STRONG TOROIDALIZATION OF BIRATIONAL MORPHISMS OF 3-FOLDS

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1. Introduction

Suppose that $f : X \to Y$ is a dominant morphism of algebraic varieties, over a field k of characteristic zero. If X and Y are nonsingular, $f : X \to Y$ is toroidal if there are simple normal crossing divisors D_X on X and D_Y on Y such that $f^{-1}(D_Y) = D_X$, and f is locally given by monomials in appropriate etale local parameters on X. The precise definition of this concept is in [AK], [KKMS] and Definition 3.2 of this paper. The problem of toroidalization is to determine, given a dominant morphism $f : X \to Y$, if there exists a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1}$$

such that Φ and Ψ are products of blow ups of nonsingular subvarieties, X_1 and Y_1 are nonsingular, and there exist simple normal crossing divisors D_{Y_1} on Y_1 and $D_{X_1} = f^{-1}(D_{Y_1})$ on X_1 such that f_1 is toroidal (with respect to D_{X_1} and D_{Y_1}). This is stated in Problem 6.2.1. of [AKMW].

A stronger form of toroidalization is also asked for in Problem 6.2.1 [AKMW], which we will call strong toroidalization. Suppose that $f : X \to Y$ is a dominant morphism of nonsingular projective varieties , D_Y is a SNC divisor on Y and $D_X = f^{-1}(D_Y)$ is a SNC divisor on X such that the locus $\mathrm{sing}(f)$ where the morphism f is not smooth is contained in D_X. The problem of strong toroidalization is to determine if there exists a commutative diagram (1) such that Φ and Ψ are products of blow ups of nonsingular centers which are supported in the preimages of D_X and D_Y respectively, and make SNCs with the respective preimages of D_X and D_Y, and f_1 is toroidal with respect to $D_{Y_1} = \Psi^{-1}(D_Y)$ and $D_{X_1} = \Phi^{-1}(D_X)$.

Toroidalization, and related concepts, have been considered earlier in different contexts, mostly for morphisms of surfaces. Strong torodialization is the strongest structure theorem which could be true for general morphisms. The concept of torodialization fails completely in positive characteristic. A simple example is shown in [C3].

In the case when Y is a curve, toroidalization follows from embedded resolution of singularities ([H]). When X and Y are surfaces, there are several proofs in print ([AkK], Corollary 6.2.3 [AKMW], [CP], [Mat]). They all make use of special properties of the birational geometry of surfaces. An outline of proofs of the above cases can be found in the introduction to [C3].

In [C3], strong toroidalization is solved in the case when X is a 3-fold and Y is a surface, In Theorem 0.1 of [C5] we prove toroidalization of birational morphisms

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of 3-folds. In this paper, we prove strong toroidalization for birational morphisms of 3-folds.

Theorem 1.1. *Suppose that $f : X \rightarrow Y$ is a birational morphism of nonsingular projective 3-folds over an algebraically closed field k of characteristic 0. Further suppose that there is a SNC divisor D_Y on Y such that $D_X = f^{-1}(D_Y)$ is a SNC divisor which contains the singular locus of the map f. Then there exists a commutative diagram of morphisms*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where Φ, Ψ are products of possible blow ups for the preimages of D_X and D_Y respectively, and f_1 is toroidal with respect to $D_{Y_1} = \Psi^{-1}(D_Y)$ and $D_{X_1} = \Phi^{-1}(D_X)$.

A possible blow up on a nonsingular 3-fold with toroidal structure is the blow up of a point or a nonsingular curve contained in the toroidal structure which makes SNCs with the toroidal structure.

As a consequence of Theorem 1.1, we find the following strong toroidalization theorem for morphisms of (possibly singular) varieties.

Theorem 1.2. *Suppose that $f : X \rightarrow Y$ is a birational morphism of 3-folds which are proper over an algebraically closed field k of characteristic 0. Further suppose that there is an equidimensional codimension 1 reduced subscheme D_Y of Y such that D_Y contains the singular locus of Y, and $D_X = f^{-1}(D_Y)$ contains the singular locus of the map f. Then there exists a commutative diagram of morphisms*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where Φ, Ψ are products of blow ups of nonsingular curves and points supported above D_X and D_Y respectively, $D_{Y_1} = \Psi^{-1}(D_Y)$ is a simple normal crossings divisor on Y_1, $D_{X_1} = f_1^{-1}(D_{Y_1})$ is a simple normal crossings divisor on X_1 and f_1 is toroidal with respect to D_{Y_1} and D_{X_1}.

The bulk of this paper is devoted to proving the following theorem.

Theorem 1.3. *Suppose that $f : X \rightarrow Y$ is a dominant morphism of nonsingular projective 3-folds over an algebraically closed field k of characteristic zero, with toroidal structures determined by SNC divisors D_Y on Y and $D_X = f^{-1}(D_Y)$ on X such that D_X contains the singular locus of f. Then there exists a commutative diagram*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that Ψ and Φ are products of possible blow ups for the preimages of D_Y, D_X respectively, such that f_1 is prepared for $D_{Y_1} = \Psi^{-1}(D_Y)$ and $D_{X_1} = \Phi^{-1}(D_X)$, and D_{X_1} is cuspidal for f_1.

The notation used in the statement of Theorem 1.3 is defined in Sections 2 and 3 of this paper. From Theorem 1.3 we easily deduce Theorems 1.1 and 1.2 from results in [C5].

Theorem 1.3 is applicable to arbitrary dominant morphisms of 3-folds, and is a significant step towards a proof of strong toroidalization of arbitrary dominant morphisms of 3-folds.

If we relax some of the restrictions in the definition of toroidalization, there are other constructions producing a toroidal morphism f_1, which are valid for arbitrary dimensions of X and Y. In [AK] it is shown that a diagram (1) can be constructed where Φ is weakened to being a modification (an arbitrary birational morphism). In [C1], [C2] and [C4], it is shown that a diagram (1) can be constructed where Φ and Ψ are locally products of blow ups of nonsingular centers and f_1 is locally toroidal, but the morphisms Φ, Ψ and f_1 may not be separated. This construction is obtained by patching local solutions, at least one of which contains the center of any given valuation.

It has been shown in [AKMW] and [W2] that weak factorization of birational morphisms holds in characteristic zero, and arbitrary dimension. That is, birational morphisms of complete varieties can be factored by an alternating sequence of blow ups and blow downs of non singular subvarieties. Weak factorization of birational (toric) morphisms of toric varieties, (and of birational toroidal morphisms) has been proven by Danilov [D1] and Ewald [E] (for 3-folds), and by Wlodarczyk [W1], Morelli [Mo] and Abramovich, Matsuki and Rashid [AMR] in general dimensions.

Our Theorem 1.1 on strong toroidalization (or the weaker Theorem 0.1 of [C5] on toroidalization), when combined with weak factorization for toroidal morphisms ([AMR]), gives a new proof of weak factorization of birational morphisms of 3-folds. We point out that our proof uses an analysis of the structure as power series of local germs of a mapping, as opposed to the entirely different proof of weak factorization, using geometric invariant theory, of [AKMW] and [W1].

The version of weak factorization which we get from Theorem 1.1 is stronger than that obtained in [AKMW], [W1] or [C5].

Corollary 1.4. *Suppose that $f : X \rightarrow Y$ is a dominant birational morphism of nonsingular projective 3-folds over an algebraically closed field k of characteristic zero, with toroidal structures determined by SNC divisors D_Y on Y and $D_X = f^{-1}(D_Y)$ on X such that D_X contains the singular locus of f. Then there exists a commutative diagram of morphisms factoring f,*

$$\begin{tikzcd}
 & Z_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & Z_3 \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_4"] & & & Z_{n-1} \arrow[dl, "\alpha_{n-1}"'] \arrow[dr, "\alpha_n"] & \\
X_1 \arrow[d, "\Phi"'] & & Z_2 & & Z_4 & \cdots & & Y_1 \arrow[d, "\Psi"] \\
X & & & & & & & Y
\end{tikzcd}$$

such that

1. *All varieties X_1, Y_1 and the Z_i are nonsingular, with toroidal structures D_{X_1}, D_{Y_1} and D_{Z_i} respectively.*
2. *There is a toroidal morphism $f_1 : X_1 \rightarrow Y_1$ making a strong toroidalization*

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

3. *The morphisms in the diagram*

$$\begin{tikzcd}
 & Z_1 \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] & & Z_3 \arrow[dl, "\alpha_3"'] \arrow[dr, "\alpha_4"] & & & Z_{n-1} \arrow[dl, "\alpha_{n-1}"'] \arrow[dr, "\alpha_n"] & \\
X_1 & & Z_2 & & Z_4 & \cdots & & Y_1
\end{tikzcd}$$

are toroidal with respect to their toroidal structures.

The proof of Corollary 1.4 is immediate from Theorem 1.1, which constructs the commutative diagram 2, and [AMR], [Mo] or [W1], which produces the diagram 3.

The problem of strong factorization, as proposed by Abhyankar [Ab2] and Hironaka [H], is to factor a birational morphism $f : X \to Y$ by constructing a diagram

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[dr] & \\
X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

where $Z \to X$ and $Z \to Y$ factor as products of blow ups of nonsingular subvarieties. Oda [O] has proposed the analogous problem for (toric) morphisms of toric varieties.

A birational morphism $f : S \to Y$ of (nonsingular) surfaces can be directly factored by blowing up points (Zariski [Z1] and Abhyankar [Ab1]), but there are examples showing that a direct factorization is not possible in general for 3-folds (Shannon [Sh] and Sally[S]).

We also obtain as an immediate corollary of Theorem 1.1 the following new result, which reduces the problem of strong factorization of 3-folds to the case of morphisms of toric varieties

Corollary 1.5. *Suppose that the Oda conjecture on strong factorization of birational morphisms of 3-dimensional toric varieties is true. Then the Abhyankar, Hironaka strong factorization conjecture of birational morphisms of complete (characteristic zero) 3-folds is true.*

Abhyankar's local factorization conjecture [Ab2], which is "strong factorization" along a valuation, follows from local monomialization (Theorem A [C2]), to reduce to a locally toroidal morphism, and local factorization for toroidal morphisms along a toroidal valuation Christensen [Ch] (for 3-folds), and Karu [K] in general dimensions. A proof in the spirit of [Ch] of local factorization of a toroidal morphism in all dimensions, using only elementary properties of determinants, is given in [CS].

2. Notation

Throughout this paper, k will be an algebraically closed field of characteristic zero. A curve, surface or 3-fold is a quasi-projective variety over k of respective dimension 1, 2 or 3. If X is a variety, and $p \in X$ is a nonsingular point, then regular parameters at p are regular parameters in $\mathcal{O}_{X,p}$. Formal regular parameters at p are regular parameters in $\hat{\mathcal{O}}_{X,p}$. If X is a variety and $V \subset X$ is a subvariety, then $\mathcal{I}_V \subset \mathcal{O}_X$ will denote the ideal sheaf of V. If V and W are subvarieties of a variety X, we denote the scheme theoretic intersection $Y = \mathrm{spec}(\mathcal{O}_X/\mathcal{I}_V + \mathcal{I}_W)$ by $Y = V \cdot W$.

Let $f : X \to Y$ be a morphism of varieties. We will denote by sing(f) the closed set of points $p \in X$ such that f is not smooth at p. If D is a Cartier divisor on Y, then $f^{-1}(D)$ will denote the reduced divisor $f^*(D)_{red}$.

Suppose that $a, b, c, d \in \mathbf{Q}$. Then we will write $(a, b) \le (c, d)$ if $a \le b$ and $c \le d$.

A toroidal structure on a nonsingular variety X is a simple normal crossing divisor (SNC divisor) D_X on X.

We will say that a nonsingular curve C which is a subvariety of a nonsingular 3-fold X with toroidal structure D_X makes simple normal crossings (SNCs) with D_X if for all $p \in C$, there exist regular parameters x, y, z at p such that $x = y = 0$ are local equations of C, and $xyz = 0$ contains the support of D_X at p.

Suppose that X is a nonsingular 3-fold with toroidal structure D_X. If $p \in D_X$ is on the intersection of three components of D_X then p is called a 3-point. If $p \in D_X$ is on the intersection of two components of D_X (and is not a 3-point) then p is called a 2-point. If $p \in D_X$ is not a 2-point or a 3-point, then p is called a 1-point. If C is an irreducible component of the intersection of two components of D_X, then C is called a 2-curve.

A possible center on a nonsingular 3-fold X with toroidal structure defined by a SNC divisor D_X, is a point on D_X or a nonsingular curve in D_X which makes SNCs with D_X. A possible center on a nonsingular surface S with toroidal structure defined by a SNC divisor D_S is a point on D_S. We will also call the blow up of a possible center a possible blow up.

Observe that if $\Phi : X_1 \rightarrow X$ is the blow up of a possible center, then $D_{X_1} = \Phi^{-1}(D_X)$ is a SNC divisor on X_1. Thus D_{X_1} defines a toroidal structure on X_1. All blow ups $\Phi : X_1 \rightarrow X$ considered in this paper will be of possible centers, and we will impose the toroidal structure on X_1 defined by $D_{X_1} = \Phi^{-1}(D_X)$.

By a general point q of a variety V, we will mean a point q which satisfies conditions which hold on some nontrivial open subset of V. The exact open condition which we require will generally be clear from context. By a general section of a coherent sheaf $\mathcal{F}$ on a projective variety X, we mean the section corresponding to a general point of the k-linear space $\Gamma(X, \mathcal{F})$.

If X is a variety, $k(X)$ will denote the function field of X. A 0-dimensional valuation ν of $k(X)$ is a valuation of $k(X)$ such that $\mathbf{k}$ is contained in the valuation ring V_ν of ν and the residue field of V_ν is k. If X is a projective variety which is birationally equivalent to X, then there exists a unique (closed) point $p_1 \in X_1$ such that V_ν dominates $\mathcal{O}_{X_1,p_1}$. p_1 is called the center of ν on X_1. If $p \in X$ is a (closed) point, then there exists a 0-dimensional valuation ν of $k(X)$ such that V_ν dominates $\mathcal{O}_{X,p}$ (Theorem 37, Section 16, Chapter VI [ZS]). For $a_1, \ldots, a_n \in k(X)$, $\nu(a_1), \ldots, \nu(a_n)$ are rationally dependent if there exist $\alpha_1, \ldots, \alpha_n \in \mathbf{Z}$ which are not all zero, such that $\alpha_1\nu(a_1) + \cdots \alpha_n\nu(a_n) = 0$ (in the value group of ν). Otherwise, $\nu(a_1), \ldots, \nu(a_n)$ are rationally independent.

3. TOROIDAL MORPHISMS AND PREPARED MORPHISMS

Suppose that X is a nonsingular variety with toroidal structure D_X. We will say that an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ is toroidal if $\mathcal{I}$ is locally generated by monomials in local equations of components of D_X.

Suppose that $q \in X$. We say that u, v, w are (formal) permissible parameters at q (for D_X) if u, v, w are regular parameters in $\hat{\mathcal{O}}_{X,q}$ such that

1. If q is a 1-point, then $u \in \mathcal{O}_{X,q}$ and $u = 0$ is a local equation of D_X at q.
2. If q is a 2-point then $u, v \in \mathcal{O}_{X,q}$ and $uv = 0$ is a local equation of D_X at q.
3. If q is a 3-point then $u, v, w \in \mathcal{O}_{X,q}$ and $uvw = 0$ is a local equation of D_X at q.

u, v, w are algebraic permissible parameters if we further have that $u, v, w \in \mathcal{O}_{X,q}$.

Definition 3.1. *Let $f : X \rightarrow Y$ be a dominant morphism of nonsingular 3-folds with toroidal structures D_Y on Y and $D_X = f^{-1}(D_Y)$ on X such that sing$(f) \subset D_X$. Suppose that u, v, w are (possibly formal) permissible parameters at $q \in Y$. Then u, v*

are **toroidal forms** *at $p \in f^{-1}(q)$ if there exist permissible parameters x, y, z in $\hat{\mathcal{O}}_{X,p}$ such that*

1. *q is a 2-point or a 3-point, p is a 1-point and*
$$u = x^a, v = x^b(\alpha + y) \tag{2}$$
where $0 \neq \alpha \in k$.
2. *q is 2-point or a 3-point, p is a 2-point and*
$$u = x^a y^b, v = x^c y^d \tag{3}$$
with $ad - bc \neq 0$.
3. *q is a 2-point or a 3-point, p is a 2-point and*
$$u = (x^a y^b)^k, v = (x^a y^b)^t(\alpha + z) \tag{4}$$
where $0 \neq \alpha \in k$, $a, b, k, t > 0$ and $\gcd(a, b) = 1$.
4. *q is a 2-point or a 3-point, p is a 3-point and*
$$u = x^a y^b z^c, v = x^d y^e z^f \tag{5}$$
where
$$\mathit{rank}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = 2.$$
5. *q is a 1-point, p is a 1-point and*
$$u = x^a, v = y \tag{6}$$
6. *q is a 1-point, p is a 2-point and*
$$u = (x^a y^b)^k, v = z \tag{7}$$
with $a, b, k > 0$ and $\gcd(a, b) = 1$

Regular parameters x, y, z as in Definition 3.1 will be called permissible parameters for u, v, w at p.

Definition 3.2. *(*[KKMS], [AK]*) A normal variety $\overline{X}$ with a SNC divisor $D_{\overline{X}}$ on $\overline{X}$ is called toroidal if for every point $p \in \overline{X}$ there exists an affine toric variety X_σ, a point $p' \in X_\sigma$ and an isomorphism of k-algebras*
$$\hat{\mathcal{O}}_{\overline{X},p} \cong \hat{\mathcal{O}}_{X_\sigma,p'}$$
such that the ideal of $D_{\overline{X}}$ corresponds to the ideal of $X_\sigma - T$ (where T is the torus in X_σ). Such a pair (X_σ, p') is called a local model at $p \in \overline{X}$. $D_{\overline{X}}$ is called a toroidal structure on $\overline{X}$.

A dominant morphism $\Phi : \overline{X} \rightarrow \overline{Y}$ of toroidal varieties with SNC divisors $D_{\overline{Y}}$ on $\overline{Y}$ and $D_{\overline{X}} = \Phi^{-1}(D_{\overline{Y}})$ on $\overline{X}$, is called toroidal at $p \in \overline{X}$, and we will say that p is a toroidal point of Φ if with $q = \Phi(p)$, there exist local models (X_σ, p') at p, (Y_τ, q') at q and a toric morphism $\Psi : X_\sigma \rightarrow Y_\tau$ such that the following diagram commutes:
$$\begin{tikzcd}
\hat{\mathcal{O}}_{\overline{X},p} & \hat{\mathcal{O}}_{X_\sigma,p'} \arrow[l] \\
\hat{\mathcal{O}}_{\overline{Y},q} \arrow[u, "\hat{\Phi}^*"] & \hat{\mathcal{O}}_{Y_\tau,q'}. \arrow[l] \arrow[u, "\hat{\Psi}^*"]
\end{tikzcd}$$
$\Phi : \overline{X} \rightarrow \overline{Y}$ is called toroidal (with respect to $D_{\overline{Y}}$ and $D_{\overline{X}}$) if Φ is toroidal at all $p \in \overline{X}$.

The following is the list of toroidal forms for a dominant morphism $f : X \rightarrow Y$ of nonsingular 3-folds with toroidal structure D_Y and $D_X = f^{-1}(D_X)$. Suppose that $p \in D_X$, $q = f(p) \in D_Y$, and f is toroidal at p. Then there exist permissible parameters u, v, w at q and permissible parameters x, y, z for u, v, w at p such that one of the following forms hold:

1. p is a 3-point and q is a 3-point,
$$\begin{aligned} u &= x^a y^b z^c \\ v &= x^d y^e z^f \\ w &= x^g y^h z^i, \end{aligned}$$
where $a, b, d, e, f, g, h, i \in \mathbf{N}$ and
$$\mathrm{Det}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.$$
2. p is a 2-point and q is a 3-point,
$$\begin{aligned} u &= x^a y^b \\ v &= x^d y^e \\ w &= x^g y^h (z + \alpha) \end{aligned}$$
with $0 \neq \alpha \in k$ and $a, b, d, e, g, h \in \mathbf{N}$ satisfy $ae - bd \neq 0$.
3. p is a 1-point and q is a 3-point,
$$\begin{aligned} u &= x^a \\ v &= x^d (y + \alpha) \\ w &= x^g (z + \beta) \end{aligned}$$
with $0 \neq \alpha, \beta \in k$, $a, d, g > 0$.
4. p is a 2-point and q is a 2-point,
$$\begin{aligned} u &= x^a y^b \\ v &= x^d y^e \\ w &= z \end{aligned}$$
with $ae - bd \neq 0$
5. p is a 1-point and q is a 2-point,
$$\begin{aligned} u &= x^a \\ v &= x^d (y + \alpha) \\ w &= z \end{aligned}$$
with $0 \neq \alpha \in k$, $a, d > 0$.
6. p is a 1-point and q is a 1-point,
$$\begin{aligned} u &= x^a \\ v &= y \\ w &= z \end{aligned}$$
with $a > 0$.

A prepared morphism $\Phi_X : X \rightarrow S$ from a nonsingular 3-fold X to a nonsingular surface S (with respect to toroidal structures D_S and $D_X = \Phi_X^{-1}(D_S)$) is defined in Definition 6.5 [C3]. A prepared morphism from a 3-fold to a 3-fold is defined in Definition 2.4 [C5]. This definition assumes that $f : X \rightarrow Y$ is birational, but this definition is perfectly valid for a generically finite morphism of 3-folds.

Remark 3.3. *Suppose that $f : X \to Y$ is a dominant proper morphism of nonsingular 3-folds with toroidal structures determined by SNC divisors D_Y on Y and $D_X = f^{-1}(D_Y)$ on X, and D_X contains the singular locus of the morphism f. With our assumptions on f, f is generically finite. Recall that the fundamental locus of a generically finite morphism $f : X \to Y$ of nonsingular proper varieties is $\{p \in Y \mid dim\ f^{-1}(p) > 0\}$. The fundamental locus is a closed set of codimension ≥ 2 in Y. Let $\overline{X}$ be the normalization of Y in the function field of X, with induced finite morphism $\lambda : \overline{X} \to Y$. The branch locus of λ is contained in the SNC divisor D_Y. Let E be an irreducible component of D_Y. By Abhyankar's lemma, the irreducible components of $\lambda^{-1}(E)$ are disjoint. Thus the irreducible components of D_X which dominate E are disjoint.*

Definition 3.4. *A dominant morphism $f : X \to Y$ of nonsingular 3-folds with toroidal structures determined by SNC divisors D_Y on Y, and $D_X = f^{-1}(D_Y)$ on X such that the singular locus of f is contained in D_X is* **prepared** *for D_Y and D_X if:*

1. *If $q \in Y$ is a 3-point, u, v, w are permissible parameters at q and $p \in f^{-1}(q)$, then u, v and w are each a unit (in $\hat{\mathcal{O}}_{X,p}$) times a monomial in local equations of the toroidal structure D_X at p . Furthermore, there exists a permutation of u, v, w such that u, v are toroidal forms at p.*
2. *If $q \in Y$ is a 2-point, u, v, w are permissible parameters at q and $p \in f^{-1}(q)$, then either*
   (a) *u, v are toroidal forms at p or*
   (b) *p is a 1-point and there exist regular parameters $x, y, z \in \hat{\mathcal{O}}_{X,p}$ such that there is an expression*
   $$\begin{aligned} u &= x^a \\ v &= x^c(\gamma(x,y) + x^d z) \\ w &= y \end{aligned}$$
   where γ is a unit series and $x = 0$ is a local equation of D_X, or
   (c) *p is a 2-point and there exist regular parameters x, y, z in $\hat{\mathcal{O}}_{X,p}$ such that there is an expression*
   $$\begin{aligned} u &= (x^a y^b)^k \\ v &= (x^a y^b)^l(\gamma(x^a y^b, z) + x^c y^d) \\ w &= z \end{aligned}$$
   where $a, b > 0$, $gcd(a, b) = 1$, $ad - bc \neq 0$, γ is a unit series and $xy = 0$ is a local equation of D_X.
3. *If $q \in Y$ is a 1-point, and $p \in f^{-1}(q)$, then there exist permissible parameters u, v, w at q such that u, v are toroidal forms at p.*

Definition 3.5. *Suppose that $f : X \to Y$ is a prepared morphism of nonsingular 3-folds with toroidal structures D_Y and $D_X = f^{-1}(D_Y)$. Then D_X is cuspidal for f if:*

1. *If E is a component of D_X which does not contain a 3-point then f is toroidal in a Zariski open neighborhood of E.*
2. *If C is a 2-curve of X which does not contain a 3-point then f is toroidal in a Zariski open neighborhood of C.*

Definition 3.6. *Suppose that X is a nonsingular 3-fold with toroidal structure determined by a SNC divisor D_X. We will say that D_X is strongly cuspidal if every component of D_X contains a 3-point and every 2-curve of D_X contains a 3-point.*

4. Preparation above 2 and 3-points

Lemma 4.1. *Suppose that $f : X \to Y$ is a dominant morphism of nonsingular projective 3-folds with toroidal structures determined by SNC divisors D_Y and $D_X = f^{-1}(D_Y)$ such that D_X contains the singular locus of f. Then there exist a commutative diagram*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that Φ and Ψ are products of blow ups of 2-curves and f_1 is toroidal above all 3-points of Y_1.

Proof. Suppose that ν is a 0-dimensional valuation of $k(X)$. We will say that ν is resolved for f if the center of ν on Y is not a 3-point or if the center of ν on Y is a 3-point, and f is toroidal at the center of ν on X.

Being resolved is an open condition on the Zariski-Riemann manifold of X, and if ν is resolved for f and

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi_1"'] & Y_1 \arrow[d, "\Psi_1"] \\ X_1 \arrow[r, "f"] & Y \end{tikzcd}$$

is a commutative diagram such that Φ_1 and Ψ_1 are products of blow ups of 2-curves, then ν is resolved for f_1.

Suppose that ν is a 0-dimensional valuation of $k(X)$ such that the center q of ν on Y is a 3-point. Let p be the center of ν on X. Let u, v, w be permissible parameters at q.

Case 1 Suppose that $\nu(u), \nu(v), \nu(w)$ are rationally independent. Since $uvw = 0$ is a local equation of D_X at p, there exist regular parameters x, y, z in $\mathcal{O}_{X,p}$ such that $xyz = 0$ contains the germ of D_X at p, and we have an expression

$$\begin{aligned} u &= x^a y^b z^c \gamma_1 \\ v &= x^d y^e z^f \gamma_2 \\ w &= x^g y^h z^i \gamma_3 \end{aligned}$$

where the γ_i are units in $\mathcal{O}_{X,p}$. Since $\nu(u), \nu(v), \nu(w)$ are rationally independent, $\nu(x), \nu(y), \nu(z)$ are also rationally independent and

$$\mathrm{Det}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0$$

which implies that p is a 3-point and f is toroidal at p. Thus ν is resolved for f.

Case 2 Suppose that $\nu(u), \nu(v)$ are rationally dependent. After possibly interchanging u, v, w we reduce to this case. Let C be the 2-curve of Y with local equation $u = v = 0$ at q. There exists a sequence of blow ups of 2-curves $\Psi_\nu : Y_\nu \to Y$ such that the center of ν on Y_ν is not a 3-point.

Y_ν is the blow up of a toroidal ideal sheaf $\mathcal{J}_\nu$ of $\mathcal{O}_Y$. Since $f^{-1}(D_Y) = D_X$, $\mathcal{J}_\nu\mathcal{O}_X$ is also a toroidal ideal sheaf. By Lemma 2.11 [C5], there exists a sequence of blow ups of 2-curves $\Phi_\nu : X_\nu \rightarrow X$ such that there is a commutative diagram of morphisms

$$\begin{tikzcd}
X_\nu \arrow[r, "f_\nu"] \arrow[d, "\Phi_\nu"'] & Y_\nu \arrow[d, "\Psi_\nu"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Thus ν is resolved for f_ν.

It follows from compactness of the Zariski Riemann manifold of X [Z], that there exists a positive integer n and commutative diagrams

$$\begin{tikzcd}
X_i \arrow[r, "f_i"] \arrow[d, "\Phi_i"'] & Y_i \arrow[d, "\Psi_i"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

for $1 \le i \le n$ such that Φ_i and Ψ_i are products of blow ups of 2-curves, and every 0-dimensional valuation ν of $k(X)$ is resolved for some f_i.

Y_i is the blow up of a toroidal ideal sheaf $\mathcal{J}_i$ of $\mathcal{O}_Y$ and X_i is the blow up of a toroidal ideal sheaf $\mathcal{I}_i$ of $\mathcal{O}_X$. Thus there exists a sequence of blow ups of 2-curves $Y^* \rightarrow Y$ such that $\prod_i \mathcal{J}_i\mathcal{O}_{Y^*}$ is invertible, by Lemma 2.11 [C5]. Y^* is thus the blow up of a toroidal ideal sheaf $\mathcal{J} \subset \mathcal{O}_Y$, so that $\mathcal{J}\mathcal{O}_X$ is also a toroidal ideal sheaf. By Lemma 2.11 [C5], there exists a sequence of blow ups of 2-curves $X^* \rightarrow X$ such that $\mathcal{J}\prod \mathcal{I}_i\mathcal{O}_{X^*}$ is invertible. Thus for $1 \le i \le n$ there exist commutative diagrams of morphisms

$$\begin{tikzcd}
X^* \arrow[r, "f^*"] \arrow[d, "\Phi_i^*"'] & Y^* \arrow[d, "\Psi_i^*"] \\
X_i \arrow[r, "f_i"] \arrow[d] & Y_i \arrow[d] \\
X \arrow[r] & Y.
\end{tikzcd}$$

Suppose that ν is a 0-dimensional valuation of $k(X)$. If the center of ν on Y^* is a 3-point, then the center of ν on Y_i is a 3-point for all i, since Ψ_i^* is toroidal. There exists an i such that ν is resolved for f_i. Thus f_i is toroidal at the center of ν on X_i. Since Φ_i^* and Ψ_i^* are toroidal, f^* is toroidal at the center of ν. Thus ν is resolved for f^*. Since all 0-dimensional valuations of $k(X)$ are resolved for f^*, it follows that f^* is toroidal above all 3-points of Y^*, and we have achieved the conclusions of the lemma. $\square$

Lemma 4.2. *Suppose that $f : X \rightarrow Y$ is a dominant morphism of nonsingular projective 3-folds, with toroidal structures determined by SNC divisors D_Y and $D_X = f^{-1}(D_Y)$ such that D_X contains the singular locus of f. Further suppose that f is toroidal above all 3-points of Y. Then there exists a commutative diagram*

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

such that Ψ and Ψ are products of blow ups of 2-curves, f_1 is toroidal above all 3-points of Y_1, and f_1 is prepared (and satisfies 2a) of Definition 3.4) above all 2-points of Y_1.

Proof. Suppose that ν is a 0-dimensional valuation of $k(X)$. We will say that ν is resolved for f if the center of ν on Y is a 1-point or if the center of ν on Y is a 2-point and f is prepared at the center of ν on X (and satisfies 2a) of Definition 3.4), or if the center of ν on Y is a 3-point, and f is toroidal at the center of ν on X.

Being resolved is an open condition on the Zariski-Riemann manifold of X. Suppose that

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a commutative diagram of morphisms such that Φ and Ψ are products of blow ups of 2-curves. If ν is a 0-dimensional valuation of $k(X)$ such that ν is resolved for f, then ν is resolved for f_1.

Suppose that $q \in Y$ is a 2-point, and ν is a 0-dimensional valuation of $k(X)$ such that q is the center of ν on Y. Let p be the center of ν on X. Let u, v, w be permissible parameters at q, so that $u = v = 0$ are local equations of the 2-curve C through q.

Case 1 Suppose that $\nu(u), \nu(v)$ are rationally independent. Since $uv = 0$ is a local equation of D_X at p, there exist regular parameters x, y, z in $\mathcal{O}_{X,p}$ such that $xyz = 0$ contains the germ of D_X in $\mathcal{O}_{X,p}$, and we have an expression

$$\begin{aligned} u &= x^a y^b z^c \gamma_1 \\ v &= x^d y^e z^f \gamma_2 \end{aligned}$$

where the γ_i are units in $\mathcal{O}_{X,p}$. Since $\nu(u), \nu(v)$ are rationally independent,

$$\text{rank}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = 2$$

which implies that p is a 2 or 3-point and f is prepared at p (and satisfies 2a) of Definition 3.4). Thus ν is resolved for f.

Case 2 Suppose that $\nu(u), \nu(v)$ are rationally dependent. There exists a sequence of blow ups of 2-curves $\Psi_\nu : Y_\nu \rightarrow Y$ such that the center of ν on Y_ν is a 1-point.

Y_ν is the blow up of a toroidal ideal sheaf $\mathcal{J}_\nu$ of $\mathcal{O}_Y$. Since $f^{-1}(D_Y) = D_X$, $\mathcal{J}_\nu$ is also a toroidal ideal sheaf. By Lemma 2.11 [C5], and induction on the number of 2-curves blown up by Ψ_ν, there exists a sequence of blow ups of 2-curves $\Phi_\nu : X_\nu \rightarrow X$ such that there is a commutative diagram of morphisms

$$\begin{tikzcd} X_\nu \arrow[r, "f_\nu"] \arrow[d, "\Phi_\nu"'] & Y_\nu \arrow[d, "\Psi_\nu"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Thus ν is resolved for f_ν.

It follows from compactness of the Zariski Riemann manifold of X [Z] that there exists a positive integer n and commutative diagrams

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "\Phi_i"'] & Y_i \arrow[d, "\Psi_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

for $1 \le i \le n$ such that Φ_i and Ψ_i are products of blow ups of 2-curves, and every valuation ν of $k(X)$ is resolved for some f_i.

Y_i is the blow up of a toroidal ideal sheaf $\mathcal{J}_i$ of $\mathcal{O}_Y$ and X_i is the blow up of a toroidal ideal sheaf $\mathcal{I}_i$ of $\mathcal{O}_X$. Thus there exists a sequence of blow ups of 2-curves $Y^* \rightarrow Y$ such that $\prod_i \mathcal{J}_i \mathcal{O}_{Y^*}$ is invertible, by Lemma 2.11 [C5]. Y^* is thus the blow up of a toroidal ideal sheaf $\mathcal{J} \subset \mathcal{O}_Y$. Thus $\mathcal{J}\mathcal{O}_X$ is also a toroidal ideal sheaf. By Lemma 2.11 [C5], there exists a sequence of blow ups of 2-curves $X^* \rightarrow X$ such that

$\mathcal{J}\prod\mathcal{I}_i\mathcal{O}_{X^*}$ is invertible. Thus for $1\le i\le n$, there exist commutative diagrams of morphisms

$$\begin{tikzcd}
X^* \arrow[r, "f^*"] \arrow[d, "\Phi_i^*"'] & Y^* \arrow[d, "\Psi_i^*"] \\
X_i \arrow[r, "f_i"] \arrow[d] & Y_i \arrow[d] \\
X \arrow[r] & Y.
\end{tikzcd}$$

Since Φ_i^* and Ψ_i^* are the blow ups of toroidal ideal sheaves, they are toroidal morphisms.

Suppose that ν is a 0-dimensional valuation of $k(X)$. If the center of ν on Y is a 3-point then f^* is resolved at the center of ν on X^*. In particular, if the center of ν on Y^* is a 3-point, then the center of ν on Y is a 3-point and ν is resolved for f^*. Suppose that the center of ν on Y^* is 2-point, and the center of ν on Y is not a 3-point. Then the center of ν on Y_i is a 2-point for all i. There exists an i such that ν is resolved for f_i. Thus f_i is prepared (and satisfies 2a) of Definition 3.4) at the center of ν on X_i. Since Φ_i^* and Ψ_i^* are toroidal, f^* is prepared (and satisfies 2a) of Definition 3.4) at the center of ν. Thus ν is resolved for f^*. Since all 0-dimensional valuations of $k(X)$ are resolved for f^*, it follows that f^* is toroidal above all 3-points of Y^*, and prepared above all 2-point of Y^*, and we have achieved the conclusions of the lemma. ☐

Lemma 4.3. *Suppose that $f:X\rightarrow Y$ satisfies the conclusions of Lemma 4.2. Suppose that H is a general hyperplane section of Y. Then f is prepared above all points of H.*

Proof. Bertini's theorem implies that H is nonsingular and makes SNCs with D_Y. Further, $H'=f^{-1}(H)$ is nonsingular and makes SNCs with D_X. Thus H contains no 3-points of Y and H' contains no 3-points.

Suppose that $q\in H\cap D_Y$ is a 1-point, and $p\in f^{-1}(q)$. Let u,v,w be regular parameters in $\mathcal{O}_{Y,q}$ such that $u=0$ is a local equation of D_Y at q, and $w=0$ is a local equation of H. Then we have regular paramaters x,y,z in $\mathcal{O}_{X,p}$ such that either p is a 1 point with $x=0$ a local equation of D_X or p is a 2-point with $xy=0$ a local equation of D_X at p. We then have an expression $u=x^a\gamma, w=z$ or $u=x^ay^b\gamma, w=z$ where γ is a unit in $\mathcal{O}_{X,p}$. Thus f is prepared at p. ☐

Corollary 4.4. *Suppose that $f:X\rightarrow Y$ satisfies the conclusions of Lemma 4.2. Then there exists a finite set of 1-points $\Omega\subset Y$ such that f is prepared above $Y-\Omega$.*

Proof. The locus of points in X where f is prepared is an open set. Since f is proper, the image Ω of the closed set of points where f is not prepared is closed in Y. Since a general hyperplane section of Y is disjoint from Ω by Lemma 4.3, Ω must be a finite set of points. ☐

Lemma 4.5. *Suppose that $f:X\rightarrow Y$ is a proper dominant morphism of nonsingular 3-folds and $\pi:Y\rightarrow S$ is a smooth dominant morphism onto a nonsingular surface S. Let $g=\pi\circ f$. Suppose that C is a nonsingular curve of S, $D=\pi^{-1}(C)$ and $D'=f^{-1}(D)$. Suppose that D' is a SNC divisor on X which contains the singular locus of g and the singular locus of f. Suppose that $\overline{q}\in C$ is a point, and that g is toroidal and prepared (with respect to C and D') away from points above finitely many points $\Omega=\{q_1,\ldots,q_m\}\subset\gamma=\pi^{-1}(\overline{q})$. Further suppose that f is finite above* *a general point of* γ. *Then there exists a commutative diagram of morphisms*

$$\begin{tikzcd} X_1 \arrow[r, "\Phi_1"] \arrow[dr, "g_1"'] & X \arrow[d, "g"] \\ & S \end{tikzcd}$$

such that Φ_1 *is a product of possible blow ups for the preimage of* D' *supported above* Ω *and* g_1 *is prepared (with respect to* C *and* $\Phi_1^{-1}(D')$*) in a neighborhood of all components* F *of* $\Phi_1^{-1}(D')$ *which dominate* D *and in a neighborhood of all components* F *of* $\Phi_1^{-1}(D')$ *which dominate a curve of* Y.

Proof. Let u, w be regular parameters in $\mathcal{O}_{S,\overline{q}}$ such that $u = 0$ is a local equation of C at $\overline{q}$. Let C' be the curve on S with local equation $w = 0$ at $\overline{q}$. Let $A = \pi^{-1}(C')$.

Since it suffices to prove the lemma above a neighborhood of $\overline{q}$ in S, we may assume that $E = C + C'$ is a SNC divisor on S whose only singular point is $\overline{q}$. Since g is toroidal away from points above Ω, we have that $g^{-1}(E)$ defines a SNC divisor on X away from points above Ω. There exists a morphism $\Phi_1 : X_1 \rightarrow X$ which is a sequence of possible blow ups for the preimage of D' supported above Ω such that with $g_1 = g \circ \Phi_1 : X_1 \rightarrow S$, $g_1^{-1}(E)$ is a SNC divisor, and $(f \circ \Phi_1)^{-1}(q_i)$ are divisors for all $q_i \in \Omega$. We may further assume that the union $\overline{A}$ of codimension 1 subvarieties of X_1 which dominate A are disjoint, since they are disjoint away from the preimage of Ω.

Let $\overline{D}$ be the union of codimension 1 subvarieties of X_1 which dominate D, so that $\overline{D}$ is a disjoint union of irreducible components of $D'' = g_1^{-1}(C)$ (by Remark 3.3).

Suppose that $p \in \overline{D}$ and $f \circ \Phi_1(p) = q_i \in \Omega$. Then p must be a 2-point or a 3-point. We have regular parameters x, y, z in $\hat{\mathcal{O}}_{X_1,p}$ such that one of the following cases hold:

1. p is a 2-point and
$$u = x^a y^b, w = y^c$$
where $x = 0$ is a local equation of $\overline{D}$, $u = 0$ is a local equation of D'' and $a, b > 0$.
2. p is a 2-point,
$$u = x^a y^b, w = y^c z$$
where $x = 0$ is a local equation of $\overline{D}$, $u = 0$ is a local equation of D'', $a, b, c > 0$ and $z = 0$ is a local equation of $\overline{A}$.
3. p is a 3-point and
$$u = x^a y^b z^c, w = y^d z^e$$
where $x = 0$ is a local equation of $\overline{D}$, $u = 0$ is a local equation of D'' and $a, b, c, d, e > 0$.

Thus g_1 is prepared in a neighborhood of $\overline{D}$.

Now suppose that F is a component of D'' which dominates a curve of Y and $p \in F$ is such that $f \circ \Phi_1(p) = q_i \in \Omega$. Then p must be a 2-point or a 3-point. By our assumption that f is finite above a general point of γ, F dominates the curve C of S. Thus we have regular parameters x, y, z in $\hat{\mathcal{O}}_{X_1,p}$ such that one of the following cases hold:

1. p is a 2-point and
$$u = x^a y^b, w = y^c$$
where $x = 0$ is a local equation of F, $u = 0$ is a local equation of D'' and $a, b > 0$.

2. p is a 2-point,
$$u = x^a y^b, w = y^c z$$
where $x = 0$ is a local equation of F, $u = 0$ is a local equation of D'', $a, b > 0$ and $z = 0$ is a local equation of $\overline{A}$.
3. p is a 3-point and
$$u = x^a y^b z^c, w = y^d z^e$$
where $x = 0$ is a local equation of F, $u = 0$ is a local equation of D'' and $a, b, c > 0$.

Thus g_1 is prepared in a neighborhood of F. ☐

Lemma 4.6. *Suppose that $f : X \rightarrow Y$ is a dominant morphism of nonsingular 3-folds with toroidal structures determined by SNC divisors D_Y and $D_X = f^{-1}(D_Y)$ such that D_X contains the singular locus of f. Further suppose that $f : X \rightarrow Y$ is toroidal and $q \in Y$ is a 2-point. Let $\Psi : Y_1 \rightarrow Y$ be the blow up of q. Then there exists a commutative diagram of morphisms*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that Φ is a sequence of possible blow ups for the preimage of D_X supported above q and f_1 is toroidal with respect to $D_{Y_1} = \Psi^{-1}(D_Y)$ and $D_{X_1} = \Phi^{-1}(D_X)$.

Proof. There exist permissible parameters u, v, w at q such that if $p \in f^{-1}(q)$ then there exist permissible parameters x, y, z for u, v, w such that if p is a 1-point, then we have a form
$$u = x^a, v = x^b(\alpha + y), w = z \tag{8}$$
with $0 \neq \alpha \in k$, and if p is a 2-point,
$$u = x^a y^b, v = x^c y^d, w = z, \tag{9}$$
with $ad - bc \neq 0$. We first show that there exists a sequence of possible blow ups
$$X_m \xrightarrow{\Phi_m} X_{m-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{\Phi_1} X \tag{10}$$
obtained by blow ups of possible centers supported above q such that the rational map $X_m \rightarrow Y_1$ is toroidal wherever it is defined, and if $\mathcal{I}_q \mathcal{O}_{X_m,p}$ is not invertible, then there exist regular parameters x, y, z in $\hat{\mathcal{O}}_{X_m,p}$ such that one of the following forms hold:

p is a 1-point
$$u = x^a, v = x^b(\alpha + y), w = x^c z \tag{11}$$
with $\alpha \neq 0$, and $c = 0$ or 1, or p is a 2-point
$$u = x^a y^b, v = x^c y^d, w = xz \tag{12}$$
with $a, c \geq 1$, or p is a 2-point
$$u = x^a y^b, v = x^c y^d, w = xyz \tag{13}$$
with $a, c \geq 1$ and $b, d \geq 1$.

The points $p \in f^{-1}(p)$ such that u, v, w do not have a form (11), (12) or (13) at p are 2-points of one of the following forms:
$$u = x^a, v = y^b, w = z, \tag{14}$$

in which case $V(x,y,z)$ is the locus in $\mathrm{spec}(\hat{\mathcal{O}}_{X,p})$ where $\mathcal{I}_q\hat{\mathcal{O}}_{X,p}$ is not invertible,

$$u = x^a, v = x^b y^c, w = z \tag{15}$$

with $b, c > 0$, in which case $V(x,z)$ is the locus in $\mathrm{spec}(\hat{\mathcal{O}}_{X,p})$ where $\mathcal{I}_q\hat{\mathcal{O}}_{X,p}$ is not invertible,

$$u = x^a y^b, v = x^c y^d, w = z \tag{16}$$

with $a, b, c, d > 0$ in which case $V(x,z) \cup V(y,z)$ is the locus in $\mathrm{spec}(\hat{\mathcal{O}}_{X,p})$ where $\mathcal{I}_q\hat{\mathcal{O}}_{X,p}$ is not invertible,

Let Z be the closed locus of points r in X such that $\mathcal{I}_q\mathcal{O}_{X,r}$ is not invertible. The isolated points p in Z have a form (14). If p is a non isolated point in Z which is a 2-point, then p has a form (15) or (16).

Suppose that E is a curve in Z such that E contains a 2-point p satisfying (15) or (16). Then a generic point of E satisfies (8) and all 2-points of E must have a form (15) or (16).

Let $\Phi_1 : X_1 \rightarrow X$ be the blow up of the finitely many points $p \in X$ of the form (14). Suppose that $p \in X$ is such a point, and $p_1 \in \Phi_1^{-1}(p)$. Without loss of generality, we may assume that $a \le b$ in (14). There are regular parameters x_1, y_1, z_1 in $\hat{\mathcal{O}}_{X_1,p_1}$ of one of the following forms:

$$x = x_1, y = x_1(y_1 + \alpha), z = x_1(z_1 + \beta) \tag{17}$$

with $\alpha, \beta \in k$,

$$x = x_1 y_1, y = y_1, z = y_1(z_1 + \alpha) \tag{18}$$

with $\alpha \in k$ or

$$x = x_1 z_1, y = y_1 z_1, z = z_1. \tag{19}$$

Suppose that (17) holds. Then u, v, w have a form

$$u = x_1^a, v = x_1^b(y_1 + \alpha)^b, w = x_1(z_1 + \beta)$$

at p_1. If $a = 1$, then $f \circ \Phi_1$ factors through Y_1 at p_1 and we have one of the following toroidal forms:

1-point maps to 2-point:

$$u_1 = u = x_1, v_1 = \frac{v}{u} = x_1^{b-1}(y_1 + \alpha)^b, w_1 = \frac{w}{u} - \beta = z_1$$

if $b > a = 1$ and $\alpha \neq 0$,

1-point maps to 1-point:

$$u_1 = u = x_1, v_1 = \frac{v}{u} - \alpha = y_1, w_1 = \frac{w}{u} - \beta = z_1$$

if $b = a = 1$, $\alpha \neq 0$,

2-point maps to 2-point:

$$u_1 = u = x_1, v_1 = \frac{v}{u} = x_1^{b-1} y_1^b, w_1 = \frac{w}{u} - \beta = z_1$$

if $a = 1$ and $\alpha = 0$.

Suppose that (17) holds and $a > 1$. If $\beta \neq 0$, we have that $\Phi_1 \circ f$ factors through Y_1 at p_1 and we have a toroidal form, obtained from a change of variable in

$$u_1 = \frac{u}{w} = x_1^{a-1}(z_1 + \beta)^{-1}, v_1 = \frac{v}{w} = x_1^{b-1}(z_1 + \beta)^{-1}(y_1 + \alpha)^b, w_1 = w = x_1(z_1 + \beta)$$

where p_1 is 1-point mapping to a 3-point if $\alpha \neq 0$ and p_1 is 2-point mapping to a 3-point if $\alpha = 0$.

If $\beta=0$ (and $a>1$) then we have

$$u=x_1^a, v=x_1^b(y_1+\alpha), w=x_1z_1$$

of the form (11) if $\alpha\neq 0$ and of the form (12) if $\alpha=0$.

Suppose that (18) holds. Then at p_1, u,v,w have a form:

$$u=x_1^ay_1^a, v=y_1^b, w=y_1(z_1+\alpha).$$

Assume $b=1$ (which implies $a=1$). then $f\circ\Phi_1$ factors through Y_1 at p_1, and there is a toroidal form:

$$u_1=\frac{u}{v}=x_1, v_1=v=y_1, w_1=\frac{w}{v}-\alpha=z_1$$

where p_1 is 2-point mapping to a 2-point.

Assume that $b>1$ and $\alpha\neq 0$. Then $f\circ\Phi_1$ factors through Y_1 at p_1, and there is a toroidal form, obtained from a change of variable in

$$u_1=\frac{u}{w}=x_1^ay_1^{a-1}(z_1+\alpha)^{-1}, v_1=\frac{v}{w}=y_1^{b-1}(z_1+\alpha)^{-1}, w_1=w=y_1(z_1+\alpha)$$

where p_1 is a 2-point mapping to a 3-point.

If $b>1$ and $\alpha=0$, then we have a form:

$$u=x_1^ay_1^a, v=y_1^b, w=y_1z_1$$

of the form (12).

Suppose that (19) holds. Then p_1 is a 3-point and u,v,w have a form

$$u=x_1^az_1^a, v=y_1^bz_1^b, w=z_1.$$

Thus $\Phi_1\circ f$ factors through Y_1 at p_1 by

$$u_1=\frac{u}{w}=x_1^az_1^{a-1}, v_1=\frac{v}{w}=y_1^bz_1^{b-1}, w_1=w=z_1,$$

where p_1 is a 3-point mapping to a 3-point.

We have thus completed the analysis of Φ_1.

We now construct (10) by induction. Each X_i will be such that the rational map $X_i\rightarrow Y_1$ is toroidal wherever it is defined, and if $p\in X_i$ is a 2-point such that $\mathcal{I}_q\mathcal{O}_{X_1,p}$ is not invertible, then there exist regular parameters x,y,z at p such that u,v,w have one of the forms (11), (12), (13), (15) or (16) at p. If a form (15) or (16) holds at p, then $\Phi_1\circ\cdots\circ\Phi_i$ is an isomorphism near p.

Each $\Phi_{i+1}:X_{i+1}\rightarrow X_i$ for $i\geq 1$ will be the blow up of a curve E_i which is a possible center and is the strict transform of a component of $Z\subset X$.

Suppose that we have constructed (10) out to X_i, and $p\in X_i$ is a 2-point such that $\mathcal{I}_q\mathcal{O}_{X_i,p}$ is not invertible, and u,v,w do not have a form (11), (12) or (13) at p. Then u,v,w have a form (15) or (16) at p. Let $E=E_i$ be a curve in the locus where $\mathcal{I}_q\mathcal{O}_{X_i}$ is not invertible which contains p. Let F be the component of D_{X_i} containing E_i We necessarily have $\mathrm{ord}_Fw=0$ and $\mathrm{ord}_Fu>0$, $\mathrm{ord}_Fv>0$. Further, $\Phi_1\circ\cdots\circ\Phi_i$ is an isomorphism near p. Thus E is the strict transform of a component of Z.

Suppose that $p'\in E_i$ is another 2-point. Then at p', since $\mathrm{ord}_Fw=0$, u,v,w must have a form (15), (16) or (12), where in this last case, $y=z=0$ is a local equation of E and $b,d\geq 1$ (since $\mathrm{ord}_Fw=0$, $\mathrm{ord}_Fu>0$ and $\mathrm{ord}_Fv>0$). If $p'\in E_i$ is a 1-point, then u,v,w have a form (8) at p', since $\mathrm{ord}_Fw=0$.

Let $\Phi_{i+1}:X_{i+1}\rightarrow X_i$ be the blow up of E and $\overline{\Phi}_{i+1}=\Phi_1\circ\cdots\circ\Phi_{i+1}$.

Suppose that $p\in E$ is a 1-point and $p_1\in\Phi_{i+1}^{-1}(p)$. Then $f\circ\overline{\Phi}_{i+1}$ is toroidal whenever it is defined, and points above p where $f\circ\overline{\Phi}_{i+1}$ does not factor through Y_i have a form (11). A detailed analysis of a case including this one is given later in the proof, after (26).

Suppose that $p\in E$ is a 2-point of the form (16) and $p_1\in\Phi_{i+1}^{-1}(p)$.

There are regular parameters x_1,y_1,z_1 in $\hat{\mathcal{O}}_{X_i,p_1}$ of one of the following forms:

$$x=x_1, z=x_1(z_1+\alpha) \tag{20}$$

with $\alpha\in k$ or

$$x=x_1z_1, z=z_1. \tag{21}$$

Suppose that (20) holds. We have that p_1 is a 2-point, and

$$u=x_1^ay^b, v=x_1^cy^d, w=x_1(z_1+\alpha).$$

If $\alpha\neq 0$, we have that $f\circ\overline{\Phi}_{i+1}$ factors through Y_1 at p_1. We have a form:

$$u_1=\frac{u}{w}=x_1^{a-1}y^b(z_1+\alpha)^{-1}, v_1=\frac{v}{w}=x_1^{c-1}y^d(z_1+\alpha)^{-1}, w_1=x_1(z_1+\alpha)$$

at the 2-point p_1, which maps to a 3-point, and thus is toroidal, after a change of variables.

If $\alpha=0$ in (20), we have

$$u=x_1^ay^b, v=x_1^cy^d, w=x_1z_1$$

of the form (12).

If (21) holds, then p_1 is a 3-point and

$$u=x_1^ay^bz_1^a, v=x_1^cy^dz_1^c, w=z_1.$$

Thus $f\circ\overline{\Phi}_{i+1}$ factors through Y_1 at p_1, and we have a toroidal form:

$$u_1=\frac{u}{w}=x_1^ay^bz_1^{a-1}, v_1=\frac{v}{w}=x_1^cy^dz_1^{c-1}, w_1=w=z_1$$

at the 3-point p_1, which maps to a 3-point.

The analysis of Φ_{i+1} above points (15) and above points satisfying (12) where $y=z=0$ are local equations of E (and $b,d\geq 1$) is similar. This last case will lead to a form (13). Since Z has only finitely many components, we inductively construct (10).

There now exists a sequence of blow ups of 2-curves $X_r\rightarrow X_m$ which are supported above q such that the rational map $X_r\rightarrow Y_1$ is toroidal where ever it is defined, and if $\mathcal{I}_q\mathcal{O}_{X_r,p}$ is not invertible, then there there exist permissible parameters x,y,z at p for u,v,w such that one of the following forms hold:

p a 1-point

$$u=x^a, v=x^b(\alpha+y), w=x^dz \tag{22}$$

with $0\neq\alpha\in k$ and $d<\min\{a,b\}$ or

p a 2-point

$$u=x^ay^b, v=x^cy^d, w=x^ey^fz \tag{23}$$

with $(e,f)<(a,b)<(c,d)$ or $(e,f)<(c,d)<(a,b)$.

We accomplish this as follows. We first consider u and v. Suppose that $p\in X_m$ is a 2-point such that $\mathcal{I}_q\mathcal{O}_{X_m,p}$ is not invertible. We have forms

$$u=x^ay^b, v=x^cy^d, w=x^ey^fz \tag{24}$$

with $e+f>0$ at 2-points p_i above p in the construction of the sequence $X_r\rightarrow X_m$. At p_i we have an invariant $(a-c)(b-d)$. This is a nonnegative integer if and only if $(a,b)\leq(c,d)$ or $(c,d)\leq(a,b)$. Further, if $(a-c)(b-d)<0$, then after blowing up the 2-curve E which has local equations $x=y=0$ at p_i, we obtain that all 2-points above p_i have a form (24), but $(a-c)(b-d)$ has increased. Further E contracts to q on Y since $e+f>0$.

After a finite number of blow ups of 2-curves above X_m (which must contract to q) we achieve that all 2-points p_i above a 2-point $p\in X_m$ such that $\mathcal{I}_q\mathcal{O}_{X_m,p}$ is not invertible have a form (24) with $(a,b)\le(c,d)$ or $(c,d)\le(a,b)$.

We now apply this algorithm to the pairs u, x^ey^f and v, x^ey^f in (24) to construct $X_r\rightarrow X_m$.

We will now inductively construct $X_n\rightarrow X_r$ so that $\mathcal{I}_q\mathcal{O}_{X_n}$ is invertible everywhere and the morphism $X_n\rightarrow Y_1$ is toroidal. We will construct a sequence of blow ups

$$X_n\rightarrow X_{n-1}\rightarrow\cdots\rightarrow X_{r+1}\rightarrow X_r \tag{25}$$

so that each $\Phi_{i+1}:X_{i+1}\rightarrow X_i$ is the blow up of a nonsingular curve λ_i which is a possible center and is contained in the locus where $\mathcal{I}_q\mathcal{O}_{X_i}$ is not invertible. We will have that the rational map $f_i:X_i\rightarrow Y_1$ is toroidal where ever it is defined, and all points $p\in X_i$ where $\mathcal{I}_q\mathcal{O}_{X_i,p}$ is not invertible have a form (22) or (23).

Suppose that we have inductively constructed (25) up to X_i and $\mathcal{I}_q\mathcal{O}_{X_i}$ is not invertible. We will construct $\Phi_{i+1}:X_{i+1}\rightarrow X_i$.

Inspection of the forms (22) and (23) shows that the locus in X_i where $\mathcal{I}_q\mathcal{O}_{X_i}$ is not invertible is a union of nonsingular curves which are possible centers. For such a curve λ, let η be a general point of λ, so that a form (22) holds at η. Let $A(\lambda)=\min\{a,b\}-d>0$.

Choose a curve λ_i which maximizes $A(\lambda)$ on X_i. Let $\Phi_{i+1}:X_{i+1}\rightarrow X_i$ be the blow up of λ_i. Suppose that $p_i\in\lambda_i$ and $p_{i+1}\in\Phi_{i+1}^{-1}(\lambda_i)$.

First suppose that p_i has the form (22). Without loss of generality, we may assume that $a\le b$. There are regular parameters x_1,y,z_1 in $\hat{\mathcal{O}}_{X_{i+1},p_{i+1}}$ satisfying

$$x=x_1, z=x_1(z_1+\beta) \tag{26}$$

or

$$x=x_1z_1, z=z_1. \tag{27}$$

Suppose that (26) holds. p_{i+1} is then a 1-point, and

$$u=x_1^a, v=x_1^b(\alpha+y), w=x_1^{d+1}(z_1+\beta). \tag{28}$$

If $d+1=a=b$ in (28), then $X_{i+1}\rightarrow Y_1$ is a morphism near p_{i+1}, which maps p_{i+1} to a 1-point, and has a toroidal form

$$u_1=u=x_1^a, v_1=\frac{v}{u}-\alpha=y_1, w_1=\frac{w}{u}-\beta=z_1.$$

If $d+1=a<b$ in (28), then $X_{i+1}\rightarrow Y_1$ is a morphism near p_{i+1}, which maps p_{i+1} to a 2-point, and has a toroidal form

$$u_1=u=x_1^a, v_1=\frac{v}{u}=x_1^{b-a}(\alpha+y_1), w_1=\frac{w}{u}-\beta=z_1.$$

If $d+1<a\le b$ and $\beta\ne0$ in (28) then $X_{i+1}\rightarrow Y_1$ is a morphism near p_{i+1}, which maps p_{i+1} to a 3-point, and has a toroidal form obtained from a change of variable in

$$u_1=\frac{u}{w}=x_1^{a-d-1}(z_1+\beta)^{-1}, v_1=\frac{v}{w}=x_1^{b-d-1}(\alpha+y_1)(z_1+\beta)^{-1}, w_1=w=x_1^{d+1}(z_1+\beta).$$

If $d+1<a\le b$ and $\beta=0$ then (28) has a form (22) with $d<d+1<\min\{a,b\}$.

Suppose that (27) holds. p_{i+1} is then a 2-point, and

$$u=x_1^az_1^a, v=x_1^bz_1^b(\alpha+y), w=x_1^dz_1^{d+1}.$$

$X_{i+1}\rightarrow Y_1$ is thus a morphism near p_{i+1}, which maps p_{i+1} to a 3-point, and has a toroidal form

$$u_1=\frac{u}{w}=x_1^{a-d}z_1^{a-d-1}, v_1=\frac{v}{w}=x_1^{b-d}z_1^{b-d-1}(\alpha+y_1), w_1=w=x_1^dz_1^{d+1}.$$

Now suppose that p_i has the form (23). After possibly interchanging u and v, we may assume that $(a,b)<(c,d)$. After possibly interchanging x and y, we may assume that there are regular parameters x_1,y,z_1 in $\hat{\mathcal{O}}_{X_{i+1},p_{i+1}}$ satisfying (26) or (27) (so that $e<a$).

Suppose that (26) holds. Then p_{i+1} is a 2-point. We have

$$u=x_1^ay^b, v=x_1^cy^d, w=x_1^{e+1}y^f(z_1+\beta). \tag{29}$$

If $(e+1,f)=(a,b)$ in (29), then $X_{i+1}\rightarrow Y_1$ is a morphism near p_{i+1}, which maps p_{i+1} to a 2-point, and has a toroidal form

$$u_1=u=x_1^ay^b, v_1=\frac{v}{u}=x_1^{c-a}y_1^{d-b}, w_1=\frac{w}{u}-\beta=z_1.$$

If $(e+1,f)<(a,b)$ and $\beta\neq 0$ in (29), then $X_{i+1}\rightarrow Y_1$ is a morphism near p_{i+1}, which maps p_{i+1} to a 3-point, and has a toroidal form obtained from a change of variable in

$$u_1=\frac{u}{w}=x_1^{a-e-1}y^{b-f}(z_1+\beta)^{-1}, v_1=\frac{v}{w}=x_1^{c-e-1}y_1^{d-f}(z_1+\beta)^{-1}, w_1=w=x_1^{e+1}y^f(z_1+\beta).$$

If $(e+1,f)<(a,b)$ and $\beta=0$ then (29) has a form (23) with $(e,f)<(e+1,f)<(a,b)$.

Suppose that (27) holds. p_{i+1} is then a 3-point, and

$$u=x_1^ay^bz_1^a, v=x_1^cy^dz_1^c, w=x_1^ey^fz_1^{e+1}.$$

$X_{i+1}\rightarrow Y_1$ is thus a morphism near p_{i+1}, which maps p_{i+1} to a 3-point, and has a toroidal form

$$u_1=\frac{u}{w}=x_1^{a-e}y^{b-f}z_1^{a-e-1}, v_1=\frac{v}{w}=x_1^{c-e}y^{d-f}z_1^{c-e-1}, w_1=w=x_1^ey^fz_1^{e+1}.$$

In summary, we have that all points where $\mathcal{I}_q\mathcal{O}_{X_{i+1}}$ is not invertible have a form (22) or (23) and if $\lambda_{i+1}\subset\Phi_i^{-1}(\lambda_i)$ is a curve such that $\mathcal{I}_q\mathcal{O}_{X_{i+1}}$ is not invertible along λ_i, we have $0<A(\lambda_{i+1})<A(\lambda_i)$. Thus after a finite number of blow ups, we construct the desired sequence (25), completing the proof of the lemma. ☐

Lemma 4.7. *Suppose that $f:X\rightarrow Y$ is a dominant morphism of nonsingular 3-folds with toroidal structures determined by SNC divisors D_Y and $D_X=f^{-1}(D_Y)$ such that D_X contains the singular locus of f. Further suppose that $f:X\rightarrow Y$ is toroidal and $C\subset Y$ is a possible center for D_Y which contains a 1-point. Let $\Psi:Y_1\rightarrow Y$ be the blow up of C. Then there exists a commutative diagram of morphisms*

$$\begin{tikzcd} \overline{X}_1 \arrow[r, "f_1"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that Φ is a sequence of possible blow ups for the preimage of D_X supported above C and f_1 is toroidal with respect to $D_{Y_1}=\Psi^{-1}(D_Y)$ and $D_{\overline{X}_1}=\Phi^{-1}(D_X)$.

Further, Φ has a factorization

$$\overline{X}_1=X_n\rightarrow X_{n-1}\rightarrow\cdots\rightarrow X_1\rightarrow X$$

where each $\Phi_{i+1}:X_{i+1}\rightarrow X_i$ is either the blow up of a section E_i over C such that $\mathcal{I}_C\mathcal{O}_{X_i}$ is not invertible, or $\Phi_{i+1}:X_{i+1}\rightarrow X_i$ is the blow up of a curve E_i which maps to a 2-point of Y and such that E_i is contained in the locus where $\mathcal{I}_C\mathcal{O}_{X_i}$ is invertible.

Proof. We follow the algorithm of Lemma 18.17 [C3] to construct Φ.

Suppose that $q\in C$ and $p\in f^{-1}(q)$. Then there are permissible parameters u,v,w for D_Y in $\mathcal{O}_{Y,q}$ and regular parameters x,y,z in $\hat{\mathcal{O}}_{X,p}$ such that one of the following cases holds:

q is a 2-point and p is a 2-point,

$$u=x^ay^b, v=x^dy^e, w=z \tag{30}$$

where $uv=0$ is a local equation of D_Y and $u=w=0$ is a local equation of C.

q is a 2-point and p is a 1-point,

$$u=x^a, v=x^b(y+\alpha), w=z \tag{31}$$

where $0\ne\alpha\in k$, $uv=0$ is a local equation of D_Y and $u=w=0$ is a local equation of C.

q is a 1-point and p is a 1-point,

$$u=x^a, v=y, w=z \tag{32}$$

where $u=0$ is a local equation of D_Y and $u=w=0$ is a local equation of C.

We will construct a sequence of morphisms

$$\cdots\rightarrow X_n\overset{\Phi_n}{\rightarrow}X_{n-1}\overset{\Phi_{n-1}}{\rightarrow}\cdots\rightarrow X_1\overset{\Phi_1}{\rightarrow}X \tag{33}$$

where each Φ_{i+1} is the blow up of a nonsingular curve E_i contained in the locus where $\mathcal{I}_C\mathcal{O}_{X_i}$ is not invertible, and for each $q\in C$ and $p\in(f\circ\Phi_1\circ\cdots\circ\Phi_i)^{-1}(q)$ such that $\mathcal{I}_C\mathcal{O}_{X_i,p}$ is not invertible, there are permissible parameters u,v,w for D_Y in $\mathcal{O}_{Y,q}$ and permissible parameters x,y,z in $\hat{\mathcal{O}}_{X,p}$ such that one of the forms (34) - (36) below hold.

q a 2-point, p a 2-point

$$u=x^ay^b, v=x^dy^e, w=x^gy^hz \tag{34}$$

with $ae-ba\ne0$, $(g,h)<(a,b)$.

q a 2-point, p a 1-point

$$u=x^a, v=x^b(y+\alpha), w=x^dz \tag{35}$$

with $0\ne\alpha\in k$, $d<a$,

q a 1-point, p a 1-point

$$u=x^a, v=y, w=x^dz \tag{36}$$

with $d<a$. Further in the locus where the rational map $X_i\rightarrow Y_1$ is a morphism, $X_i\rightarrow Y_1$ is toroidal.

Observe that the forms (30), (31) and (32) are special cases of (34), (35) and (36) respectively.

The locus of points where $\mathcal{I}_C\mathcal{O}_{X_i}$ is not invertible is a union of nonsingular curves which intersect transversally. If E is a curve in this locus, and $p'\in E$ is a general point, then u,v,w have a form (35) or (36) at p'. In either case, we define an invariant

$$\Omega(E)=a-d>0.$$

Let $\Phi_{i+1}:X_{i+1}\rightarrow X_i$ be the blow up of a curve E_i such that $\Omega(E_i)$ is maximal. Suppose that $p_1\in E_i$, $p_2\in\Phi_{i+1}^{-1}(p_1)$ and $q=(f\circ\Phi_1\circ\cdots\circ\Phi_i)(p_1)$.

Suppose that p_1 has a form (35). Then $\hat{\mathcal{O}}_{X_{i+1},p_2}$ has regular parameters x_1,y,z_1 such that

$$x=x_1, z=x_1(z_1+\beta) \tag{37}$$

with $\beta\in k$ or

$$x=x_1z_1, z=z_1. \tag{38}$$

Suppose that (37) holds. Then p_2 is a 1-point,

$$u=x_1^a, v=x_1^b(y+\alpha), w=x_1^{d+1}(z_1+\beta). \tag{39}$$

If $d+1=a$ in (39), then $X_{i+1}\rightarrow Y_1$ is a morphism near p_2, mapping p_2 to a 2-point, and at p_2, we have a toroidal form

$$u_1=u=x_1^a, v=x_1^b(y+\alpha), w_1=\frac{w}{u}-\beta=z_1.$$

If $d+1<a$ and $\beta\ne 0$ in (39) then $X_{i+1}\rightarrow Y_1$ is a morphism near p_2, mapping p_2 to a 3-point, and at p_2, we have a toroidal form obtained from a change of variable in

$$u_1=\frac{u}{w}=x_1^{a-d-1}(z_1+\beta)^{-1}, v=x_1^b(y_1+\alpha), w_1=w=x_1^{d+1}(z_1+\beta).$$

If $d+1<a$ and $\beta=0$ in (39), then we have a form (35) with d increased to $d+1$. The curve E' containing p_2 in the locus where $\mathcal{I}_C\mathcal{O}_{X_{i+1}}$ is not invertible satisfies

$$0<\Omega(E')=a-(d+1)<\Omega(E).$$

Suppose that (38) holds. Then p_2 is a 2-point.

$$u=x_1^az_1^a, v=x_1^bz_1^b(y+\alpha), w=x_1^dz_1^{d+1}. \tag{40}$$

Further, $X_{i+1}\rightarrow Y_1$ is a morphism near p_2, mapping p_2 to a 3-point, and at p_2, we have a toroidal form

$$u_1=\frac{u}{w}=x_1^{a-d}z_1^{a-d-1}, v=x_1^bz_1^b(y+\alpha), w_1=w=x_1^dz_1^{d+1}.$$

There is a similar argument if p_1 satisfies (36).

Suppose that p_1 has a form (34) and $x=z=0$ are local equations of E_i (so that $g<a$). $\hat{\mathcal{O}}_{X_{i+1},p_2}$ has regular parameters x_1, y_1, z_1 satisfying (37) or (38).

Suppose that (37) holds. Then p_2 is a 2-point,

$$u=x_1^ay^b, v=x_1^dy^e, w=x_1^{g+1}y^h(z_1+\beta). \tag{41}$$

If $(g+1,h)=(a,b)$ in (41), then $X_{i+1}\rightarrow Y_1$ is a morphism near p_2, mapping p_2 to a 2-point, and at p_2, we have a toroidal form

$$u_1=u=x_1^ay^b, v_1=v=x_1^dy^e, w_1=\frac{w}{u}-\beta=z_1.$$

If $(g+1,h)<(a,b)$ and $\beta\ne 0$ in (41), then $X_{i+1}\rightarrow Y_1$ is a morphism near p_2, mapping p_2 to a 3-point, and we have a toroidal form obtained from a change of variable in

$$u_1=\frac{u}{w}=x_1^{a-g-1}y^{b-h}(z_1+\beta)^{-1}, v=x_1^dy^e, w_1=w=x_1^{g+1}y^h(z_1+\beta).$$

If $(g+1,h)<(a,b)$ and $\beta=0$ in (41), then (41) has the form (34) with g increased to $g+1$.

Suppose that (38) holds. Then p_2 is a 3-point,

$$u=x_1^ay^bz_1^a, v=x_1^dy^ez_1^d, w=x_1^gy_1^hz_1^{g+1}.$$

$X_{i+1}\rightarrow Y_1$ is a morphism near p_2, mapping p_2 to a 3-point, and at p_2, we have a toroidal form

$$u_1=\frac{u}{w}=x_1^{a-g}y_1^{b-h}z_1^{a-g-1}, v=x_1^dy^ez_1^d, w_1=w=x_1^gy_1^hz_1^{g+1}.$$

By descending induction on $\max(\Omega(E))$, we see that the sequence (33) must terminate after a finite number of blow ups, and we complete the proof of the lemma. $\square$

5. Preparation

In this section we prove Theorem 1.3.

We may assume (after possibly blowing up points on X) that D_X is strongly cuspidal.

To prove this theorem, we may assume by Lemmas 4.1 and 4.2 that f is prepared (of type 2 a) of Definition 3.4) above 2 points and toroidal above 3-points of Y. By Corollary 4.4, f only fails to be prepared above a finite set of 1-points $\Sigma \subset Y$. Since this reduction involves only blow ups of 2-curves we continue to have the condition that D_X is strongly cuspidal.

Suppose that $q \in \Sigma$. Let D be the component of D_Y containing q. There exists a very ample effective divisor L on Y such that $q \notin L$ and $D + L \sim H$ where H is a very ample effective divisor such that $q \notin H$. Let $\alpha : Z \rightarrow Y$ be the blow up of q, with exceptional divisor E. We may replace L with a high multiple of L so that $\alpha^*H - E$ is very ample on Z. Let N be a general member of $\alpha^*H - E$. By Bertini's theorem, N is nonsingular, makes SNCs with $D_Z = \alpha^{-1}(D_Y)$, intersects 2-curves of D_Z transversally at general points, does not contain a component of the strict transform on Z of the fundamental locus of f, and is disjoint from $\alpha^{-1}(\Sigma - \{q\})$. Let $M = \alpha(N)$. Then $M \sim H$, M is nonsingular and intersects D transversally in a nonsingular curve $\overline{\gamma}$ which contains q, M contains no other points of Σ, contains no 3-points of D_Y, intersects 2-curves of D_Y transversally at general points, does not contain a component of the fundamental locus of f and by Bertini's theorem, at points which are not above q, $f^*(M)$ is nonsingular and $f^*(M) + D_X$ is a SNC dvisor. After possibly replacing L and H with effective divisors linearly equivalent to L and H respectively, we may assume that $\overline{\gamma} \cap (L + H)$ consists of 1-points of D_Y and is disjoint from the fundamental locus of f.

$U = Y - (L + H)$ is an affine neighborhood of q. Let $\gamma = \overline{\gamma} \cap U$. There exist $\overline{f}, \overline{g} \in \Gamma(Y, \mathcal{O}_Y(H))$ such that $(\overline{f}) = D + L - H$ and $(\overline{g}) = M - H$. We can thus define a morphism $\pi : U \rightarrow S = \mathbf{A}^2$ by $\pi(a) = (\overline{f}(a), \overline{g}(a))$ for $a \in U$. Let $\overline{q} = \pi(q)$. $\pi^{-1}(\overline{q}) = \gamma$ (scheme theoretically) so π is smooth in a neighborhood of γ. We may thus replace U with an open neighborhood of γ so that π is smooth.

Let $\overline{X} = f^{-1}(U)$, and $\overline{f} = f \mid \overline{X}$. Let $D_U = D_Y \cap U$, $D_U^* = D \cap U$, $D_S^* = \pi(D_U^*)$, $g = \pi \circ \overline{f} : \overline{X} \rightarrow S$, $D_{\overline{X}}^* = g^{-1}(D_S^*)$, $D_{\overline{X}} = D_X \cap \overline{X}$. The map π is toroidal with respect to D_S^* and D_U^*.

Let $D_1, \ldots, D_m$ be the components of D_Y other than D which intersect γ. Since γ intersects these components transversally, we may assume then that $\pi \mid D_i \cap U$ is étale onto its image for $1 \leq i \leq m$. We further may assume that $\Sigma \cap U = \{q\}$, and (by Bertini's theorem) for $\overline{q}' \in D_S^* - \{\overline{q}\}$, there exist regular parameters u, w at $\overline{q}'$ such that $u = 0$ is a local equation of D_S^*, and if E is the curve $w = 0$ on S, then E is nonsingular, $D_S^* + E$ is a SNC divisor, $g^{-1}(E)$ is nonsingular, and $g^{-1}(E) + D_{\overline{X}}$ is a SNC divisor on $\overline{X}$. Thus if $q' \in \pi^{-1}(\overline{q}')$, there exist permissible parameters u, v, w in $\mathcal{O}_{U,q'}$ (for D_U) such that if $p \in \overline{f}^{-1}(q')$ then there exist regular parameters x, y, z in $\hat{\mathcal{O}}_{\overline{X},p}$ such that

$$u = x^a y^b, w = z \tag{42}$$

where $u = x^a y^b = 0$ is a local equation of $D_{\overline{X}}$ at p (with $a > 0$, $b \geq 0$) if $q' \in D - \cup D_i$ and

$$u = x^a y^b, v = x^c y^d \gamma, w = z \tag{43}$$

where $\gamma\in\hat{\mathcal{O}}_{\overline{X},p}$ is a unit and $uv=x^{a+c}y^{b+d}=0$ is a local equation of $D_{\overline{X}}$ at p if $q'\in D\cap D_i$ for some i.

Since γ intersects the 2-curves $D_i\cap D\cap U$ of U at general points of the 2-curves, after possibly replacing U with a smaller open neighborhood of γ, we have that the intersection of the fundamental locus of f with U is contained in $D\cap U$.

We will now establish that g is toroidal and prepared with respect to D_S^* and $D_{\overline{X}}^*$ away from the preimages of finitely many 1-points $\Omega\subset\gamma$ of D_U.

Suppose that $q'\in(D_i-D)\cap U$, $p\in\overline{f}^{-1}(q')$, and $\overline{q}'=\pi(q')$, which implies that there exist regular parameters u,w at $\overline{q}'$, u,v,w at q' such that $v=0$ is a local equation of D_i. q' is not in the fundamental locus of $\overline{f}$, and q' is a 1-point of D_U, so by Abhyankar's lemma there exist regular parameters x,y,z in $\hat{\mathcal{O}}_{\overline{X},p}$ such that

$$u=x,v=y^b,w=z. \tag{44}$$

g is defined by $u=x,w=z$ near p, which implies that g is smooth, and thus prepared and toroidal for D_S^* and $D_{\overline{X}}^*$ at p.

Suppose that $q'\in(D-\gamma)\cap U$, $p\in\overline{f}^{-1}(q')$, $\overline{q}'=\pi(q')$. Then we have a form (42) or (43) at p, so that g is prepared and toroidal for D_S^* and $D_{\overline{X}}^*$ at p.

Let $\delta=D\cap D_i\cap U$ for some $1\le i\le m$. Suppose that $q'\in\delta\cap\gamma$ and $p\in\overline{f}^{-1}(q')$. Then $\pi(q')=\overline{q}$. Recall that q' is a general point of the 2-curve δ. There exist regular parameters u,w in $\mathcal{O}_{S,\overline{q}'}$ such that $u=0$ is a local equation of D on U, $w=0$ is a local equation of M on U, and there exists $v\in\mathcal{O}_{U,q'}$ such that $v=0$ is a local equation of D_i and u,v,w are regular parameters in $\mathcal{O}_{U,q'}$. By our choice of M, $\overline{f}^*(M)$ is nonsingular and makes SNCs with $D_{\overline{X}}$ at p. Since $uv=0$ is a local equation of $D_{\overline{X}}$ at p, and $w=0$ is a local equation of $\overline{f}^*(M)$ at p, there exist permissible parameters x,y,z in $\mathcal{O}_{\overline{X},p}$ such that

$$u=x^ay^b\gamma_1,v=x^cy^d\gamma_2,w=z$$

with γ_1,γ_2 units in $\mathcal{O}_{\overline{X},p}$. Thus g is prepared and toroidal for D_S^* and $D_{\overline{X}}^*$ at p.

Suppose that $q'\in\gamma$ is a general point. Then q' is a 1-point of D_U and $\overline{f}$ is finite above q'. There exist regular parameters u,v,w in $\mathcal{O}_{U,q'}$ such that u,w are permissible parameters for D_S^* at $\overline{q}=\pi(q')$, and if $p\in\overline{f}^{-1}(q')$, then there exist permissible parameters x,y,z in $\hat{\mathcal{O}}_{\overline{X},p}$ such that

$$u=x^a,v=y,w=z$$

by Abhyankar's lemma, which implies that g is prepared and toroidal at p for D_S^* and $D_{\overline{X}}^*$.

We conclude that g is toroidal and prepared with respect to D_S^* and $D_{\overline{X}}^*$ away from points above finitely many 1-points $\Omega\subset\gamma$ of D_U.

Recall that there are no 3-points of $\overline{X}$ supported above $D_i\cap U$ for $1\le i\le m$. After blowing up points supported above Ω, we obtain that the irreducible components F of $D_{\overline{X}}$ which do not contain a 3-point are precisely the components which dominate D_i for some i or dominate a 2-curve $D_i\cap D$ for some i, and the 2-curves C of $D_{\overline{X}}$ which do not contain a 3-point are precisely the 2-curves which dominate a 2-curve $D_i\cap D$.

Suppose that $\Lambda:Z\rightarrow U$ is a dominant morphism of 3-folds, and D_Z is a SNC divisor on U. We will say that D_Z is U cuspidal if all irreducible components F of D_Z which do not contain a 3-point dominate D_i for some i, or dominate $D_i\cap D$ for some i, and the 2-curves C of D_Z which do not contain a 3-point dominate a 2-curve $D_i\cap D$.

By Lemma 4.5, there exists a morphism $\Phi_1 : \overline{X}_1 \rightarrow \overline{X}$ such that Φ_1 is a sequence of possible blow ups for the preimage of $D^*_{\overline{X}}$ of points and nonsingular curves supported above Ω such that if $g_1 = g \circ \Phi_1 : \overline{X}_1 \rightarrow S$ and $f_1 = \overline{f} \circ \Phi_1 : \overline{X}_1 \rightarrow U$, then g_1 is prepared for D^*_S and $D^*_{\overline{X}_1} = \Phi_1^{-1}(D^*_{\overline{X}})$ in a neighborhood of all components of $D^*_{\overline{X}_1}$ which do not map to a point of Ω.

By blowing up points on components of $D^*_{\overline{X}_1}$ which dominate a point of Ω, we may suppose that $D_{\overline{X}_1} = \Phi_1^{-1}(D_{\overline{X}})$ is U cuspidal.

By 1 of Theorem 3.1 [C5], there exists a morphism $\Phi_2 : \overline{X}_2 \rightarrow \overline{X}_1$ which is a sequence of possible blow ups for the preimage of $D^*_{\overline{X}_1}$ of points and nonsingular curves supported above Ω, such that $g_2 = \pi \circ f_1 \circ \Phi_2 : \overline{X}_2 \rightarrow S$ is prepared for D^*_S and $D^*_{\overline{X}_2} = \Phi_2^{-1}(D^*_{\overline{X}_1})$. Let $f_2 = f_1 \circ \Phi_2 : \overline{X}_2 \rightarrow U$. We further have that $D_{\overline{X}_2} = \Phi_2^{-1}(D_{\overline{X}_1})$ is U cuspidal.

Now by 2 of Theorem 3.1 [C5], there exists a commutative diagram

$$\begin{tikzcd}
\overline{X}_3 \arrow[r, "g_3"] \arrow[d, "\overline{\Phi}_3"'] & S_1 \arrow[d, "\lambda_1"] \\
\overline{X}_2 \arrow[r, "g_2"] & S
\end{tikzcd}$$

such that λ_1 is a sequence of possible blow ups for the preimage of D^*_S of points supported above $\overline{q}$, $\overline{\Phi}_3$ is a sequence of possible blow ups for the preimage of $D^*_{\overline{X}_2}$ of points and nonsingular curves supported above γ, and g_3 is toroidal with respect to $D^*_{S_1} = \lambda_1^{-1}(D^*_S)$ and $D^*_{\overline{X}_3} = \overline{\Phi}_3^{-1}(D^*_{\overline{X}_2})$. We further have that $D^*_{\overline{X}_3}$ is U cuspidal. Let $f_3 = f_2 \circ \overline{\Phi}_3 : \overline{X}_3 \rightarrow U$.

Consider the commutative diagram

$$\begin{tikzcd}
\overline{X}_3 \arrow[r, "\overline{f}_3"] \arrow[d, "\overline{\Phi}"'] & \overline{Y}_1 \arrow[r, "\pi_1"] \arrow[d, "\overline{\Psi}"'] & S_1 \arrow[d, "\lambda_1"] \\
\overline{X} \arrow[r, "\overline{f}"] & U \arrow[r, "\pi"] & S
\end{tikzcd}$$

where $\overline{\Phi} = \Phi_1 \circ \Phi_2 \circ \overline{\Phi}_3$, $\overline{Y}_1 = U \times_S S_1$ and $\overline{\Psi} : \overline{Y}_1 \rightarrow U$, $\pi_1 : \overline{Y}_1 \rightarrow S_1$ are the natural projections, and $\overline{f}_3 = f_3 \times g_3$. $D^*_{\overline{Y}_1} = \overline{\Psi}^{-1}(D^*_U)$ and $D_{\overline{Y}_1} = \overline{\Psi}^{-1}(D_U)$ are SNC divisors. $\overline{Y}_1$ is nonsingular, and is obtained from U by possible blow ups for the preimage of D_U of sections over γ. Since g_3 is toroidal with respect to $D^*_{S_1}$ and $D^*_{\overline{X}_3}$, $\overline{f}_3$ is prepared with respect to $D^*_{\overline{Y}_1}$ and $D^*_{\overline{X}_3}$. Over a general point of γ, $\overline{f}_3$ is toroidal with respect to $D^*_{\overline{Y}_1}$ and $D^*_{\overline{X}_3}$. Also, over a general point of γ, $\overline{\Phi}$ is a sequence of possible blow ups for the preimages of $D^*_{\overline{X}}$ of sections over γ.

Recall that $D_Y = D + D_1 + \cdots + D_m + G$, where G consists of the components of D_Y disjoint from U, and that the $D_i \cap U$ are étale over their images in S. Let $\overline{D}_i$ be the strict transform of D_i on $\overline{Y}_1$ for $1 \leq i \leq m$.

$$D_{\overline{Y}_1} = \overline{\Psi}^{-1}(D_U) = D^*_{\overline{Y}_1} + \overline{D}_1 + \cdots + \overline{D}_m.$$

Let $D_{\overline{X}_3} = \overline{\Phi}^{-1}(D_{\overline{X}})$.

We will now verify that $D_{\overline{X}_3}$ is a U cuspidal SNC divisor on $\overline{X}_3$ and that $\overline{f}_3$ is prepared for $D_{\overline{Y}_1}$ and $D_{\overline{X}_3}$. Since $\overline{f}_3$ is prepared for $D^*_{\overline{Y}_1}$ and $D^*_{\overline{X}_3}$, we need only verify that $\overline{f}_3$ is prepared for $D_{\overline{Y}_1}$ and $D_{\overline{X}_3}$ at points $p' \in \overline{X}_3$ such that $q' = \overline{f} \circ \overline{\Phi}(p') \in D_i$ for some i.

First suppose that $q' \in D_i - \gamma$ for some i. Then $\overline{\Phi}$ and $\overline{\Psi}$ are isomorphisms near p' and q' respectively. Suppose that $q' \not\in D$. Then we have permissible parameters

v, u, w for D_U at q' which have an expression (44) at p'. Thus $\overline{f}_3$ has an expression 3 of Definition 3.4 at p'. Suppose that $q' \in D \cap D_i - \gamma$. Then q' is a 2-point of D_U, so that $\overline{f}$ is prepared above q' for D_U and $D_{\overline{X}}$. Thus $\overline{f}_3$ is prepared above q' (for $D_{\overline{Y}_1}$ and $D_{\overline{X}_3}$).

Suppose that $q' = \overline{f} \circ \overline{\Phi}(p') \in \gamma \cap D_i$ for some i. Without loss of generality, we may assume that $D_i = D_1$. Recall that $q' \in \gamma \cap D_1$ is a general point of the 2-curve $D \cap D_1$, $\overline{f}$ is prepared above q' and $\overline{f}^*(M)$ is nonsingular and makes SNCs with $D_{\overline{X}}$ above q'. Since $q' \in D \cap D_1$ is a general point, there are no 3-points in $\overline{f}^{-1}(q')$. Let D_1' be the reduced divisor on $\overline{X}$ whose components dominate D_1. The irreducible components of D_1' are disjoint by Remark 3.3.

There exist permissible parameters u, v, w in $\mathcal{O}_{U,q'}$ for the two point q' of D_U such that $u = 0$ is a local equation of D, $v = 0$ is a local equation of D_1, $w = 0$ is a local equation of M on U, and u, w are regular parameters in $\mathcal{O}_{S,\overline{q}}$ such that if $p = \overline{\Phi}(p') \in \overline{f}^{-1}(q')$, then there exist regular parameters x, y, z in $\hat{\mathcal{O}}_{\overline{X},p}$ such that one of the following prepared forms for $\overline{f}$ hold at p. u, w are toroidal forms for D_U and $D_{\overline{X}}$ in all cases.

(1) p is a 1-point of $D_{\overline{X}}$

$$\begin{aligned} u &= x^a \\ v &= x^b \gamma \\ w &= z \end{aligned} \tag{45}$$

where $\gamma \in \hat{\mathcal{O}}_{\overline{X},p}$ is a unit and $x = 0$ is a local equation of $D_{\overline{X}}$.

(2) p is a 2-point of $D_{\overline{X}}$ which is not on D_1'

$$\begin{aligned} u &= x^a y^b \\ v &= x^c y^d \gamma \\ w &= z \end{aligned} \tag{46}$$

with $a, b > 0$, $\gamma \in \hat{\mathcal{O}}_{\overline{X},p}$ is a unit and $xy = 0$ is a local equation of $D_{\overline{X}}$.

(3) p is a 2-point which is on D_1'

$$\begin{aligned} u &= x^a \\ v &= x^b y^c \\ w &= z \end{aligned} \tag{47}$$

where $xy = 0$ is a local equation of $D_{\overline{X}}$ and $y = 0$ is a local equation of D_1'.

$\overline{\Psi}$ is the sequence of monodial transforms induced by a sequence of quadratic transforms,

$$S_1 = \overline{S}_n \rightarrow \cdots \rightarrow \overline{S}_0 = S.$$

Each map $\overline{S}_{j+1} \rightarrow \overline{S}_j$ is the blow up of the ideal sheaf m_j of a point $\overline{q}_j$ above $\overline{q}$. Let

$$\overline{Y}_1 = \tilde{Y}_n \rightarrow \cdots \rightarrow \tilde{Y}_0 = U \tag{48}$$

be the induced factorization of $\overline{\Psi}$, where $\tilde{\Psi}_{j+1} : \tilde{Y}_{j+1} = U \times_S \overline{S}_{j+1} \rightarrow \tilde{Y}_j = U \times_S \overline{S}_j$, is the blow up of a curve C_j. Let $\overline{\pi}_j : \tilde{Y}_j \rightarrow \overline{S}_j$ be the natural projection.

$\overline{\Phi}$ is a sequence of morphisms

$$\overline{X}_3 = \tilde{X}_n \rightarrow \cdots \rightarrow \tilde{X}_0 = \overline{X}_2 \rightarrow \overline{X}.$$

where $\tilde{\Phi}_{j+1} : \tilde{X}_{j+1} \rightarrow \tilde{X}_j$ is a principalization of $m_j \mathcal{O}_{\tilde{X}_j}$, with natural morphism $\tilde{f}_j : \tilde{X}_j \rightarrow \tilde{Y}_j$. Let $D_{\tilde{X}_j}$, $D_{\tilde{Y}_j}$ be the respective preimages of D_U, and let $D^*_{\tilde{X}_j}$, $D^*_{\tilde{Y}_j}$ be the respective preimages of D^*_U. Let $D^*_{\overline{S}_j}$ be the preimage of D^*_S in $\overline{S}_j$. The principalizations $\tilde{\Phi}_j$ are explicitly described in the proof of Theorem 3.1 [C5]. We have a factorization

$$\tilde{X}_{j+1} = \hat{X}_{n_j,j} \rightarrow \cdots \rightarrow \hat{X}_{0,j} = \tilde{X}_j. \tag{49}$$

where each $\hat{\Phi}_{i+1,j} : \hat{X}_{i+1,j} \rightarrow \hat{X}_{i,j}$ is the blow up of a single curve or point E_{ij} which is a possible center for the preimage $D^*_{\hat{X}_{ij}}$ of D^*_U on $\hat{X}_{i,j}$. If E_{ij} is a curve, then E_{ij} is in the locus where $m_j\mathcal{O}_{\hat{X}_{i,j}}$ is not locally principal. If E_{ij} is a point, then E_{ij} is in the support of $m_j\mathcal{O}_{\hat{X}_{i,j}}$ and $m_j\mathcal{O}_{\hat{X}_{i,j},E_{ij}}$ is locally principal. Further, as is shown in the proof of Theorem 3.1 [C5], $D^*_{\tilde{X}_j}$ is U cuspidal (this is the reason for the point blow ups). Let $D_{\hat{X}_{ij}}$ be the preimage of D_U on $\hat{X}_{ij}$.

Recall that $\overline{X}_2 \rightarrow \overline{X}$ is an isomorphism above q'.

We will prove that $D_{\overline{X}_3}$ is a U cuspidal SNC divisor on $\overline{X}_3$ and $\overline{f}_3 : \overline{X}_3 \rightarrow \overline{Y}_1$ is prepared for $D_{\overline{Y}_1}$ and $D_{\overline{X}_3}$ above q' by induction on j in the morphisms $\tilde{f}_j : \tilde{X}_j \rightarrow \tilde{Y}_j$.

Recall that we have a fixed choice of regular parameters $u = u_0, v, w = w_0$ in $\mathcal{O}_{U,q'}$, which are permissible parameters for D_Y at the 2-point q', and one of the forms (45) - (47) holds at all points of $\overline{X}$ above q'.

Suppose by induction that $D_{\tilde{X}_j}$ is a U cuspidal SNC divisor, $\tilde{f}_j : \tilde{X}_j \rightarrow \tilde{Y}_j$ is prepared for $D_{\tilde{Y}_j}$ and $D_{\tilde{X}_j}$, and if $q_j \in \tilde{Y}_j$ and $\tilde{\Psi}_1 \circ \cdots \circ \tilde{\Psi}_j(q_j) = q'$, then

1. q_j is a 2-point or a 3-point of $D_{\tilde{Y}_j}$ and there exist regular parameters u_j, w_j in $\mathcal{O}_{\overline{S}_j,\overline{q}'_j}$, where $\overline{\pi}_j(q_j) = \overline{q}'_j$, such that u_j, v, w_j are permissible parameters for $D_{\tilde{Y}_j}$ in $\mathcal{O}_{\tilde{Y}_j,q_j}$.
2. If $q_j \in C_j$, then $u_j = w_j = 0$ are local equations of C_j at q_j.
3. If $p_j \in \tilde{f}_j^{-1}(q_j)$, then there exist regular parameters x_j, y_j, z_j in $\hat{\mathcal{O}}_{\tilde{X}_j,p_j}$ such that one of the following forms hold:

Case 1. q_j is a 2-point of $D_{\tilde{Y}_j}$, and $u_jv = 0$ is a local equation of $D_{\tilde{Y}_j}$ (so that $\overline{\pi}_j(q_j) = \overline{q}'_j$ is a 1-point), and p_j is a 1-point of $D_{\tilde{X}_j}$ with

$$u_j = x_j^a, v = x_j^b\gamma_j, w_j = z_j \tag{50}$$

where $x_j = 0$ is a local equation of $D_{\tilde{X}_j}$, γ_j is a unit in $\hat{\mathcal{O}}_{\tilde{X}_j,p_j}$ or p_j is a 2-point of $D_{\tilde{X}_j}$ with

$$u_j = x_j^a y_j^b, v = x_j^c y_j^d\gamma_j, w_j = z_j \tag{51}$$

where $x_jy_j = 0$ is a local equation of $D_{\tilde{X}_j}$, γ_j is a unit in $\hat{\mathcal{O}}_{\tilde{X}_j,p_j}$.

Case 2. q_j is a 3-point of $D_{\tilde{Y}_j}$, and $u_jvw_j = 0$ is a local equation of $D_{\tilde{Y}_j}$ (so that $\overline{\pi}_j(q_j) = \overline{q}'_j$ is a 2-point), and p_j is a 1-point of $D_{\tilde{X}_j}$ with

$$u_j = x_j^a, v = x_j^b\gamma_j, w_j = x_j^c(z_j + \beta) \tag{52}$$

where $x_j = 0$ is a local equation of $D_{\tilde{X}_j}$, γ_j is a unit in $\hat{\mathcal{O}}_{\tilde{X}_j,p_j}$ and $0 \neq \beta \in k$ or p_j is a 2-point of $D_{\tilde{X}_j}$ with

$$u_j = x_j^a z_j^b, v = x_j^c z_j^d\gamma_j, w_j = x_j^e z_j^f \tag{53}$$

where $af-be\ne 0$, $x_jz_j=0$ is a local equation of $D_{\tilde X_j}$, γ_j is a unit in $\hat{\mathcal{O}}_{\tilde X_j,p_j}$ or p_j is a 2-point of $D_{\tilde X_j}$ with

$$u_j=(x_j^ay_j^b)^k, v=x_j^dy_j^e\gamma_j, w_j=(x_j^ay_j^b)^t(z_j+\beta) \tag{54}$$

where $x_jy_j=0$ is a local equation of $D_{\tilde X_j}$, γ_j is a unit in $\hat{\mathcal{O}}_{\tilde X_j,p_j}$, $\gcd(a,b)=1$ and $0\ne\beta\in k$ or p_j is a 3-point of $D_{\tilde X_j}$ with

$$u_j=x_j^ay_j^bz_j^c, v=x_j^dy_j^ez_j^f\gamma_j, w_j=x_j^gy_j^hz_j^i \tag{55}$$

where $x_jy_jz_j=0$ is a local equation of $D_{\tilde X_j}$, γ_j is a unit in $\hat{\mathcal{O}}_{\tilde X_j,p_j}$ and

$$\operatorname{rank}\begin{pmatrix} a & b & c\\ g & h & i\end{pmatrix}=2.$$

We will prove that the above statements hold for $\tilde f_{j+1}:\tilde X_{j+1}\to\tilde Y_{j+1}$.

Suppose that $q_j\in C_j$ is a 2-point (and $\tilde\Psi_1\circ\cdots\circ\tilde\Psi_j(q_j)=q'$), so that Case 1 holds, and $p_j\in\tilde f_j^{-1}(q_j)$.

$\mathcal{I}_{C_j}\mathcal{O}_{\tilde X_j,p_j}$ is not invertible, and u_j,w_j satisfy (185) [C3] at p_j if (50) holds, u_j,w_j satisfy (190) [C3] at p_j if (51) holds and $a,b>0$ (so that p_j is a 2-point of $D^*_{\tilde X_j}$), u_j,w_j satisfy (185) [C3] at p_j if (51) holds and $b=0$ (so that p_j is a 1-point of $D^*_{\tilde X_j}$).

The algorithm of Lemma 18.17 [C3] (as modified after (23) in the proof of Theorem 3.1 of [C5] by adding appropriate point blow ups to ensure that $D^*_{\tilde X_{j+1}}$ is U cuspidal) is applied to construct $\tilde\Phi_{j+1}:\tilde X_{j+1}\to\tilde X_j$ and $\tilde f_{j+1}:\tilde X_{j+1}\to\tilde Y_{j+1}$ above q_j. Suppose that $q_{j+1}\in\tilde\Psi_{j+1}^{-1}(q_j)$, and $\overline\pi_{j+1}(q_{j+1})=\overline q'_{j+1}\in\overline S_{j+1}$. Then there exist regular parameters u_{j+1},w_{j+1} in $\mathcal{O}_{\overline S_{j+1},\overline q'_{j+1}}$ such that u_{j+1},v,w_{j+1} are regular parameters in $\mathcal{O}_{\tilde Y_{j+1},q_{j+1}}$ and one of the following forms hold:

$\overline q'_{j+1}$ is a 1-point of $D_{\overline S_{j+1}}$

$$u_j=u_{j+1}, w_j=u_{j+1}(w_{j+1}+\alpha) \tag{56}$$

with $\alpha\in k$, or $\overline q'_{j+1}$ is a 2-point for $D_{\overline S_{j+1}}$

$$u_j=u_{j+1}w_{j+1}, w_j=w_{j+1}. \tag{57}$$

If (56) holds at $\overline q'_{j+1}$ and $p_{j+1}\in\tilde f_{j+1}^{-1}(q_{j+1})$, then an analysis of the algorithm of Lemma 18.17 [C3] and Theorem 3.1 [C5] shows that u_{j+1},v,w_{j+1} satisfy one of the forms (50) or (51) at p_{j+1}.

If (57) holds at $\overline q'_{j+1}$, and $p_{j+1}\in\tilde f_{j+1}^{-1}(q_{j+1})$, then u_{j+1},v,w_{j+1} satisfy one of the forms (52) - (55) at p_{j+1}.

If $q_{j+1}\in C_{j+1}$, then $u_{j+1}=w_{j+1}=0$ are local equations of C_{j+1}.

Now suppose that $q_j\in C_j$ is a 3-point (and $\tilde\Psi_1\circ\cdots\circ\tilde\Psi_j(q_j)=q'$), so that Case 2 holds, and $p_j\in\tilde f_j^{-1}(q_j)$.

If $\mathcal{I}_{C_j}\mathcal{O}_{\tilde X_j,p_j}$ is not invertible, then after possibly interchanging u_j and w_j, then we have one of the following fomrs. u_j,w_j satisfy (187) [C3] at p_j if (53) holds and $a,b>0$, u_j,w_j satisfy (191) [C3] at p_j if (53) holds and $b=e=0$ (in both cases, p_j is a 2-point of $D^*_{\tilde X_j}$). u_j,w_j satisfy (187) or (191) [C3] if (55) holds (so that p_j is a 2 point of $D^*_{\tilde X_j}$). u_j,w_j satisfy (193), (194) or (195) [C3] at p_j if (55) holds

The algorithm of Lemma 18.18 [C3] is then applied to construct $\tilde\Phi_{j+1}:\tilde X_{j+1}\to\tilde X_j$ and $\tilde f_{j+1}:\tilde X_{j+1}\to\tilde Y_{j+1}$ above q_j. Suppose that $q_{j+1}\in\tilde\Psi_{j+1}^{-1}(q_j)$, and $\overline\pi(q_{j+1})=$

$\overline{q}'_{j+1}\in\overline{S}_{j+1}$. Then there exist regular parameters u_{j+1},w_{j+1} in $\mathcal{O}_{\overline{S}_{j+1},\overline{q}'_{j+1}}$ such that u_{j+1},v,w_{j+1} are regular parameters in $\mathcal{O}_{\tilde{Y}_{j+1},q_{j+1}}$ and one of the following forms hold:

$\overline{q}'_{j+1}$ is a 1-point of $D_{\overline{S}_{j+1}}$

$$u_j=u_{j+1},w_j=u_{j+1}(w_{j+1}+\alpha)\tag{58}$$

with $0\neq\alpha\in k$, or $\overline{q}'_{j+1}$ is a 2-point for $D_{\overline{S}_{j+1}}$

$$u_j=u_{j+1},w_j=u_{j+1}w_{j+1},\tag{59}$$

or $\overline{q}'_{j+1}$ is a 2-point for $D_{\overline{S}_{j+1}}$

$$u_j=u_{j+1}w_{j+1},w_j=w_{j+1}.\tag{60}$$

If (58) holds at $\overline{q}'_{j+1}$ and $p_{j+1}\in\tilde{f}^{-1}_{j+1}(q_{j+1})$, then an analysis of the algorithm of Lemma 18.18 [C3] shows that u_{j+1},v,w_{j+1} satisfy one of the forms (50) or (51) at p_{j+1}.

If (59) or (60) holds at $\overline{q}'_{j+1}$, and $p_{j+1}\in\tilde{f}^{-1}_{j+1}(q_{j+1})$, then u_{j+1},v,w_{j+1} satisfy one of the forms (52) - (55) at p_{j+1}.

We have shown that $D_{\tilde{X}_{j+1}}=\tilde{f}^{-1}_{j+1}(D_{\tilde{Y}_{j+1}})$ is a SNC divisor above q_j and that $\tilde{f}_{j+1}$ is prepared for $D_{\tilde{Y}_{j+1}}$ and $D_{\tilde{X}_{j+1}}$.

We will now verify that $D_{\tilde{X}_{j+1}}$ is U cuspidal. Since we are assuming that $D_{\tilde{X}_j}$ is U cuspidal, and we know that $D^*_{\tilde{X}_{j+1}}$ is U cuspidal, we need only verify that every 2-curve of $D_{\tilde{X}_{j+1}}$ contained in a component of $D_{\tilde{X}_{j+1}}$ which dominates D_1 contains a 3-point. We verify this by induction in the sequence (49).

Let D_1^{ij} be the union of components of $D_{\hat{X}_{ij}}$ which dominate D_1. The irreducible components of D_1^{ij} are disjoint (by Remark 3.3). Assume that every 2-curve of $D_{\hat{X}_{ij}}$ which is contained in D_1^{ij} contains a 3-point. We will show that $D_1^{i+1,j}$ also has this property. All points blown up in the construction of $\tilde{\Phi}_{j+1}$ are either 3-points of $D_{\hat{X}_{ij}}$ or are 2-points which are disjoint from D_1^{ij}. We may thus assume that the center $E_{i,j}$ blown up by $\hat{\Phi}_{i+1,j}$ is a curve.

If $E_{i,j}$ contains a 1-point of $D_{\hat{X}_{i,j}}$, then $E_{i,j}$ intersects $D_1^{i,j}$ transversally at 2-points of $D_{\hat{X}_{ij}}$, and thus all 2-curves of $D_{\hat{X}_{i+1,j}}$ contained in $D_1^{i+1,j}$ contain a 3-point. Suppose that $E_{i,j}$ is a 2-curve of $D_{\hat{X}_{i,j}}$ and Λ is an irreducible component of $D_1^{i,j}$. then either $E_{i,j}$ is contained in Λ, so that Λ contains a 3-point of $E_{i+1,j}$, as $D_{\hat{X}_{ij}}$ is by assumption U cuspidal, or else $E_{i,j}$ intersects Λ transversally at 3-points of $D_{\hat{X}_{ij}}$. In either case, all 2-curves of $D_{\hat{X}_{i+1,j}}$ contained in $D_1^{i+1,j}$ contain a 3-point.

We conclude that $D_{\tilde{X}_{j+1}}$ is U cuspidal.

We have thus established that $\overline{f}_3$ is prepared for $D_{\overline{Y}_1}$ and $D_{\overline{X}_3}$, and $D_{\overline{X}_3}$ is U cuspidal.

Recall that $\tilde{\Phi}_{i+1}:\tilde{X}_{i+1}\rightarrow\tilde{X}_i$ is a principalization of $m_i\mathcal{O}_{\tilde{X}_i}$ which in a neighborhood of a general point of γ is a sequence of blow ups of sections over γ where $m_i\mathcal{O}_{\tilde{X}_i}$ is not invertible.

Each $\tilde{\Psi}_{i+1}:\tilde{Y}_{i+1}\rightarrow\tilde{Y}_i$ is the blow up of a curve C_i which is a section over γ and is a possible center for $D_{\tilde{Y}_i}$.

We will construct $\Psi_1 : Y_1 \to Y$ such that $\Psi_1^{-1}(U) \cong \overline{Y}_1$, $\Psi_1 \mid \Psi_1^{-1}(U) = \overline{\Psi}$ and $\Psi_1^{-1}(D_Y)$ is a SNC divisor by constructing a sequence of morphisms

$$Y_1 = \hat{Y}_n \overset{\hat{\Psi}_n}{\to} \hat{Y}_{n-1} \to \cdots \overset{\hat{\Psi}_1}{\to} Y \tag{61}$$

where each $\hat{\Psi}_{i+1}$ is a product of blow ups of possible centers for the preimage of D_Y, and $\hat{\Psi}_{i+1}^{-1}(\tilde{Y}_i) \cong \tilde{Y}_{i+1}$, $\hat{\Psi}_{i+1} \mid \tilde{Y}_{i+1} = \tilde{\Psi}_{i+1}$ for all i.

We will inductively construct (61). Suppose that we have constructed $\hat{\Psi}_i : \hat{Y}_i \to \hat{Y}_{i-1}$.

Let γ_i be the Zariski closure of C_i in $\hat{Y}_i$. Then γ_{i+1} is a section over $\overline{\gamma}$, and is thus a nonsingular curve. We construct $\hat{\Psi}_{i+1}$ by first blowing up points on (the strict transform of) γ_i above $\overline{\gamma} - \gamma$ where (the strict transform of) γ_i does not make SNCs with (the preimage of) $D_{\hat{Y}_i}$, and then blowing up the strict transform of γ_i.

$\overline{X}_2 \to \overline{X}$ is an isomorphism away from the preimage of Ω. Thus the sequence of blow ups $\overline{X}_2 \to X$ extends trivially to a morphism $X_2 \to X$, so that $X_2 \to X$ is an isomorphism away from the preimage of Ω.

Now we construct $\Phi_3 : X_3 \to X_2$ such that $\Phi_3^{-1}(\overline{X}_2) = \overline{X}_3$, $\Phi_3 \mid \overline{X}_3 = \overline{\Phi}_3$ and $\Phi_3^{-1}(D_{X_3})$ is a SNC divisor by constructing a sequence of morphisms

$$X_3 = \hat{X}_n \overset{\hat{\Phi}_n}{\to} \hat{X}_{n-1} \to \cdots \overset{\hat{\Phi}_1}{\to} \hat{X}_0 = X_2$$

where $\hat{\Phi}_i^{-1}(\tilde{X}_{i-1}) \cong \tilde{X}_i$ and $\hat{\Phi}_i \mid \tilde{X}_i = \tilde{\Phi}_i$ for all i, and so that there are morphisms $\hat{X}_i \to \hat{Y}_i$ which are toroidal (with respect to the preimages of D_Y and D_X) over points of $\overline{\gamma} - \gamma$. This follows from application of Lemmas 4.6 and 4.7, and the fact that the case when γ_i is a 2-curve (or a 3-point is blown up) extends directly to a toroidal morphism.

The resulting morphism Φ_3 is an isomorphism away from the preimage of $\overline{\gamma}$.

We have constructed a diagram

$$\begin{tikzcd} X_3 \arrow[r, "f_3"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that Φ and Ψ_1 are isomorphisms away from the preimage of $\overline{\gamma}$ and f_3 is prepared with respect to $D_{Y_1} = \Psi_1^{-1}(D_Y)$ and $D_{X_3} = \Phi^{-1}(D_X)$ away from the points in $\Sigma - \{q\}$. Further, all components of D_{X_3} which do not contain a 3-point and all 2-curves of D_{X_3} which do not contain a 2-point must contract to points of $\overline{\gamma} - \gamma$. In particular, D_{X_3} is cuspidal for f_3. By induction on $|\Sigma|$, we repeat this construction to prove Theorem 1.3.

6. TOROIDALIZATION

In this section we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1

By Theorem 1.3, we can construct a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "\Phi_1"'] & Y_1 \arrow[d, "\Psi_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

such that Φ_1 and Ψ_1 are products of possible blow ups such that f_1 is prepared for $D_{Y_1} = \Psi_1^{-1}(D_Y)$ and $D_{X_1} = \Phi_1^{-1}(D_X)$ and D_{X_1} is cuspidal for f_1.

Now, we conclude the proof as in the proof of Theorem 0.1 of [C5].

By descending induction on $\tau(X_2)$ (Definition 2.9 [C5]) and by Theorems 7.11 and 8.1 [C5], there exists a commutative diagram

$$\begin{tikzcd}
X_2 \arrow[r, "f_2"] \arrow[d, "\Phi_2"'] & Y_2 \arrow[d, "\Psi_2"] \\
X_1 \arrow[r, "f_1"] & Y_1
\end{tikzcd}$$

such that Φ_2 and Ψ_2 are products of possible blow ups, f_2 is prepared for $D_{Y_2} = \Psi_2^{-1}(D_{Y_1})$ and $D_{X_2} = \Phi_2^{-1}(D_{X_1})$, D_{X_2} is cuspidal for f_2 and $\tau_{f_2}(X_2) = -\infty$.

By Theorem 8.2 [C5], f_2 is toroidal, and the conclusions of the theorem follow.

Proof of Theorem 1.2

By resolution of singularities and resolution of indeterminacy [H] (cf. Section 6.8 [C6]), and by [M], there exists a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "\Phi_1"'] & Y_1 \arrow[d, "\Psi_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where Φ_1, Ψ_1 are products of blow ups of points and nonsingular curves supported above D_Y, such that X_1 and Y_1 are nonsingular and projective, and $D_{X_1} = \Phi_1^{-1}(D_X)$ and $D_{Y_1} = \Psi_1^{-1}(D_Y)$ are SNC divisors. The proof of Theorem 1.2 now follows from Theorem 1.1.

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