GENERICALLY FINITE MORPHISMS

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1. INTRODUCTION

Suppose that k is a field, and $f : Y \rightarrow X$ is a dominant, generically finite morphism of complete k-varieties. If Y and X are complete curves, then it is classical that f is finite. If Y and X have dimensions ≥ 2 f need not be finite. The simplest example is the blowup of a nonsingular subvariety of a nonsingular projective variety.

It is however natural to ask the following question. Given a generically finite morphism $f : Y \rightarrow X$ as above, does there exist a commutative diagram

$$\begin{array}{ccc} Y_1 & \stackrel{f_1}{\rightarrow} & X_1 \\ \downarrow & & \downarrow \\ Y & \stackrel{f}{\rightarrow} & X \end{array} \tag{1}$$

such that f_1 is finite, Y_1 and X_1 are nonsingular complete k-varieties, and the vertical arrows are birational? The answer to this question is no, as is shown by a theorem of Abhyankar (Theorem 11 [2]). This theorem, (as shown in Example 6.2 of this paper) proves that such a diagram cannot always be constructed even when $f : Y \rightarrow X$ is a G-equivariant morphism of complex projective surfaces, where the extension of function fields $k(X) \rightarrow k(Y)$ is Galois with Galois group G.

In the theory of resolution of singularities a modified version of this question is important.

Question 1. With $f : Y \rightarrow X$ as above, is it possible to construct a diagram (1) such that f_1 is finite, Y_1 and X_1 are complete k-varieties such that Y_1 is nonsingular, X_1 is normal and the vertical arrows are birational?

This question has been posed by Abhyankar (with the further conditions that $Y_1 \rightarrow Y$ is a sequence of blowups of nonsingular subvarieties and Y, X are projective) explicitely on page 144 of [5], where it is called the "weak simultaneous resolution global conjecture" and implicitely in the paper [2].

As positive evidence for this conjecture, Abhyankar proves a local form of this conjecture for 2 dimensional function fields over an algebraically closed field of arbitrary characteristic [1], [3], (this is the two dimensional case of the "weak simultaneous resolution local conjecture" [5]).

An important case where Question 1 has a positive answer is for generically finite morphisms $f : Y \rightarrow X$ of projective varieties, over a field k of characteristic zero, which induce a Galois extension of function fields. We give a simple proof in Theorem 6.1. We can construct (with this Galois assumption) a diagram (1) such that the conclusions of Question 1 hold, and X_1 has normal toric singularities. This is a relative version of Theorem 7, [2].

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In Theorem 3.1 of this paper we give a counterexample to Question 1. The example is of a generically finite morphism $Y \to X$ of nonsingular, projective surfaces, defined over an algebraically closed field k of characteristic not equal to 2. This counterexample is necessarilly then a counterexample to the "weak simultaneous resolution global conjecture".

As the "weak simultaneous resolution local conjecture", posed by Abhyankar on page 144 of [5] is true in characteristic 0 (we prove it in [9] as a corollary to the local monomialization theorem, Theorem 1.1 [8], and prove a stronger statement in Theorem 4.3 of this paper), Theorem 3.1 also gives a counterexample to the philosophy that a theorem which is true in valuation theory should also be true in the birational geometry of projective varieties. This is the philosophy which led to successful proofs of resolution of singularities for surfaces and 3-folds, in characteristic zero by Zariski ([26], [28]), and in positive characteristic by Abhyankar ([1],[3],[4]). Recently there has been progress on the important problem of local uniformization in positive characteristic in higher dimensions (c.f. [18], [21],[23], [24]). Ramification of morphisms of algebraic surfaces in positive characteristic is analyzed in [10] and [11].

We prove in Theorem 5.3 that Question 1, and the "weak simultaneous resolution global conjecture" are almost true, as it always possible (over fields of characteristic zero) to construct a diagram (1) where $f_1 : Y_1 \to X_1$ is a quasi-finite morphism of integral, finite type k-schemes, Y_1 is nonsingular, X_1 has normal toric singularities, the vertical morphisms are birational and every k-valuation of $k(X)$ has a center on X_1, every k-valuation of $k(Y)$ has a center on Y_1. That is, the answer to Question 1 becomes true if we weaken the condition that the vertical arrows are proper by not insisting that these morphisms be separated.

The essential technical result used in the proof of Theorem 5.3 is the Local Monomialization Theorem, Theorem 1.1 of [8]. Local monomialization is used to prove a strengthened version of the "weak simultaneous local conjecture", Theorem 4.3 of this paper, which allows us to construct local solutions of the problem, which are patched in an arbitrary manner (this is where separatedness is lost) to construct X_1 and Y_1.

We will now give an overview of the proof of the construction of Theorem 3.1, which is the counterexample to Question 1 and the "weak simultaneous global conjecture". We will use some of the notation explained in the following section on notations.

Suppose that K^* is a finite extension of an algebraic function field K, defined over an algebraically closed field k of characteristic zero, ν^* is a k-valuation of K^* and ν is its restriction to K. By Theorem 4.3, there exists an algebraic regular local ring S with quotient field K^* dominated by ν^* and an algebraic normal local ring R with quotient field K such that S lies over R (S is the localization at a maximal ideal of the integral closure of R in K^*). This can be refined [11] to show that there exist S and R as above such that the quotient of value groups $\Gamma_{\nu^*}/\Gamma_\nu$ acts faithfully on the power series ring $\hat{S}$ by k-algebra automorphisms so that

$$\hat{R} \otimes_{R/m_R} k' \cong (\hat{S} \otimes_{S/m_S} k')^{\Gamma_{\nu^*}/\Gamma_\nu} \tag{2}$$

where k' is an algebraic closure of S/m_S. In some special cases, such as rational rank 2 valuations of algebraic function fields of dimension two, where $[K : K^*]$ is not divisible by $\mathrm{char}(k)$, this construction is stable under quadratic transforms of S. We give a direct proof in this paper.

Let k be an algebraically closed field, $\overline{\nu}$ be the rational rank 2 valuation on $\overline{K} = k(u, v)$ and L_1 be the q-cyclic extension of $\overline{K}$, where q is a prime distinct from $\mathrm{char}(k)$, constructed in Theorem 11 [2] of Abhyankar (the construction is recalled in Theorem 3.8 of this paper). $\overline{R} = k[u, v]_{(u,v)}$ is dominated by $\overline{\nu}$. The extension $\overline{K} \to L_1$ has the property that if S_1 is an algebraic regular local ring with quotient field L_1 which lies over an algebraic normal local ring R_1 with quotient field $\overline{K}$ such that R_1 is dominated by $\overline{\nu}$, and contains $\overline{R}$, then R_1 is singular.

Examining this example, we see that there is a unique extension ν_1 of $\overline{\nu}$ to L_1, and the quotient of value groups $\Gamma_{\nu_1}/\Gamma_{\overline{\nu}} \cong \mathbf{Z}_q$. By (2) we have $\hat{R}_1 \cong \hat{S}_1^{\mathbf{Z}_q}$. Since $\hat{R}_1$ is singular and $\hat{S}_1$ is a power series ring in two variables over k, the algebraic fundamental group of $\hat{R}_1$ is

$$\pi_1(\mathrm{spec}(\hat{R}_1) - m_{\hat{R}_1}) \cong \mathbf{Z}_q.$$

We now consider the extension ν_2 of $\overline{\nu}$ to a particular p-cyclic extension L_2 of $\overline{K}$ where p is a prime such that $p \neq q$ and $p \neq \mathrm{char}(k)$. We have $\Gamma_{\nu_2}/\Gamma_{\overline{\nu}} \cong \mathbf{Z}_p$, and if S_2 is an algebraic regular local ring with quotient field L_2 which is dominated by ν_2 which contains $\overline{R}$, and if there exists an algebraic normal local ring R_2 with quotient field $\overline{K}$ which lies below S_2, then $\hat{R}_2 \cong \hat{S}_2^{\mathbf{Z}_p}$ by (2). If R_2 is singular, the algebraic fundamental group of $\hat{R}_2$ is then

$$\pi_1(\mathrm{spec}(\hat{R}_2) - m_{\hat{R}_2}) \cong \mathbf{Z}_p.$$

We then construct a morphism of projective nonsingular k-surfaces $\Phi : Y \to X$ such that X has the function field $\overline{K}$ and $\overline{R}$ is the local ring of a point on X. Y is constructed in such a way that Y splits into two sheets over $\mathrm{spec}(\overline{R})$, and there are points on these two sheets which are formally the same as extensions $\overline{R} \to S_1$, $\overline{R} \to S_2$ into algebraic regular local rings with respective quotient fields L_1 and L_2 which are dominated by the respective valuations ν_1 and ν_2. We then use these formal embeddings to construct extensions $\overline{\nu}_1$ and $\overline{\nu}_2$ of ν to the function field L_0 of Y.

These extensions $\overline{\nu}_i$ have the property that if Y_1 is nonsingular and $Y_1 \to Y$ is proper birational (so that it can be factored by blowups of points) then the map $Y_1 \to X$ is formally isomorphic at the centers of the valuations $\overline{\nu}_1$ and $\overline{\nu}_2$ to the extensions of $\overline{R}$ by the corresponding sequences of quadratic transforms of the local rings S_1 and S_2 along the respective valuations ν_1 and ν_2.

Now suppose that we can construct a diagram

$$\begin{array}{ccc} Y_1 & \to & X_1 \\ \downarrow & & \downarrow \\ Y & \to & X \end{array}$$

such that $Y_1 \to X_1$ is finite, Y_1 is nonsingular, X_1 is normal, and the vertical arrows are proper and birational. Let R_1 be the local ring of the center of $\overline{\nu}$ on X_1, $S(1)$ be the local ring of the center of $\overline{\nu}_1$ on Y_1, and let $S(2)$ be the local ring of the center of $\overline{\nu}_2$ on Y_1. Since $\overline{\nu}_1$ and $\overline{\nu}_2$ both extend $\overline{\nu}$, and $\Gamma_{\overline{\nu}_1}/\Gamma_{\overline{\nu}} \cong \mathbf{Z_q}$, $\Gamma_{\overline{\nu}_2}/\Gamma_{\overline{\nu}} \cong \mathbf{Z_p}$, we must have that

$$\widehat{S(1)}^{\mathbf{Z}_q} \cong \widehat{R}_1 \cong \widehat{S(2)}^{\mathbf{Z}_p}.$$

We then have that R_1 is singular, by our contruction of $\overline{\nu}_1$, and thus the algebraic fundamental group $\pi_1(\mathrm{spec}(\hat{R}_1) - m_{\hat{R}_1})$ has simultaneously order p and order $q \neq p$ which is impossible.

2. NOTATIONS

We will denote the maximal ideal of a local ring R by m_R. We will denote the quotient field of a domain R by $QF(R)$. Suppose that $R \subset S$ is an inclusion of local rings. We will say that R dominates S if $m_S \cap R = m_R$. Suppose that K is an algebraic function field over a field k. We will say that a subring R of K is algebraic if R is essentially of finite type over k. Suppose that K^* is a finite extension of an algebraic function field K, R is a local ring with $QF(K)$ and S is a local ring with

$QF(K^*)$. We will say that S lies over R and R lies below S if S is a localization at a maximal ideal of the integral closure of R in K^*. If R is a local ring, $\hat{R}$ will denote the completion of R at its maximal ideal.

Good introductions to the valuation theory which we require in this paper can be found in Chapter VI of [30] and in [3]. A valuation ν of K will be called a k-valuation if $\nu(k)=0$. We will denote by V_ν the associated valuation ring, which necessarily contains k. A valuation ring V of K will be called a k-valuation ring if $k\subset V$. The value group of a valuation ν will be denoted by Γ_ν. If X is an integral k-scheme with function field K, then a point $p\in X$ is called a center of the valuation ν (or the valuation ring V_ν) if V_ν dominates $\mathcal{O}_{X,p}$. If R is a subring of V_ν then the center of ν (the center of V_ν) on R is the prime ideal $R\cap m_{V_\nu}$.

Suppose that R is a local domain. A monoidal transform $R\rightarrow R_1$ is a birational extension of local domains such that $R_1=R[\frac{P}{x}]_m$ where P is a regular prime ideal of R, $0\ne x\in P$ and m is a prime ideal of $R[\frac{P}{x}]$ such that $m\cap R=m_R$. $R\rightarrow R_1$ is called a quadratic transform if $P=m_R$.

If R is regular, and $R\rightarrow R_1$ is a monodial transform, then there exists a regular sustem of parameters $(x_1,\ldots,x_n)$ in R and $r\le n$ such that

$$R_1=R\left[\frac{x_2}{x_1},\ldots,\frac{x_r}{x_1}\right]_m.$$

Suppose that ν is a valuation of the quotient field R with valuation ring V_ν which dominates R. Then $R\rightarrow R_1$ is a monoidal transform along ν (along V_ν) if ν dominates R_1.

We follow the notation of [17]. In particular, we do not require that a scheme be separated.

3. A counterexample to global weak simultaneous resolution

In this section we construct the following example. This gives a counterexample to Question 1 stated in the introduction, as well as to the "weak simultaneous resolution global conjecture" stated by Abhyankar explicitely on page 144 [5] and implicitely in the paper [2]. As the "weak simultaneous resolution local conjecture" is true in characteristic 0 (We prove it in [9], and prove a stronger version in Theorem 4.3 of this paper), Theorem 3.1 also gives a counterexample to the philosophy that a theorem which is true in valuation theory should also be true in the birational geometry of projective varieties.

Theorem 3.1. *Suppose that k is an algebraically closed field of characteristic 0 or of odd prime characteristic. Then there exists a generically finite morphism $\Phi:Y\rightarrow X$ of projective nonsingular k-surfaces such that there does not exist a commutative diagram*

$$\begin{array}{ccc} Y_1 & \stackrel{\Phi_1}{\rightarrow} & X_1\\ \downarrow & & \downarrow\\ Y & \stackrel{\Phi}{\rightarrow} & X\end{array}$$

where the vertical arrows are birational and proper, Y_1 is nonsingular, X_1 is normal, and Φ_1 is finite.

Throughout this section we will suppose that k is an algebraically closed field.

Lemma 3.2. *Suppose that L is a 2 dimensional algebraic function field over k. Suppose that R is an algebraic regular local ring with quotient field L and maximal ideal $m_R=(u,v)$. Suppose that $\overline{\nu}$ is a rank 1, rational rank 2 valuation of L such that $\overline{\nu}$ dominates R, $\overline{\nu}(u),\overline{\nu}(v)>0$ and $\overline{\nu}(u),\overline{\nu}(v)$ are rationally independent. Then*

(1) *The value group of* $\overline{\nu}$ *is* $\Gamma_{\overline{\nu}} = \mathbf{Z}\overline{\nu}(u) + \mathbf{Z}\overline{\nu}(v)$.
(2) *Suppose that* $R \to R_1$ *is a sequence of quadratic transforms along* $\overline{\nu}$. *Then there exist regular parameters* (u_1, v_1) *in* R_1 *and* $a, b, c, d \in \mathbf{N}$ *such that*

$$\begin{aligned} u &= u_1^a v_1^b \\ v &= u_1^c v_1^d \end{aligned}$$

with $ad - bc = \pm 1$.
(3) *There exists a unique extension* $\hat{\nu}$ *of* $\overline{\nu}$ *to* $\hat{L} = QF(\hat{R})$ *which dominates* $\hat{R}$. *The value group of* $\hat{\nu}$ *is* $\Gamma_{\hat{\nu}} = \Gamma_{\nu}$.
(4) *If* ν_1 *is a valuation such that* ν_1 *is equivalent to* $\overline{\nu}$, *(and the value groups* $\Gamma_{\overline{\nu}}$ *and* Γ_{ν_1} *are embedded as subgroups of* $\mathbf{R}$*) then*

$$\frac{\nu_1(v)}{\nu_1(u)} = \frac{\overline{\nu}(v)}{\overline{\nu}(u)}.$$

Proof. Proof of 1. $f \in R$ implies there is an expression $f = \sum_{i+j=r}^{n-1} a_{ij}u^iv^j + h$ with $a_{ij} \in k$, $r = \mathrm{ord}(f)$, $h \in (m_R)^n$, where n is such that

$$n\overline{\nu}(m_R) > \nu(\sum_{i+j=r} a_{ij}u^iv^j).$$

Thus since $\overline{\nu}(u)$ and $\overline{\nu}(v)$ are rationally independent,

$$\overline{\nu}(f) = \overline{\nu}(\sum_{i+j=r}^{n-1} a_{ij}u^iv^j) = \min\{\overline{\nu}(u^iv^j) \mid r \le i+j \le n-1,\ \ a_{ij} \ne 0\}.$$

Thus $\Gamma_{\overline{\nu}} = \mathbf{Z}\overline{\nu}(u) + \mathbf{Z}\overline{\nu}(v)$.

Proof of 2. It suffices to prove this for a single quadratic transform. We either have that $\overline{\nu}(u) > \overline{\nu}(v)$ or $\overline{\nu}(v) > \overline{\nu}(u)$. In the first case have that

$$R_1 = R[\frac{u}{v}, v]_{(\frac{u}{v}, v)}$$

and $\overline{\nu}(\frac{u}{v})$, $\overline{\nu}(v)$ are linearly independent over $\mathbf{Q}$. In the second case we have that

$$R_1 = R[u, \frac{v}{u}]_{(u, \frac{v}{u})}$$

and $\overline{\nu}(u)$, $\overline{\nu}(\frac{v}{u})$ are linearly independent over $\mathbf{Q}$.

Proof of 3. Define an extension $\hat{\nu}$ of $\overline{\nu}$ to $\hat{L}$ by

$$\hat{\nu}(f) = \min\{i\overline{\nu}(u) + j\overline{\nu}(v) \mid a_{ij} \ne 0\}$$

if $f \in \hat{R}$, and f has the expression $f = \sum a_{ij}x^iy^j$ with $a_{ij} \in k$. $\hat{\nu}$ is a valuation since for $i, j, \alpha, \beta \in \mathbf{N}$,

$$i\overline{\nu}(u) + j\overline{\nu}(v) = \alpha\overline{\nu}(u) + \beta\overline{\nu}(v)$$

implies $i = \alpha, j = \beta$. $\hat{\nu}$ dominates $\hat{R}$ and $\Gamma_{\hat{\nu}} = \Gamma_{\nu}$.

Suppose that $\tilde{\nu}$ is an extension of $\overline{\nu}$ to $\hat{L}$ which dominates $\hat{R}$. Suppose that $f \in \hat{R}$. Write

$$f = \sum_{i+j=r}^{\infty} a_{ij}u^iv^j,$$

where $r = \mathrm{ord}(f)$, $a_{ij} \in k$. There exists n such that

$$\overline{\nu}(\sum_{i+j=r} a_{ij}u^iv^j) < n\overline{\nu}(m_R).$$

Write

$$f=\sum_{i+j=r}^{n-1}a_{ij}u^iv^j+g,$$

with $g\in m_R^n\hat R$.

$$\tilde\nu(\sum_{i+j=r}^{n-1}a_{ij}u^iv^j)=\overline{\nu}(\sum_{i+j=r}^{n-1}a_{ij}u^iv^j)=\min\{\overline{\nu}(u^iv^j)\mid a_{ij}\ne 0, r\le i+j\le n-1\}<n\overline{\nu}(m_R)$$

and $\tilde\nu(g)\ge n\overline{\nu}(m_R)$ so that

$$\tilde\nu(f)=\min\{\overline{\nu}(u^iv^j)\mid a_{ij}\ne 0\}=\hat\nu(f).$$

Proof of 4. As on page 653 [26], we consider the convergent factions $\frac{f_p}{g_p}$ of $\tau=\frac{\overline{\nu}(v)}{\overline{\nu}(u)}$. Set

$$\epsilon=f_{p-1}g_p-f_pg_{p-1}=\pm 1.$$

$\epsilon, -\tau+\frac{f_{p-1}}{g_{p-1}}, \tau-\frac{f_p}{g_p}$ have the same signs.

For arbitrary p, we can define $u_1,v_1\in L$ by

$$u=u_1^{g_p}v_1^{g_{p-1}}, v=u_1^{f_p}v_1^{f_{p-1}}.$$

$$u_1^\epsilon=\frac{u^{f_{p-1}}}{v^{g_{p-1}}}, v_1^\epsilon=\frac{v^{g_p}}{u^{f_p}}.$$

$$\begin{array}{ll}\epsilon\overline{\nu}(u_1) & =f_{p-1}\overline{\nu}(u)-g_{p-1}\overline{\nu}(v)\\ & =g_{p-1}\overline{\nu}(u)\left[\frac{f_{p-1}}{g_{p-1}}-\tau\right]\end{array}$$

which implies that $\overline{\nu}(u_1)>0$.

$$\begin{array}{ll}\epsilon\overline{\nu}(v_1) & =g_p\overline{\nu}(v)-f_p\overline{\nu}(u)\\ & =\overline{\nu}(u)g_p\left[\tau-\frac{f_p}{g_p}\right]\end{array}$$

which implies $\overline{\nu}(v_1)>0$. Thus $\nu_1(u_1),\nu_1(v_1)>0$.

$$f_{p-1}\nu_1(u)-g_{p-1}\nu_1(v)=\epsilon\nu_1(u_1),\ \ -f_p\nu_1(u)+g_p\nu_1(v)=\epsilon\nu_1(v_1)$$

imply

$$\frac{f_{p-1}}{g_{p-1}}>\frac{\nu_1(v)}{\nu_1(u)}>\frac{f_p}{g_p}$$

if $\epsilon=1$,

$$\frac{f_{p-1}}{g_{p-1}}<\frac{\nu_1(v)}{\nu_1(u)}<\frac{f_p}{g_p}$$

if $\epsilon=-1$. Since this holds for all p,

$$\frac{\nu_1(v)}{\nu_1(u)}=\frac{\overline{\nu}(v)}{\overline{\nu}(u)}.$$

□

Lemma 3.3. *Suppose that L is a 2 dimensional algebraic function field over k. Suppose that R is an algebraic regular local ring with quotient field L and maximal ideal $m_R=(u,v)$. Suppose that ν_1 is a rank 1, rational rank 2 valuation of $\hat L=QF(\hat R)$ such that $\nu_1(u)$, $\nu_1(v)>0$ are rationally independent and which dominates $\hat R$. Then $\overline{\nu}=\nu_1\mid L$ is a rank 1, rational rank 2 valuation such that*

$$\Gamma_{\nu_1}=\mathbf{Z}\overline{\nu}(u)+\mathbf{Z}\overline{\nu}(v)=\Gamma_{\overline{\nu}}.$$

Proof. By arguments as in the proof of 3. of Lemma 3.2, we see that if $f = \sum_{i+j=r}^{\infty} a_{ij}u^iv^j \in \hat{R}$ with $a_{ij} \in k$, then $\nu_1(f) = \min\{\nu_1(u^iv^j) \mid a_{ij} \neq 0\}$. Thus

$$\Gamma_{\nu_1} = \mathbf{Z}\overline{\nu}(u) + \mathbf{Z}\overline{\nu}(v) = \Gamma_{\overline{\nu}}.$$

□

Remark 3.4. *Suppose that R is an algebraic regular local ring with quotient field K. There exist many extensions of a given valuation ν of K which dominates R to $QF(\hat{R})$ which do not dominate $\hat{R}$. Let $K = k(x)$, ν be the rank 1 discrete valuation with valuation ring $V_\nu = k[x]_{(x)}$ such that $\nu(x) = 1$. Set $R = k[x]_{(x)}$.*

Choose $f_1, \ldots, f_n \in \hat{R} = k[[x]]$ such that $x, f_1, \ldots, f_n$ are algebraically independent over k, and choose $\gamma_1, \ldots, \gamma_n \in \mathbf{R}$ such that $1, \gamma_1, \ldots, \gamma_n$ are linearly independent over $\mathbf{Q}$. $K_1 = k(x, f_1, \ldots, f_n)$ is a rational function field in $n+1$ variables, so we can extend ν to a rank 1, rational rank $n+1$ valuation ν_1 of K_1 by setting $\nu_1(f_i) = \gamma_i$, $1 \leq i \leq n$. By Proposition 2.22 [3] *or Theorem 5', Section 4, Chapter VI* [30], *ν_1 extends (up to equivalence) to a valuation $\hat{\nu}$ of $QF(\hat{R})$ which we can normalize so that it is an extension of ν.*

Write $f_i = x^{m_i}\lambda_i$ where $\lambda_i \in \hat{R}$ is a unit series. $\nu_1(\lambda_i) = \gamma_i - m_i \neq 0$. Since λ_i and $\lambda_i^{-1} \in \hat{R}$, $\hat{R}$ contains elements of negative $\hat{\nu}$ value, and thus $\hat{\nu}$ does not dominate $\hat{R}$.

Lemma 3.5. *Suppose that $K \to K^*$ is a finite extension of algebraic function fields over k of dimension 2. Suppose that ν is a rank 1 rational rank 2 valuation of K, ν^* is an extension of ν to K^*. Suppose that R_0 is an algebraic regular local ring with quotient field K, maximal ideal $m_{R_0} = (u, v)$, S is an algebraic regular local ring with quotient field K^*, maximal ideal $m_S = (x, y)$ and such that S dominates R,*

$$\begin{aligned} u &= x^a y^b \delta_1 \\ v &= x^c y^d \delta_2 \end{aligned}$$

for some natural numbers a, b, c, d and units $\delta_1, \delta_2 \in S$, and such that the characteristic of k does not divide $ad - bc$.

Suppose that V_{ν^*} dominates S and $\nu(u), \nu(v)$ are rationally independent over $\mathbf{Q}$. Then

$$\Gamma_\nu = \mathbf{Z}\nu(u) + \mathbf{Z}\nu(v)$$

and ν^* is a rank 1, rational rank 2 valuation of K^* such that $\nu^*(x), \nu^*(y)$ are rationally independent over $\mathbf{Q}$, and

$$\Gamma_{\nu^*} = \mathbf{Z}\nu^*(x) + \mathbf{Z}\nu^*(y).$$

Suppose that $S \to S_1$ is a sequence of quadratic transforms along ν^*. Then S_1 has regular parameters $(\overline{x}_1, \overline{y}_1)$ such that

$$\begin{aligned} x &= \overline{x}_1^{\overline{a}} \overline{y}_1^{\overline{b}} \\ y &= \overline{x}_1^{\overline{c}} \overline{y}_1^{\overline{d}} \end{aligned}$$

with $\overline{a}\overline{d} - \overline{b}\overline{c} = \pm 1$, and there exists a (unique) algebraic regular local ring R_1 with quotient field K which lies below S_1. $\hat{R}_1 \cong \hat{S}_1^{\Gamma_{\nu^*}/\Gamma_\nu}$, where $\Gamma_{\nu^*}/\Gamma_\nu$ acts faithfully on $\hat{S}_1$ by k-algebra automorphisms, by multiplication of $\overline{x}_1, \overline{y}_1$ by roots of unity in k.

Proof. ν^* has rational rank 2 and rank 1 since K^* is finite over K (Lemmas 1 and 2 of Section 11, Chapter VI [30]). $\nu^*(x), \nu^*(y)$ are linearly independent over $\mathbf{Q}$, so Lemma 3.2 applies to ν and to ν^*. We have an expression in S_1

$$\begin{aligned} u &= \overline{x}_1^{\tilde{a}} \overline{y}_1^{\tilde{b}} \tilde{\delta}_1 \\ v &= \overline{x}_1^{\tilde{c}} \overline{y}_1^{\tilde{d}} \tilde{\delta}_2 \end{aligned}$$

with natural numbers $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ and units $\tilde{\delta}_1$ and $\tilde{\delta}_2$ in S_1 such that the characteristic of k does not divide $\tilde{a}\tilde{d}-\tilde{b}\tilde{c}$. Let

$$A=\begin{pmatrix}\tilde{a} & \tilde{b}\\ \tilde{c} & \tilde{d}\end{pmatrix},$$

$d=\mid \tilde{a}\tilde{d}-\tilde{b}\tilde{c}\mid$. There exist regular parameters $\tilde{x}_1,\tilde{y}_1$ in $\hat{S}_1$ such that

$$u=\tilde{x}_1^{\tilde{a}}\tilde{y}_1^{\tilde{b}},\quad v=\tilde{x}_1^{\tilde{c}}\tilde{y}_1^{\tilde{d}}.$$

Let ω be a d^{th} root of unity in k. $\mathbf{Z}^2/A\mathbf{Z}^2$ acts faithfully on $\hat{S}_1$ by k-algebra automorphisms. To $c\in\mathbf{Z}^2/A\mathbf{Z}^2$ the corresponding k-algebra automorphism σ_c of $\hat{S}_1$ is defined by

$$\sigma_c(\tilde{x}_1)=\omega^{<B_1,c>}\tilde{x}_1\quad \sigma_c(\tilde{y}_1)=\omega^{<B_2,c>}\tilde{y}_1$$

where B_i is the i^{th} row of $dA^{-1}=\pm\mathrm{adj}(A)$. Since $k(u,v)\rightarrow k(\tilde{x}_1,\tilde{y}_1)$ is Galois with Galois group $\mathbf{Z}^2/A\mathbf{Z}^2$, it follows that $\hat{S}_1^{\mathbf{Z}^2/A\mathbf{Z}^2}$ is the completion of a k-algebra generated by rational monomials $u^{\alpha_1}v^{\beta_1},\cdots,u^{\alpha_r}v^{\beta_r}$ (with $\alpha_i,\beta_i\in\mathbf{Z}$ for all i).

$$R_0[u^{\alpha_1}v^{\beta_1},\ldots,u^{\alpha_r}v^{\beta_r}]\subset\hat{S}_1\cap K=S_1.$$

Let $R_1=R_0[u^{\alpha_1}v^{\beta_1},\ldots,u^{\alpha_r}v^{\beta_r}]_p$ where $p=R_0[u^{\alpha_1}v^{\beta_1},\ldots,u^{\alpha_r}v^{\beta_r}]\cap m_{S_1}$. $\hat{R}_1=\hat{S}_1^{\mathbf{Z}^2/A\mathbf{Z}^2}$ is normal, so $R_1=\hat{R}_1\cap K$ is normal. Since $\sqrt{m_{R_1}S_1}=m_{S_1}$, R_1 lies below S_1 by Zariski's main Theorem (10.9 [4]). Uniqueness follows since the condition R_1 lies below S_1 implies $R_1=S_1\cap K$ by Proposition 1 (iv) [1]. ☐

Remark 3.6. *The conclusion* $\hat{R}_1\cong\hat{S}_1^{\Gamma_{\nu*}/\Gamma_\nu}$ *in Lemma 3.5 is a special case of a general result on ramification of valuations* [11].

Lemma 3.7. *Suppose that p is a prime such that p is not the characteristic of k and $\mathbf{Z}_p$ acts diagonally and faithfully on the powerseries ring $k[[x,y]]$. Set $R=k[[x,y]]^{\mathbf{Z}_p}$. Then R is a normal local ring such that either*

(1) *R is regular and the algebraic fundamental group*

$$\pi_1(spec(R)-m_R)=0$$

or

(2) *R is not regular, $R\rightarrow k[[x,y]]$ is unramified away from m_R and the algebraic fundamental group*

$$\pi_1(spec(R)-m_R)\cong\mathbf{Z}_p.$$

Proof. Let ω be a primitive p^{th} root of unity in k, σ a generator of $\mathbf{Z}_p$. There exist integers a,b with $0\le a,b<p$ such that

$$\sigma(x)=\omega^a x,\quad \sigma(y)=\omega^b y.$$

Suppose that $a=0$. Then $R=k[[x,y^p]]$ is regular. If $b=0$ then $R=k[[x^p,y]]$ is regular. In both cases,

$$\pi_1(\mathrm{spec}(R)-m_R)=\pi_1(\mathrm{spec}(R))=\pi_1(k)=0$$

by the purity of the branch locus (Theorems X 3.4, X 1.1 [14]).

Suppose that $a,b\neq 0$. Then for $1\le i\le p-1$ there exists a unique j_i such that $bj_i\equiv ai \bmod p$ with $0<j_i<p$. This implies that $x^{p-i}y^{j_i}$ is an invariant. Note that there exists an invariant of the form $x^{p-i_1}y$ for some $0<i_1<p$, so that $j_{i_1}=1$. We will show that

$$R=k[[x^p,x^{p-1}y^{j_1},\ldots,xy^{j_{p-1}},y^p]].$$

We must show that any invariant monomial in x and y is a product of powers of these $p+1$ monomials.

Suppose that x^iy^j is invariant. Then $ai+bj\equiv 0 \mod p$. Write $i=\overline{i}+\lambda p$, $j=\overline{j}+\tau p$, with $0\le \overline{i}<p$, $0\le\overline{j}<p$.

$$x^iy^j=x^{\overline{i}}y^{\overline{j}}x^{\lambda p}y^{\tau p}$$

$b\overline{j}\equiv -a\overline{i} \mod p$ implies $\overline{i}=\overline{j}=0$ or $\overline{j}=j_{p-\overline{i}}$.

Consider the finite map of normal local rings

$$\Phi: Y=\mathrm{spec}(k[[x,y]])\rightarrow X=\mathrm{spec}(R).$$

The ramification locus of Φ is defined by the 2×2 minors of

$$J(\Phi)=\begin{pmatrix} \frac{\partial(x^p)}{\partial x} & \frac{\partial(x^p)}{\partial y}\\ \frac{\partial(x^{p-1}y^{j_1})}{\partial x} & \frac{\partial(x^{p-1}y^{j_1})}{\partial y}\\ \vdots & \vdots\\ \frac{\partial(y^p)}{\partial x} & \frac{\partial(y^p)}{\partial y}\end{pmatrix}.$$

$$\mathrm{Det}\begin{pmatrix} y^{j_{p-1}} & j_{p-1}xy^{j_{p-1}-1}\\ 0 & py^{p-1}\end{pmatrix}=py^{p-1+j_{p-1}}$$

and

$$\mathrm{Det}\begin{pmatrix} px^{p-1} & 0\\ (p-i_1)x^{p-i_1-1}y & x^{p-i_1}\end{pmatrix}=px^{2p-1-i_1}$$

implies $\sqrt{I_2(J(\Phi))}=(x,y)$. Thus Φ is unramified (and étale) away from m_R.

Suppose that S is a complete normal local domain such that S is finite over R, and $R\rightarrow S$ is étale away from m_R. Let T be the normalization of the image of $S\otimes_R k[[x,y]]$ in $QF(S)\otimes_{QF(R)} QF(k[[x,y,]])$. $k[[x,y]]\rightarrow T$ is étale away from (x,y), so by the purity of the branch locus, and since k is algebraically closed, $\mathrm{spec}(T)$ is a disjoint union of copies of $\mathrm{spec}(k[x,y]])$. A choice of one of these copies gives a factorization

$$\mathrm{spec}(k[[x,y]])\rightarrow\mathrm{spec}(S)\rightarrow\mathrm{spec}(R).$$

Thus $\pi_1(X-m_R)\cong \mathbf{Z}_p$. □

Abhyankar constructs an example which shows that we cannot in general take R to be regular in general in Corollary 4.4 (and thus we cannot take R to be regular in Theorem 4.3).

Theorem 3.8. *(Abhyankar) There exists a two dimensional algebraic regular local ring $\overline{R}$ with quotient field $\overline{K}$, a valuation $\overline{\nu}$ of $\overline{K}$ which dominates $\overline{R}$, and a finite extension L_1 of $\overline{K}$ such that if $\overline{R}_1$ is an algebraic regular local ring with quotient field $\overline{K}$ such that $\overline{R}\subset\overline{R}_1$ and $V_{\overline{\nu}}$ dominates $\overline{R}_1$, then there is a unique normal algebraic local ring $\overline{S}$ with quotient field L_1 lying over $\overline{R}_1$. $\overline{S}$ is not regular.*

Proof. We give an outline of the construction, refering to Theorem 11 [2] for details.

Let $\overline{K}=k(u,v)$ be a rational function field in two variables. Let $q>3$ be a prime such that $q\ne\mathrm{char}(k)$. Set $a=q-4$, Set

$$\tau=a+\cfrac{1}{1+\cfrac{1}{a+\cfrac{1}{1+\cfrac{1}{a+\cdots}}}}\in\mathbf{R}-\mathbf{Q}.$$

Define a rank 1, rational rank 2 valuation $\overline{\nu}$ on $\overline{K}$ by setting $\overline{\nu}(u)=\tau$, $\overline{\nu}(v)=1$.

Set $\overline{R}=k[u,v]_{(u,v)}\subset V_{\overline{\nu}}$. Let

$$L_1=\overline{K}[\overline{z}]/\overline{z}^q-uv^2.$$

Let z be the image of $\overline{z}$ in L_1.

Abhyankar shows that if $\overline{R}_1$ is an algebraic regular local ring with quotient field $\overline{K}$ such that $\overline{R} \subset \overline{R}_1$, and $V_{\overline{\nu}}$ dominates $\overline{R}_1$, then there exists a unique normal algebraic local ring $\overline{S}$ with quotient field L_1 lying over $\overline{R}_1$ and $\overline{S}$ is not regular. □

By Lemma 3.2, $\Gamma_{\overline{\nu}} = \mathbf{Z} + \tau\mathbf{Z}$. We will show that there is a unique extension ν_1 of $\overline{\nu}$ to L_1. First suppose that ν_1 is a valuation of L_1 such that $V_{\nu_1} \cap \overline{K} = V_{\overline{\nu}}$. Since ν_1 must have rank 1 and rational rank 2 (Lemmas 1 and 2, Section 11, Chapter VI [30]), we can assume that the value group of ν_1 is a subgroup of $\mathbf{R}$. We can then assume that ν_1 is normalized so that $\nu_1(v) = 1$. Since $\nu_1 \mid \overline{K}$ is equivalent to $\overline{\nu}$, and $\nu_1(v) = 1$, we have $\nu_1 \mid \overline{K} = \overline{\nu}$ by Lemma 3.2. Thus ν_1 is an extension of $\overline{\nu}$. Since $\nu_1(z) = \frac{1}{q}(2+\tau)$, we have that $\Gamma_{\nu_1}/\Gamma_{\overline{\nu}} \cong \mathbf{Z}_q$, and ν_1 is the unique extension of $\overline{\nu}$ to L_1, by corollary to Theorem 25, Section 12, Chapter VI [30] and Lemma 2.18 [3].

Let p be another prime such that $p \neq q$ and $p \neq \mathrm{char}(k)$, and set

$$L_2 = \overline{K}[\overline{w}]/\overline{w}^p - uv^2.$$

Let w be the image of $\overline{w}$ in L_2. By the same analysis as for ν_1, there is a unique extension ν_2 of $\overline{\nu}$ to L_2. $\nu_2(w) = \frac{1}{p}(2+\tau)$ and $\Gamma_{\nu_2}/\Gamma_{\overline{\nu}} \cong \mathbf{Z}_p$.

We remark that

$$\tau = q - 4 + \epsilon \text{ with } 0 < \epsilon < 1. \tag{3}$$

Set $x_1(1) = \frac{v}{z}$, $y_1(1) = \frac{z^2}{v} \in L_1$.

$$\begin{aligned} \nu_1(x_1(1)) &= \nu_1(v) - \nu_1(z) = \\ &= \tfrac{q-2-\tau}{q} = \tfrac{2-\epsilon}{q} > 0 \\ \nu_1(y_1(1)) &= 2\nu_1(z) - \nu_1(v) = \tfrac{2}{q}(2+\tau) - 1 \\ &= \tfrac{q-4+2\epsilon}{q} > 0. \end{aligned}$$

Set $S_1 = k[x_1(1), y_1(1)]_{(x_1(1), y_1(1))}$. $QF(S_1) = L_1$. $\overline{R} \subset S_1 \subset V_{\nu_1}$.

$$\begin{aligned} u &= x_1(1)^{q-4} y_1(1)^{q-2} \\ v &= x_1(1)^2 y_1(1). \end{aligned} \tag{4}$$

$\Gamma_{\nu_1} = \nu_1(x_1(1))\mathbf{Z} + \nu_1(y_1(1))\mathbf{Z}$.

We will now impose the further condition that $5 \le q < p < 2q - 4$. For example, we could take $q = 11, p = 13$ or $q = 17, p = 23$. Set

$$x_1(2) = \frac{v}{w}, \;\; y_1(2) = \frac{w^2}{v} \in L_2.$$

$$\begin{aligned} \nu_2(x_1(2)) &= \nu_2(v) - \nu_2(w) \\ &= \tfrac{p-2-\tau}{p} = \tfrac{(p-q)+2-\epsilon}{p} > 0 \\ \nu_2(y_1(2)) &= 2\nu_2(w) - \nu_2(v) = \tfrac{2}{p}(2+\tau) - 1 \\ &= \tfrac{2q-p-4+2\epsilon}{p} > 0. \end{aligned}$$

Set

$$S_2 = k[x_1(2), y_1(2)]_{(x_1(2), y_1(2))}.$$

$QF(S_2) = L_2$. $\overline{R} \subset S_2 \subset V_{\nu_2}$.

$$\begin{aligned} u &= x_1(2)^{p-4} y_1(2)^{p-2} \\ v &= x_1(2)^2 y_1(2). \end{aligned} \tag{5}$$

$\Gamma_{\nu_2} = \nu_2(x_1(2))\mathbf{Z} + \nu_2(y_1(2))\mathbf{Z}$.

We will now assume that $\mathrm{char}(k) \neq 2$.

Let $k[x, y, z_1]$ be a polynomial ring in x, y, z_1,

$$f = z_1^2 - 1 + x^m y^n \in k[x, y, z_1]$$

with m, n odd and sufficiently large, as will be determined below. We will also assume that m, n are not divisible by $\mathrm{char}(k)$. Then f is irreducible. Set $S_0 = k[x, y, z_1]/(f)$. By abuse of notation, we will from now on identify x, y, z_1 with their equivalence classes in S_0. S_0 is smooth over k. Suppose that $a_1, b_1, c_1, d_1 \in \mathbf{N}$ are such that $a_1d_1 - b_1c_1$ is not divisible by $\mathrm{char}(k)$ and $a_2, b_2, c_2, d_2 \in \mathbf{N}$ are such that $a_2d_2 - b_2c_2$ is not divisible by $\mathrm{char}(k)$. We now impose the conditions

$$m > \max\{\mid a_1 - a_2 \mid, \mid c_1 - c_2 \mid\}$$

and

$$n > \max\{\mid b_1 - b_2 \mid, \mid d_1 - d_2 \mid\}.$$

Let $R = k[u, v]$ be a polynomial ring in two variables.

We define a k-algebra homomorphism

$$R \to S_0$$

by

$$\begin{array}{c} u = x^{a_1}y^{b_1}(1 - z_1) + x^{a_2}y^{b_2}(1 + z_1) \\ v = x^{c_1}y^{d_1}(1 - z_1) + x^{c_2}y^{d_2}(1 + z_1) \end{array}$$

Consider the prime ideals $P_1 = (x, y, z_1 + 1)$ and $P_2 = (x, y, z_1 - 1)$ in S_0. In the local ring $(S_0)_{P_1}$ we have $(P_1)_{P_1} = (x, y)$ since

$$(z_1 + 1) = -(z_1 - 1)^{-1}x^m y^n.$$

In the local ring $(S_0)_{P_2}$ we have $(P_2)_{P_2} = (x, y)$.

In $(S_0)_{P_1}$ we have

$$\begin{array}{ll} u & = x^{a_1}y^{b_1}(1 - z_1 - (z_1 - 1)^{-1}x^{a_2+m-a_1}y^{b_2+n-b_1}) \\ & = x^{a_1}y^{b_1}\delta_1 \\ v & = x^{c_1}y^{d_1}(1 - z_1 - (z_1 - 1)^{-1}x^{c_2+m-c_1}y^{d_2+n-d_1}) \\ & = x^{c_1}y^{d_1}\delta_2 \end{array}$$

where $\delta_1, \delta_2 \in (S_0)_{P_1}$ are units.

Since $a_1d_1-b_1c_1$ is not divisible by $\mathrm{char}(k)$, we have regular parameters $x_1(1), y_1(1) \in \widehat{(S_0)_{P_1}}$ such that

$$\begin{array}{ll} u & = x_1(1)^{a_1}y_1(1)^{b_1} \\ v & = x_1(1)^{c_1}y_1(1)^{d_1} \end{array}. \tag{6}$$

This implies that $R \subset S_0$ is an inclusion.

By a similar calculation, we have units $\epsilon_1, \epsilon_2 \in (S_0)_{P_2}$ such that

$$\begin{array}{ll} u & = x^{a_2}y^{b_2}\epsilon_1 \\ v & = x^{c_2}y^{d_2}\epsilon_2 \end{array}$$

and regular parameters $x_1(2), y_1(2) \in \widehat{(S_0)_{P_2}}$ such that

$$\begin{array}{ll} u & = x_1(2)^{a_2}y_1(2)^{b_2} \\ v & = x_1(2)^{c_2}y_1(2)^{d_2} \end{array}. \tag{7}$$

With the notation introduced in Theorem 3.8, we have $\overline{R} = R_{(u,v)}$ and $\overline{K} = QF(R)$. Set $L_0 = QF(S_0)$. L_0 is finite over $\overline{K}$ since $\overline{K} \to L_0$ is an inclusion of algebraic function fields of dimension 2.

In this construction, set

$$a_1 = q - 4, b_1 = q - 2, c_1 = 2, d_1 = 1$$

and

$$a_2 = p - 4, b_2 = p - 2, c_2 = 2, d_2 = 1,$$

where p, q are the primes chosen in Theorem 3.8, and in the paragraph following Theorem 3.8.

Let $\Phi : Y \to X$ be a morphism of smooth projective surfaces over k which extends our map $\mathrm{spec}(S_0) \to \mathrm{spec}(R)$. Such a map exists by resolution of singularities of surfaces in characteristic ≥ 0 ([4], [20], [22]). For $i = 1, 2$ we have commutative diagrams:

$$\begin{array}{lcl} \overline{R} = R_{(u,v)} & \to & (S_0)_{P_i} \\ \downarrow & & \downarrow \\ \hat{\overline{R}} = k[[u,v]] & \to & \widehat{(S_0)_{P_i}} = k[[x_1(i), y_1(i)]] \end{array} \tag{8}$$

with (by (6) and (7))

$$\begin{array}{ll} u & = x_1(i)^{a_i} y_1(i)^{b_i} \\ v & = x_1(i)^{c_i} y_1(i)^{d_i}. \end{array}$$

We further have commutative diagrams:

$$\begin{array}{lcl} \overline{R} & \to & S_i \\ \downarrow & & \downarrow \\ \hat{\overline{R}} = k[[u,v]] & \to & \hat{S}_i = k[[x_1(i), y_1(i)]] \end{array} \tag{9}$$

with (by (4) and (5))

$$\begin{array}{ll} u & = x_1(i)^{a_i} y_1(i)^{b_i} \\ v & = x_1(i)^{c_i} y_1(i)^{d_i}. \end{array}$$

Diagrams (8) and (9) patch to give commutative diagrams for $i = 1, 2$

$$\begin{array}{lcl} R_{(u,v)} & \to & (S_0)_{P_i} \\ \downarrow & & \downarrow \\ \hat{\overline{R}} & \to & \hat{S}_i \end{array} \tag{10}$$

Lemma 3.2 implies that for $i = 1, 2$, there exists a unique extension $\hat{\nu}_i$ of ν_i to $QF(\hat{S}_i)$ which dominates $\hat{S}_i$ and $\Gamma_{\hat{\nu}_i} = \Gamma_{\nu_i}$.

For $i = 1, 2$, let $\overline{\nu}_i = \hat{\nu}_i \mid L_0$ (under the inclusion $L_0 \subset QF(\hat{S}_i)$ induced by (10)). $\Gamma_{\overline{\nu}_i} \cong \Gamma_{\nu_i}$ by Lemma 3.3. $\overline{\nu}_i \mid \overline{K} = \overline{\nu}$ for $i = 1, 2$ where $\overline{\nu}$ is the valuation of $\overline{K}$ introduced in Theorem 3.8.

Suppose that there exists a diagram

$$\begin{array}{ccc} Y_1 & \stackrel{\Phi_1}{\to} & X_1 \\ \downarrow & & \downarrow \\ Y & \stackrel{\Phi}{\to} & X \end{array}$$

such that the vertical arrows are birational and proper, Y_1 is nonsingular, X_1 is normal, and Φ_1 is finite. Then $Y_1 \to Y$ is a sequence of blowups of points (Theorem II.1.1 [29]). There exist commutative diagrams

$$\begin{array}{lcl} R_1 & \to & S(1) \\ \uparrow & & \uparrow \lambda_1 \\ \overline{R} & \to & (S_0)_{P_1} \end{array} \tag{11}$$

and

$$\begin{array}{lcl} R_1 & \to & S(2) \\ \uparrow & & \uparrow \lambda_2 \\ \overline{R} & \to & (S_0)_{P_2} \end{array} \tag{12}$$

where R_1 is the center of $\overline{\nu}$ on X_1, $S(1)$ is the local ring of the center of $\overline{\nu}_1$ on Y_1 and $S(2)$ is the local ring of the center of $\overline{\nu}_2$ on Y_1. λ_1 and λ_2 are products of quadratic transforms.

By Lemma 3.5

$$\hat{R}_1 \cong \widehat{S(1)}^{\Gamma_{\overline{\nu}_1}/\Gamma_{\overline{\nu}}}. \tag{13}$$

and

$$\hat{R}_1 \cong \widehat{S(2)}^{\Gamma_{\overline{\nu}_2}/\Gamma_{\overline{\nu}}}. \tag{14}$$

By Theorem 3.8, (10) with $i = 1$ and (11), R_1 is not regular.

$$\Gamma_{\overline{\nu}_1}/\Gamma_{\overline{\nu}} \cong \mathbf{Z}_q \text{ and } \Gamma_{\overline{\nu}_2}/\Gamma_{\overline{\nu}} \cong \mathbf{Z}_p$$

by our construction.

By Lemma 3.7 and (13),

$$\pi_1(\text{spec}(\hat{R}_1) - m_{\hat{R}_1}) \cong \mathbf{Z}_q$$

and by (14)

$$\pi_1(\text{spec}(\hat{R}_1) - m_{\hat{R}_1}) \cong \mathbf{Z}_p.$$

But $p \neq q$, so we have a contradiction.

4. Ramification of valuations in algebraic function fields

Theorem 4.1. *(Monomialization; Theorem 1.1* [8]*) Let k be a field of characteristic zero, K an algebraic function field over k, K^* a finite algebraic extension of K, ν^* a k-valuation of K^*. Suppose that S^* is an algebraic regular local ring with quotient field K^* which is dominated by ν^* and R^* is an algebraic regular local ring with quotient field K which is dominated by S^*. Then there exist sequences of monoidal transforms $R^* \to R_0$ and $S^* \to S$ such that ν^* dominates S, S dominates R_0 and there are regular parameters $(x_1, ..., x_n)$ in R_0, $(y_1, ..., y_n)$ in S, units $\delta_1, \dots, \delta_n \in S$ and a matrix $A = (a_{ij})$ of nonnegative integers such that $\det(A) \neq 0$ and*

$$\begin{array}{rcl} x_1 & = & y_1^{a_{11}} \cdots y_n^{a_{1n}} \delta_1 \\ & \vdots & \\ x_n & = & y_1^{a_{n1}} \cdots y_n^{a_{nn}} \delta_n. \end{array} \tag{15}$$

The standard theorems on resolution of singularities allow one to easily find R_0 and S such that (15) holds, but, in general, we will not have the essential condition $\det(a_{ij}) \neq 0$. The difficulty in the proof of this Theorem is to achieve the condition $\det(a_{ij}) \neq 0$.

Let α_i be the images of δ_i in S/m_S for $1 \leq i \leq n$. Let $C = (a_{ij})^{-1}$, a matrix with rational coefficients. Define regular parameters $(\overline{y}_1, \dots, \overline{y}_n)$ in $\hat{S}$ by

$$\overline{y}_i = \left(\frac{\delta_1}{\alpha_1}\right)^{c_{i1}} \cdots \left(\frac{\delta_n}{\alpha_n}\right)^{c_{in}} y_i$$

for $1 \leq i \leq n$. We thus have relations

$$x_i = \alpha_i \overline{y}_1^{a_{i1}} \cdots \overline{y}_n^{a_{in}} \tag{16}$$

with $\alpha_i \in S/m_S$ for $1 \leq i \leq n$ in

$$\hat{R}_0 = R_0/m_{R_0}[[x_1, \dots, x_n]] \to \hat{S} = S/m_S[[\overline{y}_1, \dots, \overline{y}_n]].$$

Remark 4.2. *Suppose that k' is a field, $A = (a_{ij})$ is an $n \times n$ matrix of natural numbers with $det(A) \neq 0$, and we have an inclusion of lattices*

$$N' = \mathbf{Z}^n \stackrel{A}{\rightarrow} N = \mathbf{Z}^n$$

with a corresponding inclusion of dual lattices

$$M = Hom(N, \mathbf{Z}) \rightarrow M' = Hom(N', \mathbf{Z}).$$

Let σ be the cone generated by the rows of A in $N \otimes \mathbf{R}$, σ' be the cone generated by $\mathbf{N}^n$ in $N' \otimes \mathbf{R}$. Suppose that $\hat{\sigma} \cap M$ is generated by $e_1, \ldots, e_r$. By the theory of toric varieties (c.f. page 34 [12]*) we have inclusions of k'-algebras*

$$k'[\overline{x}_1, \ldots, \overline{x}_n] \rightarrow k'[\hat{\sigma} \cap M] = k'[\overline{x}^{e_1}, \ldots, \overline{x}^{e_r}] \rightarrow k'[\hat{\sigma}' \cap M'] = k'[\overline{y}_1, \ldots, \overline{y}_n]$$

with $\overline{x}_i = \overline{y}_1^{a_{i1}} \cdots \overline{y}_n^{a_{in}}$ for $1 \leq i \leq n$. $k'[\hat{\sigma} \cap M]$ is a normal ring with quotient field $k'(\overline{x}_1, \ldots, \overline{x}_n)$ and $k'[\hat{\sigma}' \cap M']$ is finite over $k[\hat{\sigma} \cap M]$.

If $k' = \mathbf{C}$, this can be expressed in a particularly nice way. $N/N' \cong \mathbf{Z}^n/A\mathbf{Z}^n$ acts on $\mathbf{C}[\overline{y}_1, \ldots, \overline{y}_n]$ by associating to $c \in N/N'$ the $\mathbf{C}$-algebra automorphism σ_c defined by

$$\sigma_c(\overline{y}_i) = \exp^{2\pi i < F_i, c >} \overline{y}_i$$

for $1 \leq i \leq n$, where

$$A^{-1} = \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix}.$$

Theorem 4.3. *Let k be a field of characteristic zero, K an algebraic function field over k, K^* a finite algebraic extension of K, ν^* a k-valuation of K^*. Suppose that S^* is an algebraic local ring with quotient field K^* which is dominated by ν^* and R^* is an algebraic local ring with quotient field K which is dominated by S^*. Then there exists a commutative diagram*

$$\begin{array}{ccccccc} R_0 & \rightarrow & R & \rightarrow & S & \subset & V_{\nu^*} \\ \uparrow & & & & \uparrow & & \\ R^* & & \rightarrow & & S^* & & \end{array} \tag{17}$$

where $S^* \rightarrow S$ and $R^* \rightarrow R_0$ are sequences of monodial transforms along ν^* such that $R_0 \rightarrow S$ have regular parameters of the form of the conclusions of Theorem 4.1, R is an algebraic normal local ring with toric singularities, which is the localization of the blowup of an ideal in R_0, and the regular local ring S is the localization at a maximal ideal of the integral closure of R in K^*.

Proof. By resolution of singularities [19] (c.f. Theorem 2.6, Theorem 2.9 [8]), we first reduce to the case where R^* and S^* are regular, and then construct, by the local monomialization theorem, Theorem 4.1 a sequence of monodial transforms along ν^*

$$\begin{array}{ccccc} R_0 & \rightarrow & S & \subset & V_{\nu^*} \\ \uparrow & & \uparrow & & \\ R^* & \rightarrow & S^* & & \end{array} \tag{18}$$

so that R_0 is a regular local ring with regular parameters $(x_1, \ldots, x_n)$, S is a regular local ring with regular parameters $(y_1, \ldots, y_n)$, there are units $\delta_1, \ldots, \delta_n$ in S, and a matrix of natural numbers $A = (a_{ij})$ with nonzero determinant d such that

$$x_i = \delta_i y_1^{a_{i1}} \cdots y_n^{a_{in}}$$

for $1 \leq i \leq n$.

Let k' be an algebraic closure of S/m_S. With the notation of (16), set $\overline{x}_i = \frac{x_i}{\alpha_i}$, so that

$$R_0 \otimes_{R_0/m_{R_0}} k' \cong k'[[\overline{x}_1, \ldots, \overline{x}_n]] \to S \otimes_{S/m_S} k' \cong k'[[\overline{y}_1, \ldots, \overline{y}_n]]$$

is defined by

$$\overline{x}_i = \overline{y}_1^{a_{i1}} \cdots \overline{y}_n^{a_{in}},$$

$1 \le i \le n$. With the notation of Remark 4.2, set $R = R_0[x^{e_1}, \ldots, x^{e_r}]_m$ where $m = (x^{e_1}, \ldots, x^{e_r})$.

$$R_0[x^{e_1}, \ldots, x^{e_r}] \subset \hat{S} \cap K^* = S$$

(by Lemma 2 [1]) and $m \subset m_{\hat{S}} \cap K^* = m_S$, so S dominates R. $\hat{R} = R_0/m_{R_0}[[\overline{x}^{e_1}, \ldots, x^{e_r}]]$ is integrally closed in its quotient field (by Remark 4.2 and Theorem 32, Section 13, Chapter VIII [27]), so $R = \hat{R} \cap K$ is integrally closed in K. After possibly reindexing the y_i, we may assume that $d = \det(A) > 0$. Let (b_{ij}) be the adjoint matrix of A. Then

$$\prod_{j=1}^{n} x_j^{b_{ij}} = \left(\prod_{j=1}^{n} \delta_j^{b_{ij}} \right) y_i^d \in R.$$

Thus $\sqrt{m_R S} = m_S$, so R lies below S by Zariski's Main Theorem (10.9 [4]).

□

As an immediate consequence, we obtain a proof in characteristic zero of the "weak simultaneous resolution local conjecture". which is stated explicitely on page 144 of [5], and is implicit in [2]. Abhyankar proves this for algebraic function fields of dimension two and any characteristic in [1] and [3]. In the paper [9], we have given a direct proof of this result, also as a consequence of Theorem 4.1 (Theorem 1.1 [8]).

Corollary 4.4. *(Corollary1)(Theorem 1.1* [9]*) Let k be a field of characteristic zero, K an algebraic function field over k, K^* a finite algebraic extension of K, ν^* a k-valuation of K^*, and S^* an algebraic regular local ring with quotient field K^* which is dominated by ν^*. Then for some sequence of monodial transforms $S^* \to S$ along ν^*, there exists a normal algebraic local ring R with quotient field K, such that the regular local ring S is the localization at a maximal ideal of the integral closure of R in K^*.*

Proof. There exists a normal algebraic local ring R^* with quotient field K such that ν^* dominates R^* (take R^* to be the local ring of the center of ν^* on a normal projective model of K). There exists a finite type k-algebra T such that the integral closure of R^* in K^* is a localization of T, and T is generated over k by $g_1, \ldots, g_m \in V^* = V_{\nu^*}$ such that $\nu^*(g_i) \ge 0$ for all i. There exists a sequence of monoidal transforms $S^* \to S_1$ along ν^* such that $T \subset S_1$ (Theorem 2.7 [8]). S_1 dominates R^*. After replacing S^* with S_1, we can assume that S^* dominates R^*. Theorem 4.3 applies to this situation, so we can construct a diagram of the form (17). □

When K^* is Galois over K, it is not difficult to construct using Galois theory and resolution of singularities a regular local ring S with quotient field K^* and a normal local ring R with quotient field K such that S lies over R (Theorem 7 [2], Theorem 6.1), although even in the Galois case the full statements of Theorem 4.3 and Corollary 4.4 do not follow from these results (Theorem 7 [2], Theorem 6.1). The general case of non Galois extensions is much more subtle, and not as well behaved, as can be seen from Theorem 3.1.

5. Generically finite morphisms

Suppose that $f : Y \to X$ is a dominant, generically finite morphism of complete k-varieties. In this section we construct a commutative diagram

$$\begin{array}{ccc} \overline{Y} & \stackrel{f_1}{\to} & \overline{X} \\ \downarrow & & \downarrow \\ Y & \stackrel{f}{\to} & X \end{array}$$

of the form of (1) such that $\overline{Y}$ is nonsingular, f_1 is "close" to being finite, and the vertical arrows are birational and "close" to being proper.

Let K be an algebraic function field over a field k of characteristic zero. In his work on resolution of singularities, Zariski [27] constructed for each k-valuation ring V of K a projective model X_V of K such that the center of V is nonsingular on X_V. By the quasi-compactness of the Zariski-Riemann manifold of valuations of K, there exists a finite number of the X_V, $\{X_{V_1}, \dots, X_{V_n}\}$ such that every valuation ring V of K has a nonsingular center on at least one of the X_{V_i}.

In dimension ≤ 3, Zariski [28] was able to patch the open nonsingular locus of appropriate birational transforms of the X_{V_i} to produce a nonsingular projective model X of K.

The only part of Zariski's proof of the existence of a nonsingular model which does not extend to arbitrary dimension is the final step where nonsingular open subsets U_i of X_{V_i} are patched (after appropriate birational transforms) to produce a projective variety. Hironaka observes in Chapter 0, Section 6 of [17] that we can always patch the nonsingular loci U_i of the X_{V_i} along open sets where they are isomorphic, to produce an integral finite type scheme X such that X is nonsingular and every valuation ring V of K has a center on X, but X will in general not be separated. Hironaka calls such schemes "complete".

If such an X is separated, then the following Lemma shows that X is in fact complete in the usual sense, that is X is a proper k-scheme.

Lemma 5.1. *Suppose that X is a separated, integral, finite type k-scheme, such that every k-valuation of $k(X)$ has a center on X. Then X is a proper k-scheme.*

The proof of this Lemma is immediate from (ii) of Corollary II 7.3.10 [15] (Recall that in the notation of [15], a scheme is a separated pre-scheme), or can be deduced directly from the valuative criterion of properness (Theorem II 7.3.8 [15] or Theorem II 4.7 [17]).

One may expect a scheme X which satisfies all of the conditions of the Lemma above except the separatedness condition to be universally closed. This is false, as is shown by the following example.

Example 5.2. *Suppose that k is a field. There exists a (nonseparated) integral finite type k-scheme such that every k-valuation ring of $k(X)$ has a center on X, but X is not universally closed over k.*

Proof. Let ϕ be an imbedding of $\mathbf{P}^1$ in $\mathbf{P}^3$. Let $Z = \phi(\mathbf{P}^1 - \{\infty\})$, $x_0 = \phi(\infty)$. Let $\pi : X_1 \to \mathbf{P}^3$ be the blowup of $\overline{Z} = \phi(\mathbf{P}^1)$, $X_2 = \mathbf{P}^3 - \{x_0\}$. We can construct an integral finite type k-scheme X by glueing X_1 to X_2 along the open sets $X_1 - \pi^{-1}(\overline{Z})$ and $X_2 - Z$. By construction, every k-valuation of $k(X)$ has a center on X. ϕ induces an isomorphism of $\mathbf{P}^1 - \{\infty\}$ with the closed subset $Z \subset X_1$. Z is closed in X. Thus the induced morphism $\phi : \mathrm{spec}(k(\mathbf{P}^1) \to X$ does not extend to a morphism $\mathrm{spec}(\mathcal{O}_{\mathbf{P}^1,\infty}) \to X$.

Suppose that X is universally closed over k. Let $U = \mathrm{spec}(k(\mathbf{P}^1))$, $T = \mathrm{spec}(\mathcal{O}_{\mathbf{P}^1,\infty})$, with natural morphism $i : U \to T$. Let $t_1 \in T$ be the generic point, $t_0 \in T$ the special point.

Let A be the closure of $\phi \times i(U)$ in $X \times T$. $\pi_2(A)$ is closed since the projection $\pi_2 : X \times T \to T$ is closed by assumption. So there exists $y_0 \in A$ such that $\pi_2(y_0) = t_0$. By Lemma II 4.4 [17] we have an extension $T \to X \times T$ of $U \to X \times T$ which projects to an extension $\mathrm{spec}(T) \to X$ of $U \to X$, and thus an extension of ϕ to $\mathrm{spec}(\mathcal{O}_{\mathbf{P}^1,\infty}) \to X$, a contradiction. □

Theorem 5.3. *Suppose that $f : Y \to X$ is a dominant, generically finite morphism of complete varieties over a field k of characteristic zero. Then there exists a commutative diagram of integral, finite type k-schemes*

$$\begin{array}{ccc} \overline{Y} & \to & \overline{X} \\ \downarrow & & \downarrow \\ Y & \to & X \end{array}$$

such that $\overline{Y}$ is nonsingular, $\overline{X}$ has normal toric singularities, the vertical arrows are birational morphisms, $\overline{Y} \to \overline{X}$ is quasi-finite and every k-valuation ring of $k(X)$ has a center on $\overline{X}$, every k-valuation ring of $k(Y)$ has a center on $\overline{Y}$.

Remark 5.4. *Theorem 3.1 (and Lemma 5.1) show that we cannot take the vertical arrows in Theorem 5.3 to be proper.*

Proof. Let $K = k(X)$ be the function field of X, $K^* = k(Y)$ be the function field of Y. By assumption, $k(Y)$ is finite over $k(X)$. By resolution of singularities [19], we may assume that Y and X are nonsingular. Let V^* be a k-valuation ring of K^* such that $\mathrm{trdeg}_k V^*/m_{V^*} = 0$, $V = V^* \cap K$. Let p be the center of V on X, q the center of V^* on Y.

By Theorem 4.3, there exists a sequence of the form (17),

$$\begin{array}{cccc} R_0 & \to R \to & S & \subset \;\; V^* \\ \uparrow & & \uparrow & \\ \mathcal{O}_{X,p} & \to & \mathcal{O}_{Y,q} & \end{array},$$

such that S is regular and R has toric singularities.

Let $N = \mathbf{Z}^n$, σ be the cone generated by the rows of $A = (a_{ij})$ (with the notation of (15) in $N \otimes \mathbf{R}$. Let M be the dual lattice of N, $\hat{\sigma}$ be the dual cone of σ. By the proof of Theorem 4.3 (and Remark 4.2), if $\hat{\sigma} \cap M$ is generated by $e_1, \ldots, e_r$, then $R = R_0[x^{e_1}, \ldots, x^{e_r}]_m$. There is a natural inclusion

$$k[\hat{\sigma} \cap M] = k[x^{e_1}, \ldots, x^{e_r}] \to R.$$

$U_\sigma = \mathrm{spec}(k[\hat{\sigma} \cap M])$ is a normal affine toric variety.

Thus there exist affine open sets U_p of p in X and $\overline{U}_q$ of q in Y and affine rings R_V with quotient fields K and S_{V^*} with quotient field K^* with the following properties:

(1) R_V is normal and S_{V^*} is regular.
(2) If p_1 is the center of V on R_V and q_1 is the center of V^* on S_{V^*}, then $(R_V)_{p_1} = R$, $(S_{V^*})_{q_1} = S$.
(3) There is a commutative diagram

$$\begin{array}{ccc} R_V & \to & S_{V^*} \\ \uparrow & & \uparrow \\ \Gamma(U_p, \mathcal{O}_X) & \to & \Gamma(\overline{U}_q, \mathcal{O}_Y) \end{array} \qquad (19)$$

such that $R_V\rightarrow S_{V^*}$ is quasi-finite,

$$k[\hat\sigma\cap M]=k[x^{e_1},\ldots,x^{e_r}]\rightarrow R_V$$

is étale, so that R_V has normal toric singularites, and the vertical arrows are birational morphisms.

Let $Z(K)$ be the Zariski-Riemann manifold of k-valuation rings of K, $Z(K^*)$ be the Zariski-Riemann manifold of k-valuation rings of K^*. These spaces have natural topologies with respect to which they are quasi-compact (Theorem 40 Section 17, Chapter VI, [30]). There is a natural continuous map $\Phi: Z(K^*)\rightarrow Z(K)$ defined by $\Phi(V^*)=V^*\cap K$. For each $V^*\in Z(K^*)$ such that $\text{trdeg}_k V^*/m_{V^*}=0$, let Y_{V^*} be a projective variety which contains $\text{spec}(S_{V^*})$ as an open set, and let X_V be a projective variety which contains $\text{spec}(R_V)$ as an open set and such that the birational rational maps $Y_{V^*}\rightarrow X_V$, $Y_{V^*}\rightarrow Y$ and $X_V\rightarrow X$ are morphisms. Then there are commutative diagrams of continuous maps

$$\begin{array}{lll} Z(K^*) & \rightarrow & Z(K)\\ \downarrow\pi_{V^*} & & \downarrow\pi_V\\ Y_{V^*} & \rightarrow & X_V\\ \downarrow & & \downarrow\\ Y & \rightarrow & X\end{array}.$$

Let $\overline{W}_{V^*}=\pi_{V^*}^{-1}(\text{spec}(S_{V^*}))$, an open neighborhood of V^* in $Z(K^*)$, $W_V=\pi_V^{-1}(\text{spec}(R_V))$, an open neighborhood of V in $Z(K)$. Suppose that $V'\in Z(K^*)$ and $\text{trdeg}_k V'/m_{V'}>0$. Let $\pi: V'\rightarrow V'/m_{V'}$ be the residue map. By corollary 3 to Theorem 5, Section 4, Chapter VI [30], there exists a k-valuation ring V_1 with quotient field $V'/m_{V'}$ such that $\text{trdeg}_k V_1/m_{V_1}=0$. Set $V^*=\pi^{-1}(V_1)$, a k-valuation ring of K^* such that V^* is a specialization of V' and $\text{trdeg}_k V^*/m_{V^*}=0$ (page 57 [3]). Since V^* dominates a local ring S of W_{V^*}, V' must dominate a localization of S, which is the local ring of a point of W_{V^*}. Thus $V'\in\overline{W}_{V^*}$. $\{\overline{W}_{V^*}\}_{V^*\in Z(K^*)}$ is thus an open cover of $Z(K^*)$. $\{W_V\}_{V\in Z(K)}$ is also an open cover of $Z(K)$.

Since $Z(K^*)$ and $Z(K)$ are quasi-compact, there is a finite set of valuation rings $V_1^*,\ldots,V_m^*\in Z(K^*)$ with $\text{trdeg}_k V_i^*/m_{V_i^*}=0$ for all i such that if $V_i=V_i^*\cap K$ then $\{\overline{W}_{V_1^*},\ldots,\overline{W}_{V_m^*}\}$ is an open cover of $Z(K^*)$ and $\{W_{V_1},\ldots,W_{V_m}\}$ is an open cover of $Z(K)$.

For $1\le i\le m$ we have commutative diagrams

$$\begin{array}{rcr} D_i:=\text{spec}(S_{V_i^*}) & \rightarrow & C_i:=\text{spec}(R_{V_i})\\ b_i\downarrow & & a_i\downarrow\\ \overline{U}_{q_i} & \rightarrow & U_{p_i}\end{array},$$

where q_i is the center of V_i^* on Y, p_i is the center of V_i on X. Let $A_i\subset X$ be nontrivial open sets where a_i is an isomorphism and let $B_i\subset f^{-1}(A_i)$ be nontrivial open sets where b_i is an isomorphism. Then define $\overline{Y}$ to be the finite type k-scheme obtained by patching D_i to D_j for $i\ne j$ along the nontrivial open set $B_i\cap B_j$, and define $\overline{X}$ to be the finite type k-scheme obtained by patching C_i to C_j for $i\ne j$ along the nontrivial open set $A_i\cap A_j$.

By construction, $\overline{Y}$ and $\overline{X}$ are integral, $\overline{Y}$ is nonsingular, $\overline{X}$ has normal toric singularities and $\overline{f}:\overline{Y}\rightarrow\overline{X}$ is quasi-finite. □

6. Galois extensions

For Galois extensions, Question 1 of the introduction has a positive answer. Suppose that $K\rightarrow K^*$ is a finite Galois extension of algebraic function fields of characteristic zero. The existence of a finite map of normal projective k-varieties $Y\rightarrow X$, where Y is nonsingular, $k(X) = K$, $k(Y) = K^*$ has been proven by Abhyankar in Theorem 7 [2]. We prove a relative version of this result in Theorem 6.1.

Theorem 6.1. *Suppose that $\Phi : Y \rightarrow X$ is a dominant morphism of projective varieties over an algebraically closed field k of characteristic zero such that $k(Y)$ is a finite Galois extension of $k(X)$. Then there exists a commutative diagram*

$$\begin{array}{ccc} \overline{Y} & \rightarrow & \overline{X} \\ \downarrow & & \downarrow \\ Y & \rightarrow & X \end{array}$$

such that $\overline{Y} \rightarrow Y$, $\overline{X} \rightarrow X$ are birational morphisms of projective k-varieties, $\overline{Y}$ is nonsingular, $\overline{X}$ is normal, $\overline{Y} \rightarrow \overline{X}$ is finite and $\overline{X}$ has normal toric singularities.

Proof. Let X_0 be the normalization of X in $k(X)$, Y_0 be the normalization of X_0 in $k(Y)$. Let $G = \mathrm{Gal}(k(Y)/k(X))$. G acts on Y_0 and $Y_0/G = X_0$. Let D_0 be the branch locus of $Y_0 \rightarrow X_0$. Let $\pi_1 : X_1 \rightarrow X_0$ be a resolution of singularities so that $D_1 = \pi_1^{-1}(D_0)_{red}$ is a simple normal crossings divisor. If X_1 is the blowup of an ideal sheaf $\mathcal{I}_0$ in X_0, let Y_1 be the normalization of the blowup of $\mathcal{I}_0\mathcal{O}_{Y_0}$. $f_1 : Y_1 \rightarrow X_1$ is finite, and G acts on Y_1. The branch locus of f_1 which is a divisor supported on D_1 (by the purity of the branch locus) has simple normal crossings, so by Abhyankar's Lemma ([6], XIII 5.3 [13]) Y_1 has normal toric singularities, and (by Lemma 7 [1]) if $p \in Y_1$ is a closed point, the stabilizer

$$G^s(p) = \{\sigma \in G \mid \sigma(p) = p\}$$

is Abelian. Suppose that $\mathcal{I}_1 \subset \mathcal{O}_{Y_1}$ is an ideal sheaf such that the blowup of $\mathcal{I}_1$ in Y_1 dominates Y. Let $\mathcal{J} = \prod_{g \in G} g(\mathcal{I}_1)$, and $\tilde{Y}_2$ be the blowup of $\mathcal{J}$. Then G acts on $\tilde{Y}_2$ and the rational map $\tilde{Y}_2 \rightarrow Y$ is a morphism. Let $Y_2 \rightarrow \tilde{Y}_2$ be a G-equivariant resolution of singularities of $\tilde{Y}_2$ ([7], [25]), with composed map $\pi_2 : Y_2 \rightarrow Y_1$. G acts on Y_2 and for $q \in Y_2$, $G^s(q) < G^s(\pi_2(q))$ so that $G^s(q)$ is Abelian. Let $\overline{Y} = Y_2$, $\overline{X} = Y_2/G$, a projective k-variety (c.f. page 126, [16]), which has normal toric singularities (by Lemma 7, [1]). $\square$

Example 6.2. *Even if $k(Y)$ is Galois over $k(X)$, with Galois group G, and $Y \rightarrow X$ is G-equivariant, we cannot take both $\overline{X}$ and $\overline{Y}$ to be nonsingular in Theorem 5.3 or in Theorem 6.1.*

This is an immediate consequence of Abhyankar's example, Theorem 11, [2] (restated in Theorem 3.8 of this paper.) Let k be an algebraically closed field of characteristic zero. With the notations of Theorem 3.8, L_1 is a Galois extension of $\overline{K} = k(u, v)$ with Galois group $\mathbf{Z}_q$. $\mathbf{Z}_q$ acts on $S = k[u, v, z]/z^q - uv^2$ and its invariant ring is $R = k[u, v]$. With the notation of Theorem 3.8, $\overline{R} = R_{(u,v)}$. By equivariant resolution of singularites [7], [25] (applied to the normalization of $X = \mathbf{P}^2$ in L_1) there exists a dominant $\mathbf{Z}_q$ equivariant morphism of nonsingular projective k-surfaces $Y \rightarrow X$ such that $k(X) = \overline{K}$, $k(Y) = L_1$ and there exists a point $p \in X$ such that $\mathcal{O}_{X,p} = \overline{R}$.

Suppose that there exists a diagram

$$\begin{array}{ccc} \overline{Y} & \stackrel{\overline{\Phi}}{\rightarrow} & \overline{X} \\ \downarrow & & \downarrow \\ Y & \rightarrow & X \end{array}$$

as in the conclusions of Theorem 5.3 such that $\overline{X}$ is nonsingular. Let $\overline{\nu}$ be the valuation of $\overline{K}$ constructed in Theorem 3.8. Let ν_1 be the (unique) extension of $\overline{\nu}$ to L_1. Let $q \in \overline{Y}$ be a center of ν_1, $p = \overline{\Phi}(q)$. $B = \mathcal{O}_{\overline{Y},q}$ dominates $A = \mathcal{O}_{\overline{X},p}$. Since $\overline{\Phi}$ is quasi-finite, B lies over A by Zariski's Main Theorem (10.9 [4]). Since A is regular and $\overline{\nu}$ dominates A, B is not regular by Theorem 3.8, a contradiction.

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