_i)-(q_{\nu_i-1}(\sigma_i)-q_{\nu_i-2}(\sigma_i))} \\ &= R_i^{q_{\nu_i}(\sigma_i)} = R_i^{m_i}. \end{aligned}$$

In either case, we can express $R_i(1), R_i(2), \dots, O_i$ as powers of R_i.

The left arm of the first horizontal branch of $\Gamma(f)$ is $\Delta(\sigma, w)$ with $\sigma = (t_{1,1}, t_{1,2}, \dots, t_{1,2r-1})$, $r = \left[\frac{\nu_1+1}{2}\right]$,

$$w = \begin{cases} 1 + t_{1,\nu_1} & \nu_1 \text{ even} \\ wt(O_1) & \nu_1 \text{ odd.} \end{cases}$$

We have a natural isomorphism of $G(\Delta(\sigma, w))$ with the subgroup of $G(\Gamma)$ generated by $L_1(k)$, $1 \le k \le b_{1,2\left[\frac{\nu_1+1}{2}\right]} - 1$.

By arguments similar to the one for the right arms of the graph, we get $O_1 = L_1^{n_1}$, and all the intermediate nodes are powers of L_1.

Proof. (of Lemma 7.1)

The proof is by induction on $j = b_i + k$, $1 \le j \le b_r$.

We first prove the formula for $j = 1$ and 2. $N(2) = N(1)^{\mathrm{wt}(N(1))}$. Let $j = 1$. If $b_1 = 0$, then b_1+1 is the first node. $N(b_1+1) = N(1) = N(1)^{p_1+1p_2}$ since $p_1 = a_1 = 0$ and $p_2 = 1 + a_1 a_2 = 1$.

If $b_1 \ne 0$, then $N(1) = N(b_0 + 1) = N(1)^{p_{-1}+1p_0}$.

Now, if $b_i \ne 1$ for $i = 1$ or 2,

$$\begin{aligned} N(2) = N(1)^2 &= N(1)^{p_{-1}+2p_0} \quad \text{if } b_1 \ne 0 \\ &= N(1)^{p_1+2p_2} \quad \text{if } b_1 = 0. \end{aligned}$$

If $1 = b_i$ for $i = 1$ or 2, and hence $a_{2i-1} = 1$, then $p_{2i-2} = 1$, $p_{2i-1} = 1$, $p_{2i} = 1 + a_{2i}$. So

$$N(1) = N(b_i) = N(1)^1 = N(1)^{p_{2i-1}}$$

and

$$N(2) = N(b_i + 1) = N(1)^{2+a_{2i}} = N(1)^{p_{2i-1}+p_{2i}}.$$

Thus we have the formula for $j = 1$ and 2.

Assume, by induction, that we know the formula for j and $j-1$, for some $j < b_r$. Then

$$j = b_i + k \;\; 0 \le k < a_{2i+1} \text{ for some } i \text{ and } k.$$

<u>Case 1</u> $k > 0$.

Then the weight at j is 2. We have the relation $N(j+1)N(j-1)N(j)^{-2} = 1$. So $N(j+1) = N(j)^2 N(j)^{-1}$.

$$\begin{aligned} N(j+1) &= N(1)^{2(p_{2i-1}+kp_{2i})} N(1)^{-(p_{2i-1}+(k-1)p_{2i})} \\ &= N(1)^{p_{2i-1}+(k+1)p_{2i}} \end{aligned}$$

as desired.

<u>Case 2</u> $k = 0$.

Then the j^{th} node has weight $2 + a_{2i}$, so

$$N(j+1) = N(j)^{2+a_{2i}} N(j-1)^{-1}.$$

By induction,

$$N(j) = N(1)^{p_{2i-1}},$$

$$\begin{aligned} N(j-1) &= N(1)^{p_{2i-3}+(a_{2i-1}-1)p_{2i-2}} \\ &= N(1)^{p_{2i-1}-p_{2i-2}} \end{aligned}$$

so

$$\begin{aligned} N(j+1) &= N(1)^{(2+a_{2i})p_{2i-1}-p_{2i-1}+p_{2i-2}} \\ &= N(1)^{p_{2i-1}+p_{2i-2}+a_{2i}p_{2i-1}} \\ &= N(1)^{p_{2i-1}+p_{2i}} \end{aligned}$$

as desired. Finally, since $p_{2i-1} + a_{2i+1}p_{2i} = p_{2i+1}$, we have the result for all $j \le b_r$. $N(b_r) = N(1)^{p_{2r-1}} = N(1)^n$ by Lemma 1.4, and

$$N(b_r - 1) = N(1)^{p_{2r-3}+(a_{2r-2}-1)p_{2r-2}} = N(1)^{p_{2r-1}-p_{2r-2}}.$$

□

Our Main Theorem, Theorem 0.1 will follow from Theorem 6.2 and Theorem 7.3 below.

Theorem 7.3. *Let $\Gamma = \Gamma(f)$ be the resolution graph of f. The group $G(\Gamma)$ is generated by $2g+1$ generators P_i, Q_i, $1 \le i \le g$ and Q_0 with the relations*

$$\begin{array}{lcll} Q_0 & = & 1 & \\ Q_i^{m_i} & = & P_i^{\overline{n}_i} Q_{i-1}^{m_{i-1}m_i} & 1 \le i \le g \\ P_{i+1}P_i^{y_i}Q_{i-1}^{m_{i-1}x_i} & = & Q_i^{x_i} & 1 \le i \le g-1 \end{array}$$

where (m_i, n_i) are the characteristic pairs of f and $\overline{n}_i = n_i - m_i n_{i-1}, x_i, y_i$ satisfy

$$x_i\overline{n}_i - y_i m_i = 1$$

furthermore,

$$\begin{array}{lcl} x_i & = & \left(\frac{1-(-1)^\nu}{2}\right) q_{\nu_i}(\sigma_i) + (-1)^{\nu_i} q_{\nu_i-1}(\sigma_i) \\ y_i & = & \left(\frac{1-(-1)^\nu}{2}\right) p_{\nu_i}(\sigma_i) + (-1)^{\nu_i} p_{\nu_i-1}(\sigma_i) \end{array}$$

where

$$\frac{\overline{n}_i}{m_i} = [t_{i,1}, \ldots, t_{i,\nu_i}]$$

and

$$\sigma_i = (t_{i,1}, \ldots, t_{i,\nu_i}).$$

Proof. Recall that R_i is the right most node on the ith branch where as L_i is the left extreme and O_i is the node which is joined by an edge to $Li+1$ of the (i+1)th branch.

Let Q_i be the generator for R_i for each i. We have, by Remark 7.2, for each i with $1 \le i \le g$, all the nodes to the right of O_i in terms of Q_i, and $O_i = Q_i^{m_i}$.

Let $P_1 = L_1$ and

$$P_i = L_i O_{i-1}^{-(\mathrm{wt}(O_{i-1})-1)} \; 2 \le i \le g.$$

Since L_i commutes with O_{i-1}, P_i commutes with L_i and O_{i-1}. By Remark 7.2, all the nodes on the 1^{st} branch to the left of O_1 are obtained in terms of P_1 and $O_1 = P_1^{n_1} = Q_1^{m_1}$.

In order to prove that G is generated by the P_i's and Q_i's, it just remains to get the nodes to the left of O_i in terms of P_j's and O_j's. When $t_{i1} = 0$ and $\nu_i = 2$, $L_i = O_i = R_i(b_{i,\nu_i})$ and there is nothing to the left of O_i. So we may assume that $t_{i1} > 0$ or $\nu_i > 2$.

Consider the following formula

$$\begin{array}{lcll} L_i(r) & = & P_i^r O_{i-1} & 1 \le r \le b_{i,1} \\ & = & P_i^{p_{2j-1}(\sigma_i)+kp_{2j}(\sigma_i)} O_{i-1}^{q_{2j-1}(\sigma_i)+kq_{2j}(\sigma_i)} & \text{if } r = b_{i,2j-1}+k, b_{i,2j-1} \le r \le b_{i,2j+1} \end{array} \tag{18}$$

We verify formula (18) by induction on r, for

$$b_{i,2j-1} \le r \le b_{i,2j+1}, 0 \le j \le \left[\frac{\nu_i+1}{2}\right] - 1.$$

Now, if $b_{i,1} = 0$, then $L_i(r)$ does not exist for $r \le b_{i,1}$. If $b_{i,1} \ne 0$, then $\mathrm{wt}(O_{i-1}) = 2$ and

$$L_i = P_i O_{i-1}$$

by definition. Suppose $L_i(k) = P_i^k O_{i-1}$ for all $k \le t < b_{i,1}$. Since P_i commutes with O_{i-1}, we have

$$\begin{array}{rcl} L_i(t+1) & = & L_i(t)^2 L_i(t-1)^{-1} \\ & = & P_i^{2t} O_{i-1}^2 P_i^{-t+1} O_{i-1}^{-1} \\ & = & P_i^{t+1} O_{i-1} \end{array}$$

so $L_i(r) = P_i^r O_{i-1}$, $1 \le r \le b_{i,1}$.

Thus we have verified (18) for $j = 1$, $r \le b_{i,1}$. Suppose that (18) is proved for all $r \le b_{i,2j-1}$, $2j-1 < \nu_i - 1$. Then let $r = b_{i,2j-1}$.

$$\begin{array}{rcl} L_i(r+1) & = & L_i(r)^{2+t_{i,2j}} L_i(r-1)^{-1} \\ & = & P_i^{(2+t_{i,2j})p_{2j-1}(\sigma_i)-(p_{2j-1}(\sigma_i)-p_{2j-2}(\sigma_i))} O_{i-1}^{(2+t_{i,2j})q_{2j-1}(\sigma_i)-(q_{2j-1}(\sigma_i)-q_{2j-2}(\sigma_i))} \\ & = & P_i^{p_{2j-1}(\sigma_i)+p_{2j}(\sigma_i)} O_i^{q_{2j-1}(\sigma_i)+q_{2j}(\sigma_i)}. \end{array}$$

Thus (18) is verified for $b_{i,2j-1}+1$. Suppose it is true for $r \le b_{i,2j-1} + k < b_{i,2j+1}$. Then

$$\begin{array}{rcl} L_i(r+1) & = & L_i(r)^2 L_i(r-1)^{-1} \\ & = & P_i^{2p_{2j-1}(\sigma_i)+2kp_{2j}(\sigma_i)-(p_{2j-1}(\sigma_i)+(k-1)p_{2j}(\sigma_i))} O_{i-1}^{2q_{2j-1}(\sigma_i)+2kq_{2j}(\sigma_i)-(q_{2j-1}(\sigma_i)+(k-1)q_{2j}(\sigma_i))} \\ & = & P_i^{p_{2j-1}(\sigma_i)+(k+1)p_{2j}(\sigma_i)} O_{i-1}^{q_{2j-1}(\sigma_i)+(k+1)q_{2j}(\sigma_i)} \end{array}$$

Thus (18) is verified for all $r \le b_{i,2j-1}$, $j \le \left[\frac{\nu_i+1}{2}\right]$.

This proves that G is generated by the P_i's and Q_i's.

Now, if $\nu_i = 2j-1$, then by Remark 7.2 and (18),

$$Q_i^{m_i} = O_i = L_i(b_{i,\nu_i}) = P_i^{p_{\nu_i}(\sigma_i)} O_{i-1}^{q_{\nu_i}(\sigma_i)}.$$

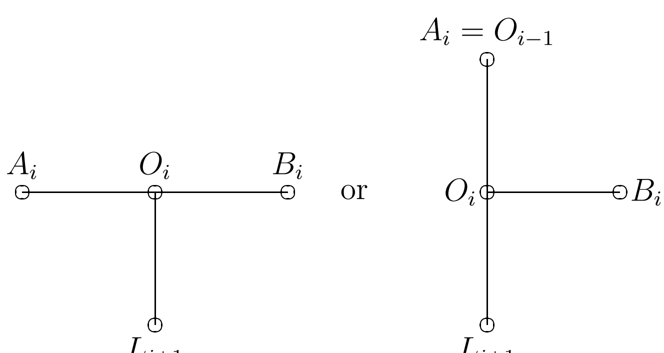
If $\nu_i = 2j$, then $O_i = L_i(b_{i,2j-1})^{1+t_{i,2j}} L_i(b_{i,2j-1}-1)^{-1}$. So

$$\begin{array}{rcl} Q_i^{m_i} & = & O_i = P_i^{(1+t_{i,2j})p_{2j-1}(\sigma_i)-(p_{2j-1}(\sigma_i)-p_{2j-2}(\sigma_i))} O_{i-1}^{(1+t_{i,2j})q_{2j-1}(\sigma_i)-(q_{2j-1}(\sigma_i)-q_{2j-2}(\sigma_i))} \\ & = & P_i^{p_{2j-2}(\sigma_i)+t_{i,2j}p_{2j-1}(\sigma_i)} O_{i-1}^{q_{2j-2}(\sigma_i)+t_{i,2j}q_{2j-1}(\sigma_i)} \\ & = & P_i^{p_{2j}(\sigma_i)} O_{i-1}^{q_{2j}(\sigma_i)} \end{array}$$

In either case,

$$\begin{array}{rcl} Q_i^{m_i} & = & P_i^{p_{\nu_i}(\sigma_i)} O_{i-1}^{q_{\nu_i}(\sigma_i)} \\ & = & P_i^{\overline{n}_i} Q_{i-1}^{m_{i-1}m_i} \qquad 1 \le i \le g. \end{array}$$

Finally, all that remains is to consider the relations contributed by the nodes O_i, $1 \le i < g$. We have



In any case, we can order the nodes adjacent to O_i in the relation (13) for O_i so that

$$L_{i+1}A_i = O_i^{\mathrm{wt}(O_i)}B_i^{-1}.$$

Therefore

$$O_i^{-(\mathrm{wt}(O_i)-1)}L_{i+1}A_i = O_iB_i^{-1}$$

and thus

$$P_{i+1}A_i = Q_i^{m_i}B_i^{-1}. \tag{19}$$

By Remark 7.2 and Lemma 7.1,

$$\begin{array}{rcll} B_i & = & Q_i^{q_{\nu_i}(\sigma_i)-q_{\nu_i-1}(\sigma_i)} = Q_i^{m_i-x_i} \text{ where } x_i = q_{\nu_i-1}(\sigma_i) & \text{if } \nu_i \text{ is even} \\ & = & Q_i^{q_{\nu_i-1}(\sigma_i)} = Q_i^{m_i-x_i} \text{ where } x_i = q_{\nu_i}(\sigma_i) - q_{\nu_i-1}(\sigma_i) & \text{if } \nu_i \text{ is odd,} \end{array}$$

so,

$$P_{i+1}A_i = Q_i^{x_i}. \tag{20}$$

Suppose that we are not in the case $t_{i,1} = 0$ and $\nu_i = 2$. Then by (18)

$$\begin{array}{rcl} A_i & = & P_i^{p_{\nu_i-1}(\sigma_i)}O_{i-1}^{q_{\nu_i-1}(\sigma_i)} = P_i^{y_i}O_{i-1}^{x_i} = P_i^{y_i}Q_{i-1}^{m_{i-1}x_i} \text{ where } x_i = q_{\nu_i-1}(\sigma_i), y_i = p_{\nu_i-1}(\sigma_i) \\ & & \text{if } \nu_i \text{ is even} \\ & = & P_i^{p_{\nu_i}(\sigma_i)-p_{\nu_i-1}(\sigma_i)}O_{i-1}^{q_{\nu_i}(\sigma_i)-q_{\nu_i-1}(\sigma_i)} = P_i^{y_i}O_{i-1}^{x_i} = P_i^{y_i}Q_{i-1}^{m_{i-1}x_i} \\ & & \text{where } x_i = q_{\nu_i}(\sigma_i) - q_{\nu_i-1}(\sigma_i), y_i = p_{\nu_i}(\sigma_i) - p_{\nu_i-1}(\sigma_i) \text{ if } \nu_i \text{ is odd} \end{array} \tag{21}$$

If $A_i = O_{i-1}$, then $t_{i,1} = 0$, $\nu_i = 2$. So

$$\begin{array}{rcl} x_i & = & q_{\nu_i-1}(\sigma_i) = q_1(\sigma_i) = 1 \\ y_i & = & p_{\nu_i-1}(\sigma_i) = p_1(\sigma_i) = t_{i,1} = 0. \end{array}$$

Therefore,

$$\begin{array}{rcl} A_i & = & O_{i-1} = P_i^0Q_{i-1}^{m_{i-1}} \\ & = & P_i^{y_i}Q_{i-1}^{m_{i-1}x_i} \end{array} \tag{22}$$

From (20), (21) and (22),

$$P_{i+1}P_i^{y_i}Q_{i-1}^{m_{i-1}x_i} = Q_i^{x_i} \quad 1 \le i \le g-1.$$

This completes all the relations. The relation at O_g simply expresses the distinguished node O_{g+1} in terms of O_g, A_g, B_g.

Thus we have the group generated by P_i, Q_i with relations,

$$\begin{array}{lcll} Q_0 & = & 1 & \\ Q_i^{m_i} & = & P_i^{\overline{n}_i}Q_{i-1}^{m_{i-1}m_i} & 1 \le i \le g \\ P_{i+1}P_i^{y_i}Q_{i-1}^{m_{i-1}x_i} & = & Q_i^{x_i} & 1 \le i \le g-1 \\ P_iQ_{i-1}^{m_{i-1}} & = & Q_{i-1}^{m_{i-1}}P_i & 1 \le i \le g \end{array}$$

The rest of the commutator relations are obvious. Finally, the commutativity of P_i and $O_i = Q_{i-1}^{m_{i-1}}$ can be deduced by induction from the first three relations.

In fact, when $i = 1$ P_1 does commute with $Q_0 = 1$. Using induction, we see that,

$$\begin{aligned}
P_{i+1}Q_i^{m_i} &= P_{i+1}P_i^{\overline{n}_i}Q_{i-1}^{m_{i-1}m_i} \\
&= Q_i^{x_i}Q_{i-1}^{-m_{i-1}x_i}P_i^{-y_i}P_i^{\overline{n}_i}Q_{i-1}^{m_{i-1}m_i} \\
&= Q_i^{x_i}P_i^{\overline{n}_i}Q_{i-1}^{m_{i-1}m_i}Q_{i-1}^{-m_{i-1}x_i}P_i^{-y_i} \\
&= Q_i^{m_i+x_i}Q_{i-1}^{-m_{i-1}x_i}P_i^{-y_i} \\
&= Q_i^{m_i}P_{i+1}
\end{aligned}$$

Removing these redundant relations, we get the statement of the theorem. The fact that x_i and y_i satisfy the relation $x_i\overline{n}_i - y_im_i = 1$ follows from our Lemma 1.3.
This finishes the proof of the theorem. □

Remark 7.4. *It is interesting to note that our method of proof yields values for x_i and y_i precisely in terms of the characteristic pairs. This is not found in the topological results of Brauner or Zariski. However, it can be seen that any values of x_i and y_i that satisfy $x_i\overline{n}_i - y_im_i = 1$ will indeed result in exactly the same set of relations. This is so because P_i commutes with $Q_{i-1}^{m_{i-1}}$ and hence with $Q_i^{m_i}$ as well.*

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