MONOMIALIZATION OF STRONGLY PREPARED MORPHISMS FROM NONSINGULAR N-FOLDS TO SURFACES

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1 Introduction

Monomialization of morphisms is the problem of transforming a mapping into a monomial mapping by blowing up a chain of nonsingular subvarieties in its domain and image.

Consider the following basic example. Let $\Phi : A^n_{\mathbf{k}} \to A^m_{\mathbf{k}}$ be a morphism of affine spaces over field $\mathbf{k}$. Then Φ is given by a collection of polynomials $f_1, \dots, f_m$ in n variables:

$$\begin{aligned} y_1 &= f_1(x_1, \dots, x_n) \\ &\vdots \\ y_m &= f_m(x_1, \dots, x_n). \end{aligned}$$

The simplest structure of Φ is obtained when $f_1, \dots, f_m$ are monomials and

$$\begin{aligned} y_1 &= x_1^{a_{1\,1}} \cdots x_n^{a_{1\,n}} \\ &\vdots \\ y_m &= x_1^{a_{m\,1}} \cdots x_n^{a_{m\,n}}. \end{aligned}$$

If, moreover, Φ is a dominant morphism the matrix $(a_{i\,j})$ is forced to satisfy the nondegeneracy condition $\mathrm{rank}(a_{i\,j}) = m$.

In case of a general dominant morphism $\Phi : X \to Y$ between two $\mathbf{k}$-varieties X and Y we would like to get such a nice description locally .

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Definition 1.1. Suppose that $\Phi : X \to Y$ is a dominant morphism of nonsingular $\mathbf{k}-$varieties. Φ is called monomial if for all points $p \in X$ there exist an étale neighborhood U of p, uniformizing parameters $(x_1, \ldots, x_n)$ on U, regular parameters $y_1, \ldots, y_m$ in $\mathcal{O}_{Y,\Phi(p)}$ and a matrix (a_{ij}) of nonnegative integers with rank m such that

$$\begin{aligned} y_1 &= x_1^{a_{11}} \cdots x_n^{a_{1n}} \\ &\vdots \\ y_m &= x_1^{a_{m1}} \cdots x_n^{a_{mn}}. \end{aligned}$$

The natural question arises.

Question. Suppose that $\Phi : X \to Y$ is a dominant morphism of $\mathbf{k}$-varieties. Does there exist a monomialization of Φ? Or, more precisely, given a dominant morphism $\Phi : X \to Y$ does there exist a monomial morphism $\Phi_1 : X_1 \to Y_1$ such that the following diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Phi} & Y \end{array}$$

and all vertical maps are products of blowups of nonsingular subvarieties in X and Y?

The answer is "yes" over a characteristic zero field when Y is a curve or when Y is a surface and $\dim(X) \leq 3$.

Suppose that k is algebraically closed field of characteristic zero. If $\Phi : X \to C$ is a dominant morphism from a $\mathbf{k}$-variety to a curve the existence of monomialization follows from resolution of singularities. If $\Phi : P \to S$ is a dominant morphism of surfaces one proof of monomialization (over $\mathbf{C}$) is given by Akbulut and King [3]. And the last known case, when $\Phi : X \to S$ is a dominant morphism from a 3-fold to a surface, is done by the first author in [4]. This proof of monomialization breaks down into two key steps:

1) obtain a diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Phi} & Y \end{array}$$

where Φ_1 is a strongly prepared morphism and the vertical maps are the products of blowups of nonsingular subvarieties;

2) monomialize the strongly prepared morphism $\Phi_1 : X_1 \to Y_1$.

The natural next case to consider is monomialization of morphisms from n-folds to surfaces. A proof would follow from the two steps above when X is an n-fold and Y is a surface. In this paper we complete step 2). Our main result is

Theorem 1.2. *Suppose that $\Phi : X \to S$ is a strongly prepared morphism from a nonsingular n-fold X to a nonsingular surface S.*

Then there exists a finite sequence of quadratic transforms $\pi_1 : S_1 \to S$ and monoidal transforms centered at nonsingular varieties of codimension 2 $\pi_2 : X_1 \to X$ such that the induced morphism $\bar{\Phi} : X_1 \to S_1$ is monomial.

From here we deduce that it is possible to toroidalize a strongly prepared morphism.

Theorem 1.3. *Suppose that $\Phi : X \to S$ is a strongly prepared morphism from a nonsingular n-fold X to a nonsingular surface S.*

Then there exists a finite sequence of quadratic transforms $\pi_1 : S_1 \to S$ and monoidal transforms centered at nonsingular varieties of codimension 2 $\pi_2 : X_1 \to X$ such that the induced morphism $\bar{\Phi} : X_1 \to S_1$ is toroidal.

The definition of a strongly prepared morphism is given in Section 3, definition 3.2. The class of strongly prepared morphisms is rather restrictive. However, a natural example of such a morphism can be obtained as follows.

Let $\Phi : X \to S$ be a monomial mapping from an n-fold to a surface and $\pi : X_1 \to X$ be a finite sequence of blowups of points. Then the composition map $\Phi \circ \pi : X_1 \to S$ is strongly prepared, but not necessarily monomial.

2 Notations

We will suppose that $\mathbf{k}$ is an algebraically closed field of characteristic zero. By a variety we will mean a separated, integral finite type $\mathbf{k}$-scheme. A point of a variety will mean a closed point. By a generic point on a variety we will mean a point which satisfies a good condition which holds on an open set of points.

Suppose that Z is a variety and $p \in Z$ is a point. Then m_p will denote the maximal ideal of $\mathcal{O}_{Z,p}$.

Suppose that $P(x)=\sum_{i=0}^{\infty}c_ix^i\in\mathbf{k}[[x]]$ is a series. Given $e\in\mathbb{N}$, $P_e(x)$ will denote the polynomial $P_e(x)=\sum_{i=0}^{e}c_ix^i$. Given a series $f(x_1,\ldots,x_n)\in\mathbf{k}[[x_1,\ldots,x_n]]$, $ord\,f$ will denote the order of f (with $ord\,0=\infty$).

If $x\in\mathbb{Q}$, $[x]$ will denote the greatest integer $n\in\mathbb{N}$ such that $n\leq x$. The greatest common divisor of $a_1,\ldots,a_n\in\mathbb{N}$ will be denoted by $(a_1,\ldots,a_n)$.

Definition 2.1. A reduced divisor D on a nonsingular variety X of dimension n is called a simple normal crossing divisor (SNC divisor) if all components of D are nonsingular and the following condition holds.

> Suppose that $p\in X$ is a point and $D_1,\ldots,D_s$ are the components of D containing p. Then $s\leq n$ and there exist regular parameters $(x_1,\ldots,x_n)$ in $\mathcal{O}_{X,p}$ such that $D_1,\ldots,D_s$ have at p local equations $x_1=0,\ldots,x_s=0$, respectively.

Definition 2.2. A codimension 2 subvarieties $C_1,\ldots,C_m$ of a nonsingular dimension n variety X make simple normal crossings (SNCs) if C_i is nonsingular for all $i=1,\ldots,m$ and the following condition holds.

> Suppose that $p\in X$ is a point and $C_1,\ldots,C_s$ are the subvarieties containing p. Then $s\leq\binom{n}{2}$ and there exist regular parameters $(x_1,\ldots,x_n)$ in $\mathcal{O}_{X,p}$ such that for all $i=1,\ldots,s$ $x_{l_i}=x_{k_i}=0$ are local equations of C_i at p and $1\leq l_i<k_i\leq n$.

Definition 2.3. Suppose that $\Phi:X\to Y$ is a dominant morphism of nonsingular $\mathbf{k}-$varieties. Φ is called monomial if for all points $p\in X$ there exist an étale neighborhood U of p, uniformizing parameters $(x_1,\ldots,x_n)$ on U, regular parameters $y_1,\ldots,y_m$ in $\mathcal{O}_{Y,\Phi(p)}$ and a matrix (a_{ij}) of nonnegative integers with rank m such that

$$\begin{aligned}y_1&=x_1^{a_{11}}\cdots x_n^{a_{1n}}\\&\vdots\\y_m&=x_1^{a_{m1}}\cdots x_n^{a_{mn}}.\end{aligned}$$

3 Monomialization

Definition 3.1. Suppose that $\Phi:X\to S$ is a dominant morphism from a nonsingular variety X to a nonsingular variety S with reduced SNC divisors

D_S on S and E_X on X such that $\Phi^{-1}(D_S)_{red} = E_X$. Let $sing(\Phi)$ be the locus of singular points of Φ. We will say that Φ is quasi prepared (with respect to D_S) if $sing(\Phi) \subset E_X$.

Suppose that $\Phi : X \to S$ is a quasi prepared morphism from a nonsingular n-fold X to a nonsingular surface S. If $p \in E_X$ we will say that p is a $1, 2, \ldots, n$ point depending on if p is contained in $1, 2, \ldots, n$ components of E_X. $q \in D_S$ will be called a 1 or 2 point depending on if q is contained in 1 or 2 components of D_S.

Regular parameters $(u, v) \in \mathcal{O}_{S,q}$ for $q \in D_S$ are permissible if:

1) q is a 1 point and $u = 0$ is a local equation of D_S or

2) q is a 2 point and $uv = 0$ is a local equation of D_S.

Definition 3.2. Suppose that $\Phi : X \to S$ is a quasi prepared morphism from a nonsingular n-fold X to a nonsingular surface S. We will say that Φ is strongly prepared at $p \in X$ (with respect to D_S) if there exist permissible parameters (u, v) at $\Phi(p)$ and regular parameters $(x_1, \ldots, x_n)$ in $\hat{\mathcal{O}}_{X,p}$ such that one of the following forms holds:

(1) $1 \leq k \leq n-1$

p is a k point, $u = 0$ is a local equation of E_X and

$$\begin{aligned} u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\ v &= P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} x_{k+1} \end{aligned}$$

where $m > 0$, $a_1, \ldots, a_k > 0$ with $(a_1, \ldots, a_k) = 1$, $b_1, \ldots, b_k \geq 0$ and P is a series;

(2) $2 \leq k \leq n$

p is a k point, $u = 0$ is a local equation of E_X and

$$\begin{aligned} u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\ v &= P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} \end{aligned}$$

where $m > 0$, $a_1, \ldots, a_k > 0$ with $(a_1, \ldots, a_k) = 1$, $b_1, \ldots, b_k \geq 0$, $rank \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$ and P is a series;

(3) $2 \leq k \leq n$

p is a k point, $uv = 0$ is a local equation of E_X and

$$
\begin{aligned}
u &= x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} \\
v &= x_2^{b_2} \cdots x_k^{b_k}
\end{aligned}
$$

where $a_2, \ldots, a_{k-1}, b_2, \ldots, b_{k-1} \geq 0$, $a_1, b_k > 0$ and $a_i + b_i > 0$ for all $i = 2, \ldots, k-1$.

In this case the regular parameters $(x_1, \ldots, x_n)$ in $\hat{\mathcal{O}}_{X,p}$ are called $*$-permissible parameters at p for (u, v) and the permissible parameters (u, v) are called prepared.

$\Phi : X \to S$ is strongly prepared if it is strongly prepared at every point $p \in X$.

We will now assume that $\Phi : X \to S$ is a strongly prepared morphism from a nonsingular n-fold X to a nonsingular surface S.

Lemma 3.3. *Suppose that $\mathcal{O}_{X,p} \to R$ is finite étale and there exist $x_1, \ldots, x_n \in R$ such that $(x_1, \ldots, x_n)$ are regular parameters in R_q for all primes $q \subset R$ such that $q \cap \mathcal{O}_{X,p} = m_p$. Then there exists an étale neighborhood U of p such that $(x_1, \ldots, x_n)$ are uniformizing parameters on U.*

Proof. There exists an affine neighborhood $V_1 = \text{Spec }(A)$ of $p \in X$ and a finite étale extension B of A such that $B \otimes_A A_{m_p} \cong R$. Set $U_1 = \text{Spec }(B)$. Let $\pi : U_1 \to V_1$ be the natural map. There exists an open neighborhood U_2 of $\pi^{-1}(p)$ such that $(x_1, \ldots, x_n)$ are uniformizing parameters on U_2. Let $Z = U_1 - U_2$ and $W = \pi(Z)$. Set $U = U_1 - \pi^{-1}(W)$, then $U \to V = V_1 - W$ is finite étale. Thus there exists an étale neighborhood U of p where $(x_1, \ldots, x_n)$ are uniformizing parameters. □

Lemma 3.4. *Suppose that $\Phi : X \to S$ is strongly prepared at $p \in E_X$. Then there exist prepared parameters (u, v) for $\Phi(p)$ and $*$-permissible parameters $(x_1, \ldots, x_n)$ for (u, v) at p such that $(x_1, \ldots, x_n)$ are uniformizing parameters on an étale neighborhood of p and one of the following forms holds:*

(1) $1 \leq k \leq n-1$

p is a k point, $u = 0$ is a local equation of E_X and

$$
\begin{aligned}
u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\
v &= P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} x_{k+1}
\end{aligned}
\tag{I}
$$

where $m>0$, $a_1,\ldots,a_k>0$ *with* $(a_1,\ldots,a_k)=1$, $b_1,\ldots,b_k\ge 0$ *and either* $P\equiv 0$ *or* P *is a polynomial of order* $\le \max_{1\le i\le k}\{\frac{b_i}{a_i}\}$ *such that if* $\Phi(p)$ *is a 1 point then* $m\nmid ordP$*;*

(2) $2\le k\le n$

p is a k point, $u=0$ *is a local equation of* E_X *and*

$$\begin{aligned} u &= (x_1^{a_1}\cdots x_k^{a_k})^m \\ v &= P(x_1^{a_1}\cdots x_k^{a_k})+x_1^{b_1}\cdots x_k^{b_k} \end{aligned} \tag{II}$$

where $m>0$, $a_1,\ldots,a_k>0$ *with* $(a_1,\ldots,a_k)=1$, $b_1,\ldots,b_k\ge 0$, *rank* $\begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k\end{pmatrix}=2$ *and either* $P\equiv 0$ *or* P *is a polynomial of order* $\le \max_{1\le i\le k}\{\frac{b_i}{a_i}\}$ *such that if* $\Phi(p)$ *is a 1 point then* $m\nmid ordP$*;*

(3) $2\le k\le n$

p is a k point, $uv=0$ *is a local equation of* E_X *and*

$$\begin{aligned} u &= x_1^{a_1}\cdots x_{k-1}^{a_{k-1}} \\ v &= x_2^{b_2}\cdots x_k^{b_k} \end{aligned} \tag{III}$$

where $a_2,\ldots,a_{k-1},b_2,\ldots,b_{k-1}\ge 0$, $a_1,b_k>0$ *and* $a_i+b_i>0$ *for all* $i=2,\ldots,k-1$.

Proof. Let (u,v) be prepared parameters for $\Phi(p)$.

Suppose first that there exist regular parameters $(x_1,\ldots,x_n)\in\hat{\mathcal{O}}_{X,p}$ such that

$$\begin{aligned} u &= (x_1^{a_1}\cdots x_k^{a_k})^m \\ v &= P(x_1^{a_1}\cdots x_k^{a_k})+x_1^{b_1}\cdots x_k^{b_k}x_{k+1}. \end{aligned}$$

Then there exist $y_1,\ldots,y_k\in\mathcal{O}_{X,p}$ and units $\alpha_1,\ldots,\alpha_k\in\hat{\mathcal{O}}_{X,p}$ such that $x_i=\alpha_i y_i$ for all $i=1\cdots k$.

Set $\gamma=(\alpha_1^{a_1}\cdots\alpha_k^{a_k})^{\frac{1}{a_1}}$ and $R=\mathcal{O}_{X,p}[\gamma]$. R is finite étale over $\mathcal{O}_{X,p}$. Let L be the quotient field of R.

Set $e=\max_{1\le i\le k}\{[\frac{b_i}{a_i}]\}$ and

$$y_{k+1}=\alpha_1^{b_1}\cdots\alpha_k^{b_k}x_{k+1}+\frac{P(\alpha_1^{a_1}\cdots\alpha_k^{a_k}y_1^{a_1}\cdots y_k^{a_k})-P_e(\alpha_1^{a_1}\cdots\alpha_k^{a_k}y_1^{a_1}\cdots y_k^{a_k})}{y_1^{b_1}\cdots y_k^{b_k}}.$$

Then $y_{k+1}=\frac{v-P_e(\alpha_1^{a_1}\cdots\alpha_k^{a_k}y_1^{a_1}\cdots y_k^{a_k})}{y_1^{b_1}\cdots y_k^{b_k}}\in\hat R_q\cap L=R_q$ for all maximal ideals q of R. Thus $y_{k+1}\in\bigcap R_q=R$.

Set $\bar y_1=\gamma y_1$ and $\bar y_{k+1}=\gamma^{-b_1}y_{k+1}$, so that

$$\begin{aligned}u&=(\bar y_1^{a_1}y_2^{a_2}\cdots y_k^{a_k})^m\\ v&=P_e(\bar y_1^{a_1}y_2^{a_2}\cdots y_k^{a_k})+\bar y_1^{b_1}y_2^{b_2}\cdots y_k^{b_k}\bar y_{k+1}.\end{aligned}$$

For $i=k+2\cdots n$ choose $y_i\in\mathcal{O}_{X,p}$ such that $y_i\equiv x_i \mod m_p^2\hat{\mathcal{O}}_{X,p}$. Then $\bar y_1,y_2,\ldots,y_k,\bar y_{k+1},y_{k+2},\ldots,y_n\in R$ are regular parameters at all maximal ideals of R. Since R is finite étale over $\mathcal{O}_{X,p}$, by Lemma 3.3 there exists an étale neighborhood U of p such that $(\bar y_1,y_2,\ldots,y_k,\bar y_{k+1},y_{k+2},\ldots,y_n)$ are uniformizing on U.

To finish the analysis of this case when $\Phi(p)$ is a 1 point and $ordP_e<\infty$ we only need to ensure that $m\nmid ordP$ in the formula for v. Suppose that $P_e(t)=\sum_{i=1}^{i=e}\lambda_i(x_1^{a_1}\cdots x_k^{a_k})^i$. Set

$$\bar v=v-\sum_{i=1}^{i=[\frac{e}{m}]}\lambda_{im}u^i$$

and

$$\bar P=P_e-\sum_{i=1}^{i=[\frac{e}{m}]}\lambda_{im}(x_1^{a_1}\cdots x_k^{a_k})^{im}$$

so that regular parameters $(u,\bar v)$ at $\Phi(p)$ and regular parameters $(\bar y_1,y_2,\ldots,y_k,\bar y_{k+1},y_{k+2},\ldots,y_n)$ at p satisfy the conditions of the lemma.

Suppose now that there exist regular parameters $(x_1,\ldots,x_n)$ in $\hat{\mathcal{O}}_{X,p}$ such that

$$\begin{aligned}u&=(x_1^{a_1}\cdots x_k^{a_k})^m\\ v&=P(x_1^{a_1}\cdots x_k^{a_k})+x_1^{b_1}\cdots x_k^{b_k},\end{aligned}$$

where by permuting $x_1,\ldots,x_k$ we can assume that $f=a_1b_2-a_2b_1>0$.

Then there exist $y_1,\ldots,y_k\in\mathcal{O}_{X,p}$ and units $\alpha_1,\ldots,\alpha_k\in\hat{\mathcal{O}}_{X,p}$ such that $x_i=\alpha_iy_i$ for all $i=1,\ldots,k$.

Set $\gamma=(\alpha_1^{a_1}\cdots\alpha_k^{a_k})^{\frac{1}{f}}$ and $R=\mathcal{O}_{X,p}[\gamma]$. R is finite étale over $\mathcal{O}_{X,p}$. Let L be the quotient field of R.

Set $e = \max_{1\le i\le k}\{[\frac{b_i}{a_i}]\}$ and

$$\omega = \alpha_1^{b_1}\cdots\alpha_k^{b_k} + \frac{P(\alpha_1^{a_1}\cdots\alpha_k^{a_k}y_1^{a_1}\cdots y_k^{a_k}) - P_e(\alpha_1^{a_1}\cdots\alpha_k^{a_k}y_1^{a_1}\cdots y_k^{a_k})}{y_1^{b_1}\cdots y_k^{b_k}}.$$

Then ω is a unit and $\omega = \frac{v-P_e(\alpha_1^{a_1}\cdots\alpha_k^{a_k}y_1^{a_1}\cdots y_k^{a_k})}{y_1^{b_1}\cdots y_k^{b_k}} \in \hat{R}_q \cap L = R_q$ for all maximal ideals q of R. Thus $\omega \in \bigcap R_q = R$.

Let $R_1 = R[w^{-\frac{1}{f}}]$. R_1 is finite étale over R and, therefore, it is finite étale over $\mathcal{O}_{X,p}$.

Set $\bar{y}_1 = \gamma^{b_2}\omega^{-\frac{a_2}{f}}y_1$ and $\bar{y}_2 = \gamma^{-b_1}\omega^{\frac{a_1}{f}}y_2$, so that

$$\begin{aligned} u &= (\bar{y}_1^{a_1}\bar{y}_2^{a_2}y_3^{a_3}\cdots y_k^{a_k})^m \\ v &= P_e(\bar{y}_1^{a_1}\bar{y}_2^{a_2}y_3^{a_3}\cdots y_k^{a_k}) + \bar{y}_1^{b_1}\bar{y}_2^{b_2}y_3^{b_3}\cdots y_k^{b_k}. \end{aligned}$$

For $i = k+1,\ldots,n$ choose $y_i \in \mathcal{O}_{X,p}$ such that $y_i \equiv x_i \mod m_p^2\hat{\mathcal{O}}_{X,p}$. Then $\bar{y}_1, \bar{y}_2, y_3, \ldots, y_n \in R_1$ are regular parameters at all maximal ideals of R_1. Since R_1 is finite étale over $\mathcal{O}_{X,p}$, by Lemma 3.3 there exists an étale neighborhood U of p such that $(\bar{y}_1, \bar{y}_2, y_3\cdots y_n)$ are uniformizing on U.

To finish the analysis of this case if $\Phi(p)$ is a 1 point we change v to $\bar{v}$ and P_e to $\bar{P}$ in the same way as we did above, so that regular parameters $(u,\bar{v})$ at $\Phi(p)$ and regular parameters $(\bar{y}_1, \bar{y}_2, y_3, \ldots, y_n)$ at p satisfy the conditions of the lemma.

Finally suppose that there exist regular parameters $(x_1,\ldots,x_n)$ in $\hat{\mathcal{O}}_{X,p}$ such that

$$\begin{aligned} u &= x_1^{a_1}\cdots x_{k-1}^{a_{k-1}} \\ v &= x_2^{b_2}\cdots x_k^{b_k}. \end{aligned}$$

Then there exist $y_1,\ldots,y_k \in \mathcal{O}_{X,p}$ and units $\alpha_1,\ldots,\alpha_k \in \hat{\mathcal{O}}_{X,p}$ such that $x_i = \alpha_i y_i$ for all $i = 1,\ldots,k$.

Set $\gamma = (\alpha_1^{a_1}\cdots\alpha_{k-1}^{a_{k-1}})^{\frac{1}{a_1}}$, $\omega = (\alpha_2^{b_2}\cdots\alpha_k^{b_k})^{\frac{1}{b_k}}$ and $R = \mathcal{O}_{X,p}[\gamma,\omega]$. R is finite étale over $\mathcal{O}_{X,p}$.

Set $\bar{y}_1 = \gamma y_1$ and $\bar{y}_k = \omega y_k$, so that

$$\begin{aligned} u &= \bar{y}_1^{a_1}y_2^{a_2}\cdots y_{k-1}^{a_{k-1}} \\ v &= y_2^{b_2}\cdots y_{k-1}^{b_{k-1}}\bar{y}_k^{b_k}. \end{aligned}$$

For $i = k+1, \ldots, n$ choose $y_i \in \mathcal{O}_{X,p}$ such that $y_i \equiv x_i \bmod m_p^2\hat{\mathcal{O}}_{X,p}$. Then $\bar{y}_1, y_2, \ldots, y_{k-1}, \bar{y}_k, y_{k+1}, \ldots, y_n \in R$ are regular parameters at all maximal ideals of R. Since R is finite étale over $\mathcal{O}_{X,p}$, by Lemma 3.3 there exists an étale neighborhood U of p such that $(\bar{y}_1, y_2, \ldots, y_{k-1}, \bar{y}_k, y_{k+1}, \ldots, y_n)$ are uniformizing on U.

This completes the proof. $\square$

Suppose that $p \in E_X$ and (u, v) are permissible parameters for $\Phi(p)$. (u, v) will be called strongly prepared at $\Phi(p)$ and $*$-permissible parameters $(x_1, \ldots, x_n)$ for (u, v) at p will be called strongly permissible if they satisfy the conditions of Lemma 3.4

Definition 3.5. Suppose that $\Phi : X \to S$ is strongly prepared with respect to D_S. Suppose that $p \in E_X$. We will say that p is a good point for Φ if there exist permissible parameters (u, v) at $\Phi(p)$ and $*$-permissible parameters $(x_1, \ldots, x_n)$ at p for (u, v) such that one of the following forms holds:

(1a) $1 \le k \le n-1$

p is a k point, $u = 0$ is a local equation of E_X and

$$
\begin{aligned}
u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\
v &= \alpha(x_1^{a_1} \cdots x_k^{a_k})^t + (x_1^{a_1} \cdots x_k^{a_k})^t x_{k+1}
\end{aligned}
\tag{G.Ia}
$$

where $m > 0$, $t \ge 0$, $a_1, \ldots, a_k > 0$ with $(a_1, \ldots, a_k) = 1$ and $\alpha \in \mathbf{k}$;

(1b) $2 \le k \le n-1$

p is a k point, $u = 0$ is a local equation of E_X and

$$
\begin{aligned}
u &= x_1^{a_1} \cdots x_k^{a_k} \\
v &= x_1^{b_1} \cdots x_k^{b_k} x_{k+1}
\end{aligned}
\tag{G.Ib}
$$

where $a_1, \ldots, a_k > 0$, $b_1, \ldots, b_k \ge 0$ and $rank \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$;

(2) $2 \le k \le n$

p is a k point, $u = 0$ is a local equation of E_X and

$$
\begin{aligned}
u &= x_1^{a_1} \cdots x_k^{a_k} \\
v &= x_1^{b_1} \cdots x_k^{b_k}
\end{aligned}
\tag{G.II}
$$

where $a_1, \ldots, a_k > 0$, $b_1, \ldots, b_k \geq 0$ and $rank \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$;

(3) $2 \leq k \leq n$

p is a k point, $uv = 0$ is a local equation of E_X and

$$\begin{aligned} u &= x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} \\ v &= x_2^{b_2} \cdots x_k^{b_k} \end{aligned} \tag{G.III}$$

where $a_2, \ldots, a_{k-1}, b_2, \ldots, b_{k-1} \geq 0$, $a_1, b_k > 0$ and $a_i + b_i > 0$ for all $i = 2, \ldots, k-1$.

p will be called a bad point if p is not a good point.

Remark 3.6. If $p \in E_X$ is a good point then following the proof of Lemma 3.4 we can always find strongly prepared parameters (u, v) at $\Phi(p)$ and strongly permissible parameters $(x_1, \ldots, x_n)$ at p for (u, v) such that one of the forms (G.Ia), (G.Ib), (G.II) or (G.III) holds.

Lemma 3.7. *Suppose that $p \in X$ is a 1 point and (u,v) are strongly prepared parameters at $\Phi(p)$, $(x_1, \ldots, x_n)$ are strongly permissible parameters at p for (u, v) such that (I) holds and*

$$\begin{aligned} u &= x_1^{a_1} \\ v &= P(x_1) + x_1^{c_1} x_2, \quad ordP = d_1. \end{aligned}$$

Suppose that $(\bar{u}, \bar{v})$ are also strongly prepared parameters at $\Phi(p)$ and $(y_1, \ldots, y_n)$ are strongly permissible parameters at p for $(\bar{u}, \bar{v})$ such that (I) holds and

$$\begin{aligned} \bar{u} &= y_1^{a_2} \\ \bar{v} &= Q(y_1) + y_1^{c_2} y_2, \quad ordQ = d_2. \end{aligned}$$

If $\Phi(p)$ is a 1 point then $a_1 = a_2$, $c_1 = c_2$ and $d_1 = d_2$.
If $\Phi(p)$ is a 2 point then $d_1, d_2 < \infty$ and $c_1 - d_1 = c_2 - d_2$, $a_1 + d_1 = a_2 + d_2$.

Proof. This follows from the discussion before Definition 18.7 in [4]. □

Suppose that E is a component of E_X, $p \in E$, $f \in \hat{\mathcal{O}}_{X,p}$ and $x = 0$ is a local equation of E at p. Then we define

$$\nu_E(f) = \max\{n \text{ such that } x^n | f\}.$$

Suppose that $p \in X$ is a 1 point and E is the component of E_X containing p. Suppose that (u, v) are strongly prepared parameters at $\Phi(p)$ such that $u = 0$ is a local equation of E at p and $(x_1, \ldots, x_n)$ are strongly permissible parameters for (u, v) at p with

$$\begin{aligned} u &= x_1^a \\ v &= P(x_1) + x_1^c x_2, \quad ordP = d. \end{aligned}$$

Then $\nu_E(v) = d$ if $d < \infty$ and $\nu_E(v) = c$ if $d = \infty$. Thus, in view of Lemma 3.7, $c - \nu_E(v)$ and $a + \nu_E(v)$ are independent of the choice of strongly prepared parameters (u, v) at $\Phi(p)$ and they are also independent of the choice of strongly permissible parameters for (u, v) at p.

Definition 3.8. Let $p \in X$, $E \subset E_X$, regular parameters (u, v) in $\mathcal{O}_{S,\Phi(p)}$ and regular parameters $(x_1, \ldots, x_n)$ in $\hat{\mathcal{O}}_{X,p}$ be as above with

$$\begin{aligned} u &= x_1^a \\ v &= P(x_1) + x_1^c x_2. \end{aligned}$$

Define $A(\Phi, p) = c - \nu_E(v)$.
If $A(\Phi, p) > 0$ define $C(\Phi, p) = (c - \nu_E(v),\ a + \nu_E(v))$.

Notice that if $p \in X$ is a 1 point then p is a good point if and only if $A(\Phi, p) = 0$ or, equivalently, $ordP = d \geq c$.

Lemma 3.9. *Suppose that $p \in X$ is a 1 point and E is the component of E_X containing p.*

Then there exists an open neighborhood U of p such that $A(\Phi, p') = A(\Phi, p)$ for all $p' \in E \cap U$ and if $A(\Phi, p) > 0$ then $C(\Phi, p') = C(\Phi, p)$ for all $p' \in E \cap U$.

Proof. There exist strongly prepared parameters (u, v) at $\Phi(p)$ and strongly permissible parameters $(x_1, \ldots, x_n)$ at p for (u, v) such that

$$\begin{aligned} u &= x_1^a \\ v &= P(x_1) + x_1^c x_2 \text{ and } ordP = d. \end{aligned}$$

If U is an étale neighborhood of p where $(x_1,\ldots,x_n)$ are uniformizing parameters then for any $p' \in U\cap E$ there exist $\alpha_2,\ldots,\alpha_n \in \mathbf{k}$ such that $(x_1, \bar{x}_2 = x_2+\alpha_2,\ldots,\bar{x}_n = x_n+\alpha_n)$ are strongly permissible parameters at p' for strongly prepared parameters $(u,\bar{v})$ at $\Phi(p')$ and $\bar{v}=v$ if $\Phi(p')$ is a 2 point or $a\nmid c$, $\bar{v} = v+\alpha_2 u^{\frac{c}{a}}$ if $\Phi(p')$ is a 1 point and $a|c$.

Suppose that the first assumption holds and $\bar{v}=v$ then at p'

$$\begin{aligned} u &= x_1^a\\ \bar{v} &= P(x_1) - \alpha_2 x_1^c + x_1^c(x_2+\alpha_2) = P_1(x_1) + x_1^c\bar{x}_2. \end{aligned}$$

Thus if $d<c$ then $ordP_1 = d$ and $A(\Phi,p') = c-d = A(\Phi,p) > 0$, $C(\Phi,p') = (c-d, a+d) = C(\Phi,p)$. If $d\geq c$ then $ordP_1\geq c$ and $A(\Phi,p') = 0 = A(\Phi,p)$.

Suppose that the second assumption holds and $\bar{v} = v+\alpha_2 u^{\frac{c}{a}}$ then at p'

$$\begin{aligned} u &= x_1^a\\ \bar{v} &= P(x_1) + x_1^c(x_2+\alpha_2) = P(x_1) + x_1^c\bar{x}_2. \end{aligned}$$

Thus $A(\Phi,p') = A(\Phi,p)$ and $C(\Phi,p') = C(\Phi,p)$ if $A(\Phi,p)>0$.

□

Now we can define $A(\Phi,E) = A(\Phi,p)$ for $p\in E$ a 1 point. If $A(\Phi,E)>0$ define $C(\Phi,E) = C(\Phi,p)$.

We will call $E\subset E_X$ a good component of E_X if $A(\Phi,E) = 0$. E will be called a bad component if it is not a good component or, equivalently, if $A(\Phi,E)>0$.

Lemma 3.10. *Suppose* $2\leq k\leq n$.

Suppose that $p\in X$ *is a* k *point and* $E_1,\ldots,E_k$ *are the components of* E_X *containing* p. *Suppose that* (u,v) *are strongly prepared at* $\Phi(p)$ *and* $(x_1,\ldots,x_n)$ *are strongly permissible parameters for* (u,v) *at* p *with* $x_i=0$ *being a local equation of* E_i *for* $i=1,\ldots,k$.

If

$$\begin{aligned} u &= (x_1^{a_1}\cdots x_k^{a_k})^m\\ v &= P(x_1^{a_1}\cdots x_k^{a_k}) + x_1^{b_1}\cdots x_k^{b_k}x_{k+1} \end{aligned} \tag{I}$$

or

$$\begin{aligned} u &= (x_1^{a_1}\cdots x_k^{a_k})^m\\ v &= P(x_1^{a_1}\cdots x_k^{a_k}) + x_1^{b_1}\cdots x_k^{b_k} \end{aligned} \tag{II}$$

then $A(\Phi, E_i) = b_i - \nu_{E_i}(v)$ *for* $i = 1, \ldots, k$ *and if* $A(\Phi, E_i) > 0$
$C(\Phi, E_i) = (b_i - \nu_{E_i}(v),\, a_i m + \nu_{E_i}(v))$.
If

$$u = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} \qquad \text{(III)}$$
$$v = x_2^{b_2} \cdots x_k^{b_k}$$

then $A(\Phi, E_i) = 0$ *for all* $i = 1, \ldots, k$.

Proof. Suppose that $p \in X$ is a k point satisfying (I), (u, v) are strongly prepared parameters at $\Phi(p)$ and $(x_1, \ldots, x_n)$ are strongly permissible parameters for (u, v) at p such that

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} x_{k+1} \text{ and } ordP = d.$$

After possibly permuting $x_1, \ldots, x_k$ we can assume that $i = 1$ and prove only that $A(\Phi, E_1) = b_1 - \nu_{E_1}(v)$ and $C(\Phi, E_1) = (b_1 - \nu_{E_1}(v), a_1 m + \nu_{E_1}(v))$ if $A(\Phi, E_1) > 0$.

Suppose that U is an étale neighborhood of p where $(x_1, \ldots, x_n)$ are uniformizing parameters and $p' \in U \cap E_1$ is a 1 point. Then there exist $\alpha_2, \ldots, \alpha_k \in \mathbf{k} - \{0\}$ and $\alpha_{k+1}, \ldots, \alpha_n \in \mathbf{k}$ such that $(x_1, \bar{x}_2 = x_2 - \alpha_2, \ldots, \bar{x}_n = x_n - \alpha_n)$ are regular parameters at p'.

Set $\gamma = ((\bar{x}_2 + \alpha_2)^{a_2} \cdots (\bar{x}_k + \alpha_k)^{a_k})^{\frac{1}{a_1}}$, for $i = 2 \cdots k$ set $f_i = a_1 b_i - a_i b_1$ and $\omega = ((\bar{x}_2 + \alpha_2)^{f_2} \cdots (\bar{x}_k + \alpha_k)^{f_k})^{\frac{1}{a_1}}$. Notice that $\gamma^{a_1}, \omega^{a_1} \in \mathcal{O}_{U,p'}$ are units in $\mathcal{O}_{U,p'}$ and therefore $\mathcal{O}_{U,p'}[\gamma, \omega]$ is finite étale over $\mathcal{O}_{U,p'}$.

Set $\bar{x}_1 = \gamma x_1$ and $\tilde{x}_{k+1} = \bar{x}_{k+1}\omega + \alpha_{k+1}\omega - \alpha_{k+1}(\alpha_2^{f_2} \cdots \alpha_k^{f_k})^{\frac{1}{a_1}}$ then

$$u = \bar{x}_1^{a_1 m}$$
$$v = P(\bar{x}_1^{a_1}) + \bar{x}_1^{b_1}\tilde{x}_{k+1} + \alpha \bar{x}_1^{b_1}, \text{ where } \alpha = \alpha_{k+1}(\alpha_2^{f_2} \cdots \alpha_k^{f_k})^{\frac{1}{a_1}} \in \mathbf{k}.$$

Assume first that $\Phi(p)$ is a 2 point or $a_1 m \nmid b_1$, then (u, v) are strongly prepared parameters at $\Phi(p')$.

If $a_1 d \le b_1$ then $(\bar{x}_1, \ldots, \bar{x}_k, \tilde{x}_{k+1}, \bar{x}_{k+2}, \ldots, \bar{x}_n)$ are strongly permissible parameters at p' for (u, v) and

$$u = \bar{x}_1^{a_1 m}$$
$$v = P_1(\bar{x}_1) + \bar{x}_1^{b_1}\tilde{x}_{k+1}$$

with $ordP_1 = a_1d$ if $a_1d < b_1$ and $ordP_1 \ge b_1$ if $a_1d = b_1$. Thus $A(\Phi, E_1) = A(\Phi, p') = b_1 - \nu_{E_1}(v)$ in this case and if $A(\Phi, p') > 0$ or, equivalently, if $a_1d < b_1$ then $C(\Phi, E_1) = C(\Phi, p') = (b_1 - a_1d, a_1m + a_1d) = (b_1 - \nu_{E_1}(v), a_1m + \nu_{E_1}(v))$.

If $a_1d > b_1$ then set $\bar{x}_{k+1} = \tilde{x}_{k+1} + \frac{P(\bar{x}_1^{a_1})}{\bar{x}_1^{b_1}}$ to get strongly permissible parameters $(\bar{x}_1, \ldots, \bar{x}_n)$ for (u, v) at p', so that

$$u = \bar{x}_1^{a_1m}$$
$$v = \bar{x}_1^{b_1}\bar{x}_{k+1} + \alpha\bar{x}_1^{b_1}$$

and $A(\Phi, E_1) = A(\Phi, p') = 0 = b_1 - \nu_{E_1}(v)$.

Assume now that $\Phi(p)$ is a 1 point and $a_1m|b_1$, then $(u, \bar{v} = v - \alpha u^{\frac{b_1}{a_1m}})$ are strongly prepared parameters at $\Phi(p')$.

If $a_1d \le b_1$ then $(\bar{x}_1, \ldots, \bar{x}_k, \tilde{x}_{k+1}, \bar{x}_{k+2}, \ldots, \bar{x}_n)$ are strongly permissible parameters at p' for (u, v) and

$$u = \bar{x}_1^{a_1m}$$
$$v = P(\bar{x}_1^{a_1}) + \bar{x}_1^{b_1}\tilde{x}_{k+1} = P_1(\bar{x}_1) + \bar{x}_1^{b_1}\tilde{x}_{k+1}.$$

Thus $A(\Phi, E_1) = A(\Phi, p') = b_1 - a_1d = b_1 - \nu_{E_1}(v)$ in this case and if $A(\Phi, p') > 0$ then $C(\Phi, E_1) = C(\Phi, p') = (b_1 - a_1d, a_1m + a_1d) = (b_1 - \nu_{E_1}(v), a_1m + \nu_{E_1}(v))$.

If $a_1d > b_1$ then set $\bar{x}_{k+1} = \tilde{x}_{k+1} + \frac{P(\bar{x}_1^{a_1})}{\bar{x}_1^{b_1}}$ to get strongly permissible parameters $(\bar{x}_1, \ldots, \bar{x}_n)$ for (u, v) at p' so that

$$u = \bar{x}_1^{a_1m}$$
$$v = \bar{x}_1^{b_1}\bar{x}_{k+1}$$

and $A(\Phi, E_1) = A(\Phi, p') = 0 = b_1 - \nu_{E_1}(v)$.

Suppose that $p \in X$ is a k point satisfying (II), (u, v) are strongly prepared parameters at $\Phi(p)$ and $(x_1, \ldots, x_n)$ are strongly permissible parameters for (u, v) at p such that

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} \text{ and } ordP = d.$$

After possibly permuting $x_1,\dots,x_k$ we can assume that $i=1$ and prove only that $A(\Phi,E_1)=b_1-\nu_{E_1}(v)$ and $C(\Phi,E_1)=(b_1-\nu_{E_1}(v),a_1+\nu_{E_1}(v))$ if $A(\Phi,E_1)>0$.

Suppose that U is an étale neighborhood of p where $(x_1,\dots,x_n)$ are uniformizing parameters and $p'\in U\cap E_1$ is a 1 point. Then there exist $\alpha_2,\dots,\alpha_k\in\mathbf{k}-\{0\}$ and $\alpha_{k+1},\dots,\alpha_n\in\mathbf{k}$ such that $(x_1,\bar{x}_2=x_2-\alpha_2,\dots,\bar{x}_n=x_n-\alpha_n)$ are regular parameters at p'.

Notice that since $rank\begin{pmatrix}a_1 & \dots & a_k\\ b_1 & \dots & b_k\end{pmatrix}=2$ there exists $j\in\{2,\dots,k\}$ such that $a_1b_j-a_jb_1\neq 0$. So, after possibly permuting $x_2,\dots,x_k$ we can assume that $j=2$, that is $a_1b_2-a_2b_1\neq 0$.

Set $\gamma=((\bar{x}_2+\alpha_2)^{a_2}\cdots(\bar{x}_k+\alpha_k)^{a_k})^{\frac{1}{a_1}}$, for $i=2,\dots,k$ set $f_i=a_1b_i-a_ib_1$, $\omega=((\bar{x}_3+\alpha_3)^{f_3}\cdots(\bar{x}_k+\alpha_k)^{f_k})^{\frac{1}{a_1}}$ and $\delta=(\bar{x}_2+\alpha_2)^{\frac{f_2}{a_1}}$. Notice that $\gamma^{a_1},\omega^{a_1},\delta^{a_1}\in\mathcal{O}_{U,p'}$ are units in $\mathcal{O}_{U,p'}$ and therefore $\mathcal{O}_{U,p'}[\gamma,\omega\delta]$ is finite étale over $\mathcal{O}_{U,p'}$.

Set $\bar{x}_1=\gamma x_1$ and $\tilde{x}_2=\delta\omega-(\alpha_2^{f_2}\cdots\alpha_k^{f_k})^{\frac{1}{a_1}}$ then

$$u=\bar{x}_1^{a_1m}$$
$$v=P(\bar{x}_1^{a_1})+\bar{x}_1^{b_1}\tilde{x}_2+\alpha\bar{x}_1^{b_1},\ \text{where } \alpha=(\alpha_2^{f_2}\cdots\alpha_k^{f_k})^{\frac{1}{a_1}}\in\mathbf{k}-\{0\}.$$

Now the same analysis as above with $\tilde{x}_2$ playing the role of $\tilde{x}_{k+1}$ above shows that $A(\Phi,E_1)=b_1-\nu_{E_1}(v)$ and $C(\Phi,E_1)=(b_1-\nu_{E_1}(v),a_1m+\nu_{E_1}(v))$ if $A(\Phi,E_1)>0$.

Suppose that $p\in X$ is a k point satisfying (III), (u,v) are strongly prepared parameters at $\Phi(p)$ and $(x_1,\dots,x_n)$ are strongly permissible parameters for (u,v) at p such that

$$u=x_1^{a_1}\cdots x_{k-1}^{a_{k-1}}$$
$$v=x_2^{a_2}\cdots x_k^{b_k}.$$

Since $a_1,b_k>0$ and $a_j+b_j>0$ for all $j=2,\dots,k-1$, after possibly permuting $x_1,\dots,x_k$ and u,v we can assume that $i=1$ and

$$u=x_1^{a_1}\cdots x_{k-1}^{a_{k-1}}$$
$$v=P(x_1^{a_1}\cdots x_{k-1}^{a_{k-1}})+x_1^{b_1}\cdots x_k^{b_k},\ \text{where } a_1,b_k>0 \text{ and } P(t)\equiv 0.$$

In this notations the proof of the required statement repeats the proof for case (II). $\square$

Theorem 3.11. *Suppose that $\Phi : X \to S$ is strongly prepared. Then the locus of bad points in X is a Zariski closed set of pure codimension 1, consisting of the union of all bad components of E_X.*

Proof. Let Z be the union of all bad components of E_X and let p be a good point on E_X, q be a bad point on E_X. Then it suffices to show that $q \in Z$ while $p \in E_X - Z$.

Suppose that p is a good k point, (u, v) are strongly prepared parameters at $\Phi(p)$, $(x_1, \ldots, x_n)$ are strongly permissible parameters for (u, v) at p such that one of the following forms holds

(1a)

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = \alpha(x_1^{a_1} \cdots x_k^{a_k})^t + (x_1^{a_1} \cdots x_k^{a_k})^t x_{k+1};$$

(1b)

$$u = x_1^{a_1} \cdots x_k^{a_k}$$
$$v = x_1^{b_1} \cdots x_k^{b_k} x_{k+1};$$

(2)

$$u = x_1^{a_1} \cdots x_k^{a_k}$$
$$v = x_1^{b_1} \cdots x_k^{b_k};$$

(3)

$$u = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}}$$
$$v = x_2^{b_2} \cdots x_k^{b_k}.$$

In all these cases $x_1 = 0, \ldots, x_k = 0$ are local equations of the components of E_X containing p. So we can assume that for all $i = 1, \ldots, k$ $x_i = 0$ is a local equation of $E_i \subset E$, a component of E_X.

By Lemma 3.10 we compute $A(\Phi, E_i)$ as follows

(1a) $A(\Phi, E_i) = a_i t - \nu_{E_i}(v) = a_i t - a_i t = 0$

(1b) $A(\Phi, E_i) = b_i - \nu_{E_i}(v) = b_i - b_i = 0$

(2) $A(\Phi, E_i) = b_i - \nu_{E_i}(v) = b_i - b_i = 0$

(3) $A(\Phi, E_i) = 0$.

Thus all components of E_X containing p are good, so p does not lie in Z.

Suppose that q is a bad k point, (u, v) are strongly prepared parameters at $\Phi(q)$, $(x_1, \ldots, x_n)$ are strongly permissible parameters for (u, v) at q and (I) holds, that is

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} x_{k+1}, \text{ where } P(t) \neq 0.$$

Let $x_1 = 0, \ldots, x_k = 0$ be local equations of the components $E_1, \ldots, E_k \subset E_X$ containing q.

By Lemma 3.10 we can compute $A(\Phi, E_i)$ for all $i = 1, \ldots, k$ as follows

$$A(\Phi, E_i) = b_i - \nu_{E_i}(v) = \begin{cases} b_i - a_i ordP > 0, & \text{if } a_i ordP < b_i; \\ 0, & \text{otherwise.} \end{cases}$$

To prove the statement of the theorem we need to find such $j \in \{1, \ldots, k\}$ that E_j is a bad component or, equivalently, $A(\Phi, E_j) > 0$.

Assume the contrary. Let $a_i ordP \geq b_i$ for all $i = 1, \ldots, k$ then $v = x_1^{b_1} \cdots x_k^{b_k}(x_{k+1} + \frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}})$ and $\frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}} \in \hat{\mathcal{O}}_{X,q}$.

If $ord \frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}} \geq 1$ then set $\bar{x}_{k+1} = x_{k+1} + \frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}}$ so that $(x_1, \ldots, x_k, \bar{x}_{k+1}, x_{k+2}, \ldots, x_n)$ are strongly permissible parameters for (u, v) at q and

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = x_1^{b_1} \cdots x_k^{b_k} \bar{x}_{k+1}.$$

Thus one of the forms (G.Ia) or (G.Ib) holds for q and, therefore, q is a good point while q was originally chosen to be a bad point.

If $ord \frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}} = 0$ then there exist $\alpha \in \mathbf{k} - \{0\}$ and $y \in m_q \hat{\mathcal{O}}_{X,q}$ such that $\frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}} = y + \alpha$. Set $\bar{x}_{k+1} = x_{k+1} + y$ so that $(x_1, \ldots, x_k, \bar{x}_{k+1}, x_{k+2}, \ldots, x_n)$ are strongly permissible parameters for (u, v) at q and

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = x_1^{b_1} \cdots x_k^{b_k}(\bar{x}_{k+1} + \alpha).$$

Now if $rank \begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix} < 2$ then (G.Ia) holds and, therefore, q is a good point. This contradicts the choice of q.

If $rank \begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix} = 2$ then after possibly permuting $x_1, \dots, x_k$ we can assume that $f = a_1 b_2 - a_2 b_1 \neq 0$. Set

$$\begin{aligned} \bar{x}_1 &= x_1(\bar{x}_{k+1} + \alpha)^{\frac{-a_2}{f}} \\ \bar{x}_2 &= x_2(\bar{x}_{k+1} + \alpha)^{\frac{a_1}{f}} \end{aligned}$$

then $(\bar{x}_1, \bar{x}_2, x_3, \dots, x_k, \bar{x}_{k+1}, x_{k+2}, \dots, x_n)$ are $*$-permissible parameters for (u, v) at q and

$$\begin{aligned} u &= (\bar{x}_1^{a_1} \bar{x}_2^{a_2} x_3^{a_3} \cdots x_k^{a_k})^m \\ v &= \bar{x}_1^{b_1} \bar{x}_2^{b_2} x_3^{b_3} \cdots x_k^{b_k}. \end{aligned}$$

Thus (G.II) holds for q and, therefore, q is a good point while q was originally chosen to be a bad point.

This shows that if q is a bad point on E_X and (I) holds at q then there exists such a component E of E_X containing q that $A(\Phi, E) > 0$. So $E \subset Z$ and, therefore, $q \in Z$.

Suppose that q is a bad k point, (u, v) are strongly prepared parameters at $\Phi(q)$, $(x_1, \dots, x_n)$ are strongly permissible parameters for (u, v) at q and (II) holds, that is

$$\begin{aligned} u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\ v &= P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k}, \text{ where } P(t) \neq 0. \end{aligned}$$

Let $x_1 = 0, \dots, x_k = 0$ be local equations of the components $E_1, \dots, E_k \subset E_X$ containing q.

By Lemma 3.10 we can compute $A(\Phi, E_i)$ for all $i = 1, \dots, k$ as follows
$$A(\Phi, E_i) = b_i - \nu_{E_i}(v) = \begin{cases} b_i - a_i ordP > 0, & \text{if } a_i ordP < b_i; \\ 0, & \text{otherwise.} \end{cases}$$

To prove the statement of the theorem we need to find such $j \in \{1, \dots, k\}$ that E_j is a bad component or, equivalently, $A(\Phi, E_j) > 0$.

Assume the contrary. Let $a_i ordP \geq b_i$ for all $i = 1, \dots, k$ then $v = x_1^{b_1} \cdots x_k^{b_k}(1 + \frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}})$. Since $rank \begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix} = 2$ there exists $l \in \{1, \dots, k\}$ such that $a_l ordP > b_l$, so $\frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}} \in m_q \hat{\mathcal{O}}_{X,q}$.

After possibly permuting $x_1,\ldots,x_k$ we can assume that $f=a_1b_2-a_2b_1\neq 0$. Set

$$\bar{x}_1 = x_1(1+\frac{P(x_1^{a_1}\cdots x_k^{a_k})}{x_1^{b_1}\cdots x_k^{b_k}})^{\frac{-a_2}{f}}$$
$$\bar{x}_2 = x_2(1+\frac{P(x_1^{a_1}\cdots x_k^{a_k})}{x_1^{b_1}\cdots x_k^{b_k}})^{\frac{a_1}{f}}$$

so that $(\bar{x}_1,\bar{x}_2,x_3,\ldots,x_n)$ are $*$-permissible parameters for (u,v) at q and

$$u = (\bar{x}_1^{a_1}\bar{x}_2^{a_2}x_3^{a_3}\cdots x_k^{a_k})^m$$
$$v = \bar{x}_1^{b_1}\bar{x}_2^{a_2}x_3^{a_3}\cdots x_k^{b_k}.$$

Thus (G.II) holds for q and, therefore, q is a good point while q was originally chosen to be a bad point.

This shows that if q is a bad point on E_X and (II) holds at q then there exists such a component E of E_X containing q that $A(\Phi,E)>0$. So $E\subset Z$ and, therefore, $q\in Z$.

□

Lemma 3.12. *Suppose that $\Phi: X\to S$ is strongly prepared, $q\in D_S$ and $p\in\Phi^{-1}(q)$ is a k point on X such that one of the forms (I), (II) or (III) holds at p. Then $m_q\mathcal{O}_{X,p}$ is not invertible if and only if one of the following holds:*

(1a) $1\le k\le n-1$

$$u = (x_1^{a_1}\cdots x_k^{a_k})^m \qquad \text{(N.Ia)}$$
$$v = (x_1^{a_1}\cdots x_k^{a_k})^t x_{k+1}$$

where $m>0$, $a_1,\ldots,a_k>0$ with $(a_1,\ldots,a_k)=1$ and $0\le t<m$;

(1b) $2\le k\le n-1$

$$u = (x_1^{a_1}\cdots x_k^{a_k})^m \qquad \text{(N.Ib)}$$
$$v = (x_1^{b_1}\cdots x_k^{b_k})x_{k+1}$$

where $a_1,\ldots,a_k>0$ with $(a_1,\ldots,a_k)=1$, $b_1,\ldots,b_k\ge 0$,
rank $\begin{pmatrix} a_1 & \ldots & a_k\\ b_1 & \ldots & b_k\end{pmatrix}=2$ *and* $\min_{1\le i\le k}\{\frac{b_i}{a_i}\}<m$*;*

(1c) $2 \le k \le n-1$

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m \qquad \text{(N.Ic)}$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} x_{k+1}$$

where $a_1, \ldots, a_k > 0$ *with* $(a_1, \ldots, a_k) = 1$, $b_1, \ldots, b_k \ge 0$, *rank* $\begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$ *and* $\min_{1 \le i \le k}\{\frac{b_i}{a_i}\} < ordP < \max_{1 \le i \le k}\{\frac{b_i}{a_i}\}$, $\min_{1 \le i \le k}\{\frac{b_i}{a_i}\} < m$;

(2a) $2 \le k \le n$

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m \qquad \text{(N.IIa)}$$
$$v = (x_1^{b_1} \cdots x_k^{b_k})$$

where $a_1, \ldots, a_k > 0$ *with* $(a_1, \ldots, a_k) = 1$, $b_1, \ldots, b_k \ge 0$, *rank* $\begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$ *and* $\min_{1 \le i \le k}\{\frac{b_i}{a_i}\} < m < \max_{1 \le i \le k}\{\frac{b_i}{a_i}\}$;

(2b) $2 \le k \le n$

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m \qquad \text{(N.IIb)}$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k}$$

where $a_1, \ldots, a_k > 0$ *with* $(a_1, \ldots, a_k) = 1$, $b_1, \ldots, b_k \ge 0$, *rank* $\begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$ *and* $\min_{1 \le i \le k}\{\frac{b_i}{a_i}\} < ordP < \max_{1 \le i \le k}\{\frac{b_i}{a_i}\}$, $\min_{1 \le i \le k}\{\frac{b_i}{a_i}\} < m$;

(3) $2 \le k \le n$

$$u = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} \qquad \text{(N.III)}$$
$$v = x_2^{b_2} \cdots x_k^{b_k}$$

where $a_2, \ldots, a_{k-1}, b_2, \ldots, b_{k-1} \ge 0$, $a_1, b_k > 0$ *and* $a_i + b_i > 0$ *for all* $i = 2, \ldots, k-1$.

Proof. Suppose that (I) holds at p. First consider the case when $rank\begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix} < 2$. Then there exist $t \geq 0$ such that

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + (x_1^{a_1} \cdots x_k^{a_k})^t x_{k+1}.$$

If $d = ordP \leq t$ then $v = (x_1^{a_1} \cdots x_k^{a_k})^d \gamma$, where $\gamma \in \hat{\mathcal{O}}_{X,p}$ is a unit. So either u is a multiple of v if $d \leq m$ or v is a multiple of u if $m \leq d$. Thus we may assume that $ordP > t$.

Set

$$\bar{x}_{k+1} = x_{k+1} + \frac{P(x_1^{a_1} \cdots x_k^{a_k})}{(x_1^{a_1} \cdots x_k^{a_k})^t} \in \mathcal{O}_{\hat{X},p}$$

to get strongly permissible parameters $(x_1, \dots, x_k, \bar{x}_{k+1}, x_{k+2}, \dots, x_n)$ for (u, v) at p such that

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = (x_1^{a_1} \cdots x_k^{a_k})^t \bar{x}_{k+1}.$$

So $(u, v)\mathcal{O}_{X,p}$ is not invertible if and only if $t < m$ and we get case (N.Ia) of the lemma.

Suppose now that $rank\begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix} = 2$. If $P(t) \equiv 0$ in the formula for v then

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = x_1^{b_1} \cdots x_k^{b_k} x_{k+1}$$

and v is not a multiple of u if and only if $a_i m > b_i$ for some $i \in \{1, \dots, k\}$, that is if $m > \min_{1 \leq i \leq k}\{\frac{b_i}{a_i}\}$. Thus we meet case (N.Ib) of the lemma.

Suppose that $P(t) \neq 0$ then $d = ordP \leq \max_{1 \leq i \leq k}\{\frac{b_i}{a_i}\}$. If $d = \max_{1 \leq i \leq k}\{\frac{b_i}{a_i}\}$ then

$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} x_{k+1} = (x_1^{b_1} \cdots x_k^{b_k})(\frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}} + x_{k+1})$$

where the smallest degree term of $\frac{P(x_1^{a_1} \cdots x_k^{a_k})}{x_1^{b_1} \cdots x_k^{b_k}}$ is $x_1^{a_1 d - b_1} \cdots x_k^{a_k d - b_k}$. $x_1^{a_1 d - b_1} \cdots x_k^{a_k d - b_k} \in m_p \hat{\mathcal{O}}_{X,p}$ since $d \geq \frac{b_i}{a_i}$ for all $i = 1, \dots, k$ and $d \neq \frac{b_j}{a_j}$ for at least one $j \in \{1,\dots,k\}$ due to maximality of the *rank* $\begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix}$. So by setting $\bar{x}_{k+1} = x_{k+1} + \frac{P(x_1^{a_1}\cdots x_k^{a_k})}{x_1^{b_1}\cdots x_k^{b_k}}$ we return to the situation when $P(t) \equiv 0$ in the formula for v.

Therefore, we may restrict our considerations to $d = ordP < \max_{1\le i\le k}\{\frac{b_i}{a_i}\}$. If $d \le \min_{1\le i\le k}\{\frac{b_i}{a_i}\}$ then $v = (x_1^{a_1}\cdots x_k^{a_k})^d\gamma$, where

$$\gamma = \frac{P(x_1^{a_1}\cdots x_k^{a_k})}{(x_1^{a_1}\cdots x_k^{a_k})^d} + x_1^{b_1-a_1d}\cdots x_k^{b_k-a_kd}x_{k+1} \in \hat{\mathcal{O}}_{X,p}$$

is a unit. So either v is a multiple of u if $m \le d$ or u is a multiple of v if $d \le m$. On the other hand if $ordP > \min_{1\le i\le k}\{\frac{b_i}{a_i}\}$ we denote by c_i the minimum of a_id and b_i to present v as follows

$$v = x_1^{c_1}\cdots x_k^{c_k}(\frac{P(x_1^{a_1}\cdots x_k^{a_k})}{x_1^{c_1}\cdots x_k^{c_k}} + x_1^{b_1-c_1}\cdots x_k^{b_k-c_k}x_{k+1}) = x_1^{c_1}\cdots x_k^{c_k}\gamma,$$

where $\gamma \in m_p\mathcal{O}_{X,p}$ and $x_i \nmid \gamma$ for all $i = 1,\dots,k$. Then u cannot be a multiple of v and v is a multiple of u if and only if

$$m \le \min_{1\le i\le k}\{\frac{c_i}{a_i}\} = \min_{1\le i\le k}\min\{d, \frac{b_i}{a_i}\} = \min_{1\le i\le k}\{\frac{b_i}{a_i}\}.$$

Thus, with the assumption $m > \min_{1\le i\le k}\{\frac{b_i}{a_i}\}$ $(u,v)\mathcal{O}_{X,p}$ is not invertible and this is case (N.Ic) of the lemma.

Suppose that (II) holds at p. Assume first that $P(t) \equiv 0$ in the formula for v so that

$$u = (x_1^{a_1}\cdots x_k^{a_k})^m$$
$$v = x_1^{b_1}\cdots x_k^{b_k}$$

Then v is not a multiple of u if and only if there exist $i \in \{1,\cdots k\}$ such that $a_im > b_i$, that is if $m > \min_{1\le i\le k}\{\frac{b_i}{a_i}\}$. The symmetric condition for u not being a multiple of v gives the restriction $m < \max_{1\le i\le k}\{\frac{b_i}{a_i}\}$. Thus we meet case (N.IIa) of the lemma.

Let $P(t) \neq 0$ then $d = ordP \le \max_{1\le i\le k}\{\frac{b_i}{a_i}\}$. If $d = \max_{1\le i\le k}\{\frac{b_i}{a_i}\}$ then

$$v = P(x_1^{a_1}\cdots x_k^{a_k}) + x_1^{b_1}\cdots x_k^{b_k} = (x_1^{b_1}\cdots x_k^{b_k})(\frac{P(x_1^{a_1}\cdots x_k^{a_k})}{x_1^{b_1}\cdots x_k^{b_k}} + 1) = x_1^{b_1}\cdots x_k^{b_k}\gamma$$

where the smallest degree term of $\frac{P(x_1^{a_1}\cdots x_k^{a_k})}{x_1^{b_1}\cdots x_k^{b_k}}$ is $x_1^{a_1 d-b_1}\cdots x_k^{a_k d-b_k}$. So $\gamma \in \hat{\mathcal{O}}_{X,p}$ is a unit since $d \geq \frac{b_i}{a_i}$ for all $i = 1,\dots,k$ and $d \neq \frac{b_j}{a_j}$ for at least one $j \in \{1,\dots,k\}$ due to maximality of the *rank* $\begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix}$.

After possibly permuting regular coordinates $x_1,\dots,x_n$ we can assume that $f = a_1b_2 - a_2b_1 \neq 0$. Then by setting

$$\bar{x}_1 = x_1\gamma^{-\frac{a_2}{f}}$$
$$\bar{x}_2 = x_2\gamma^{\frac{a_1}{f}}$$

we get strongly permissible parameters $(\bar{x}_1,\bar{x}_2,x_3,\dots,x_n)$ for (u,v) at p such that

$$u = (\bar{x}_1^{a_1}\bar{x}_2^{a_2}x_3^{a_3}\cdots x_k^{a_k})^m$$
$$v = \bar{x}_1^{b_1}\bar{x}_2^{b_2}x_3^{b_3}\cdots x_k^{b_k}.$$

Thus we may assume that $ordP < \max_{1\leq i\leq k}\{\frac{b_i}{a_i}\}$.

If $d \leq \min_{1\leq i\leq k}\{\frac{b_i}{a_i}\}$ then $v = (x_1^{a_1}\cdots x_k^{a_k})^d\gamma$, where

$$\gamma = \frac{P(x_1^{a_1}\cdots x_k^{a_k})}{(x_1^{a_1}\cdots x_k^{a_k})^d} + x_1^{b_1-a_1d}\cdots x_k^{b_k-a_kd} \in \hat{\mathcal{O}}_{X,p}$$

is a unit. So either v is a multiple of u if $m \leq d$ or u is a multiple of v if $d \leq m$. On the other hand if $ordP > \min_{1\leq i\leq k}\{\frac{b_i}{a_i}\}$ we denote by c_i the minimum of a_id and b_i to present v as follows

$$v = x_1^{c_1}\cdots x_k^{c_k}(\frac{P(x_1^{a_1}\cdots x_k^{a_k})}{x_1^{c_1}\cdots x_k^{c_k}} + x_1^{b_1-c_1}\cdots x_k^{b_k-c_k}) = x_1^{c_1}\cdots x_k^{c_k}\gamma,$$

where $\gamma \in m_p\mathcal{O}_{X,p}$ and $x_i \nmid \gamma$ for all $i = 1,\dots,k$. Then u cannot be a multiple of v and v is a multiple of u if and only if

$$m \leq \min_{1\leq i\leq k}\{\frac{c_i}{a_i}\} = \min_{1\leq i\leq k}\min\{d,\frac{b_i}{a_i}\} = \min_{1\leq i\leq k}\{\frac{b_i}{a_i}\}.$$

Thus, with the assumption $m > \min_{1\leq i\leq k}\{\frac{b_i}{a_i}\}$ $(u,v)\mathcal{O}_{X,p}$ is not invertible and this is case (N.IIb) of the lemma.

Suppose that (III) holds at p. Then

$$u = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}}$$
$$v = x_1^{b_2} \cdots x_k^{b_k}, \text{ with } a_1, b_k > 0.$$

So $m_q \mathcal{O}_{X,p}$ is not invertible and this describes case (N.III) of the lemma.

□

Lemma 3.13. *Suppose that $\Phi : X \to S$ is strongly prepared. Let $\pi_1 : S_1 \to S$ be the blow up of S at a point $q \in D_S$.*

Let U be the largest open set of X such that the rational map $X \to S_1$ is a morphism $\Phi_1 : U \to S_1$ on U. Then Φ_1 is strongly prepared with respect to $\pi_1^{-1}(D_S)$, and if all points of U are good for Φ then all points of U are good for Φ_1 also.

Proof. This follows from the analysis of the proof of Lemma 3.12. In fact, the conclusions of the Lemma are clear if $u \mid v$ in $\mathcal{O}_{X,p}$. If $v \mid u$ and (1) or (2) holds in Definition 3.2, we can make a change of variables in the x_i, replacing x_i with $\gamma_i x_i$, where γ_i is a unit series for $1 \le i \le k$, and making an appropriate change of x_{k+1} to get an expression of the form (1) or (2) of Definition 3.2, with u and v interchanged. If p is a good point, the new expressions of v and u will have the good expressions of (1a) or (2) of Definition 3.5. □

Theorem 3.14. *Suppose that $\Phi : X \to S$ is strongly prepared, $p \in X$ is a 1 point and the rational map Φ_1 from X to the blow up S_1 of $q = \Phi(p)$ is a morphism in a neighborhood of p.*

Then $A(\Phi_1, p) \le A(\Phi, p)$ and if $A(\Phi_1, p) = A(\Phi, p) > 0$ then $C(\Phi_1, p) < C(\Phi, p)$.

Proof. Let $(x_1, \ldots, x_n)$ be strongly permissible parameters at p for strongly prepared parameters (u, v) at $\Phi(p)$, then

$$u = x_1^a$$
$$v = P(x_1) + x_1^b x_2$$

with $ordP = d \le \infty$.

Suppose first that $P(x) \equiv 0$. Then $b \ge a$ since $(u, v)\mathcal{O}_{X,p}$ is principal and there exist strongly prepared parameters (u_1, v_1) at $\Phi_1(p)$ such that

$$u = u_1, \quad v = u_1 v_1.$$

Thus

$$u_1 = x_1^a$$
$$v_1 = x_1^{b-a}x_2$$

and $A(\Phi_1, p) = 0 = A(\Phi, p)$.

Now suppose that $P(x) \neq 0$, so $d \leq b$. If $d \geq a$ then there exist $\alpha \in \mathbf{k}$ and strongly prepared parameters (u_1, v_1) at $\Phi_1(p)$ such that

$$u = u_1, \quad , v = u_1(v_1 + \alpha).$$

Thus

$$u_1 = x_1^a$$
$$v_1 = \frac{P(x_1)}{x_1^a} - \alpha + x_1^{b-a}x_2$$

and $A(\Phi_1, p) \leq (b-a) - (d-a) = b - d = A(\Phi, p)$, where the equality holds if and only if $d > a$. In this case $C(\Phi_1, p) = (b-d, d-a+a) = (b-d, d) < (b-d, d+a) = C(\Phi, p)$.

If $d < a$ then there exist strongly prepared parameters (u_1, v_1) at $\Phi_1(p)$ and strongly permissible parameters $(\bar{x}_1, \bar{x}_2, x_3, \ldots, x_n)$ at p such that

$$u = u_1 v_1, \quad v = u_1$$

and

$$u_1 = \bar{x}_1^d$$
$$v_1 = \frac{\bar{P}(\bar{x}_1)}{\bar{x}_1^d} + \bar{x}_1^{b+a-2d}\bar{x}_2, \text{ with } ord\bar{P} = a.$$

Thus $A(\Phi_1, p) = (b+a-2d) - (a-d) = b - d = A(\Phi, p)$ and $C(\Phi_1, p) = (b-d, a-d+d) = (b-d, a) < (b-d, d+a) = C(\Phi, p)$.

□

Suppose that $\Phi : X \to S$ is strongly prepared. We will denote by $Z(\Phi)$ the locus of bad points in X. If $q \in D_S$ denote by $N_q(\Phi)$ the locus of points in X where Φ does not factor through the blowup of q. Then $N(\Phi)$ will denote the union of $N_q(\Phi)$ for all $q \in D_S$.

We will denote by $B_2(X)$ the set of all 2 points in X. Let $\bar{B}_2(X)$ be the Zariski closure of $B_2(X)$. We will also say that a codimension 2 subvariety $C \subset X$ is a 2-variety if $C = E_1 \cap E_2$ for some components E_1 and E_2 of E_X.

Remark 3.15. $\bar{B}_2(X)$ is the union of all 2-varieties on X.

Lemma 3.16. *Suppose that $\Phi : X \to S$ is strongly prepared and $q \in D_S$ is such that $Z(\Phi) \cap N_q(\Phi) \neq \emptyset$. Then $Z(\Phi) \cap N_q(\Phi)$ is a Zariski closed set of pure codimension 2, consisting of the union of all 2-varieties in $\Phi^{-1}(q)$ with a generic point in the form of (N.IIb).*

Suppose that C is a component of $Z(\Phi) \cap N_q(\Phi)$ and $\pi : X_1 \to X$ is the blowup of C with exceptional variety $E = \pi^{-1}(C)_{red}$. Then $\Phi_1 = \Phi \circ \pi$ is strongly prepared and $A(\Phi_1, E) < A(\Phi)$.

Proof. $Z(\Phi)$ and $N_q(\Phi)$ are both closed, so to prove the first statement of the theorem it suffices to show that any bad point $p \in N_q(\Phi)$ lies on a 2-variety C such that a generic point $p' \in C$ is in the form of (N.IIb) and $p' \in \Phi^{-1}(q)$. Notice also that if $p \in Z(\Phi) \cap N_q(\Phi)$ then either (N.Ic) or (N.IIb) holds at p.

Suppose that $p \in N_q(\Phi)$ is a k point and (N.Ic) holds at p, (u, v) are strongly prepared parameters at q and $(x_1, \ldots, x_n)$ are strongly permissible parameters at p for (u, v), then

$$
\begin{aligned}
u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\
v &= P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} x_{k+1}.
\end{aligned}
$$

After possibly permuting $x_1, \ldots, x_k$ we can assume that $\frac{b_1}{a_1} < ordP < \frac{b_2}{a_2}$ and $\frac{b_1}{a_1} < m$.

Suppose that $E_1, \ldots, E_k$ are the components of E_X containing p with local equations $x_1 = 0, \ldots, x_k = 0$ respectively. Let U be an étale neighborhood of p where $(x_1, \ldots, x_n)$ are uniformizing parameters.

Set $C = E_1 \cap E_2$ and fix a 2 point $p' \in U \cap C$ away from the vanishing locus of x_{k+1}. Then there exist $\alpha_3, \ldots, \alpha_{k+1} \in \mathbf{k} - \{0\}$ and $\alpha_{k+2}, \ldots, \alpha_n \in \mathbf{k}$ such that $(x_1, x_2, \bar{x}_3 = x_3 - \alpha_3, \bar{x}_4 = x_4 - \alpha_4, \ldots, \bar{x}_n = x_n - \alpha_n)$ are regular parameters at p'.

Since $f = a_1 b_2 - a_2 b_1 \neq 0$ we can set $\gamma = ((\bar{x}_3 + \alpha_3)^{a_3} \cdots (\bar{x}_k + \alpha_k)^{a_k})^{\frac{1}{f}}$ and $\omega = ((\bar{x}_3 + \alpha_3)^{b_3} \cdots (\bar{x}_k + \alpha_k)^{b_k} (\bar{x}_{k+1} + \alpha_{k+1}))^{\frac{1}{f}}$. Then $\gamma^f, \omega^f \in \mathcal{O}_{U,p'}$ are units in $\mathcal{O}_{U,p'}$ and, therefore, $\mathcal{O}_{U,p'}[\gamma, \omega]$ is finite étale over $\mathcal{O}_{U,p'}$.

Set $\bar{x}_1 = \gamma^{b_2} \omega^{-a_2} x_1$ and $\bar{x}_2 = \gamma^{-b_1} \omega^{a_1} x_2$ so that $(\bar{x}_1, \ldots, \bar{x}_n)$ are strongly permissible parameters at p' for strongly prepared parameters (u, v) at q and

$$
\begin{aligned}
u &= (\bar{x}_1^{a_1} \bar{x}_2^{a_2})^m \\
v &= P(\bar{x}_1^{a_1} \bar{x}_2^{a_2}) + \bar{x}_1^{b_1} \bar{x}_2^{b_2}.
\end{aligned}
$$

Thus (N.IIb) holds at a generic point p' of C and $\Phi(p') = q$.

If $\pi : X_1 \to X$ is the blow up of C then $\Phi_1 = \Phi \circ \pi$ is strongly prepared above p. If p lies in the intersection of more than 2 components $E_1, \ldots, E_k$ of E_X then $\pi^{-1}(p)$ does not contain any 1 point. Assume that p is a 2 point. If $s \in \pi^{-1}(p)$ is a 1 point then $(x_1, \bar{x}_2, x_3, \ldots, x_n)$ are strongly permissible parameters at s where $\bar{x}_2$ is defined by $x_2 = x_1(\bar{x}_2 + \alpha)$ for some nonzero $\alpha \in \mathbf{k}$.

After setting $\bar{x}_1 = x_1(\bar{x}_2 + \alpha)^{\frac{a_2}{a_1+a_2}}$ and $\bar{x}_3 = x_3(\bar{x}_2 + \alpha)^{\frac{f}{a_1+a_2}}$ the following equalities hold

$$u = (x_1^{a_1} x_2^{a_2})^m = \bar{x}_1^{(a_1+a_2)m}$$
$$v = P(x_1^{a_1} x_2^{a_2}) + x_1^{b_1} x_2^{b_2} x_3 = P(\bar{x}_1^{a_1+a_2}) + \bar{x}_1^{b_1+b_2} \bar{x}_3.$$

If $(a_1+a_2) ordP \geq (b_1+b_2)$ then s is a good point, $A(\Phi_1, E) = A(\Phi_1, s) = 0$ and $A(\Phi) > 0$ since the locus of bad points $Z(\Phi)$ is not empty. So, assume that $(a_1 + a_2) ordP < (b_1 + b_2)$. Since $b_1 - a_1 ordP < 0$

$$A(\Phi_1, E) = A(\Phi_1, s) = b_1 + b_2 - (a_1 + a_2) ordP < b_2 - a_2 ordP =$$
$$= A(\Phi, E_2) \leq A(\Phi).$$

Suppose that $p \in N_q(\Phi)$ is a k point and (N.IIb) holds at p, (u, v) are strongly prepared parameters at q and $(x_1, \ldots, x_n)$ are strongly permissible parameters at p for (u, v), then

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k}.$$

After possibly permuting $x_1, \ldots, x_k$ we can assume that $\frac{b_1}{a_1} < ordP < \frac{b_2}{a_2}$ and $\frac{b_1}{a_1} < m$.

Suppose that $E_1, \ldots, E_k$ are the components of E_X containing p with local equations $x_1 = 0, \ldots, x_k = 0$ respectively. Let U be an étale neighborhood of p where $(x_1, \ldots, x_n)$ are uniformizing parameters.

Set $C = E_1 \cap E_2$ and fix a 2 point $p' \in U \cap C$. Then there exist $\alpha_3, \ldots, \alpha_k \in \mathbf{k} - \{0\}$ and $\alpha_{k+1}, \ldots, \alpha_n \in \mathbf{k}$ such that $(x_1, x_2, \bar{x}_3 = x_3 - \alpha_3, \bar{x}_4 = x_4 - \alpha_4, \ldots, \bar{x}_n = x_n - \alpha_n)$ are regular parameters at p'.

Since $f = a_1 b_2 - a_2 b_1 \neq 0$ we can set $\gamma = ((\bar{x}_3 + \alpha_3)^{a_3} \cdots (\bar{x}_k + \alpha_k)^{a_k})^{\frac{1}{f}}$ and $\omega = ((\bar{x}_3 + \alpha_3)^{b_3} \cdots (\bar{x}_k + \alpha_k)^{b_k})^{\frac{1}{f}}$. Then $\gamma^f, \omega^f \in \mathcal{O}_{U,p'}$ are units in $\mathcal{O}_{U,p'}$ and, therefore, $\mathcal{O}_{U,p'}[\gamma, \omega]$ is finite étale over $\mathcal{O}_{U,p'}$.

Set $\bar{x}_1 = \gamma^{b_2}\omega^{-a_2}x_1$ and $\bar{x}_2 = \gamma^{-b_1}\omega^{a_1}x_2$ so that $(\bar{x}_1, \ldots, \bar{x}_n)$ are strongly permissible parameters at p' for strongly prepared parameters (u, v) at q and

$$\begin{aligned} u &= (\bar{x}_1^{a_1}\bar{x}_2^{a_2})^m \\ v &= P(\bar{x}_1^{a_1}\bar{x}_2^{a_2}) + \bar{x}_1^{b_1}\bar{x}_2^{b_2}. \end{aligned}$$

Thus (N.IIb) holds at a generic point p' of C and $\Phi(p) = q$.

If $\pi : X_1 \to X$ is the blow up of C then $\Phi_1 = \Phi \circ \pi$ is strongly prepared above p. If p lies in the intersection of more than 2 components $E_1, \ldots, E_k$ of E_X then $\pi^{-1}(p)$ does not contain any 1 point. Assume that p is a 2 point. If $s \in \pi^{-1}(p)$ is a 1 point then $(x_1, \bar{x}_2, x_3, \ldots, x_n)$ are strongly permissible parameters at s where $\bar{x}_2$ is defined by $x_2 = x_1(\bar{x}_2 + \alpha)$ for some nonzero $\alpha \in \mathbf{k}$.

After setting $\bar{x}_1 = x_1(\bar{x}_2 + \alpha)^{\frac{a_2}{a_1+a_2}}$ and $\tilde{x}_2 = (\bar{x}_2 + \alpha)^{\frac{f}{a_1+a_2}} - \alpha^{\frac{f}{a_1+a_2}}$ the following equalities hold

$$\begin{aligned} u &= (x_1^{a_1}x_2^{a_2})^m = \bar{x}_1^{(a_1+a_2)m} \\ v &= P(x_1^{a_1}x_2^{a_2}) + x_1^{b_1}x_2^{b_2} = P(\bar{x}_1^{a_1+a_2}) + \bar{x}_1^{b_1+b_2}\tilde{x}_2 + \alpha^{\frac{f}{a_1+a_2}}\bar{x}_1^{b_1+b_2}. \end{aligned}$$

If $(a_1 + a_2)ordP \geq (b_1 + b_2)$ then s is a good point and $A(\Phi_1, E) = A(\Phi_1, s) = 0 < A(\Phi)$. So, assume that $(a_1 + a_2)ordP < (b_1 + b_2)$.

Since $b_1 - a_1 ordP < 0$

$$\begin{aligned} A(\Phi_1, E) = A(\Phi_1, s) &= b_1 + b_2 - (a_1 + a_2)ordP < b_2 - a_2 ordP = \\ &= A(\Phi, E_2) \leq A(\Phi). \end{aligned}$$

□

Theorem 3.17. *Suppose that $\Phi : X \to S$ is strongly prepared and $Z(\Phi) \cap N(\Phi) \neq \varnothing$. Then $Z(\Phi) \cap N(\Phi)$ is a Zariski closed set of pure codimension 2, consisting of the union of all 2-varieties with a generic point in the form of (N.IIb).*

Suppose that C is a component of $Z(\Phi) \cap N(\Phi)$ and $\pi : X_1 \to X$ is the blowup of C with exceptional variety $E = \pi^{-1}(C)_{red}$. Then $\Phi_1 = \Phi \circ \pi$ is strongly prepared and $A(\Phi_1, E) < A(\Phi)$.

Proof. This theorem follows from Lemma 3.16 due to finiteness of the number of 2-varieties in X. □

Remark 3.18. With the notation of Lemma 3.16, $\Phi(Z(\Phi) \cap N_q(\Phi)) = \{q\}$ and each component of $Z(\Phi) \cap N_q(\Phi)$ is the intersection of a good component E_1 with a bad component E_2.

Lemma 3.19. *Suppose that $p \in Z(\Phi) \cap N_q(\Phi)$ is a 2 point and (u, v) are strongly prepared parameters at q, $(x_1, \ldots, x_n)$ are strongly permissible parameters at p for (u, v) such that (N.IIb) holds and*

$$u = (x_1^{a_1} x_2^{a_2})^m$$
$$v = P(x_1^{a_1} x_2^{a_2}) + x_1^{b_1} x_2^{b_2},$$

where $\frac{b_1}{a_1} < ordP < \frac{b_2}{a_2}$ and $d = ordP$. q is a 1-point.

Suppose that $(\bar{u}, \bar{v})$ are also strongly prepared parameters at q and $(y_1, \ldots, y_n)$ are strongly permissible parameters at p for $(\bar{u}, \bar{v})$ such that (N.IIb) holds and

$$\bar{u} = (y_1^{a'_1} y_2^{a'_2})^{m'}$$
$$\bar{v} = Q(y_1^{a'_1} y_2^{a'_2}) + y_1^{b'_1} y_2^{b'_2},$$

where $\frac{b'_1}{a'_1} < ordQ < \frac{b'_2}{a'_2}$ and $d' = ordQ$.

Then $a_1 = a'_1$, $a_2 = a'_2$, $b_1 = b'_1$, $b_2 = b'_2$, $d = d'$, $m = m'$.

Proof. In order to decide whether q is a 1 or 2 point we will compare the varieties given by local equations $u = 0$ and $uv = 0$ on X. According to the assumption on d, uv can be presented as

$$uv = x_1^{a_1 m + b_1} x_2^{a_2 m + a_2 d}(\alpha x_1^{a_1 d - b_1} + x_2^{b_2 - a_2 d} + x_1 x_2 \frac{P(x_1^{a_1} x_2^{a_2}) - \alpha (x_1^{a_1} x_2^{a_2})^d}{x_1^{a_1 d + 1} x_2^{a_2 d + 1}}),$$

where $0 \neq \alpha \in \mathbf{k}$, $x_1^{a_1 d - b_1}$, $x_2^{b_2 - a_2 d} \in m_p \hat{\mathcal{O}}_{X,p}$, $\frac{P(x_1^{a_1} x_2^{a_2}) - (x_1^{a_1} x_2^{a_2})^d}{x_1^{a_1 d + 1} x_2^{a_2 d + 1}} \in \hat{\mathcal{O}}_{X,p}$. Thus $uv = 0$ defines a variety with at least 3 irreducible components at the 2 point p. Therefore $uv = 0$ cannot be a local equation of D_S. So q is a 1 point.

This implies that every permissible change of coordinates at q will translate u into αu for some unit series $\alpha \in \hat{\mathcal{O}}_{X,p}$. Thus

$$(x_1^{a_1} x_2^{a_2})^m = \alpha (y_1^{a'_1} y_2^{a'_2})^{m'}, \text{ where } \alpha \text{ is a unit series.}$$

The powers of irreducible factors on the left hand side $a_1 m$ and $a_2 m$ are equal to the powers of irreducible factors on the right hand side $a'_1 m'$ and

$a_2'm'$, possibly in reverse order. And since $(a_1, a_2) = 1$ and $(a_1', a_2') = 1$ we can claim that $m = m'$ and $\{a_1, a_2\} = \{a_1', a_2'\}$.

Denote by E_1 and E_2 the components of E_X containing p with local equations $x_1 = 0$ and $x_2 = 0$ respectively. Then by Lemma 3.10 $A(\Phi, E_1) = 0$ and $A(\Phi, E_2) = b_2 - a_2 ordP > 0$. So E_1 is a good component while E_2 is a bad component.

Since $\bar{u} = 0$ is a local equation of E_X, $y_1 = 0$ and $y_2 = 0$ are local equations of E_1 and E_2, possibly in reverse order. Then by Lemma 3.10 the invariant A of the component of E_X with local equation $y_1 = 0$ is equal to 0. So $y_1 = 0$ is a local equation of the good component E_1, while $y_2 = 0$ is a local equation of E_2. From here and equality of the sets $\{a_1, a_2\}$ and $\{a_1', a_2'\}$ it follows that $a_1 = a_1'$ and $a_2 = a_2'$.

Suppose that U is an étale neighborhood of p where $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are uniformizing parameters. Fix a 1 point $p' \in U \cap E_2$. Following the proof of Lemma 3.10 we can find strongly permissible parameters $(\bar{x}_1, \ldots, \bar{x}_n)$ and strongly permissible parameters $(\bar{y}_1, \ldots, \bar{y}_n)$ at p' such that

$$\begin{aligned} u &= \bar{x}_2^{a_2 m} \\ v &= \bar{P}(\bar{x}_2) + \bar{x}_2^{b_2}\bar{x}_1, \text{ with } ord\bar{P} = a_2 d \end{aligned}$$

and

$$\begin{aligned} \bar{u} &= \bar{y}_2^{a_2' m'} \\ \bar{v} &= \bar{Q}(\bar{y}_2) + \bar{y}_2^{b_2'}\bar{y}_1, \text{ with } ord\bar{Q} = a_2' d'. \end{aligned}$$

So by Lemma 3.7 $b_2 = b_2'$, $a_2 d = a_2' d'$, and therefore, $d = d'$.

To show that $b_1 = b_1'$ we fix a 1 point $p' \in U \cap E_1$. Then following the proof of Lemma 3.10 we find strongly permissible parameters $(\bar{x}_1, \ldots, \bar{x}_n)$ and strongly permissible parameters $(\bar{y}_1, \ldots, \bar{y}_n)$ at p' such that

$$\begin{aligned} u &= \bar{x}_1^{a_1 m} \\ v &= \alpha_1 \bar{x}_1^{b_1} + \bar{x}_1^{b_1}\bar{x}_2, \text{ with } \alpha_1 \in \mathbf{k} \end{aligned}$$

and

$$\begin{aligned} \bar{u} &= \bar{y}_1^{a_1' m} \\ \bar{v} &= \alpha_2 \bar{y}_1^{b_1} + \bar{y}_1^{b_1'}\bar{y}_1, \text{ with } \alpha_2 \in \mathbf{k}. \end{aligned}$$

So by Lemma 3.7 $b_1 = b_1'$. ☐

If α, β are real numbers, define

$$S(\alpha, \beta) = \max\{(\alpha, \beta), (\beta, \alpha)\}$$

where the maximum is in the Lexicographic ordering.

Definition 3.20. Suppose that $\Phi : X \to S$ is strongly prepared and $p \in E_X$ is a 2 point such that (II) holds at p. Define

$$\sigma(p) = \begin{cases} S(|b_1 - a_1 ordP|, |b_2 - a_2 ordP|), & \text{if } p \in N(\Phi) \cap Z(\Phi); \\ 0, & \text{otherwise.} \end{cases}$$

If C is a 2-variety in E_X containing a 2 point p in the form of (II), set $\sigma(C) = \sigma(p)$. Set $\sigma(C) = 0$, otherwise.

Finally, define $\sigma(\Phi) = \max\{\sigma(C) | C \subset E_X \text{ is a 2-variety}\}$.

Remark 3.21. In view of Lemmas 3.16 and 3.19, at every 2 point $p \in N(\Phi) \cap Z(\Phi)$, where (II) holds, $b_1 - a_1 ordP$ and $b_2 - a_2 ordP$ are independent of the choice of strongly prepared parameters (u, v) at $\Phi(p)$ and they are also independent of the choice of strongly permissible parameters for (u, v) at p. So, $\sigma(p)$ is well defined at every 2 point $p \in E_X$ in the form of (II). To justify the definition of $\sigma(C)$ for a 2-variety C we will prove the following

Lemma 3.22. *Suppose that $p \in E_X$ is a 2 point in the form of (N.IIb) and C is a 2-variety containing p.*

Then there exists an open neighborhood U of p such that $\sigma(p) = \sigma(p')$ for all $p' \in U \cap C$.

Proof. There exist strongly prepared parameters (u, v) at $\Phi(p)$ and strongly permissible parameters $(x_1, \ldots, x_n)$ at p such that

$$\begin{aligned} u &= (x_1^{a_1} x_2^{a_2})^m \\ v &= P(x_1^{a_1} x_2^{a_2}) + x_1^{b_1} x_2^{b_2} \end{aligned} \qquad (*)$$

and $\frac{b_1}{a_1} < d = ordP < \frac{b_2}{a_2}$, $\frac{b_1}{a_1} < m$.

Let U be an étale neighborhood of p where $(x_1, \ldots, x_n)$ are uniformizing parameters. Since $x_1 = x_2 = 0$ are local equations of C, for any $p' \in U \cap C$ there exist $\alpha_3, \ldots, \alpha_n \in \mathbf{k}$ such that $(x_1, x_2, \bar{x}_3 = x_3 + \alpha_3, \ldots, \bar{x}_n = x_n + \alpha_n)$ are strongly permissible parameters at p' for strongly prepared parameters (u, v) at $\Phi(p')$. Then the same equations $(*)$ hold at p' and, therefore, $\sigma(p') = S(|b_1 - a_1 ordP|, |b_2 - a_2 ordP|) = \sigma(p)$.

□

Theorem 3.23. *Suppose that $\Phi : X \to S$ is strongly prepared. Then there exists a sequence of blowups of 2-varieties $X_1 \to X$ such that the induced map $\Phi_1 : X_1 \to S$ is strongly prepared, $A(\Phi_1, E) < A(\Phi_1) = A(\Phi)$ if E is an exceptional component of E_{X_1} for $X_1 \to X$ and $Z(\Phi_1) \cap N(\Phi_1) = \varnothing$.*

Proof. $Z(\Phi) \cap N(\Phi) = \varnothing$ if and only if $\sigma(\Phi) = 0$.

Suppose that $\sigma(\Phi) > 0$ and $C \subset Z(\Phi) \cap N(\Phi)$ is a 2-variety such that $\sigma(C) = \sigma(\Phi)$. Let $\pi : X_1 \to X$ be the blowup of C. Then by Theorem 3.17 $\Phi_1 = \Phi \circ \pi$ is strongly prepared and $A(\Phi_1, E) < A(\Phi)$, so we will show that at every 2 point $s \in \pi^{-1}(C)$ in the form of (II) $\sigma(s) < \sigma(\Phi)$.

Suppose that $p \in C$ is a k point and (N.Ic) holds at p, (u, v) are strongly prepared parameters at $\Phi(p)$ and $(x_1, \ldots, x_n)$ are strongly permissible parameters at p for (u, v) such that

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k} x_{k+1}$$

and $x_1 = x_2 = 0$ are local equations of C. Then there will not be any 2 point in the form of (II) in $\pi^{-1}(p)$.

Suppose that $p \in C$ is a k point and (N.IIb) holds at p, (u, v) are strongly prepared parameters at $\Phi(p)$ and $(x_1, \ldots, x_n)$ are strongly permissible parameters at p for (u, v) such that

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = P(x_1^{a_1} \cdots x_k^{a_k}) + x_1^{b_1} \cdots x_k^{b_k}$$

and $x_1 = x_2 = 0$ are local equations of C.

Then after possibly permuting x_1 and x_2 we can assume that $\frac{b_1}{a_1} < ordP < \frac{b_2}{a_2}$ and $\frac{b_1}{a_1} < m$.

Suppose that $E_1, \ldots, E_k$ are the components of E_X containing p with local equations $x_1 = 0, \ldots, x_k = 0$ respectively. If p lies in the intersection of more than 3 components $E_1, \ldots, E_k$ of E_X there will not be any 2 point in $\pi^{-1}(p)$.

Assume first that p is a 2 point with

$$u = (x_1^{a_1} x_2^{a_2})^m$$
$$v = P(x_1^{a_1} x_2^{a_2}) + x_1^{b_1} x_2^{b_2} \text{ and } d = ordP.$$

Then $\sigma(\Phi) = \sigma(C) = \sigma(p) = S(a_1 d - b_1, b_2 - a_2 d)$.

Suppose that a 2 point $s \in \pi^{-1}(p)$ has $*$-permissible parameters $(x_1, \bar{x}_2, x_3, \ldots, x_n)$ such that $x_2 = x_1\bar{x}_2$, then

$$u = (x_1^{a_1+a_2}\bar{x}_2^{a_2})^m$$
$$v = P(x_1^{a_1+a_2}\bar{x}_2^{a_2}) + x_1^{b_1+b_2}\bar{x}_2^{b_2}.$$

Since $d < \frac{b_2}{a_2}$, following the proof of Lemma 3.4 we can find strongly permissible parameters $(y_1, \ldots, y_n)$ at s such that

$$u = (y_1^{a_1+a_2}y_2^{a_2})^m$$
$$v = P(y_1^{a_1+a_2}y_2^{a_2}) + y_1^{b_1+b_2}y_2^{b_2}.$$

Then $\sigma(s) > 0$ if and only if $(u, v)\mathcal{O}_{X_1,s}$ is not invertible and (N.IIb) holds at p. Let it be the case, then $d > \frac{b_1+b_2}{a_1+a_2}$ and $(a_1 + a_2)d - (b_1 + b_2) < a_1d - b_1$ since $a_2d - b_2 < 0$. Thus

$$\sigma(s) = S((a_1 + a_2)d - (b_1 + b_2), b_2 - a_2d) < S(a_1d - b_1, b_2 - a_2d) = \sigma(\Phi).$$

Suppose that a 2 point $s \in \pi^{-1}(p)$ has $*$-permissible parameters $(\bar{x}_1, x_2, x_3, \ldots, x_n)$ such that $x_1 = \bar{x}_1x_2$ then

$$u = (\bar{x}_1^{a_1}x_2^{a_1+a_2})^m$$
$$v = P(\bar{x}_1^{a_1}x_2^{a_1+a_2}) + \bar{x}_1^{b_1}x_2^{b_1+b_2}.$$

If $d \geq \frac{b_1+b_2}{a_1+a_2}$ there exist strongly permissible parameters $(y_1, \ldots, y_n)$ at s such that

$$u = (y_1^{a_1}y_2^{a_1+a_2})^m$$
$$v = y_1^{b_1}y_2^{b_1+b_2}.$$

Thus $\sigma(s) = 0$ in this case.

Assume that $d < \frac{b_1+b_2}{a_1+a_2}$, then following the proof of Lemma 3.4 we can find strongly permissible parameters $(y_1, \ldots, y_n)$ at s such that

$$u = (y_1^{a_1}y_2^{a_1+a_2})^m$$
$$v = P(y_1^{a_1}y_2^{a_1+a_2}) + y_1^{b_1}y_2^{b_1+b_2}.$$

Suppose that $\sigma(s) > 0$, that is (N.IIb) holds at s. Then since $b_1 - a_1d < 0$, $(b_1 + b_2) - (a_1 + a_2)d < b_2 - a_2d$ and

$$\sigma(s) = S(a_1d - b_1, (b_1 + b_2) - (a_1 + a_2)d) < S(a_1d - b_1, b_2 - a_2d) = \sigma(\Phi).$$

Assume now that p is a 3 point with

$$u = (x_1^{a_1}x_2^{a_2}x_3^{a_3})^m$$
$$v = P(x_1^{a_1}x_2^{a_2}x_3^{a_3}) + x_1^{b_1}x_2^{b_2}x_3^{b_3} \text{ and } d = ordP.$$

Suppose that $s \in \pi^{-1}(p)$ is a 2 point, then s has regular parameters $(x_1, \bar{x}_2, x_3, \ldots, x_n)$ defined by $x_2 = x_1(\bar{x}_2 + \alpha)$ for some nonzero $\alpha \in \mathbf{k}$ and

$$u = (x_1^{a_1+a_2}(\bar{x}_2 + \alpha)^{a_2}x_3^{a_3})^m$$
$$v = P(x_1^{a_1+a_2}(\bar{x}_2 + \alpha)^{a_2}x_3^{a_3}) + x_1^{b_1+b_2}(\bar{x}_2 + \alpha)^{b_2}x_3^{b_3}.$$

If $rank \begin{pmatrix} a_1 + a_2 & a_3 \\ b_1 + b_2 & b_3 \end{pmatrix} < 2$ then $\sigma(s) = 0$ since (II) cannot hold at s. So, consider the case when $rank \begin{pmatrix} a_1 + a_2 & a_3 \\ b_1 + b_2 & b_3 \end{pmatrix} = 2$.

Set $h = (a_1 + a_2)b_3 - a_3(b_1 + b_2)$ and

$$\bar{x}_1 = x_1(\bar{x}_2 + \alpha)^{\frac{a_2b_3 - a_3b_2}{h}}$$
$$\bar{x}_3 = x_3(\bar{x}_2 + \alpha)^{\frac{a_1b_2 - a_2b_1}{h}}$$

to get $*$-permissible parameters $(\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \ldots, x_n)$ at s with

$$u = (\bar{x}_1^{a_1+a_2}\bar{x}_3^{a_3})^m$$
$$v = P(\bar{x}_1^{a_1+a_2}\bar{x}_3^{a_3}) + \bar{x}_1^{b_1+b_2}\bar{x}_3^{b_3}.$$

If $d \geq \max\{\frac{b_1+b_2}{a_1+a_2}, \frac{b_3}{a_3}\}$ there exist strongly permissible parameters $(y_1, \ldots, y_n)$ at s such that

$$u = (y_1^{a_1+a_2}y_3^{a_3})^m$$
$$v = y_1^{b_1+b_2}y_3^{b_3}.$$

Thus $\sigma(s) = 0$ in this case.

Assume that $d < \max\{\frac{b_1+b_2}{a_1+a_2}, \frac{b_3}{a_3}\}$, then following the proof of Lemma 3.4 we can find strongly permissible parameters $(y_1, \ldots, y_n)$ at s such that

$$u = (y_1^{a_1+a_2}y_3^{a_3})^m$$
$$v = P(y_1^{a_1+a_2}y_3^{a_3}) + y_1^{b_1+b_2}y_3^{b_3}.$$

Suppose that $\sigma(s) > 0$, so (N.IIb) holds at s and $\min\{\frac{b_1+b_2}{a_1+a_2}, \frac{b_3}{a_3}\} < d < \max\{\frac{b_1+b_2}{a_1+a_2}, \frac{b_3}{a_3}\}$.

If $\frac{b_1+b_2}{a_1+a_2} < d < \frac{b_3}{a_3}$ then according to the proof of Lemma 3.16 the 2-variety $E_1 \cap E_3$ lies in $Z(\Phi) \cap N(\Phi)$ and $\sigma(E_1 \cap E_3) = S(a_1 d - b_1, b_3 - a_3 d)$. Thus since $a_2 d - b_2 < 0$, $(a_1 + a_2)d - (b_1 + b_2) < a_1 d - b_1$ and

$$\begin{aligned}\sigma(s) &= S((a_1 + a_2)d - (b_1 + b_2), b_3 - a_3 d) < S(a_1 d - b_1, b_3 - a_3 d) = \\ &= \sigma(E_1 \cap E_3) \leq \sigma(\Phi).\end{aligned}$$

If $\frac{b_3}{a_3} < d < \frac{b_1+b_2}{a_1+a_2}$ then notice that $\sigma(s) > 0$ implies that $\frac{b_3}{a_3} < m$. So according to the proof of Lemma 3.16 the 2-variety $E_2 \cap E_3$ lies in $Z(\Phi) \cap N(\Phi)$ and $\sigma(E_2 \cap E_3) = S(a_3 d - b_3, b_2 - a_2 d)$. Thus since $b_1 - a_1 d < 0$, $(b_1 + b_2) - (a_1 + a_2)d < b_2 - a_2 d$ and

$$\begin{aligned}\sigma(s) &= S(a_3 d - b_3, (b_1 + b_2) - (a_1 + a_2)d) < S(a_3 d - b_3, b_2 - a_2 d) = \\ &= \sigma(E_2 \cap E_3) \leq \sigma(\Phi).\end{aligned}$$

By Theorem 3.17, induction on the number of 2-varieties $C \in X$ with $\sigma(C) = \sigma(\Phi)$ and induction on $\sigma(\Phi)$ we achieve the conclusions of the theorem. □

Theorem 3.24. *Suppose that $\Phi : X \to S$ is strongly prepared and $q \in S$. Suppose also that $N(\Phi)$ does not contain any bad point. If $N_q(\Phi) \neq \varnothing$ then $N_q(\Phi)$ is a pure codimension 2 subscheme which makes SNCs with $\bar{B}_2(X)$.*

Suppose that C is a component of $N_q(\Phi)$ and $\pi : X_1 \to X$ is the blowup of C, $E = \pi^{-1}(C)_{red}$ and $\Phi_1 = \Phi \circ \pi$. Then Φ_1 is strongly prepared, $Z(\Phi_1) \cap N(\Phi_1) = \varnothing$ and $A(\Phi_1, E) = 0$.

Proof. Suppose that $p \in N_q(\Phi)$ is a k point, (u, v) are strongly prepared parameters at q and $(x_1, \ldots, x_n)$ are strongly permissible parameters for (u, v) at p. Let U be an étale neighborhood of p where $(x_1, \ldots, x_n)$ are uniformizing parameters.

Denote by $E_1, \ldots, E_k$ the components of E_X containing p with local equations $x_1 = 0, \ldots, x_k = 0$, respectively.

The assumption that $N(\Phi)$ does not contain bad points implies that there is no point of the form (N.Ic) or (N.IIb) in $N_q(\Phi)$.

Suppose that (N.Ia) holds at p :

$$\begin{aligned}u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\ v &= (x_1^{a_1} \cdots x_k^{a_k})^t x_{k+1}.\end{aligned}$$

If $p' \in E_1 \cap U$ then there exist $\alpha_2, \ldots, \alpha_n \in \mathbf{k}$ such that $(x_1, \bar{x}_2 = x_2 - \alpha_2, \ldots, \bar{x}_n = x_n - \alpha_n)$ are regular parameters at p' and

$$u = (x_1^{a_1}(\bar{x}_2 + \alpha_2)^{a_2} \cdots (\bar{x}_k + \alpha_k)^{a_k})^m$$
$$v = (x_1^{a_1}(\bar{x}_2 + \alpha_2)^{a_2} \cdots (\bar{x}_k + \alpha_k)^{a_k})^t(\bar{x}_{k+1} + \alpha_{k+1}).$$

From here it follows that $m_q\mathcal{O}_{X,p'}$ is not invertible if and only if $(\bar{x}_{k+1}+\alpha_{k+1})$ is not a unit, i.e. if $\alpha_{k+1} = 0$. Thus $N_q(\Phi) \cap E_1 \cap U = V(x_1, x_{k+1})$ and by symmetry $N_q(\Phi) \cap U = V(x_1, x_{k+1}) \cup V(x_2, x_{k+1}) \cup \cdots \cup V(x_k, x_{k+1})$. So, in the neighborhood of p $N_q(\Phi)$ is a union of codimension 2 varieties which make SNCs with $\bar{B}_2$.

Let $\pi : X_1 \to X$ be the blowup of a component C of $N_q(\Phi)$ passing through p, then $\Phi_1 = \Phi \circ \pi$ is strongly prepared above p. If p lies in more than 1 component of E_X, $\pi^{-1}(p)$ does not contain any 1 point.

Assume that p is a 1 point and the equations $u = x_1^{a_1 m}$, $v = x_1^{a_1 t}x_2$ hold at p. Then C is defined by $x_1 = x_2 = 0$ in the neighborhood of p.

If $s \in \pi^{-1}(p)$ is a 1 point then s has $*$-permissible parameters $(x_1, \bar{x}_2, x_3, \ldots, x_n)$, where $x_2 = x_1(\bar{x}_2 + \alpha)$ for some $\alpha \in \mathbf{k}$, and

$$u = x_1^{a_1 m}$$
$$v = \alpha x_1^{a_1 t+1} + x_1^{a_1 t+1}\bar{x}_2.$$

Thus s is a good point and $A(\Phi_1, E) = A(\Phi, s) = 0$.

Finally, notice that $Z(\Phi_1) \cap N(\Phi_1) \subset (\pi^{-1}(N(\Phi)) \cup E) \cap (Z(\Phi_1)) \subset \pi^{-1}(N(\Phi) \cap Z(\Phi))$, since E is a good component of E_{X_1}. So $Z(\Phi_1) \cap N(\Phi_1) = \varnothing$.

Suppose that (N.Ib) holds at p :

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = x_1^{b_1} \cdots x_k^{b_k} x_{k+1}.$$

If $p' \in E_1 \cap U$ then there exist $\alpha_2, \ldots, \alpha_n \in \mathbf{k}$ such that $(x_1, \bar{x}_2 = x_2 - \alpha_2, \ldots, \bar{x}_n = x_n - \alpha_n)$ are regular parameters at p' and

$$u = (x_1^{a_1}(\bar{x}_2 + \alpha_2)^{a_2} \cdots (\bar{x}_k + \alpha_k)^{a_k})^m$$
$$v = x_1^{b_1}(\bar{x}_2 + \alpha_2)^{b_2} \cdots (\bar{x}_k + \alpha_k)^{b_k}(\bar{x}_{k+1} + \alpha_{k+1}).$$

Thus $m_q\mathcal{O}_{X,p'}$ is not invertible if and only if at least one of the following holds:

1) $(\bar{x}_{k+1}+\alpha_{k+1})$ is not a unit and $a_i m > b_i$ for some i such that $\alpha_i \neq 0$.

2) $(\bar{x}_j+\alpha_j)$ is not a unit and $\min\{\frac{b_1}{a_1}, \frac{b_j}{a_j}\} < m < \max\{\frac{b_1}{a_1}, \frac{b_j}{a_j}\}$ for some $j \in \{2, \ldots, k\}$.

Fix $i \in \{1, \ldots, k\}$ and denote by J_i the set of all j which satisfy the inequality $\min\{\frac{b_i}{a_i}, \frac{b_j}{a_j}\} < m < \max\{\frac{b_i}{a_i}, \frac{b_j}{a_j}\}$, then

$$N_q(\Phi) \cap E_1 \cap U = \begin{cases} \bigcup_{j \in J_1} V(x_1, x_j) \cup V(x_1, x_{k+1}) & \text{if } b_1 < a_1 m \\ \bigcup_{j \in J_1} V(x_1, x_j) & \text{if } b_1 \geq a_1 m \end{cases}$$

and by symmetry

$$N_q(\Phi) \cap U = \left(\bigcup_{i=1}^{k} \bigcup_{j \in J_i} V(x_i, x_j) \right) \cup \left(\bigcup_{i \in I} V(x_i, x_{k+1}) \right)$$

with $I = \{i \mid b_i < a_i m\}$. So, in the neighborhood of p $N_q(\Phi)$ is a union of codimension 2 varieties which make SNCs with $\bar{B}_2$.

Let $\pi : X_1 \to X$ be the blowup of a component C of $N_q(\Phi)$ passing through p, then $\Phi_1 = \Phi \circ \pi$ is strongly prepared above p. If p lies in more than 2 components of E_X, $\pi^{-1}(p)$ does not contain any 1 point.

Suppose that p is a 2 point and

$$u = (x_1^{a_1} x_2^{a_2})^m$$
$$v = x_1^{b_1} x_2^{b_2} x_3, \text{ with } \frac{b_1}{a_1} < \frac{b_2}{a_2}.$$

If C is defined by $x_1 = x_3 = 0$ or $x_2 = x_3 = 0$ in the neighborhood of p then there is no 1 point in $\pi^{-1}(p)$. So we may assume that $\frac{b_1}{a_1} < m < \frac{b_2}{a_2}$ and $C = V(x_1, x_2)$.

If $s \in \pi^{-1}(p)$ is a 1 point then $\hat{\mathcal{O}}_{X_1,s}$ has regular parameters $(x_1, \bar{x}_2, x_3, \ldots, x_n)$, where $x_2 = x_1(\bar{x}_2 + \alpha)$ for some nonzero $\alpha \in \mathbf{k}$, and

$$u = x_1^{(a_1+a_2)m} (\bar{x}_2 + \alpha)^{a_2 m}$$
$$v = x_1^{b_1+b_2} (\bar{x}_2 + \alpha)^{b_2} x_3.$$

Set $\bar{x}_1 = x_1(\bar{x}_2+\alpha)^{\frac{a_2}{a_1+a_2}}$ and $\bar{x}_3 = x_3(\bar{x}_2+\alpha)^{\frac{a_1 b_2 - a_2 b_1}{a_1+a_2}}$, so that $(\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4 \cdots, x_n)$ are $*$-permissible parameters at p satisfying the equalities

$$u = \bar{x}_1^{(a_1+a_2)m}$$
$$v = \bar{x}_1^{b_1+b_2} \bar{x}_3.$$

Thus s is a good point and $A(\Phi_1, E) = A(\Phi, s) = 0$.

Arguing as above we also conclude that $Z(\Phi_1) \cap N(\Phi_1) = \varnothing$.

Suppose that (N.IIa) holds at p :

$$u = (x_1^{a_1} \cdots x_k^{a_k})^m$$
$$v = x_1^{b_1} \cdots x_k^{b_k}.$$

If $p' \in E_1 \cap U$ then there exist $\alpha_2, \ldots, \alpha_n \in \mathbf{k}$ such that $(x_1, \bar{x}_2 = x_2 - \alpha_2, \ldots, \bar{x}_n = x_n - \alpha_n)$ are regular parameters at p' and

$$u = (x_1^{a_1}(\bar{x}_2 + \alpha_2)^{a_2} \cdots (\bar{x}_k + \alpha_k)^{a_k})^m$$
$$v = x_1^{b_1}(\bar{x}_2 + \alpha_2)^{b_2} \cdots (\bar{x}_k + \alpha_k)^{b_k}.$$

Thus $m_q\mathcal{O}_{X,p'}$ is not invertible if and only if $(\bar{x}_j + \alpha_j)$ is not a unit and $\min\{\frac{b_1}{a_1}, \frac{b_j}{a_j}\} < m < \max\{\frac{b_1}{a_1}, \frac{b_j}{a_j}\}$ for some $j \in \{2, \ldots, k\}$.

Fix $i \in \{1, \ldots, k\}$ and denote by J_i the set of all j which satisfy the inequality $\min\{\frac{b_i}{a_i}, \frac{b_j}{a_j}\} < m < \max\{\frac{b_i}{a_i}, \frac{b_j}{a_j}\}$, then

$$N_q(\Phi) \cap E_1 \cap U = \bigcup_{j \in J_1} V(x_1, x_j)$$

and by symmetry

$$N_q(\Phi) \cap U = \bigcup_{i=1}^{k} \bigcup_{j \in J_i} V(x_i, x_j).$$

So, in the neighborhood of p $N_q(\Phi)$ is a union of 2- varieties, in particular $N_q(\Phi)$ makes SNCs with $\bar{B}_2$.

Let $\pi : X_1 \to X$ be the blowup of a component C of $N_q(\Phi)$ passing through p, then $\Phi_1 = \Phi \circ \pi$ is strongly prepared above p. If p lies in more than 2 components of E_X, $\pi^{-1}(p)$ does not contain any 1 point.

Assume that p is a 2 point and

$$u = (x_1^{a_1} x_2^{a_2})^m$$
$$v = x_1^{b_1} x_2^{b_2} \text{ with } \frac{b_1}{a_1} < m < \frac{b_2}{a_2}.$$

Then C is defined by $x_1 = x_2 = 0$ in the neighborhood of p.

If $s \in \pi^{-1}(p)$ is a 1 point then $\hat{\mathcal{O}}_{X_1,s}$ has regular parameters $(x_1, \bar{x}_2, x_3, \dots, x_n)$, where $x_2 = x_1(\bar{x}_2 + \alpha)$ for some nonzero $\alpha \in \mathbf{k}$, and

$$\begin{aligned} u &= x_1^{(a_1+a_2)m}(\bar{x}_2+\alpha)^{a_2 m} \\ v &= x_1^{b_1+b_2}(\bar{x}_2+\alpha)^{b_2}. \end{aligned} \tag{$**$}$$

Set $f = a_1b_2 - a_2b_1$, $\bar{x}_1 = x_1(\bar{x}_2+\alpha)^{\frac{a_2}{a_1+a_2}}$ and $\tilde{x}_2 = (\bar{x}_2+\alpha)^{\frac{f}{a_1+a_2}} - \alpha^{\frac{f}{a_1+a_2}}$. Then $(\bar{x}_1, \tilde{x}_2, \bar{x}_3, x_4 \cdots, x_n)$ are $*$-permissible parameters at p satisfying the equalities

$$\begin{aligned} u &= \bar{x}_1^{(a_1+a_2)m} \\ v &= \alpha^{\frac{f}{a_1+a_2}}\bar{x}_1^{b_1+b_2} + \bar{x}_1^{b_1+b_2}\tilde{x}_2. \end{aligned}$$

Thus s is a good point and $A(\Phi_1, E) = A(\Phi, s) = 0$.

Arguing as above we also conclude that $Z(\Phi_1) \cap N(\Phi_1) = \varnothing$.

Suppose that (N.III) holds at p :

$$\begin{aligned} u &= x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} \\ v &= x_2^{b_2} \cdots x_k^{b_k}. \end{aligned}$$

So if we set $m = (a_1, \dots, a_{k-1}) > 0$, $\bar{a}_i = a_i/m$ for all $i = 1, \dots, k-1$ and $\bar{a}_k = b_1 = 0$, we obtain the required statement by going through the analysis of the previous case. □

Remark 3.25. Suppose that C is a component of $N_q(\Phi)$ passing through a point $p \in E_X$, where p is a 2 point in the form of (N.IIa) or (N.III). Let $\pi : X_1 \to X$ be the blowup of C. Then formula $(**)$ shows that at every 1 point $s \in \pi^{-1}(p)$, $m_{\Phi(p)}\mathcal{O}_{X_1,s}$ is invertible. In particular, (N.Ia) cannot hold at s.

Moreover, since every 1 point $s \in \pi^{-1}(C)$ can only lie in $\pi^{-1}(p)$ for some 2 point $p \in C$, this implies that (N.Ia) does not hold at any 1 point $s \in \pi^{-1}(C)$.

Suppose that $\Phi : X \to S$ is strongly prepared and $q \in D_S$. Suppose that $p \in N_q(\Phi)$ is a 1 point, (u, v) are strongly prepared parameters at q and $(x_1, \dots, x_n)$ are strongly permissible parameters at p such that

$$\begin{aligned} u &= x_1^m \\ v &= x_1^t x_2. \end{aligned}$$

Let U be an étale neighborhood of p where $(x_1, \ldots, x_n)$ are uniformizing parameters.

Suppose that $C \subset N_q(\Phi)$ is a codimension 2 variety containing p. Then $C = V(x_1, x_2)$ in the neighborhood of p and for every point $p' \in C \cap U$ there exist $(\alpha_3, \ldots, \alpha_n) \in \mathbf{k}$ and strongly permissible parameters $(x_1, x_2, \bar{x}_3 = x_3 - \alpha_3, \ldots, \bar{x}_n = x_n - \alpha_n)$ such that

$$\begin{aligned} u &= x_1^m \\ v &= x_1^t x_2. \end{aligned}$$

For any 1 point $p \in N_q(\Phi)$ define $\Omega_q(p) = m - t > 0$. If $C \subset N_q(\Phi)$ is a codimension 2 variety define

$$\Omega_q(C) = \begin{cases} \Omega_q(p), & \text{if there exists a 1 point } p \in C; \\ 0, & \text{otherwise.} \end{cases}$$

Set $\Omega_q(\Phi) = \max\{\Omega_q(C) | C \text{ is a codimension 2 variety in } N_q(\Phi)\}$.

Remark 3.26. In view of Lemma 3.7 $\Omega_q(p)$ is well defined for every 1 point $p \in N_q(\Phi)$. If $C \subset N_q(\Phi)$ is a variety of codimension 2 $\Omega_q(C)$ is also well defined since $\Omega_q(p) = \Omega_q(p')$ for all 1 points p and p' in C.

Lemma 3.27. *Suppose that $\Phi : X \to S$ is strongly prepared and $q \in D_S$. Suppose also that $N(\Phi)$ does not contain any bad point.*

Then there exists a sequence of blowups of nonsingular varieties of codimension 2 $X_1 \to X$ which are not 2-varieties, such that the induced map $\Phi_1 : X_1 \to S$ is strongly prepared, $Z(\Phi_1) \cap N(\Phi_1) = \varnothing$, $A(\Phi_1, E) = 0$ if E is an exceptional component of E_{X_1} for $X_1 \to X$ and $N_q(\Phi_1)$ contains only 2-varieties.

Proof. $N_q(\Phi)$ contains only 2-varieties if and only if $\Omega_q(\Phi) = 0$.

Suppose that $\Omega_q(\Phi) > 0$ and $C \in N_q(\Phi)$ is a codimension 2 variety with $\Omega_q(C) = \Omega_q(\Phi)$. Let $\pi : X_1 \to X$ be the blowup of C. Then by Theorem 3.24 we only need to verify that $\Omega_q(s) < \Omega_q(\Phi)$ for every 1 point $s \in N_q(\Phi_1) \cap E$.

Suppose that $p \in C$ is such that there exist a 1 point $s \in \pi^{-1}(p)$. Then p is necessarily a 1 point. If (u, v) are strongly prepared parameters at q and $(x_1, \ldots, x_n)$ are strongly permissible parameters at p, then s has $*$-permissible parameters $(x_1, \bar{x}_2, x_3, \ldots, x_n)$ such that $x_2 = x_1(\bar{x}_2 + \alpha)$ for some $\alpha \in k$ and

$$\begin{aligned} u &= x_1^m \\ v &= x_1^{t+1}(\bar{x}_2 + \alpha). \end{aligned}$$

Thus $m_q\mathcal{O}_{X_1,s}$ is not invertible if and only if $\alpha = 0$ and in this case

$$\Omega_q(s) = m - (t+1) < m - t = \Omega_q(p) = \Omega_q(C) = \Omega_q(\Phi).$$

By induction on the number of codimension 2 varieties in $N_q(\Phi)$ such that $\Omega_q(\Phi) = \Omega_q(C)$ and induction on $\Omega_q(\Phi)$ we achieve the conclusions of the lemma. $\square$

Definition 3.28. Suppose that $\Phi : X \to S$ is strongly prepared and $q \in S$. Suppose also that $p \in \Phi^{-1}(q)$ is a 2 point. Define

$$\omega(p) = \begin{cases} S(|b_1 - a_1 m|, |b_2 - a_2 m|), & \text{if (N.IIa) holds at } p; \\ S(a_1, b_2), & \text{if (N.III) holds at } p; \\ 0, & \text{otherwise.} \end{cases}$$

If C is a 2-variety in $\Phi^{-1}(q)$ containing a 2 point p in one of the forms (N.IIa) or (N.III), set $\omega(C) = \omega(p)$. Set $\omega(C) = 0$, otherwise.

Define $\omega_q(\Phi) = \max\{\omega(C) | C \subset \Phi^{-1}(q) \text{ is a 2-variety}\}$.

Remark 3.29. It is not hard to see that $\omega(p)$ is well defined for every 2 point $p \in \Phi^{-1}(q)$. Using the arguments similar to the proof of Lemma 3.22 we can also show that $\omega(C)$ is well defined for every 2-variety $C \subset \Phi^{-1}(q)$.

Lemma 3.30. *Suppose that $\Phi : X \to S$ is strongly prepared and $q \in S$. Suppose also that $N(\Phi)$ does not contain any bad point and $N_q(\Phi)$ consists of a union of 2-varieties.*

Then there exists a sequence of blowups of 2-varieties $X_1 \to X$ such that the induced map $\Phi_1 : X_1 \to S$ is strongly prepared, $Z(\Phi_1) \cap N(\Phi_1) = \varnothing$, $A(\Phi_1, E) = 0$ if E is an exceptional component of E_{X_1} for $X_1 \to X$ and $N_q(\Phi_1) = \varnothing$.

Proof. Under the assumptions of the lemma $N_q(\Phi) = \varnothing$ if and only if $\omega_q(\Phi) = 0$.

Suppose that $\omega_q(\Phi) > 0$ and $C \subset N_q(\Phi)$ is a 2-variety such that $\omega(C) = \omega_q(\Phi)$. Let $\pi : X_1 \to X$ be the blowup of C. Then by Theorem 3.24 and Remark 3.25 we only need to verify that $\omega(s) < \omega_q(\Phi)$ at every 2 point $s \in \pi^{-1}(C)$.

Suppose that $p \in C$ is a k point and (N.IIa) holds at p, (u,v) are strongly prepared parameters at $\Phi(p)$ and $(x_1,\ldots,x_n)$ are strongly permissible parameters at p for (u,v) such that

$$\begin{aligned} u &= (x_1^{a_1}\cdots x_k^{a_k})^m \\ v &= x_1^{b_1}\cdots x_k^{b_k}. \end{aligned}$$

and $x_1 = x_2 = 0$ are local equations of C. Then after possibly permuting x_1 and x_2 we can assume that $\frac{b_1}{a_1} < m < \frac{b_2}{a_2}$.

Suppose that $E_1,\ldots,E_k$ are the components of E_X containing p with local equations $x_1 = 0,\ldots,x_k = 0$ respectively. If p lies in the intersection of more than 3 components $E_1,\ldots,E_k$ of E_X there will not be any 2 point in $\pi^{-1}(p)$.

Assume first that p is a 2 point with

$$\begin{aligned} u &= (x_1^{a_1}x_2^{a_2})^m \\ v &= x_1^{b_1}x_2^{b_2}. \end{aligned}$$

Then $\omega_q(\Phi) = \omega(C) = \omega(p) = S(a_1m - b_1, b_2 - a_2m)$.

Suppose that a 2 point $s \in \pi^{-1}(p)$ has strongly permissible parameters $(x_1,\bar{x}_2,x_3,\ldots,x_n)$ such that $x_2 = x_1\bar{x}_2$, then

$$\begin{aligned} u &= (x_1^{a_1+a_2}\bar{x}_2^{a_2})^m \\ v &= x_1^{b_1+b_2}\bar{x}_2^{b_2}. \end{aligned}$$

So, $\omega(s) > 0$ if and only if $(u,v)\mathcal{O}_{X_1,s}$ is not invertible and (N.IIa) holds at p. Let it be the case, then $m > \frac{b_1+b_2}{a_1+a_2}$ and $(a_1+a_2)m - (b_1+b_2) < a_1m - b_1$ since $a_2d - b_2 < 0$. Thus

$$\omega(s) = S((a_1+a_2)m - (b_1+b_2), b_2 - a_2m) < S(a_1m - b_1, b_2 - a_2m) = \omega_q(\Phi).$$

Suppose that a 2 point $s \in \pi^{-1}(p)$ has strongly permissible parameters $(\bar{x}_1,x_2,x_3,\ldots,x_n)$ such that $x_1 = \bar{x}_1x_2$, then

$$\begin{aligned} u &= (\bar{x}_1^{a_1}x_2^{a_1+a_2})^m \\ v &= \bar{x}_1^{b_1}x_2^{b_1+b_2}. \end{aligned}$$

So, $\omega(s) > 0$ if and only if $(u,v)\mathcal{O}_{X_1,s}$ is not invertible, that is if $m < \frac{b_1+b_2}{a_1+a_2}$. Then $(b_1+b_2) - (a_1+a_2)m < b_2 - a_2m$, since $b_1 - a_1m < 0$, and

$$\omega(s) = S(a_1m - b_1, (b_1+b_2) - (a_1+a_2)m) < S(a_1m - b_1, b_2 - a_2m) = \omega_q(\Phi).$$

Assume now that p is a 3 point with

$$u = (x_1^{a_1}x_2^{a_2}x_3^{a_3})^m$$
$$v = x_1^{b_1}x_2^{b_2}x_3^{b_3}.$$

Suppose that $s \in \pi^{-1}(p)$ is a 2 point, then s has regular parameters $(x_1, \bar{x}_2, x_3, \ldots, x_n)$ defined by $x_2 = x_1(\bar{x}_2 + \alpha)$ for some nonzero $\alpha \in \mathbf{k}$ and

$$u = (x_1^{a_1+a_2}(\bar{x}_2 + \alpha)^{a_2}x_3^{a_3})^m$$
$$v = x_1^{b_1+b_2}(\bar{x}_2 + \alpha)^{b_2}x_3^{b_3}.$$

If $rank\begin{pmatrix} a_1+a_2 & a_3 \\ b_1+b_2 & b_3 \end{pmatrix} < 2$ then $\omega(s) = 0$ since $(u,v)\mathcal{O}_{X_1,s}$ is invertible. So, consider the case when $rank\begin{pmatrix} a_1+a_2 & a_3 \\ b_1+b_2 & b_3 \end{pmatrix} = 2$.

Set $h = (a_1+a_2)b_3 - a_3(b_1+b_2)$ and

$$\bar{x}_1 = x_1(\bar{x}_2 + \alpha)^{\frac{a_2b_3 - a_3b_2}{h}}$$
$$\bar{x}_3 = x_3(\bar{x}_2 + \alpha)^{\frac{a_1b_2 - a_2b_1}{h}}$$

to get $*$-permissible parameters $(\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \ldots, x_n)$ at s with

$$u = (\bar{x}_1^{a_1+a_2}\bar{x}_3^{a_3})^m$$
$$v = \bar{x}_1^{b_1+b_2}\bar{x}_3^{b_3}.$$

Following the proof of Lemma 3.4 we can find strongly permissible parameters $(y_1, \ldots, y_n)$ at s such that

$$u = (y_1^{a_1+a_2}y_3^{a_3})^m$$
$$v = y_1^{b_1+b_2}y_3^{b_3}.$$

Assume that $\omega(s) > 0$, then $\min\{\frac{b_1+b_2}{a_1+a_2}, \frac{b_3}{a_3}\} < m < \max\{\frac{b_1+b_2}{a_1+a_2}, \frac{b_3}{a_3}\}$.

If $\frac{b_1+b_2}{a_1+a_2} < m < \frac{b_3}{a_3}$ then according to the proof of Theorem 3.24 the 2-variety $E_1 \cap E_3$ lies in $N_q(\Phi)$ and $\omega(E_1 \cap E_3) = S(a_1m - b_1, b_3 - a_3m)$. Thus since $a_2m - b_2 < 0$, $(a_1+a_2)m - (b_1+b_2) < a_1m - b_1$ and

$$\omega(s) = S((a_1+a_2)m - (b_1+b_2), b_3 - a_3m) < S(a_1m - b_1, b_3 - a_3m) = \\ = \omega(E_1 \cap E_3) \leq \omega_q(\Phi).$$

If $\frac{b_3}{a_3} < m < \frac{b_1+b_2}{a_1+a_2}$ then according to the proof of Theorem 3.24 the 2-variety $E_2 \cap E_3$ lies in $N_q(\Phi)$ and $\omega(E_2 \cap E_3) = S(a_3 m - b_3, b_2 - a_2 m)$. Thus since $b_1 - a_1 m < 0$, $(b_1 + b_2) - (a_1 + a_2)m < b_2 - a_2 m$ and

$$\begin{aligned}\omega(s) &= S(a_3 m - b_3, (b_1 + b_2) - (a_1 + a_2)m) < S(a_3 m - b_3, b_2 - a_2 m) = \\ &= \omega(E_2 \cap E_3) \leq \omega_q(\Phi).\end{aligned}$$

Suppose that $p \in C$ is a k point and (N.III) holds at p, (u, v) are strongly prepared parameters at $\Phi(p)$ and $(x_1, \ldots, x_n)$ are strongly permissible parameters at p for (u, v) such that

$$\begin{aligned}u &= x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} \\ v &= x_2^{b_2} \cdots x_k^{b_k}.\end{aligned}$$

If we set $m = (a_1, \ldots, a_{k-1}) > 0$, $\bar{a}_i = a_i/m$ for all $i = 1, \ldots, k_1$ and $\bar{a}_k = b_1 = 0$, we obtain the required statement by going through the analysis of the previous case.

By Theorem 3.24, induction on the number of 2-varieties $C \in N_q(\Phi)$ with $\omega(C) = \omega_q(\Phi)$ and induction on $\omega_q(\Phi)$ we achieve the conclusions of the lemma.

□

Theorem 3.31. *Suppose that $\Phi : X \to S$ is a strongly prepared morphism from a nonsingular n-fold X to a nonsingular surface S.*

Then there exists a finite sequence of quadratic transforms $\pi_1 : S_1 \to S$ and monoidal transforms centered at nonsingular varieties of codimension 2 $\pi_2 : X_1 \to X$ such that the induced morphism $\bar{\Phi} : X_1 \to S_1$ is monomial.

Proof. $\Phi : X \to S$ is monomial if and only if all points of X are good for Φ, that is if $A(\Phi) = 0$.

Suppose that $A(\Phi) > 0$ and E is a component of E_X such that $C(\Phi, E) = C(\Phi)$. Since $A(\Phi, E) > 0$ $\Phi(E)$ is a point $q \in D_S$.

Let $\pi_1 : S_1 \to S$ be the blowup of q. Then by Theorem 3.23, Lemma 3.27, Lemma 3.30 and Lemma 3.13 there exists a sequence of blowups of nonsingular codimension 2 varieties $\pi_2 : X_1 \to X$ such that $\Phi_1 : X_1 \to S$ is strongly prepared, $A(\Phi_1, \bar{E}) < A(\Phi_1) = A(\Phi)$ if $\bar{E}$ is the exceptional divisor for Φ_1 and the induced morphism $\Phi_2 : X_1 \to S_1$ is strongly prepared.

Thus if $\tilde{E}$ is the strict transform of E on X_1 by Theorem 3.14 $C(\Phi_2, \tilde{E}) < C(\Phi_1) = C(\Phi)$.

By induction on the number of components E of E_X with $C(\Phi, E) = C(\Phi)$ and induction on $C(\Phi)$ we achieve the conclusion of the theorem. $\square$

4 Toroidalization

Definition 4.1. A normal variety X with a SNC divisor E_X on X is called toroidal if for every point $p \in X$ there exists an affine toric variety X_σ, a point $p' \in X_\sigma$ and an isomorphism of $\mathbf{k}$ algebras $\hat{\mathcal{O}}_{X,p} \cong \hat{\mathcal{O}}_{X_\sigma,p'}$ such that the ideal of E_X corresponds to the ideal of $X_\sigma - T$ (where T is the torus in X_σ). Such a pair is called a model at p.

A dominant morphism $\Phi : X \to Y$ of toroidal varieties with SNC divisors E_X on X and D_Y on Y satisfying $\Phi^{-1}(D_Y) \subset E_X$ is called toroidal at $p \in X$ if there exist local models (X_σ, p') at p, (Y_τ, q') at $q = \Phi(p)$ and a toric morphism $\Psi : X_\sigma \to Y_\tau$ such that the following diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,p} & \overset{\sim}{\leftarrow} & \hat{\mathcal{O}}_{X_\sigma,p'} \\ \hat{\Phi}^* \uparrow & & \uparrow \hat{\Psi}^* \\ \hat{\mathcal{O}}_{Y,q} & \overset{\sim}{\leftarrow} & \hat{\mathcal{O}}_{Y_\tau,q'}. \end{array}$$

$\Phi : X \to Y$ is called toroidal (with respect to E_X and D_Y) if Φ is toroidal at every point $p \in X$.

From now on we will assume that $\Phi : X \to S$ is a strongly prepared morphism from a nonsingular n-fold X to a nonsingular surface S, and all points of E_X are good for Φ. We will also say that p is a toroidal point for Φ if Φ is toroidal at p. A point p which is not toroidal for Φ will be called nontoroidal.

Lemma 4.2. *Suppose that $\Phi : X \to S$ is a morphism from a nonsingular n-fold X to a nonsingular surface S, D_S is a SNC divisor on S such that $E_X = \Phi^{-1}(D_S)$ is a SNC divisor on X and $p \in X$ is a k point. Then Φ is toroidal at p if and only if there exist regular parameters $(x_1, \ldots, x_n)$ in $\hat{\mathcal{O}}_{X,p}$ and (u, v) in $\mathcal{O}_{S,p}$ such that one of the following forms holds:*

(1z) $1 \le k \le n-1$

$u = 0$ is a local equation of D_S, $x_1 \cdots x_k = 0$ is a local equation of E_X and

$$u = x_1^{a_1} \cdots x_k^{a_k} \qquad \text{(T.Iz)}$$
$$v = x_{k+1}$$

where $a_1, \ldots, a_k > 0$*;*

(1n) $1 \leq k \leq n-1$

$uv = 0$ *is a local equation of* D_S*,* $x_1 \cdots x_k = 0$ *is a local equation of* E_X *and*

$$\begin{aligned} u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\ v &= \alpha (x_1^{a_1} \cdots x_k^{a_k})^t + (x_1^{a_1} \cdots x_k^{a_k})^t x_{k+1} \end{aligned} \tag{T.In}$$

where $a_1, \ldots, a_k > 0$*,* $m, t > 0$ *and* $\alpha \in \mathbf{k} - \{0\}$*;*

(2) $2 \leq k \leq n$

$uv = 0$ *is a local equation of* D_S*,* $x_1 \cdots x_k = 0$ *is a local equation of* E_X *and*

$$\begin{aligned} y_1 &= x_1^{a_1} \cdots x_k^{a_k} \\ y_2 &= x_1^{b_1} \cdots x_k^{b_k} \end{aligned} \tag{T.II}$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k \geq 0$*,* $a_i + b_i > 0$ *for all* $i = 1, \ldots, k$ *and* *rank* $\begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$.

Proof. Let $X_{\sigma,l}$ be the n-dimensional nonsingular affine toric variety $\operatorname{Spec} \mathbf{k}[z_1, \ldots, z_n, z_{l+1}^{-1}, \ldots, z_n^{-1}]$ (where $l \in \{1, \ldots, n\}$) with the torus $T_n = \operatorname{Spec} \mathbf{k}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$ and $Z_{\sigma,l} = X_{\sigma,l} - T_n$. Let Y_τ be the 2-dimensional nonsingular affine toric variety $\operatorname{Spec} \mathbf{k}[y_1, y_2]$ with the torus $T_2 = \operatorname{Spec} \mathbf{k}[y_1, y_2, y_1^{-1}, y_2^{-1}]$ and $Z_\tau = Y_\tau - T_2$. Then any dominant toric morphism $\Psi : X_{\sigma,l} \to Y_\tau$ satisfying $Z_{\sigma,l} = \Psi^{-1}(Z_\tau)$ is given by the equations

$$\begin{aligned} y_1 &= z_1^{a_1} \cdots z_n^{a_n} \\ y_2 &= z_1^{b_1} \cdots z_n^{b_n} \end{aligned}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are integers, $a_i, b_i \geq 0$ and $a_i + b_i > 0$ for all $i = 1, \ldots, l$, *rank* $\begin{pmatrix} a_1 & \ldots & a_n \\ b_1 & \ldots & b_n \end{pmatrix} = 2$.

We will describe the map $\Psi : X_{\sigma,l} \to Y_\tau$ locally.

If $p' \in X_{\sigma,l}$ is a k point (with respect to $Z_{\sigma,l}$), then $k \leq l$ and after possibly permuting $z_1, \ldots, z_l$ we can find nonzero $\alpha_{k+1}, \ldots, \alpha_n \in \mathbf{k}$ such that

$(z_1, \ldots, z_k, \bar{z}_{k+1} = z_{k+1} - \alpha_{k+1}, \ldots, \bar{z}_n = z_n - \alpha_n)$ are regular parameters at p' and $z_1 \cdots z_k = 0$ is a local equation of $Z_{\sigma,l}$ at p'.

Assume first that $b_1 = \ldots = b_k = 0$, then $a_1, \ldots, a_k > 0$, $(y_1, \bar{y}_2 = y_2 - \alpha_{k+1}^{b_{k+1}} \cdots \alpha_n^{b_n})$ are regular parameters at $q' = \Psi(p')$ and $y_1 = 0$ is a local equation of Z_τ at q'. Set $\bar{z}_k = z_k((\bar{z}_{k+1} + \alpha_{k+1})^{a_{k+1}} \cdots (\bar{z}_n + \alpha_n)^{a_n})^{\frac{1}{a_k}}$ and $\tilde{z}_{k+1} = \bar{y}_2$ to get regular parameters $(z_1, \ldots, z_{k_1}, \bar{z}_k, \tilde{z}_{k+1}, \bar{z}_{k+2}, \ldots, \bar{z}_n)$ in $\mathcal{O}_{X_{\sigma,l},p'}$ such that

$$
\begin{aligned}
y_1 &= z_1^{a_1} \cdots z_{k-1}^{a_{k-1}} \bar{z}_k^{a_k} \\
\bar{y}_2 &= \tilde{z}_{k+1}.
\end{aligned}
$$

Assume now that at least one of $b_1, \ldots, b_k$ is greater than 0 and $rank \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 1$. Let $m = (a_1, \ldots, a_k)$ and $\bar{a}_i = a_i/m$, then $b_i = \bar{a}_i t$ for some $t > 0$ and $a_i, b_i > 0$ for all $i = 1, \ldots, k$, (y_1, y_2) are regular parameters at $q' = \Psi(p')$ and $y_1 y_2 = 0$ is a local equation of Z_τ at p'. Set

$$
\begin{aligned}
\alpha &= \alpha_{k+1}^{b_{k+1} - \frac{t}{m} a_{k+1}} \cdots \alpha_n^{b_n - \frac{t}{m} a_n} \\
\bar{z}_k &= z_k((\bar{z}_{k+1} + \alpha_{k+1})^{a_{k+1}} \cdots (\bar{z}_n + \alpha_n)^{a_n})^{\frac{1}{a_k}} \\
\tilde{z}_{k+1} &= (\bar{z}_{k+1} + \alpha_{k+1})^{b_{k+1} - \frac{t}{m} a_{k+1}} \cdots (\bar{z}_n + \alpha_n)^{b_n - \frac{t}{m} a_n}
\end{aligned}
$$

to get regular parameters $(z_1, \ldots, z_{k_1}, \bar{z}_k, \tilde{z}_{k+1}, \bar{z}_{k+2}, \ldots, \bar{z}_n)$ in $\hat{\mathcal{O}}_{X_{\sigma,l},p'}$ such that

$$
\begin{aligned}
y_1 &= (z_1^{\bar{a}_1} \cdots z_{k-1}^{\bar{a}_{k-1}} \bar{z}_k^{\bar{a}_k})^m \\
y_2 &= \alpha(z_1^{\bar{a}_1} \cdots z_{k-1}^{\bar{a}_{k-1}} \bar{z}_k^{\bar{a}_k})^t + (z_1^{\bar{a}_1} \cdots z_{k-1}^{\bar{a}_{k-1}} \bar{z}_k^{\bar{a}_k})^t \tilde{z}_{k+1}.
\end{aligned}
$$

Finally assume that $rank \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$. Then after possibly permuting $z_1, \ldots, z_k$ we can suppose that $f = a_{k-1} b_k - a_k b_{k-1} \neq 0$. (y_1, y_2) are regular parameters at $q' = \Psi(p')$ and $y_1 y_2 = 0$ is a local equation of Z_τ at q' in this case. Set

$$
\begin{aligned}
\bar{z}_{k-1} &= z_{k-1}(\bar{z}_{k+1} + \alpha_{k+1})^{\frac{a_{k+1} b_k - a_k b_{k+1}}{f}} \cdots (\bar{z}_n + \alpha_n)^{\frac{a_n b_k - a_k b_n}{f}} \\
\bar{z}_k &= z_k(\bar{z}_{k+1} + \alpha_{k+1})^{\frac{a_{k-1} b_{k+1} - a_{k+1} b_{k-1}}{f}} \cdots (\bar{z}_n + \alpha_n)^{\frac{a_{k-1} b_n - a_n b_{k-1}}{f}}
\end{aligned}
$$

to get regular parameters $(z_1, \ldots, z_{k-2}, \bar{z}_{k-1}, \ldots, \bar{z}_n)$ in $\hat{\mathcal{O}}_{X_{\sigma,l},p'}$ such that

$$\begin{aligned} y_1 &= z_1^{a_1} \cdots z_{k-2}^{a_{k-2}}\, \bar{z}_{k-1}^{a_{k-1}}\, \bar{z}_k^{a_k} \\ y_2 &= z_1^{b_1} \cdots z_{k-2}^{b_{k-2}}\, \bar{z}_{k-1}^{b_{k-1}}\, \bar{z}_k^{b_k}. \end{aligned}$$

By the definition Φ is toroidal at p if and only if there exist $k \le l \le n$, a k point $p' \in X_{\sigma,l}$ and a toric morphism $\Psi : X_{\sigma,l} \to Y_\tau$ such that Φ has the same local description at p as the morphism Ψ has at p', that is if one of the forms (T.Iz), (T.In) or (T.II) holds. □

Remark 4.3. If $p \in E_X$ is a toroidal point, arguing as in the proof of Lemma 3.4 we can always find strongly permissible parameters $(x_1, \ldots, x_n)$ at p such that one of the forms (T.Iz), (T.In) or (T.II) holds.

Remark 4.4. Suppose that $\Phi : X \to S$ is a strongly prepared morphism with $Z(\Phi) = \varnothing$ and a k point $p \in X$ is not toroidal for Φ. Then there exist strongly prepared parameters (u, v) at $q = \Phi(p)$ and strongly permissible parameters $(x_1, \ldots, x_n)$ at p such that one of the following forms holds:

(1a) $1 \le k \le n-1$

$u = 0$ is a local equation of D_S, $x_1 \cdots x_k = 0$ is a local equation of E_X and

$$\begin{aligned} u &= (x_1^{a_1} \cdots x_k^{a_k})^m \\ v &= \alpha (x_1^{a_1} \cdots x_k^{a_k})^t + (x_1^{a_1} \cdots x_k^{a_k})^t x_{k+1} \end{aligned} \tag{NT.Ia}$$

where $a_1, \ldots, a_k > 0$, $m, t > 0$ and $\alpha \in \mathbf{k}$;

(1b) $2 \le k \le n-1$

$u = 0$ is a local equation of D_S, $x_1 \cdots x_k = 0$ is a local equation of E_X and

$$\begin{aligned} u &= x_1^{a_1} \cdots x_k^{a_k} \\ v &= x_1^{b_1} \cdots x_k^{b_k} x_{k+1} \end{aligned} \tag{NT.Ib}$$

where $a_1, \ldots, a_k > 0$, $b_1, \ldots, b_k \ge 0$ and $rank \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$;

(2) $2 \le k \le n$

$u = 0$ is a local equation of D_S, $x_1 \cdots x_k = 0$ is a local equation of E_X and

$$\begin{aligned} u &= x_1^{a_1} \cdots x_k^{a_k} \\ v &= x_1^{b_1} \cdots x_k^{b_k} \end{aligned} \tag{NT.II}$$

where $a_1, \dots, a_k > 0$, $b_1, \dots, b_k \ge 0$ and $rank \begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix} = 2$.

Suppose that $p \in E_X$ is a 1 point such that $q = \Phi(p)$ is a 1 point on D_S. Let E be the component of E_X containing p. Suppose that (u, v) are strongly prepared parameters at q such that $u = 0$ is a local equation of D_S and $(x_1, \dots, x_n)$ are strongly permissible parameters at p with

$$\begin{aligned} u &= x_1^a \\ v &= x_1^c (x_{k+1} + \alpha), \quad \alpha \in \mathbf{k}. \end{aligned}$$

By Lemma 3.7 $a = \nu_E(u)$ and $c = \nu_E(v)$ are independent of the choice of parameters at p and q, so we can define an invariant $I(\Phi, p) = c - a$. Moreover, following the proof of Lemma 3.9 we see that $\nu_E(u)$ and $\nu_E(v)$ evaluated at p are equal to $\nu_E(u)$ and $\nu_E(v)$, respectively, evaluated at any 1 point $p' \in E$. Therefore, $I(\Phi, E) = I(\Phi, p)$ is a well defined notion.

In the above notations p is a toroidal point if and only if $c = 0$. Thus either all 1 points on E are toroidal or all of them are nontoroidal. Define E to be a toroidal component if at least one 1 point on E is toroidal, define E to be nontoroidal otherwise. Set

$$I(\Phi) = \max\{I(\Phi, E) | E \text{ is a nontoroidal component of } E_X\}.$$

Theorem 4.5. *Suppose that $\Phi : X \to S$ is strongly prepared with $Z(\Phi) = \varnothing$. Then the locus of nontoroidal points on X is a Zariski closed set of pure codimension 1, consisting of all nontoroidal components of E_X.*

Proof. Let Y be the union of all nontoroidal components of E_X. We will show that any nontoroidal point of E_X lies on Y and there is no toroidal point lying on Y.

Suppose that p is k point, $q = \Phi(p)$. Suppose that (u, v) are strongly prepared parameters at q and $(x_1, \dots, x_n)$ are strongly permissible parameters at p. Let $E_1, \dots, E_k$ be the components of E_X containing p with local equations $x_1 = 0, \ldots, x_k = 0$, respectively, and let U be a neighborhood of p where $(x_1, \ldots, x_n)$ are uniformizing parameters.

We will assume that p is a toroidal point for Φ and verify that E_1 is toroidal, that is contains a toroidal 1 point. By the symmetry this will imply that all components $E_1, \ldots, E_k$ containing p are toroidal and $p \notin Y$.

Suppose first that (T.Iz) holds at p then a 1 point $p' \in E_1 \cap V(x_{k+1}) \cap U$ is toroidal since $\Phi(p') = \Phi(p)$ is a 1 point and there exist strongly permissible parameters $(\bar{x}_1, \ldots, \bar{x}_k, x_{k+1}, \bar{x}_{k+2}, \ldots, \bar{x}_n)$ at p' such that

$$\begin{aligned} u &= \bar{x}_1^{a_1} \\ v &= x_{k+1}. \end{aligned}$$

Suppose that (T.In) holds at p then a 1 point $p' \in E_1 \cap V(x_{k+1}) \cap U$ is toroidal since $\Phi(p') = \Phi(p)$ is a 2 point and there exist strongly permissible parameters $(\bar{x}_1, \ldots, \bar{x}_k, x_{k+1}, \bar{x}_{k+2}, \ldots, \bar{x}_n)$ at p' such that

$$\begin{aligned} u &= \bar{x}_1^{a_1 m} \\ v &= \alpha \bar{x}_1^{a_1 t} + \bar{x}_1^{a_1 t} x_{k+1}. \end{aligned}$$

Suppose that (T.II) holds at p. After possibly interchanging u and v we can assume that $a_1 > 0$ and $b_1 \geq 0$. Furthermore, since $rank \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix} = 2$ there exist $i \in \{1, \ldots, k\}$ such that $b_i - \frac{b_1}{a_1} a_i \neq 0$. Thus after possibly permuting $x_2, \ldots, x_k$ we can assume that $b_2 - \frac{b_1}{a_1} a_2 \neq 0$.

Let $p' \in E_1 \cap U$ be a 1 point, then there exist nonzero $\alpha_2, \ldots, \alpha_k \in \mathbf{k}$ and regular parameters $(\bar{x}_1, \ldots, \bar{x}_n)$ at p' such that

$$\begin{aligned} u &= \bar{x}_1^{a_1} \\ v &= \bar{x}_1^{b_1} (\bar{x}_2 + \alpha_2)^{b_2 - \frac{b_1}{a_1} a_2} \cdots (\bar{x}_k + \alpha_k)^{b_k - \frac{b_1}{a_1} a_k}. \end{aligned}$$

Set $\alpha = \alpha_2^{b_2 - \frac{b_1}{a_1} a_2} \cdots \alpha_k^{b_k - \frac{b_1}{a_1} a_k}$ and $\tilde{x}_2 = (\bar{x}_2 + \alpha_2)^{b_2 - \frac{b_1}{a_1} a_2} \cdots (\bar{x}_k + \alpha_k)^{b_k - \frac{b_1}{a_1} a_k} - \alpha$ to get $*$-permissible parameters $(\bar{x}_1, \tilde{x}_2, \bar{x}_3, \ldots, \bar{x}_k, x_{k+1}, \ldots, x_n)$ at p' such that

$$\begin{aligned} u &= \bar{x}_1^{a_1} \\ v &= \bar{x}_1^{b_1} (\tilde{x}_2 + \alpha). \end{aligned}$$

If $b_1 > 0$ then $\Phi(p') = \Phi(p)$ is a 2 point and p' is toroidal since (T.In) holds at p'. If $b_1 = 0$ then $(u, \bar{v} = v - \alpha)$ are strongly prepared parameters at $\Phi(p')$ and p' is a toroidal point since either (T.Iz) or (T.II) holds at p'.

Assume now that p is a nontoroidal point for Φ and find a component E_j containing p which is not toroidal, that is contains a nontoroidal 1 point. This will imply $p \in Y$.

Suppose first that (NT.Ia) holds at p then a 1 point $p' \in E_1 \cap V(x_{k+1}) \cap U$ is nontoroidal since $\Phi(p') = \Phi(p)$ is a 1 point and there exist strongly permissible parameters $(\bar{x}_1, \ldots, \bar{x}_k, x_{k+1}, \bar{x}_{k+2}, \ldots, \bar{x}_n)$ at p' such that

$$
\begin{aligned}
u &= \bar{x}_1^{a_1 m} \\
v &= \alpha \bar{x}_1^{a_1 t} + \bar{x}_1^{a_1 t} x_{k+1}.
\end{aligned}
$$

Suppose that (NT.Ib) holds at p. After possibly interchanging $x_1, \ldots, x_k$ we can assume that $b_1 > 0$. Then a 1 point $p' \in E_1 \cap V(x_{k+1}) \cap U$ is nontoroidal since $\Phi(p') = \Phi(p)$ is a 1 point and there exist strongly permissible parameters $(\bar{x}_1, \ldots, \bar{x}_n)$ at p' such that (NT.Ia) holds:

$$
\begin{aligned}
u &= \bar{x}_1^{a_1} \\
v &= \bar{x}_1^{b_1} x_{k+1}.
\end{aligned}
$$

Suppose that (NT.II) holds at p. After possibly interchanging $x_1, \ldots, x_k$ we can assume that $b_1 > 0$ and $b_2 - \frac{b_1}{a_1} a_2 \neq 0$.

Let $p' \in E_1 \cap U$ be a 1 point, then there exist nonzero $\alpha_2, \ldots, \alpha_k \in \mathbf{k}$ and regular parameters $(\bar{x}_1, \ldots, \bar{x}_n)$ at p' such that

$$
\begin{aligned}
u &= \bar{x}_1^{a_1} \\
v &= \bar{x}_1^{b_1} (\bar{x}_2 + \alpha_2)^{b_2 - \frac{b_1}{a_1} a_2} \cdots (\bar{x}_k + \alpha_k)^{b_k - \frac{b_1}{a_1} a_k}.
\end{aligned}
$$

Set $\alpha = \alpha_2^{b_2 - \frac{b_1}{a_1} a_2} \cdots \alpha_k^{b_k - \frac{b_1}{a_1} a_k}$ and $\tilde{x}_2 = (\bar{x}_2 + \alpha_2)^{b_2 - \frac{b_1}{a_1} a_2} \cdots (\bar{x}_k + \alpha_k)^{b_k - \frac{b_1}{a_1} a_k} - \alpha$ to get strongly permissible parameters $(\bar{x}_1, \tilde{x}_2, \bar{x}_3, \ldots, \bar{x}_k, x_{k+1}, \ldots, x_n)$ at p' such that

$$
\begin{aligned}
u &= \bar{x}_1^{a_1} \qquad (***) \\
v &= \bar{x}_1^{b_1} (\tilde{x}_2 + \alpha).
\end{aligned}
$$

Thus $\Phi(p') = \Phi(p)$ is a 1 point and p' is nontoroidal since (NT.Ia) holds at p'.

□

Remark 4.6. Suppose that $p \in E_X$ is a nontoroidal k point in the form of (NT.II). In the notations of Theorem 4.5 we have

$$u = x_1^{a_1} \cdots x_k^{b_1}$$
$$v = x_1^{b_1} \cdots x_k^{b_k}$$

at p. Then formula $(***)$ shows that $I(\Phi, E_1) = b_1 - a_1$.

Analogously $I(\Phi, E_i) = b_i - a_i$ for all $i \in \{1, \dots, k\}$ such that $b_i > 0$.

Theorem 4.7. *Suppose that $\Phi : X \to S$ is strongly prepared and $Z(\Phi) = \varnothing$, $p \in X$ is a 1 point such that $q = \Phi(p)$ is a 1 point. Suppose that $\pi : S_1 \to S$ is the blow up of q and the rational map $\Phi_1 : X \to S_1$ is a morphism in a neighborhood of p.*

If $I(\Phi, p) > 0$ then $I(\Phi_1, p) < I(\Phi, p)$. If $I(\Phi, p) \leq 0$ then Φ_1 is toroidal at p.

Proof. Let (u, v) be strongly prepared parameters at q and $(x_1, \dots, x_n)$ be strongly permissible parameters at p. Since q is a 1 point and $m_q\mathcal{O}_{X,p}$ is invertible at p, (NT.Ia) holds at p:

$$u = x_1^a$$
$$v = \alpha x_1^c + x_1^c x_2$$

and either $c \geq a$ or $c < a$ and $\alpha \neq 0$. Notice also that by the definition of strongly permissible parameters $\alpha = 0$ if $c = a$.

Assume that $I(\Phi, p) = c - a \geq 0$, then there exist strongly prepared parameters (u_1, v_1) at $\Phi_1(p)$ such that

$$u = u_1, \quad v = u_1 v_1.$$

In case when $c > a$ we have

$$u_1 = x_1^a$$
$$v_1 = \alpha x_1^{c-a} + x_1^{c-a} x_2$$

and $I(\Phi_1, p) = c - 2a < c - a = I(\Phi, p)$.

If $c = a$ then $u_1 = x_1^a$ and $v_1 = x_2$ at p. Thus p is a toroidal point for Φ_1.

Assume that $I(\Phi, p) < 0$ and, therefore, $\alpha \neq 0$. Then there exist prepared parameters (u_1, v_1) at $\Phi_1(p)$ and strongly permissible parameters $(\bar{x}_1, \bar{x}_2, x_3, \dots, x_n)$ at p such that

$$u = u_1 v_1, \quad v = v_1$$

and

$$u_1 = \bar{x}_1^{a-c}$$
$$v_1 = \alpha^{\frac{a}{a-c}}\bar{x}_1^c + \bar{x}_1^c\bar{x}_2.$$

Thus $\Phi_1(p)$ is a 2 point and p is toroidal for Φ_1. $\square$

Lemma 4.8. *Suppose that $\Phi : X \to S$ is strongly prepared, $Z(\Phi) = \varnothing$ and $q \in S$ is a 1 point. Suppose that a variety C of codimension 2 is a component of $N_q(\Phi)$ and C is not a 2-variety.*

Let $\pi : X_1 \to X$ be the blowup of C, $E = \pi^{-1}(C)$ and $\Phi_1 = \Phi \circ \pi$. Then Φ_1 is strongly prepared, $Z(\Phi_1) = \varnothing$ and $I(\Phi_1, E) \leq 0$.

Proof. Theorem 3.24 implies that Φ_1 is strongly prepared and $Z(\Phi_1) = \varnothing$.

Suppose that p is a point on C, (u, v) are strongly prepared parameters at q and $(x_1, \ldots, x_n)$ are strongly prepared parameters at p. Analyzing the proof of theorem 3.24 we see that if $\pi^{-1}(p)$ contains a 1 point, either p is a 1 point satisfying (N.Ia) or p is a 2 point satisfying (N.Ib). Since a generic point on C is a 1 point and $I(\Phi_1, E) = I(\Phi_1, s)$ for any 1 point $s \in E$, it suffices to verify that $I(\Phi_1, s) \leq 0$ if $s \in \pi^{-1}(p)$ is a 1 point and p is 1 point satisfying (N.Ia).

In this case

$$u = x_1^m$$
$$v = x_1^t x_2 \text{ with } t < m$$

and C is defined by $x_1 = x_2 = 0$ in the neighborhood of p. Then s has strongly permissible parameters $(x_1, \bar{x}_2, x_3, \ldots, x_n)$, where $x_2 = x_1(\bar{x}_2 + \alpha)$ for some $\alpha \in \mathbf{k}$, and

$$u = x_1^m$$
$$v = \bar{\alpha}x_1^{t+1} + x_1^{t+1}\bar{x}_2$$

with $\bar{\alpha} = \alpha$ if $t + 1 \neq m$ and $\bar{\alpha} = 0$ if $t + 1 = m$.

Thus $I(\Phi_1, s) = t + 1 - m \leq 0$. $\square$

Lemma 4.9. *Suppose that $\Phi : X \to S$ is strongly prepared, $Z(\Phi) = \varnothing$ and $q \in S$ is a 1 point. Suppose that a 2-variety C is a component of $N_q(\Phi)$.*

Let $\pi : X_1 \to X$ be the blowup of C, $E = \pi^{-1}(C)$ and $\Phi_1 = \Phi \circ \pi$. Then Φ_1 is strongly prepared, $Z(\Phi_1) = \varnothing$ and $I(\Phi_1, E) < I(\Phi)$.

Proof. Theorem 3.24 implies that Φ_1 is strongly prepared and $Z(\Phi_1) = \varnothing$.

Suppose that p is a point on C, (u, v) are strongly prepared parameters at q and $(x_1, \ldots, x_n)$ are strongly prepared parameters at p. Analyzing the proof of theorem 3.24 we see that if $\pi^{-1}(p)$ contains a 1 point, p is a 2 point satisfying (N.IIa):

$$\begin{aligned} u &= (x_1^{a_1} x_2^{a_2})^m \\ v &= x_1^{b_1} x_2^{b_2} \text{ with } \frac{b_1}{a_1} < m < \frac{b_2}{a_2} \end{aligned}$$

and C is defined by $x_1 = x_2 = 0$ in the neighborhood of p. Then $\hat{\mathcal{O}}_{X_1, s}$ has regular parameters $(x_1, \bar{x}_2, x_3, \ldots, x_n)$, where $x_2 = x_1(\bar{x}_2 + \alpha)$ for some nonzero $\alpha \in \mathbf{k}$, and

$$\begin{aligned} u &= x_1^{(a_1+a_2)m} (\bar{x}_2 + \alpha)^{a_2 m} \\ v &= x_1^{b_1+b_2} (\bar{x}_2 + \alpha)^{b_2}. \end{aligned}$$

Set $f = a_1 b_2 - a_2 b_1$, $\bar{x}_1 = x_1(\bar{x}_2 + \alpha)^{\frac{a_2}{a_1+a_2}}$ and $\tilde{x}_2 = (\bar{x}_2 + \alpha)^{\frac{f}{a_1+a_2}} - \alpha^{\frac{f}{a_1+a_2}}$. Then $(\bar{x}_1, \tilde{x}_2, \bar{x}_3, x_4 \cdots, x_n)$ are $*$-permissible parameters at p satisfying the equalities

$$\begin{aligned} u &= \bar{x}_1^{(a_1+a_2)m} \\ v &= \alpha^{\frac{f}{a_1+a_2}} \bar{x}_1^{b_1+b_2} + \bar{x}_1^{b_1+b_2} \tilde{x}_2. \end{aligned}$$

Thus

$$I(\Phi_1, E) = I(\Phi, s) = (b_1+b_2) - (a_1+a_2)m = (b_2 - a_2 m) + (b_1 - a_1 m) < b_2 - a_2 m.$$

Denote by E_2 the component of E_X with local equation $x_2 = 0$, then by Remark 4.6

$$I(\Phi_1, E) < b_2 - a_2 m = I(\Phi, E_2) \leq I(\Phi).$$

□

Remark 4.10. It follows from the above proof that if $N_q(\Phi)$ contains a 2-variety then $I(\Phi) > 0$.

Lemma 4.11. *Suppose that $\Phi : X \to S$ is strongly prepared and $Z(\Phi) = \varnothing$. Then there exists a finite sequence of quadratic transforms $\pi_1 : S_1 \to S$ and monoidal transforms centered at nonsingular varieties of codimension 2 $\pi_2 : X_1 \to X$ such that the induced map $\bar{\Phi} : X_1 \to S_1$ is strongly prepared, $Z(\bar{\Phi}) = \varnothing$ and $I(\bar{\Phi}) \leq 0$.*

Proof. Suppose that $I(\Phi) > 0$ and E is a component of E_X such that $I(\Phi, E) = I(\Phi)$. Then $\Phi(E)$ is a 1 point $q \in D_S$.

Let $\pi_1 : S_1 \to S$ be the blowup of q. Then by Lemma 3.27, Lemma 4.8, Lemma 3.30, Lemma 4.9 and Lemma 3.13 there exists a sequence of blowups of nonsingular codimension 2 varieties $\pi_2 : X_1 \to X$ such that $\Phi_1 : X_1 \to S$ is strongly prepared with $Z(\Phi_1) = \varnothing$, $I(\Phi_1, \bar{E}) < I(\Phi_1) = I(\Phi)$ if $\bar{E}$ is the exceptional divisor for Φ_1 and the induced morphism $\Phi_2 : X_1 \to S_1$ is strongly prepared with $Z(\Phi_2) = \varnothing$.

Thus if $\tilde{E}$ is the strict transform of E on X_1 by Theorem 4.7 $I(\Phi_2, \tilde{E}) < I(\Phi_1) = I(\Phi)$.

By induction on the number of components E of E_X with $I(\Phi, E) = I(\Phi)$ and induction on $I(\Phi)$ we achieve the conclusion of the theorem. □

Theorem 4.12. *Suppose that $\Phi : X \to S$ is a strongly prepared morphism from a nonsingular n-fold X to a nonsingular surface S.*

Then there exists a finite sequence of quadratic transforms $\pi_1 : S_1 \to S$ and monoidal transforms centered at nonsingular varieties of codimension 2 $\pi_2 : X_1 \to X$ such that the induced morphism $\bar{\Phi} : X_1 \to S_1$ is toroidal.

Proof. From Theorem 3.31 and Lemma 4.11 we obtain a finite sequence of quadratic transforms $S_1 \to S$ and monoidal transforms centered at nonsingular codimension 2 varieties $X_1 \to X$ such that the induced map $\Phi_1 : X_1 \to S_1$ is strongly prepared, $Z(\Phi_1) = \varnothing$ and $I(\Phi_1) \leq 0$.

If Φ_1 is not toroidal consider the set

$$T(\Phi_1) = \{q \in D_S | q = \Phi_1(E) \text{ where } E \text{ is a nontoroidal component of } E_X\}.$$

$T(\Phi_1)$ is a finite set, containing only 1 points.

Let $q \in T(\Phi_1)$ and $\pi_1 : S_2 \to S_1$ be the blowup of q. Then by Remark 4.10 $N_q(\Phi_1)$ contains only nonsingular codimension 2 varieties which are not 2-varieties. Thus by Lemma 3.27 and Lemma 4.8 there exists a sequence of blowups of nonsingular codimension 2 varieties $\pi_2 : X_2 \to X_1$ such that $\Phi_2 : X_2 \to S_1$ is strongly prepared with $Z(\Phi_2) = \varnothing$ and $I(\Phi_2) \leq 0$.

By Lemma 3.13 and Theorem 4.7 the induced morphism $\Phi_3 : X_2 \to S_2$ is strongly prepared, $Z(\Phi_3) = 0$ and all points in $\Phi_1^{-1}(q)$ are toroidal. Therefore, $T(\Phi_3) = T(\Phi_1) - \{q\}$.

By induction on the number of points in $T(\Phi_1)$ we achieve the conclusion of the theorem. □

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