MONOMIAL RESOLUTIONS OF MORPHISMS OF ALGEBRAIC SURFACES

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1 Introduction.

Let k be a perfect field and L/K be a finite separable field extension of one-dimensional function fields over k. A classical result (c.f. I.6, [Ha]) states that K (resp. L) has a unique proper and smooth model C (resp. D), and that there is a unique morphism of curves $f: D \rightarrow C$ inducing the field inclusion $K \subset L$ at the generic points of C and D. It has the following properties:

(i) f is a finite morphism.

(ii) f is *monomial* on its tamely ramified locus; let $\beta \in D$ be any point, with $\alpha := f(\beta) \in C$, such that the extension of discrete valuation rings $\mathcal{O}_{Y,\beta}/\mathcal{O}_{X,\alpha}$ is tamely ramified. There exists a local-étale ring extension R of $\mathcal{O}_{Y,\beta}$ and regular parameters u of $\mathcal{O}_{X,\alpha}$ and $\overline{x}$ of R such that $u = \overline{x}^a$ for some a prime to the characteristic of k.

In this paper, we investigate a two-dimensional version of this statement, that is, L/K is a finite separable field extension of two-dimensional function fields over k. By birational resolution of singularities ([Ab4], [H], [Li2]) and elimination of indeterminacies (theorem 26.1, [Li]), there exists a proper and smooth

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model X (resp. Y) of K (resp. L), together with a morphism $f: Y \to X$ inducing the field inclusion $K \subset L$ at the generic points of X and Y. Such a morphism is in general neither finite nor monomial.

In [Ab2], the question is raised of whether this can be arranged by blowing-up: does there exist compositions of point blow-ups $Y' \to Y$ and $X' \to X$, together with a map $f': Y' \to X'$ such that f' is finite and/or monomial? It is actually shown (theorem 12, [Ab2]) that f' finite cannot in general be achieved. The obstruction is local for the Riemann-Zariski manifold of L/k.

This leaves open the question of whether f' can be taken to be monomial. For complex surfaces, a positive answer has been given in [AKi] (theorem 7.4.1). Their method however does not generalize to positive characteristic, due to the lack of canonical forms for the equations defining f (assertion 7.4.1.1, [AKi]). In general, it is not possible to monomialize an arbitrary morphism in $\mathrm{char}(p) > 0$, even for a morphism of curves. We give a simple example later on in this introduction. The obstruction to monomialization is the appearence of wild ramification. In the presence of wild ramification, monomialization is possible only in some very special cases.

We present a quite general solution to this problem: any *proper, tamely ramified* morphism $f: Y \to X$ of surfaces (which are separated but not necessarily proper over k), inducing the field inclusion $K \subset L$ at the generic points of X and Y, can, after performing suitable compositions of point blow-ups $Y' \to Y$ and $X' \to X$, be arranged to a monomial morphism $f': Y' \to X'$. Moreover, there is a unique *minimal* such f'.

Our method is constructive. That is, we give an algorithm, which, starting from an arbitrary proper f as above, produces its associated minimal f'. This algorithm is explained in section 4. An easy reduction (proposition 8) shows that it can be assumed that both of the critical locus C_f and the branch locus of f are divisors with strict normal crossings.
In section 3, we then attach to every vertical component E of C_f a nonnegative integer, its complexity i_E (definition 4), which is zero if f is monomial at all points of E (compare with that used in [AKi], p.222). That our algorithm eventually makes it drop, which is the main technical point, follows from propositions 6 and 7.
The main theorem is stated in section 2, together with the appropriate notions of tamely monomial and of tamely ramified (not necessarily finite) morphism.

In Section 5, we give a proof that a tamely ramified f can be made toroidal when k is an algebraically closed field. Although this result is known, for instance it is implicit in [AKi] in the case when k is algebraically closed of characteristic zero, we include it as an interesting point in the general theory of resolution of morphisms of surfaces. Our proof constructs a minimal toroidal model.

There is a local formulation of a monomial resolution for a mapping. Suppose that $f : Y \to X$ is a morphism of varieties over a field k. If $f(p) = q$, we have an induced homomorphism of local rings

$$R = \mathcal{O}_{X,q} \subset S = \mathcal{O}_{Y,p}$$

We will say that $R \to S$ is a monomial mapping if there are regular parameters $(x_1, ..., x_m)$ in R, $(y_1, ..., y_n)$ in S (with $m \le n$), units $\delta_1, \ldots, \delta_n \in S'$ and a matrix (a_{ij}) of nonnegative integers such that (a_{ij}) has rank m, and

$$\begin{array}{lcl} x_1 & = & y_1^{a_{11}} .....y_n^{a_{1n}}\delta_1 \\ & & \vdots \\ x_m & = & y_1^{a_{m1}} .....y_n^{a_{mn}}\delta_m. \end{array} \tag{0.1}$$

Suppose that V is a valuation ring of the quotient field K of S, such that V dominates S. Then we can ask if there are sequences of monoidal transforms $R \to R'$ and $S \to S'$ such that V dominates S', S' dominates R', and $R' \to S'$ has an especially good form.

$$\begin{array}{ccc} R' & \to & S' \subset V \\ \uparrow & & \uparrow \\ R & \to & S \end{array} \tag{0.2}$$

Zariski's Local Uniformization Theorem [Z1] says that (when char$(k) = 0$) there exists a diagram (0.2) such that R' and S' are regular.

In Theorem 1.1 [C] we obtain a diagram (0.2) making $R' \to S'$ a monomial mapping whenever the quotient field of S is a finite extension of the quotient field of R, and the characteristic of k is 0.

If $R' \to S'$ is a mapping of the form (0.1), and the characteristic of k is zero, there exists a local etale extension $S' \to S''$ such that S'' has regular parameters $\overline{y}_1, \ldots \overline{y}_n$ such that

$$\begin{array}{lcl} x_1 & = & \overline{y}_1^{a_{11}} .....\overline{y}_n^{a_{1n}} \\ & & \vdots \\ x_m & = & \overline{y}_1^{a_{m1}} .....\overline{y}_n^{a_{mn}}. \end{array} \tag{0.3}$$

In char $p > 0$, the form (0.3) is not possible to obtain from a monomial mapping by an étale extension in general. Already in dimension 1,

$$x = y^p + y^{p+1}$$

gives a simple counterexample. However, the above example is a monomial mapping. In fact, if R and S are regular local rings of dimension 1, then $R \subset S$ is a monomial mapping, since R and S are Dedekind domains.

If R and S have dimension 2, k is a field of characteristic $p > 0$,and V is a valuation ring dominating S, then we ask if it is possible to obtain a diagram

(0.2) making $R' \to S'$ a monomial mapping. From our theorem 1, we deduce a positive answer whenever p does not divide the order of a Galois closure of the quotient field of S over the quotient field of R.

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2 Preliminaries and statement of main result.

All along this article, k denotes a perfect field of characteristic $p \geq 0$, and K/k a finitely generated field extension. L/K is a finite separable field extension.

By an algebraic k-scheme, we mean a Noetherian separated k-scheme, all whose local rings are essentially of finite type over k. The function field of an integral algebraic k-scheme X is denoted by $K(X)$. If α is a closed point of such a scheme, its ideal sheaf is denoted by M_α. If R is a local ring, its residue field is denoted by $\kappa(R)$.

Definition 1 *A proper, generically finite morphism of integral algebraic k-schemes $f:\ Y \to X$ is called a model of the field extension L/K if $K(X) = K$, $K(Y) = L$, $\dim X$ $(= \dim Y) = \mathrm{tr.deg}_k K$, and if the following diagram commutes:*

$$\begin{array}{ccc} Spec L & \longrightarrow & Spec K \\ \big\downarrow & & \big\downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

A model is said to be proper if X/k (and hence Y/k as well) is proper, and nonsingular if both of X and Y are nonsingular. Models are partially ordered by domination, where a model $f':\ Y' \to X'$ dominates another model $f:\ Y \to X$ if there exist *proper* maps $\pi:\ Y' \to Y$ and $\eta:\ X' \to X$ such that the following diagram commutes (and is compatible with the maps of definition 1):

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \eta}\big\downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Recall that a k-valuation ring V $(k \subset V)$ of L, with $V \subset L = K(V)$ is said to be *divisorial* if its group is isomorphic to $\mathbf{Z}$, and if $\mathrm{tr.deg}_k \kappa(V) =$

$\text{tr.deg}_k L - 1$ (divisorial valuations are called *prime divisors* in [ZS2], p.88). A generically finite inclusion $W \subset V$ of divisorial k-valuation rings is said to be tamely ramified if $\text{char}k = 0$, or if $\text{char}k = p > 0$, its ramification index is not divisible by p, and the residue field extension $\kappa(V)/\kappa(W)$ is separable.

Definition 2 *A model $f : Y \to X$ is said to be tamely ramified if for every divisorial k-valuation ring V of L, with $K(V) = L$, having a center in Y, the extension of valuation rings $V/V \cap K$ is tamely ramified.*

Remark: since the models we are considering are not necessarily finite, a notion of tame ramification involving *all* divisorial valuations rings having a center in Y is needed. For finite morphisms, the usual definition (2.2.2 of [GM], or p.41 of [Mi]) only involves those divisorial valuations as above whose center in X has codimension one.
This raises the following problem: if $f : Y \to X$ is a model, and if the induced finite map

$$\overline{f} : \mathbf{Spec} f_* \mathcal{O}_Y \to X$$

is tamely ramified in the sense of [GM], under which conditions is it true that f is tamely ramified according to definition 2?

From now on, it will be assumed that $tr.deg_k K = 2$. All models therefore are proper, generically finite morphisms of integral surfaces.

Given a *nonsingular* model $f : Y \to X$, its critical locus is denoted by C_f. A scheme structure on C_f is given by the vanishing of the Jacobian determinant. Since L/K is separable, C_f is a divisor on Y. There exist well defined effective divisors R_f, S_f on Y such that $C_f = R_f + S_f$, the induced map $R_f \to f(R_f)$ is finite, and $f(S_f)$ is a finite set. Let $B_f := f(R_f)_{\text{red}}$. By the Zariski-Nagata theorem on the purity of the branch locus, Theorem X.3.1 [SGA], B_f is a divisor on X.

Definition 3 *A nonsingular model $f : Y \to X$ is said to be tamely monomial if for every $\beta \in Y$, with $\alpha := f(\beta) \in X$, there exist regular systems of parameters (r.s.p. for short) (u, v) of $\mathcal{O}_{X,\alpha}$ and (x, y) of $\mathcal{O}_{Y,\beta}$ such that*

(i) If $\alpha \in \text{Supp}(B_f)$, B_f is locally at α defined by $u = 0$ or $uv = 0$.

(ii) Either

(1) $$\begin{cases} u = \gamma x^a y^b \\ v = \delta x^c y^d \end{cases},$$

where $\gamma\delta$ is a unit in $\mathcal{O}_{Y,\beta}$ and p does not divide $ad - bc$, or

(2) $$\begin{cases} u = \gamma x^a \\ v = \delta x^c \end{cases},$$

where both of $\gamma\delta$ and $a\gamma\dfrac{\partial\delta}{\partial y} - c\delta\dfrac{\partial\gamma}{\partial y}$ are units in $\mathcal{O}_{Y,\beta}$.

A tamely monomial model dominating a given model is called a tamely monomial resolution.

Proposition 1 *A tamely monomial model $f: Y \to X$ has the following properties.*

(i) *B_f, f^*B_f and S_f are divisors with strict normal crossings.*

(ii) *For every $\beta \in Y$, with $\alpha := f(\beta) \in X$, and regular parameters (u, v) in $\mathcal{O}_{X,\alpha}$ as in (ii) of Definition 3, there exists an affine neighborhood U of β, and an étale cover V of U such that there are uniformizing parameters $(\overline{x}, \overline{y})$ on V with*

$$u = \overline{x}^a \overline{y}^b$$
$$v = \overline{x}^c \overline{y}^d$$

for some natural numbers a, b, c, d such that p does not divide $ad - bc$.

Proof: (i) directly follows from definition 3.

We will prove (ii), under the assumption that case (2) of Definition 3 holds. There exists an affine neighborhood U of β such that (x, y) are uniformizing parameters on U, and γ, δ, and $a\gamma\frac{\partial\delta}{\partial y} - c\delta\frac{\partial\gamma}{\partial y}$ are units in $\Gamma(U, \mathcal{O}_Y)$. p cannot divide both a and c. Without loss of generality, p does not divide a. Set $d = 1$, $b = 0$,

$$R = \Gamma(U, \mathcal{O}_Y)[\gamma^{\frac{1}{a}}, \delta^{\frac{1}{a}}],$$

$V = \mathrm{Spec}(R)$. V is an étale cover of U. Set

$$\overline{x} = x\gamma^{\frac{1}{a}}, \overline{y} = \delta\gamma^{\frac{-c}{a}}.$$

$(\overline{x}, \overline{y})$ are uniformizing parameters on V since

$$\frac{\partial\overline{y}}{\partial y} = \frac{1}{a}\gamma^{\frac{-a}{c}-1}[a\gamma\frac{\partial\delta}{\partial y} - c\delta\frac{\partial\gamma}{\partial y}]$$

is a unit in $\Gamma(V, \mathcal{O}_V)$.

Our main result is

Theorem 1 *Given a model f of L/K, the following properties are equivalent.*

(i) *f admits a minimal (w.r.t. domination) tamely monomial resolution.*

(ii) *f admits a tamely monomial resolution.*

(iii) *f is tamely ramified.*

Theorem 1 will be proved at the end of section 4. Note that, if $\mathrm{char} k = 0$, any model of L/K is tamely ramified.

Remark: In case the given model $f: Y \to X$ is *finite*, there is an easier proof of theorem 1, using Abhyankar's lemma ([Ab5], 2.3.4, [GM]). This can be seen

as follows; first reduce to f finite and ramified over a divisor with strict normal crossings. By Abhyankar's lemma, f can be locally described as a Kummer covering, after a local-étale change of coordinates on X. Let Y' be the *minimal* resolution of singularities of Y. By explicit computations, it is now seen that $Y' \to X$ is a tamely monomial morphism.
From this result, one deduces the *existence* of a tamely monomial resolution of a given f as in theorem 1, since any model can be dominated by a finite one. However, the tamely monomial resolution thus obtained is not in general the minimal one.

Corollary 1 *Assume that* $\mathrm{char} k = 0$ *or* $\mathrm{char} k = p > 0$ *and* p *does not divide the degree of the Galois closure* $\overline{L}/K$ *of* L/K*. Then any model of* L/K *admits a minimal tamely monomial resolution.*

Proof: Assume that $\mathrm{char} k = p > 0$. Let V be a divisorial valuation ring of L and $\overline{V}$ be a divisorial valuation ring of $\overline{L}$ such that $V = \overline{V} \cap L$. Let $W := V \cap K$ and e and f be the ramification index and residual degree of the extension $\overline{V}/W$. By V.9.22 of [ZS1], $[\overline{L} : K] = efg$, where g is the number of conjugates of $\overline{V}$ under the action of $\mathrm{Gal}(\overline{L}/K)$. Hence p does not divide ef. This implies that $\overline{V}/W$ is tamely ramified. Consequently, V/W is tamely ramified.

3 The complexity.

In this section, we only consider nonsingular models $f : Y \to X$ such that both of B_f and f^*B_f are divisors with strict normal crossings.

Let α be a point in X. A r.s.p. (u, v) of $\mathcal{O}_{X,\alpha}$ is said to be *admissible* if $\alpha \notin \mathrm{Supp}(B_f)$, or if $\alpha \in \mathrm{Supp}(B_f)$ and B_f is locally at α defined by $u = 0$ or $uv = 0$ (see *(i)* in definition 3).

Definition 4 *Given a reduced irreducible component* E *of* S_f*, with* $\alpha := f(E) \in X$*, the complexity* i_E *of* E *is defined by the following formula:*

$$i_E := \nu_E(S_f) + 1 - \max_{(u,v)\ adm.} \nu_E(uv) \geq 0,$$

where ν_E *is the divisorial valuation associated with* E*, and the maximum is taken over all admissible r.s.p. at* α*.*

Remark: it follows from this definition that if $f : Y \to X$ and $f' : Y' \to X'$ are two nonsingular models as above, and E (resp. E') is a reduced irreducible component of S_f (resp. $S_{f'}$) such that

(i) $\mathcal{O}_{Y,E} = \mathcal{O}_{Y',E'}$, and

(ii) $\mathcal{O}_{X,\alpha} = \mathcal{O}_{X',\alpha'}$, where $\alpha := f(E)$ and $\alpha' := f'(E')$,

then $i_E = i_{E'}$. This fact will be repeatedly used in this section.

We first recall the following classical birational fact (theorem 3, [Ab1]), together with its global counterpart (theorem 4.1, [Li]).

Proposition 2 *Let R be a two-dimensional regular local ring with quotient field K and S be a regular local ring birationally dominating R. Assume that S is either two-dimensional or a divisorial k-valuation ring.*
There exists a unique sequence

$$R = R_0 \subset R_1 \subset \ldots \subset R_n = S$$

such that, for $1 \leq i \leq n$, R_i is a quadratic transform of R_{i-1}.

Proposition 3 *Let R be a two-dimensional regular local ring with quotient field K and $X \to \mathrm{Spec}R$ be a proper birational map with X regular.*
There exists a sequence

$$X = X_n \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0 = \mathrm{Spec}R$$

such that, for $1 \leq i \leq n$, X_i is the blow-up of a closed point of X_{i-1}.

Proposition 2 implies the following.

Corollary 2 *Let $f:\ Y \to X$ be a nonsingular model as in the beginning of this section, and let E be a reduced irreducible component of S_f, with $\alpha := f(E) \in X$. Assume that $\alpha \notin \mathrm{Supp}(B_f)$. Let $\beta \in f^{-1}(\alpha)$. Then,*

(i) If $\beta \notin \mathrm{Supp}(S_f)$, then $M_\alpha \mathcal{O}_{Y,\beta} = M_\beta$.

(ii) If $\beta \in \mathrm{Supp}(S_f)$, then $M_\alpha \mathcal{O}_{Y,\beta}$ is a principal ideal.

Proof: Let $\overline{R}$ be the integral closure of $\mathcal{O}_{X,\alpha}$ in L. By X.3.1 [SGA], $\overline{R}$ is a regular semilocal ring which is unramified over $\mathcal{O}_{X,\alpha}$. One has $\overline{R} \subseteq \mathcal{O}_{Y,\beta}$ in either case.
If $\beta \notin \mathrm{Supp}(S_f)$, $\mathcal{O}_{Y,\beta} = \overline{R}_{M_\beta \cap \overline{R}}$ by proposition 2. Hence $M_\alpha \mathcal{O}_{Y,\beta} = M_\beta$.
If $\beta \in \mathrm{Supp}(S_f)$, $\mathcal{O}_{Y,\beta}$ dominates a quadratic transform R_1 of $\overline{R}_{M_\beta \cap \overline{R}}$ by proposition 2. Hence $M_\alpha \mathcal{O}_{Y,\beta}$ is a principal ideal.

Proposition 4 *Let $f:\ Y \to X$ be a nonsingular model as above, and let $\beta \in Y$, with $\alpha := f(\beta)$. There exists an admissible r.s.p. (u,v) at α such that for every reduced irreducible component E of S_f passing through β,*

$$\nu_E(uv) = \max_{(u',v')\ adm.} \nu_E(u'v').$$

Proof: Since f^*B_f (and hence S_f as well) is a divisor with strict normal crossings, there exist $s_\beta \leq 2$ components of S_f passing through β. The above statement is trivial unless $s_\beta = 2$, which we now assume. Since B_f is a divisor with strict normal crossings, there exist $r_\alpha \leq 2$ components of B_f passing through α. The above statement is trivial if $r_\alpha = 2$, and we hence assume $r_\alpha \leq 1$. Let E be an irreducible component of S_f passing through β. We consider two cases:

First assume that $r_\alpha = 0$. By proposition 2 and Theorem X.3.1 [SGA], there exists a regular local ring R, essentially of finite type and unramified over $\mathcal{O}_{X,\alpha}$, and a succession of quadratic transforms

$$R = R_0 \subset R_1 \subset \ldots \subset R_n = \mathcal{O}_{Y,\beta},$$

with $n \geq 1$. Let $u \in \mathcal{O}_{X,\alpha}$ be a regular parameter. Let $t_i \in R_i$, $1 \leq i \leq n$, be a regular parameter such that $\mathrm{ht}\left((t_i) \cap R_{i-1}\right) = 2$. Define by induction on i, $0 \leq i \leq n$, elements $u_i \in R_i$ by $u_0 = u$, and if $i \geq 1$:

$$\begin{cases} u_{i-1} &= t_i u_i \quad \text{if } u_{i-1}R_i \neq R_i \\ u_{i-1} &= u_i \quad \text{if } u_{i-1}R_i = R_i \end{cases}.$$

Let m_u, $1 \leq m_u \leq n$, be the largest integer m such that $u_{m-1} = t_m u_m$. We have:

$$\nu_E(u) = \sum_{i=1}^{m_u} \nu_E(t_i).$$

In particular, $\nu_E(u)$ is a non decreasing function of m_u. Also notice that for general u, $m_u = 1$. A r.s.p. (u, v) satisfying the conclusion of the proposition is then obtained by taking v maximizing m_v, and any transversal u.

Assume now that $r_\alpha = 1$. Let $u = 0$ be a local equation of B_f at α, and $xy = 0$ be a local equation of $(S_f)_{\mathrm{red}}$ at β. Let $v, w \in \mathcal{O}_{X,\alpha}$ be such that both of (u, v) and (u, w) are (admissible) r.s.p. We have

$$\begin{cases} u &= \gamma x^a y^b \\ v &= x^c y^d v' \\ w &= x^{c'} y^{d'} w' \end{cases},$$

where γ is a unit in $\mathcal{O}_{Y,\beta}$, $a, b, c, d, c', d' > 0$, and neither x nor y divides $v'w'$. Assuming that $c' > c$, we will prove that $d' \geq d$ and the conclusion will follow. By the Weierstrass preparation theorem, there exists a power series $P(u) \in \kappa(\alpha)[[u]]$ such that

$$w = unit \times (v - P(u)) \in \widehat{\mathcal{O}}_{X,\alpha} \simeq \kappa(\alpha)[[u, v]]. \tag{3.1}$$

Let $\lambda \in \mathcal{O}_{X,\alpha}$ be a unit such that

$$P(u) \equiv \lambda u^m \mod u^{m+1},$$

where $m = \mathrm{ord}_u P$. Since $c' = \mathrm{ord}_x w > c = \mathrm{ord}_x v$, (3.1) implies that $c = ma$. This gives the congruence

$$v' y^d \equiv \lambda \gamma^m y^{mb} \mod x$$

in $\mathcal{O}_{Y,\beta}$. Hence $d \leq mb$. It then follows from (3.1) that

$$d' = \mathrm{ord}_y w \geq \min\{\mathrm{ord}_y v, mb\} = d.$$

This concludes the proof.

Proposition 4 leads to the following definition of the local complexity on Y.

Definition 5 *Let $f : Y \to X$ be a nonsingular model as above, and $\beta \in \mathrm{Supp}(S_f)$. The complexity i_β of f at β is defined by*

$$i_\beta := \max_E i_E \geq 0,$$

where the maximum is taken over all reduced irreducible components of S_f passing through β.

Lemma 1 *Let $f : Y \to X$ be a nonsingular model as above, and $\beta \in \mathrm{Supp}(R_f)$, with $\alpha := f(\beta)$. Let $x = 0$ be a local equation of a reduced component D of R_f passing through β, and $u = 0$ be an equation of $\Delta := f(D)$ at α. Write $u = x^a u' \in \mathcal{O}_{Y,\beta}$, where $a \geq 2$, and x does not divide u'.*

The extension of divisorial valuation rings $\mathcal{O}_{Y,D}/\mathcal{O}_{X,\Delta}$ is tamely ramified if and only if $\mathrm{ord}_D R_f = a - 1$.

Proof: Choose a r.s.p. (u, v) at α, and a r.s.p. (x, y) at β. A local equation at β of C_f is given by

$$\mathrm{Jac}_\beta(f) = x^{a-1}\left(au' \frac{\partial v}{\partial y} + x \mathrm{Jac}(u', v)\right).$$

Then $\mathrm{ord}_D R_f = a - 1$ if and only if p does not divide a and x does not divide $\dfrac{\partial v}{\partial y}$.

Since (k is perfect) α (resp. β) is a smooth point of Δ (resp. D) (ex. II.8.1, [Ha]), $\Omega^1_{\Delta/k}$ (resp. $\Omega^1_{D/k}$) is generated at α (resp. β) by dv (resp. dy). The inclusion $\kappa(\mathcal{O}_{X,\Delta}) \subseteq \kappa(\mathcal{O}_{Y,D})$ is separable if and only if $\Omega^1_{D/\Delta}$ is a torsion sheaf (II.8.6.A, [Ha]). Let $\overline{f} : D \to \Delta$ be the finite map induced by f. There is an exact sequence for differentials on D (II.8.11, [Ha])

$$f^* \Omega^1_{\Delta/k} \xrightarrow{d\overline{f}} \Omega^1_{D/k} \longrightarrow \Omega^1_{D/\Delta} \longrightarrow 0.$$

Clearly, $\Omega^1_{D/\Delta}$ is a torsion sheaf if and only if $(\Omega^1_{D/\Delta})_\beta$ is a torsion $\mathcal{O}_{Y,\beta}$-module. The tangent map $d\overline{f}$ is given at β by

$$dv \mapsto \left(\frac{\partial v}{\partial y} \mod x\right) dy.$$

It follows that the inclusion $\kappa(\mathcal{O}_{X,\Delta}) \subseteq \kappa(\mathcal{O}_{Y,D})$ is separable if and only if x does not divide $\dfrac{\partial v}{\partial y}$.

Summing up, $\mathrm{ord}_D R_f = a-1$ if and only the extension of divisorial valuation rings $\mathcal{O}_{Y,D}/\mathcal{O}_{X,\Delta}$ is tamely ramified as required.

The following proposition characterizes a tamely monomial model by way of its maximal complexity.

Proposition 5 *Let $f: Y \to X$ be a nonsingular model as above, and let $\beta \in Y$. f is tamely monomial at β (i.e. has the local form (1) or (2) of definition 3) w.r.t. some admissible r.s.p. at $\alpha := f(\beta)$ if and only if*

(i) For every irreducible component D of R_f passing through β, with $\Delta := f(D)$, the extension of divisorial valuation rings $\mathcal{O}_{Y,D}/\mathcal{O}_{X,\Delta}$ is tamely ramified.

(ii) $i_\beta = 0$ if $\beta \in \mathrm{Supp}(S_f)$.

In particular, f is a tamely monomial model if and only if (i) holds for all components of R_f and (ii) holds for all $\beta \in \mathrm{Supp}(S_f)$.

Proof: Choose an admissible r.s.p. (u, v) at α. If $\beta \in \mathrm{Supp}(S_f)$, assume furthermore that (u, v) achieves $i_\beta = 0$ (proposition 4). We first prove the if part. We consider six cases.

Case 1. $\beta \notin \mathrm{Supp}(C_f)$. Consequently $M_\alpha \mathcal{O}_{Y,\beta} = M_\beta$.

For cases 2 to 6, assume in addition that (x, y) is chosen such that $(C_f)_{\mathrm{red}}$ has local equation $x = 0$ or $xy = 0$ at β.

Case 2. β is a smooth point of $\mathrm{Supp}(R_f)$ *and* $\beta \notin \mathrm{Supp}(S_f)$. Then $x = 0$ is a local equation of $D := (R_f)_{\mathrm{red}}$ at β and, say, $u = 0$ is a local equation of $f(D)$ at α. We have $u = x^a u'$, where $a \geq 2$ and x does not divide u'. The local equation at β of C_f is given by

$$\mathrm{Jac}_\beta(f) = x^{a-1}\left(au'\frac{\partial v}{\partial y} + x\mathrm{Jac}(u', v)\right). \tag{3.2}$$

By lemma 1, $\mathrm{ord}_D R_f = a-1$. Since C_f is a divisor with strict normal crossings, $au'\frac{\partial v}{\partial y}$ is a unit. f is then reduced at β to the monomial form (1)

$$\begin{cases} u &= \gamma x^a \\ v &= y \end{cases},$$

where γ is a unit and p does not divide a.

Case 3. β is a singular point of $\mathrm{Supp}(R_f)$. Then $xy = 0$ is a local equation of $D_1 + D_2 = (R_f)_{\mathrm{red}}$ at β. Suppose, if possible, that $u = 0$ is a local equation of $f(D_1 \cup D_2)$ at α. Hence $u = x^a y^b u'$, where $a, b \geq 2$ and neither x nor y divides u'. The local equation at β of C_f is given by

$$\mathrm{Jac}_\beta(f) = x^{a-1} y^{b-1} \left(ayu' \frac{\partial v}{\partial y} - bxu' \frac{\partial v}{\partial x} + xy\mathrm{Jac}(u', v) \right).$$

By lemma 1, $\mathrm{ord}_{D_1} R_f = a - 1$ and $\mathrm{ord}_{D_2} R_f = b - 1$. Since C_f is a divisor with strict normal crossings, one gets that

$$ayu' \frac{\partial v}{\partial y} - bxu' \frac{\partial v}{\partial x} + xy\mathrm{Jac}(u', v)$$

is a unit: a contradiction. So $uv = 0$ is a local equation of $f(D_1 \cup D_2)$ at α, and f is then reduced at β to the monomial form (1)

$$\begin{cases} u &= \gamma x^a \\ v &= \delta y^d \end{cases}.$$

The local equation of C_f at β is given by

$$\mathrm{Jac}_\beta(f) = x^{a-1} y^{d-1} (ad\gamma\delta + g), \tag{3.3}$$

where g is a nonunit. Hence $\gamma\delta$ is a unit and p does not divide ad.

Case 4. β is a smooth point of $\mathrm{Supp}(S_f)$ *and* $\beta \notin \mathrm{Supp}(R_f)$. Then $x = 0$ is a local equation of $E := (S_f)_{\mathrm{red}}$ at β. We have $u = x^a u'$ and $v = x^c v'$, where $a, c \geq 1$ and x does not divide $u'v'$. The local equation at β of C_f is given by

$$\mathrm{Jac}_\beta(f) = x^{a+c-1} \left(au' \frac{\partial v'}{\partial y} - cv' \frac{\partial u'}{\partial y} + x\mathrm{Jac}(u', v') \right). \tag{3.4}$$

By assumption *(ii)*, $\mathrm{ord}_E S_f = a + c - 1$. Since C_f is a divisor with strict normal crossings, it follows that $au' \frac{\partial v'}{\partial y} - cv' \frac{\partial u'}{\partial y}$ is a unit.
Suppose $u'v'$ is not a unit, say, v' is not. Hence $au' \frac{\partial v'}{\partial y}$ is a unit. f is then reduced at β to the monomial form (1)

$$\begin{cases} u &= \gamma x^a \\ v &= x^c y \end{cases},$$

where γ is a unit and p does not divide a.
Suppose $u'v'$ is a unit. f is then reduced at β to the monomial form (2).

Case 5. $\beta \in \mathrm{Supp}(R_f) \cap \mathrm{Supp}(S_f)$. Then $x = 0$ (resp. $y = 0$) is a local equation of $D := (R_f)_{\mathrm{red}}$ (resp. $E := (S_f)_{\mathrm{red}}$) at β. We have $u = x^a y^b u'$ and $v = y^d v'$, where $a \geq 2$, $b, d \geq 1$ and neither x nor y divides $u'v'$. The local equation at β of C_f is given by

$$\mathrm{Jac}_\beta(f) = x^{a-1} y^{b+d-1} \left(adu'v' + ayu' \frac{\partial v'}{\partial y} + xg \right) \tag{3.5}$$

for some g. By lemma 1, $\mathrm{ord}_D R_f = a-1$. By assumption *(ii)*, $\mathrm{ord}_E S_f = b+d-1$. Since C_f is a divisor with strict normal crossings, one gets that $adu'v'$ is a unit. f is then reduced at β to the monomial form (1)

$$\begin{cases} u &= \gamma x^a y^b \\ v &= \delta y^d \end{cases},$$

where $\gamma\delta$ is a unit and p does not divide ad.

Case 6. β is a singular point of $\mathrm{Supp}(S_f)$. Then $xy = 0$ is a local equation of $E_1 + E_2 = (S_f)_{\mathrm{red}}$ at β. Hence $u = x^a y^b u'$ and $v = x^c y^d v'$, where $a, b, c, d \geq 1$ and neither x nor y divides $u'v'$. The local equation at β of C_f is given by

$$\mathrm{Jac}_\beta(f) = x^{a+c-1} y^{b+d-1} \left((ad-bc)u'v' + g\right), \tag{3.6}$$

where g is a nonunit. By assumption *(ii)*, $\mathrm{ord}_{E_1} S_f = a + c - 1$ and $\mathrm{ord}_{E_2} S_f = b + d - 1$. Since C_f is a divisor with strict normal crossings, one gets that $(ad - bc)u'v'$ is a unit. f is then reduced at β to the monomial form (1).

The only if part of the proposition easily follows by applying formulas (3.2) to (3.6) to the monomial expression (1) or (2). The last statement is obvious. This completes the proof.

In order to construct a tamely monomial model dominating a given model as above, it is necessary to study the behaviour of the complexity i_β under blow-up. This is achieved in propositions 6 and 7 below.

Proposition 6 *Let $f: Y \to X$ be a nonsingular model as above, and let E be a reduced irreducible component of S_f, with $\alpha := f(E) \in X$.*
Assume that $M_\alpha \mathcal{O}_Y$ is locally principal. Let $\eta: X' \to X$ be the blowing-up of α, and $f': Y \to X'$ be the induced map.
Assume in addition that $\alpha' := f'(E) \in X'$ is a point, and let i_E (resp. i'_E) be the complexity of E w.r.t. f (resp. f'). Then $i'_E \leq i_E$.

Proof: Pick an admissible r.s.p. (u, v) at α achieving i_E. Say, $\nu_E(u) = \min_{t\in M_\alpha}\{\nu_E(t)\}$. Then $u = 0$ is a local equation of the exceptional divisor of η at α'. Pick v' such that (u, v') is an admissible r.s.p. at α' with $\nu_E(v')$ maximal. Consequently

$$\begin{array}{rcrcccl} i'_E & \leq & \nu_E(S_{f'}) & + & 1 & - & \nu_E(uv') \\ & = & \nu_E(S_f) - \nu_E(u) & + & 1 & - & \nu_E(uv') \quad = i_E + \nu_E(v) - \nu_E(uv') \end{array}.$$

If $\nu_E(u) = \nu_E(v)$, then $\nu_E(v) - \nu_E(uv') < 0$ and $i'_E < i_E$.
If $\nu_E(u) < \nu_E(v)$, then $(u, \frac{v}{u})$ is an admissible r.s.p. at α' and consequently $\nu_E(v) = \nu_E(u\frac{v}{u}) \leq \nu_E(uv')$, i.e. $i'_E \leq i_E$.

Lemma 2 *Let $f: Y \to X$ be a nonsingular model as above, and $\beta \in \mathrm{Supp}(S_f)$, with $\alpha := f(\beta)$. Let $x = 0$ be a local equation of a reduced component E of S_f passing through β, and (u,v) be a r.s.p. at α achieving i_E. Write*

$$\begin{cases} u &= x^a u' \\ v &= x^c v' \end{cases},$$

where $a, c \geq 1$, and x does not divide $u'v'$. Let $\delta := \mathrm{g.c.d.}(a,c)$.
Assume that ${u'}^{\frac{c}{\delta}}$ divides ${v'}^{\frac{a}{\delta}}$, and that $\frac{{v'}^{\frac{a}{\delta}}}{{u'}^{\frac{c}{\delta}}}$ is not a unit in $\mathcal{O}_{Y,\beta}$.
The extension of divisorial valuation rings $\mathcal{O}_{Y,E}/\mathcal{O}_{Y,E} \cap K$ is tamely ramified if and only if $i_E = 0$.

Proof: Choose a r.s.p. (x,y) at β. A local equation at β of C_f is given as in (3.4) by

$$\mathrm{Jac}_\beta(f) = x^{a+c-1}\left(au'\frac{\partial v'}{\partial y} - cv'\frac{\partial u'}{\partial y} + x\mathrm{Jac}(u',v')\right). \tag{3.7}$$

Then $i_E = 0$ if and only if x does not divide $au'\frac{\partial v'}{\partial y} - cv'\frac{\partial u'}{\partial y}$. Let $\varphi := \frac{v^{\frac{a}{\delta}}}{u^{\frac{c}{\delta}}}$. By assumption, $\varphi \in \mathcal{O}_{Y,\beta}$ and φ is not a unit.

Let I be the integral closure of the ideal $(u^{\frac{c}{\delta}}, v^{\frac{a}{\delta}})$. Then I is a *simple* complete M_α-primary ideal (p.385, appendix 5 of [ZS2]). There are local inclusions

$$\mathcal{O}_{X,\alpha} \subset R_\mathcal{Q} \subseteq \mathcal{O}_{Y,\beta}\ , \tag{3.8}$$

where $R = \mathcal{O}_{X,\alpha}\left[\frac{I}{u^{\frac{c}{\delta}}}\right]$ and $\mathcal{Q} := M_\beta \cap R$. By construction, $\varphi \in R$ and is neither a unit nor is divisible by x in $\mathcal{O}_{Y,\beta}$. This implies that $\mathrm{ht}((x) \cap R) = 1$.

Remark: In case E is the unique component of S_f passing through β, the ring $R_\mathcal{Q}$ is the local ring *lying below* $\mathcal{O}_{Y,\beta}$ according to Abhyankar's terminology (cf. prop. 2 and def. 4 of [Ab4]).

By Zariski's theory of complete ideals in two-dimensional regular local rings ((E) p.391, [ZS2]), there is a 1-1 correspondance between simple complete M_α-primary ideals of $\mathcal{O}_{X,\alpha}$ and divisorial valuation rings of K dominating $\mathcal{O}_{X,\alpha}$; the reduced exceptional divisor of the blow-up $\overline{X} := \mathrm{Proj}(\bigoplus_{n \geq 0} I^n) \to \mathrm{Spec}\mathcal{O}_{X,\alpha}$ is an irreducible curve F and $V := \mathcal{O}_{\overline{X},F}$ is a divisorial valuation ring (proposition 21.3 and remark following, [Li]). By what preceeds,

$$V = R_{(x)\cap R} = \mathcal{O}_{Y,E} \cap K.$$

Let t be a uniformizing parameter of V. Since I is a monomial ideal, the value group of V is generated by the values $\mathrm{ord}_t u$ and $\mathrm{ord}_t v$; this follows from [Sp], lemmas 8.1 and 8.2, and corollary 8.5, where k needs not be algebraically closed in the special case of a monomial ideal. Since $\mathrm{ord}_t\varphi = 0$, this implies that $\mathrm{ord}_t u = \frac{a}{\delta}$ and $\mathrm{ord}_t v = \frac{c}{\delta}$. Hence $IV = (t)^{\frac{a}{\delta}\frac{c}{\delta}}$. On the other hand, $I\mathcal{O}_{Y,E} = (x)^{\frac{ac}{\delta}}$. Hence the ramification index of the extension of divisorial valuation rings $\mathcal{O}_{Y,E}/V$ is equal to δ.

The ideal I is generated by all monomials $u^m v^n$ such that

$$\frac{m}{\frac{c}{\delta}} + \frac{n}{\frac{a}{\delta}} \geq 1.$$

Since g.c.d.$(\frac{a}{\delta}, \frac{c}{\delta}) = 1$, one gets that

$$\nu_E(u^m v^n) = ma + nc > \frac{ac}{\delta} = \nu_E(u^{\frac{c}{\delta}})$$

for all such monomials provided $(m, n) \notin \{(\frac{c}{\delta}, 0), (0, \frac{a}{\delta})\}$. This proves that

$$\frac{R}{(x) \cap R} = \kappa(\alpha)[\overline{\varphi}],$$

where $\overline{\varphi}$ is the image of φ in the ring to the left. Let $\overline{\alpha}$ be the point of F corresponding to $\mathcal{Q}$. Then $\overline{\varphi}$ is a regular parameter of $\mathcal{O}_{F,\overline{\alpha}}$. By (3.8), the rational map $Y \cdots \to \overline{X}$ is defined at β. Besides, β (resp. $\overline{\alpha}$) is a smooth point of E (resp. F). Arguing as in lemma 1, one deduces that the residue field extension $\kappa(\mathcal{O}_{Y,E})/\kappa(V)$ $(= K(E)/K(F))$ is separable if and only if x does not divide $\frac{\partial\varphi}{\partial y}$ in $\mathcal{O}_{Y,\beta}$. We have

$$u'v'\frac{\partial\varphi}{\partial y} = \varphi\left(\frac{a}{\delta}u'\frac{\partial v'}{\partial y} - \frac{c}{\delta}v'\frac{\partial u'}{\partial y}\right).$$

Summing up, $\mathcal{O}_{Y,E}/V$ is tamely ramified if and only if x does not divide $au'\frac{\partial v'}{\partial y} - cv'\frac{\partial u'}{\partial y}$. By (3.7), this is equivalent to $i_E = 0$.

Proposition 7 *Let $f : Y \to X$ be a nonsingular model as above, and let $\beta \in Y$, with $\alpha := f(\beta) \in X$. Assume that $M_\alpha\mathcal{O}_{Y,\beta}$ is not a principal ideal. Let $Y' \to Y$ be the blowing-up of β, with exceptional divisor E', and $f' : Y' \to X$ be the composed map. Let $i_{E'}$ be the complexity of E' w.r.t. f'.*
Assume in addition that either f is monomial at β (i.e. has the local form (1) or (2) of definition 3) w.r.t. some admissible r.s.p. at α, or that the map $f^{(\alpha)} : Y^{(\alpha)} \to \mathrm{Spec}\mathcal{O}_{X,\alpha}$ obtained from f by the base change $\mathrm{Spec}\mathcal{O}_{X,\alpha} \to X$ is tamely ramified (definition 2). The following holds

(i) if $\beta \notin \mathrm{Supp}(S_f)$, then $i_{E'} = 0$.

(ii) if $\beta \in \mathrm{Supp}(S_f)$, then $i_{E'} \leq i_\beta$, and $i_{E'} < i_\beta$ if $i_\beta > 0$.

Proof: First assume that $\alpha \notin \mathrm{Supp}(B_f)$. By corollary 2, this implies that $M_\alpha\mathcal{O}_{Y,\beta} = M_\beta$, since $M_\alpha\mathcal{O}_{Y,\beta}$ is not a principal ideal. We have $i_{E'} = 1+1-2 =$

0 in this case.

Now assume that $\alpha \in \mathrm{Supp}(B_f)$. Pick an admissible r.s.p. (u, v) at α achieving i_E for every reduced irreducible component E of S_f passing through β. We consider six cases as in proposition 5.

Case 1. We have $M_\alpha \mathcal{O}_{Y,\beta} = M_\beta$ and $i_{E'} = 0$ as above.

Case 2. By definition or by proposition 5, f is reduced at β to the monomial form (1)

$$\begin{cases} u & = & \gamma x^a \\ v & = & y \end{cases},$$

where γ is a unit and p does not divide a. Hence $i_{E'} = a + 1 - (a + 1) = 0$.

Case 3. By definition or by proposition 5, f is reduced at β to the monomial form (1)

$$\begin{cases} u & = & \gamma x^a \\ v & = & \delta y^d \end{cases},$$

where $\gamma\delta$ is a unit and p does not divide ad. Hence $i_{E'} = a+d-1+1-(a+d) = 0$.

Case 4. First assume that f is monomial at β. By proposition 5, $i_\beta = 0$ and f is reduced at β to the monomial form (1)

$$\begin{cases} u & = & \gamma x^a \\ v & = & x^c y \end{cases},$$

where γ is a unit and p does not divide a. Hence $i_{E'} = a+c+1-(a+c+1) = 0$.
Assume now that $f^{(\alpha)}$ is tamely ramified. Write

$$\begin{cases} u & = & x^a u' \\ v & = & x^c v' \end{cases},$$

where $a, c \geq 1$ and x does not divide $u'v'$. Then $i_{E'} \leq i_\beta + 1 - \mathrm{ord}_\beta(u'v')$. Since $M_\alpha \mathcal{O}_{Y,\beta}$ is not a principal ideal, $u'v'$ is not a unit. Hence $i_{E'} \leq i_\beta$. Assume equality holds. Then u' or v' is a unit, say u', and v' is not. Hence lemma 2 applies, and gives that $i_\beta = 0$.

Case 5. First assume that f is monomial at β. By proposition 5, $i_\beta = 0$ and f is reduced at β to the monomial form (1)

$$\begin{cases} u & = & \gamma x^a y^b \\ v & = & \delta y^d \end{cases},$$

where $\gamma\delta$ is a unit and p does not divide ad. Hence $i_{E'} = a + b + d - 1 + 1 - (a + b + d) = 0$.
Assume now that $f^{(\alpha)}$ is tamely ramified. Write

$$\begin{cases} u & = & x^a u' \\ v & = & x^c y^d v' \end{cases},$$

where $a, c \geq 1$, $d \geq 2$, and neither x nor y divides $u'v'$. Let D be the reduced component of R_f with equation $y = 0$. By lemma 1, $\mathrm{ord}_D R_f = d - 1$. Then $i_{E'} \leq i_\beta - \mathrm{ord}_\beta(u'v') \leq i_\beta$. Assume $i_{E'} = i_\beta$. Then $u'v'$ is a unit. By lemma 2, this implies that $i_\beta = 0$.

Case 6. First assume that f is monomial at β. By proposition 5, $i_\beta = 0$ and f is reduced at β to the monomial form (1)

$$\begin{cases} u &= \gamma x^a y^b \\ v &= \delta x^c y^d \end{cases},$$

where $\gamma\delta$ is a unit and p does not divide $ad - bc$. Hence $i_{E'} = a + b + c + d - 1 + 1 - (a + b + c + d) = 0$.
Assume now that $f^{(\alpha)}$ is tamely ramified. Write

$$\begin{cases} u &= x^a y^b u' \\ v &= x^c y^d v' \end{cases},$$

where $a, b, c, d \geq 1$, and neither x nor y divides $u'v'$. Let E_1 (resp. E_2) be the reduced component of S_f with equation $x = 0$ (resp. $y = 0$). Then

$$i_{E'} \leq i_{E_1} + i_{E_2} - \mathrm{ord}_\beta(u'v'). \tag{3.9}$$

Since $\alpha \in \mathrm{Supp}(B_f)$ and (u, v) is admissible, $u = 0$ is a local equation of a component of $\mathrm{Supp}(B_f)$. Since $f^* B_f$ is a divisor with strict normal crossings, u' is a unit.
Suppose that v' is not a unit. By possibly permuting x and y, it can be assumed that $ad - bc \geq 0$. Lemma 2 hence applies, and we get $i_{E_1} = 0$. Hence $i_\beta = i_{E_2}$. By (3.9), this gives $i_{E'} < i_\beta$.
Suppose that v' is a unit. Since $M_\alpha \mathcal{O}_{Y,\beta}$ is not a principal ideal, $ad - bc \neq 0$. After possibly permuting u and v, and x and y, lemma 2 applies w.r.t. both of E_1 and E_2. Hence $i_\beta = i_{E_1} = i_{E_2} = 0$ and this gives $i_{E'} = 0$ by (3.9).

4 The algorithm

Let $f : Y \to X$ be a *nonsingular* model of L/K, and α a point in X. We define a new nonsingular model f_α dominating f as follows: let $X_\alpha \to X$ be the blowing-up of α, and $Y_\alpha \to Y$ be the minimal composition of point blowing-ups such that $M_\alpha \mathcal{O}_{Y_\alpha}$ is locally invertible (i.e. the minimal resolution of singularities of the blow-up of Y along the ideal $M_\alpha \mathcal{O}_Y$). By the universal property of blow-up, there exists a map $f_\alpha : \ Y_\alpha \to X_\alpha$.

Lemma 3 *The above model f_α is the minimal (w.r.t. domination) nonsingular model $f' : Y' \to X'$ of L/K dominating f, and such that the center on X' of the M_α-adic valuation of K is a curve.*

Proof: Let $f' : Y' \to X'$ be a nonsingular model dominating f such that the center on X' of the M_α-adic valuation of K is a curve. By proposition 3, $X' \to X$ factors through the blow-up X_α of X at α. Since $M_\alpha \mathcal{O}_{Y'}$ is locally invertible, $Y' \to Y$ factors through Y_α.

Proposition 8 *There exists a minimal nonsingular model $\widetilde{f}$ dominating any given model f of L/K, and such that both of $B_{\widetilde{f}}$ and $\widetilde{f}^* B_{\widetilde{f}}$ are divisors with strict normal crossings. Any nonsingular model dominating $\widetilde{f}$ has the same property.*

Proof: Since any algebraic k-surface admits a minimal resolution of singularities (A. p.155, [Li2]), X can be replaced with its minimal resolution X' and Y by the minimal resolution Y' of the normalization of X in L. It hence can be assumed that $f : Y \to X$ is nonsingular. Let $f' : Y' \to X'$ be a nonsingular model dominating f. Let $\alpha \in X$ be a point of $\mathrm{Supp}(B_f)$ which is not a strict normal crossing. If $B_{f'}$ is a divisor with strict normal crossings, f' is not an isomorphism above α. By lemma 3 and proposition 3, f' dominates f_α.
By embedded resolution of curves in surfaces (V.3.9, [Ha]), we hence may assume that B_f has strict normal crossings. Suppose that $f^* B_f$ does not have only strict normal crossings. Let $\eta : \widetilde{Y} \to Y$ be the minimal composition of point blow-ups such that $\eta^* C_f$ has only strict normal crossings. Set $\widetilde{f} : \widetilde{Y} \to X$ to be the morphism induced by f.

Lemma 4 *A tamely monomial model $f : Y \to X$ is tamely ramified.*

Proof: Let V be a divisorial k-valuation ring of L having a center in Y.
Case 1: the center of V in X is a curve. By proposition 5*(i)*, $V/V \cap K$ is tamely ramified.
Case 2 : the center of V in X is a point α. Consider the model $f_\alpha : Y_\alpha \to X_\alpha$. By propositions 5, 6 and 7, f_α also is a tamely monomial model. If the center of V in X_α is a curve, we are done by case 1. If it is a point α_1, then $\mathcal{O}_{X,\alpha} \subset \mathcal{O}_{X_\alpha,\alpha_1}$, and we apply again case 2 to f_{α_1}.
This process terminates after a finite number of steps by proposition 2 applied to $R = \mathcal{O}_{X,\alpha}$ and $S = V \cap K$.

Proof of theorem 1 (stated at the end of section 2): $(i) \Longrightarrow (ii)$ is trivial and $(ii) \Longrightarrow (iii)$ has been proved in lemma 4 above. We prove $(iii) \Longrightarrow (i)$.

By proposition 8, it can be assumed that f is a nonsingular model and that both of B_f and $f^* B_f$ are divisors with strict normal crossings. Hence the results of section 3 apply.

The algorithm: Assume furthermore that f is *not* a tamely monomial model. By proposition 5, there exists a reduced irreducible component E of S_f with $i_E > 0$. Choose such an E with i_E maximal, and let $\alpha := f(E) \in X$. We get a new nonsingular model $f_\alpha :\ Y_\alpha \to X_\alpha$ dominating f and such that both of B_{f_α} and $f_\alpha^* B_{f_\alpha}$ are divisors with strict normal crossings by proposition 8 above.
Iterate the process if f_α is not tamely monomial. This gives rise to a sequence of nonsingular models $f, f_{\alpha_1}, \dots, f_{\alpha_i}, \dots$, such that f_{α_i} dominates $f_{\alpha_{i-1}}$ for $i \geq 1$. It will be proved below that for some $i \geq 1$, f_{α_i} is the minimal tamely monomial resolution of f.

Proof of theorem 1 continued: Let E be a reduced irreducible component of S_f such that $i_E > 0$. Let $\alpha := f(E)$. We first claim that any tamely monomial model $f' :\ Y' \to X'$ dominating f (if there exists one) dominates f_α as well. By lemma 3, it is sufficient to show that the center on X' of the M_α-adic valuation of K is a curve. Since X' is nonsingular, it is also sufficient by proposition 3 to prove that $X' \to X$ is not an isomorphism above α. Assume the contrary. Let $f^{(\alpha)}$ (resp. ${f'}^{(\alpha)}$) be the map obtained from f (resp. f') by the base change $\mathrm{Spec}\mathcal{O}_{X,\alpha} \hookrightarrow X$. There is a commutative diagram with proper maps

$$\begin{array}{ccc} {Y'}^{(\alpha)} & \xrightarrow{{f'}^{(\alpha)}} & \mathrm{Spec}\mathcal{O}_{X,\alpha} \\ {\scriptstyle\pi}\downarrow & & \downarrow\wr \\ Y^{(\alpha)} & \xrightarrow{f^{(\alpha)}} & \mathrm{Spec}\mathcal{O}_{X,\alpha} \end{array} .$$

Let E' be the strict transform of E in ${Y'}^{(\alpha)}$. By definition, $i_{E'} = i_E$. But ${f'}^{(\alpha)}$ is tamely monomial by assumption and thus $i_{E'} = 0$ by proposition 5. This is a contradiction, since $i_E > 0$, and the claim is proved.

Let

$$I_f := \max_{\beta \in \mathrm{Supp}(S_f)} i_\beta \ ,$$

and

$$\Sigma_f := \{\mathcal{O}_{Y,E} \mid E \text{ is a reduced irreducible component of } S_f \text{ with } i_E = I_f\}.$$

To conclude the proof, it must be shown that for some $i \geq 1$, the model f_{α_i} in the algorithm above is tamely monomial, i.e. $I_{f_{\alpha_i}} = 0$. Assume not. By lemma 5 below, $(I_{f_{\alpha_j}}, \Sigma_{f_{\alpha_j}})$ is constant for large enough j. Pick a divisorial valuation ring $V \in \bigcap_{j \geq 0} \Sigma_{f_{\alpha_j}}$ of L, such that for infinitely many values of j, α_j

is the center of V in X_{α_j}. This gives rise to an increasing sequence of quadratic transforms $(\mathcal{O}_{X_{\alpha_j},\alpha_j})$ dominated by $V \cap K$. But any such sequence must be finite by proposition 2.

Lemma 5 *With notations as above, assume $I_f > 0$ and let E be a reduced irreducible component of S_f with $i_E = I_f$. Let $\alpha := f(E)$.*

Then $(I_{f_\alpha}, \Sigma_{f_\alpha}) \leq (I_f, \Sigma_f)$ for the lexicographical ordering, where the second summand is (partially) ordered by inclusion.

Proof: Let $f' : Y_\alpha \to X$. By proposition 7, $(I_{f'}, \Sigma_{f'}) = (I_f, \Sigma_f)$. By proposition 6, $(I_{f_\alpha}, \Sigma_{f_\alpha}) \leq (I_{f'}, \Sigma_{f'})$.

5 Toroidalization of morphisms of surfaces

Suppose that k is an algebraically closed field. We recall the definitions of toroidal varieties and morphisms from [KKMS] and [AKa].

Suppose that X is a normal k-variety, with an open subset U_X. The embedding $U_X \subset X$ is *toroidal* if for every $x \in X$ there exists an affine toric variety X_σ, a point $s \in X_\sigma$, and an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X_\sigma,s}$ such that the ideal of $X - U_X$ corresponds to the ideal of $X_\sigma - T$, where T is the torus in X_σ. Such a pair (X_σ, s) is called a local model at $x \in X$.

A dominant morphism $f : (U_X \subset X) \to (U_B \subset B)$ of toroidal embeddings is called *toroidal* if for every closed point $x \in X$ there exist local models (X_σ, s) at x, (X_τ, t) at $f(x)$ and a toric morphism $g : X_\sigma \to X_\tau$ such that the following diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,x} & \leftarrow & \hat{\mathcal{O}}_{X_\sigma,s} \\ \hat{f}^* \uparrow & & \uparrow \hat{g}^* \\ \hat{\mathcal{O}}_{B,f(x)} & \leftarrow & \hat{\mathcal{O}}_{X_\tau,t} \end{array}$$

By a k-surface, we mean a proper, 2 dimensional, integral, normal k-variety.

Suppose that X is a nonsingular k-surface, and D_X is a SNC (Simple Normal Crossings) divisor on X. Then the embedding $X - D_X \subset X$ is toroidal.

In this section, we will consider tamely ramified morphisms $f : Y \to X$, where X and Y are k-surfaces with respective (Weil) divisors D_X and D_Y such that $f^{-1}(D_X) = D_Y$, set theoretically. If $\eta : X_1 \to X$ is a birational proper morphism of k-surfaces, we can define a divisor $D_{X_1} = \eta^{-1}(D_X)$ on X_1. If D_X is a SNC divisor, and X_1 is nonsingular, then D_{X_1} is a SNC divisor.

We will say that $f : Y \to X$ is toroidal relative to D_Y and D_X if

$$f : (Y - D_Y \subset Y) \to (X - D_X \subset X)$$

is toroidal.

We will prove that tamely ramified morphisms of k-surfaces can be made toroidal. While this result is known to be true, for instance it is implicit in

[AKi], the result is of sufficient interest that we give a statement of the theorem, and an outline of a proof. Recall that, in this section, k is an algebraically closed field of characteristic zero.

Theorem 2 *Suppose that $f : Y \to X$ is a dominant tamely ramified morphism of k-surfaces, and D_X, D_Y are respective divisors on X and Y, such that $f^{-1}(D_X) = D_Y$ and D_Y contains all singular points of f and of Y. Then there exist projective birational morphisms $\beta : Y' \to Y$ and $\alpha : X' \to X$ such that Y' and X' are nonsingular and $f' : Y' \to X'$ is a toroidal morphism, relative to $\beta^{-1}(D_Y)$ and $\alpha^{-1}(D_X)$.*

We need to generalize the notion of a tamely monomial model defined in Definition 3 of Section 2, to incorporate information about the divisors D_X and D_Y.

Definition 6 *A nonsingular model $f : Y \to X$ is said to be tamely monomial with respect to SNC divisors D_Y on Y and D_X on X if $f^{-1}(D_X) = D_Y$ set theoretically, $C_f \subset D_Y$, and if for every $\beta \in Y$, with $\alpha := f(\beta) \in X$, there exist regular systems of parameters (u, v) of $\mathcal{O}_{X,\alpha}$ and (x, y) of $\mathcal{O}_{Y,\beta}$ such that*

(i) If $\alpha \in \mathrm{Supp}(D_X)$, D_X is locally at α defined by $u = 0$ or $uv = 0$.

(ii) Either

$$\left\{\begin{array}{lll} u & = & \gamma x^a y^b \\ v & = & \delta x^c y^d \end{array}\right. ,$$

where $\gamma\delta$ is a unit in $\mathcal{O}_{Y,\beta}$, p does not divide $ad - bc$ and D_Y is locally at β defined by $xy = 0$, or

$$\left\{\begin{array}{lll} u & = & \gamma x^a \\ v & = & \delta x^c \end{array}\right. ,$$

where both of $\gamma\delta$ and $a\gamma\frac{\partial\delta}{\partial y} - c\delta\frac{\partial\gamma}{\partial y}$ are units in $\mathcal{O}_{Y,\beta}$, and D_Y is locally at β defined by $x = 0$.

Theorem 3 *Suppose that $f : Y \to X$ is a tamely ramified morphism of k-surfaces. Suppose that D_X and D_Y are divisors on X and Y repectively, such that $f^{-1}(D_X) = D_Y$, and D_Y contains all singular points of the mapping f, and all singular points of Y. Then there exist projective birational morphisms $\tau : Y_1 \to Y$ and $\sigma : X_1 \to X$ such that Y_1 and X_1 are nonsingular, the divisors $D_{Y_1} = \tau^{-1}(D_Y)$ and $D_{X_1} = \sigma^{-1}(D_X)$ are SNC divisors, and $f_1 : Y_1 \to X_1$ is tamely monomial with respect to D_{Y_1} and D_{X_1}.*

By an argument as is section 4, we see that in fact the morphism $f_1 : Y_1 \to X_1$ constructed in the proof of Theorem 3 is the minimal tamely monomial morphism with respect to the pullback of the divisors D_Y and D_X.

The proof of Theorem 3 is a variation on the proof of the existence of a tamely monomial resolution, given in the preceeding sections. Note that any model $f : Y \to X$ has divisors D_X and D_Y as in the assumptions of the theorem.

After performing projective birational morphisms on X and Y, we can assume that X and Y are nonsingular, and D_X and D_Y are SNC divisors. We must make a change in the definition of admissibility of Section 3.

Let α be a point in X. A r.s.p. (u,v) of $\mathcal{O}_{X,\alpha}$ is said to be *admissible* if $\alpha \notin \mathrm{Supp}(D_X)$, or if $\alpha \in \mathrm{Supp}(D_X)$ and D_X is locally at α defined by $u=0$ or $uv=0$ (see *(i)* in definition 6).

The results of Chapters 3 and 4 can now be easily modified to produce a proof of Theorem 3.

By Theorem 3, we may suppose that X, Y are nonsingular k-surfaces, and that $f: Y \to X$ is tamely monomial with respect to divisors D_X on X and D_Y on Y. Thus D_X and D_Y have simple normal crossings, $D_Y = f^{-1}(D_X)$ and $C_f \subset D_Y$. We further have that for all $p \in D_X$ and $q \in f^{-1}(p)$ there exist regular parameters (u,v) in $\mathcal{O}_{X,p}$ and (x,y) in $\hat{\mathcal{O}}_{Y,q}$ such that one of the following holds.

Case 1 $D_{X,p} = V(uv)$, $D_{Y,q} = V(xy)$, $u = x^a y^b$, $v = x^c y^d$, $ad - bc \neq 0$.

Case 2 $D_{X,p} = V(uv)$, $D_{Y,q} = V(x)$, $u = x^a$, $v = x^c(y+\alpha)$, $0 \neq \alpha \in k$.

Case 3 $D_{X,p} = V(u)$, $D_{Y,q} = V(xy)$, $u = x^a y^b$, $v = x^c y^d$, $ad - bc \neq 0$.

Case 4 $D_{X,p} = V(u)$, $D_{Y,q} = V(x)$, $u = x^a$, $v = x^c(y+\alpha)$, $\alpha \in k$.

We will call cases 2 and 4 1-points, cases 1 and 3 2-points. Regular parameters as above will be called permissible.

The morphism is toroidal (relative to D_X and D_Y) if all points satisfy cases 1, 2 or 4*, where 4* is

Case 4* $D_{X,p} = V(u)$, $D_{Y,q} = V(x)$, $u = x^a$, $v = y$.

We will call a point $q \in Y$ good (or bad) if f is toroidal (not toroidal) at q. By direct calculation, we see that

Lemma 6 *The locus of bad points of Y is closed of pure codimension 1 in Y.*

The set of image points in X of bad points is finite.

Let

$$G_f = \{q \in Y | q \in f^{-1}(p) \text{ is a 1-point such that } p \in X \text{ is the image of a bad point}\}.$$

If $q \in G_f$, and (x,y), (u,v) are permissible parameters at q and p, we have an expression

$$u = x^a$$
$$v = x^c(\gamma + y)$$

for some $\gamma \in k$, and $D_{X,p} = V(u)$, $D_{Y,q} = V(x)$. We can define an invariant for $q \in G_f$ by

$$I(q,X) = \max\{c-a|(x,y),(u,v) \text{ are permissible parameters at } q \text{ and } p\}$$

We then further define a global invariant

$$r(Y,X) = \max\{I(q,X)|q \in G_f\}.$$

Suppose that $r(Y,X) > 0$, and that $p \in X$ is such that there exists $q \in f^{-1}(p)$ with $I(q,X) = r(Y,X)$.

Let $\pi : X_1 \to X$ be the blowup of p. Let $f_1 : Y \to X_1$ be the induced rational map, with $D_{X_1} = \pi^{-1}(D_X)$.

The following two lemmas are obtained by direct calculation of the effect of a quadratic transform at $q \in Y$ or at $p = f(q) \in X$.

Lemma 7 *Suppose that $q \in f^{-1}(p)$ is such that f_1 is a morphism near q. Suppose that q is a 1-point. If $I(q,X) \le 0$, then q is a good point for f_1. If $I(q,X) > 0$, then $I(q,X_1) < I(q,X)$.*

The points where f_1 is not a morphism have one of the following forms.

$$u = x^a, v = x^c y, \text{ with } c < a, (D_Y = V(x), D_X = V(u)), \text{ or} \tag{3}$$

$$u = x^a y^b, v = x^c y^d, \text{ with } a < c, b > d \text{ or } a > c, b < d, (D_Y = V(xy), D_X = V(u)). \tag{4}$$

Lemma 8 *Suppose that $q \in f^{-1}(p)$ is such that f_1 is not a morphism near q. Let $\tau : Y_1 \to Y$ be the blowup of q, $f_2 = f_1 \circ \tau$.*

Suppose that q is a 1-point, so that q has the form of (3). Suppose that $q' \in \tau^{-1}(q)$. Then f_2 is a morphism at q' and f_2 is toroidal at q' relative to $D_{Y_1} = \tau^{-1}(D_Y)$ and D_{X_1}, unless $\hat{\mathcal{O}}_{Y_1,q'}$ has regular parameters x_1, y_1 such that

$$u = x_1^a, v = x_1^{c+1} y_1,$$

with $D_{Y_1} = V(x_1), D_X = V(u)$, and

$$I(q,X) < I(q',X) < 0.$$

Suppose that q is a 2-point, so that q has the form of (4). We can assume that $a > c$ and $b < d$. Suppose that $q' \in \tau^{-1}(q)$.

If q' is a 1-point, then f_2 is a morphism at q' and either f_2 is toroidal at q' relative to $D_{Y_1} = \tau^{-1}(D_Y)$, and D_{X_1}, or we have $a + b < c + d$, so that $\hat{\mathcal{O}}_{Y_1,q'}$ has regular parameters (x_1, y_1), $\mathcal{O}_{X_1,f_2(q')}$ has regular parameters (u_1, v_1), such that

$$u_1 = u = x_1^{a+b}, v_1 = \frac{v}{u} = x_1^{c+d-(a+b)}(\gamma + y_1),$$

with $\gamma \ne 0$, $D_{Y_1} = V(x_1), D_{X_1} = V(u_1)$. In this case we have

$$I(q', X_1) = (c+d) - 2(a+b) < (d-b) - 1 < r(Y,X).$$

By the above two lemmas, we can conclude

Proposition 9 *There exist sequences of quadratic transforms* $\alpha : Y' \to Y$ *and* $\beta : X' \to X$ *such that* $f' : Y' \to X'$ *is a tamely monomial mapping, and* $r(Y', X') \le 0$*, relative to* $\alpha^{-1}(D_Y)$ *and* $\beta^{-1}(D_X)$*.*

Thus, we may assume that $r(Y, X) \le 0$. Suppose that $p \in X$ is the image of a bad point. Then $v \mid u$ at all 2-points above p. If q is a 1-point above p, then

$$u = x^a, v = x^c(\gamma + y)$$

with $\gamma \in k$ and $c \le a$. Let $\pi : X_1 \to X$ be the blowup of p, $f_1 : Y \to X_1$ be the induced rational map. Then f_1 is a morphism and is toroidal at all 2-points of $f^{-1}(p)$, and at all 1-points with $\gamma \ne 0$, and at all 1-points with $\gamma = 0$ and $c = a$. The only points q of $f^{-1}(p)$ where f_1 is not a morphism (and is not toroidal) are 1-points of the form

$$u = x^a, v = x^c y$$

with $I(q, X) = c - a < 0$. Let $\tau : Y_1 \to Y$ be the blowup of such a q. Let $f_2 = f_1 \circ \tau$. Then f_2 is a morphism and is toroidal at all points of $\tau^{-1}(q)$ except possibly at a point q' which has regular parameters (x_1, y_1) satisfying $x = x_1, y = x_1 y_1$,

$$u = x_1^a, v = x_1^{c+1} y_1$$

with

$$I(q, X) < I(q', X) < 0$$

By ascending induction on the negative number $I(q, X)$, we eventually construct a toroidalization. We thus have attained the conclusions of Theorem 2.

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