ABHYANKAR'S PROOF OF UNIFORMIZATION IN p-CYCLIC GALOIS EXTENSIONS

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The purpose of this note is to expound the following fundamental theorem of Abhyankar.

Theorem 0.1. Let K be a two dimensional algebraic function field over an algebraically closed field k of characteristic $p\neq 0$, let K^* be a Galois extension of K of degree p, and let ω be a rational nondiscrete valuation of K/k having only one extension ω^* to K^*. Assume that ω can be uniformized. Then ω^* can be uniformized.

This theorem is stated in Theorem 4 of [1] (and later in [2]) and is a critical part of Abhyankar's proof of local uniformization of a valuation of a two dimensional algebraic function field over an algebraically closed field. Abhyankar makes use of ramification theory to reduce to the case of Theorem 0.1.

The statement "ω can be uniformized" means that there exists a regular local ring R, with quotient field K, such that R is a localization of a finite type k-algebra (R is an algebraic local ring of K) and ω dominates R (ω is nonnegative on R and is positive on the maximal ideal m_R of R, or equivalently, if V is the valuation ring of ω, consisting of the elements of K which have nonnegative value, then $R\subset V$ and $m_V\cap R=m_R$ where m_V is the maximal ideal of V). The statement that ω has only one extension to K^* is sometimes also said as "ω does not split in K^*". The statement that ω is a rational nondiscrete valuation means that the value group of ω is (order isomorphic to) a subset of $\mathbf{Q}$ which is not isomorphic to $\mathbf{Z}$.

Suppose that R is a regular algebraic local ring of K which is dominated by ω. Let x,y be a regular system of parameters in R. let R_1 be the local ring of the blowup of the maximal ideal m_R of R (a quadratic transformation of R) which is dominated by ω. Then R_1 has a regular system of parameters x_1,y_1 of one of the following types:

$$x=x_1,\ y=x_1(y_1+\alpha_1) \tag{1}$$

with $\alpha_1\in k$, or

$$x=x_1y_1,\ y=y_1. \tag{2}$$

We can continue to blow up maximal ideals to construct a sequence of regular algebraic local rings,

$$R\rightarrow R_1\rightarrow R_2\rightarrow\cdots \tag{3}$$

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along (dominated by) ω, where each R_{i+1} has regular parameters (x_{i+1}, y_{i+1}) of one of the following types:

$$x_i = x_{i+1}, y_i = x_{i+1}(y_{i+1} + \alpha_{i+1}) \tag{4}$$

with $\alpha_{i+1} \in k$, or

$$x_i = x_{i+1}y_{i+1}, y_i = y_{i+1}. \tag{5}$$

The fact that ω is rational nondiscrete tells us that we must obtain a form (4) with $\alpha_{i+1} \neq 0$ infinitely many times in the sequence, and we must obtain a form (5) infinitely many times in the sequence.

We know from "embedded principalization of ideals" in regular local rings of dimension two, and since ω is rational nondiscrete, that if $f \in R$, then there exists an index i in the sequence (3) such that $f = x_i^n \delta$ where $n \in \mathbf{N}$ and δ is a unit in R_i

We now introduce a construction which will be used in the proof of the theorem. Since K^* is a galois extension of K of degree p, it is an Artin-Schreier extension. Thus $K^* = K[\overline{z}']$ where the minimal polynomial $g'(z)$ of $\overline{z}'$ over K has the form $g'(z) = z^p + z + d$. Since ω can be uniformized, there exists a regular algebraic local ring R of K such that ω dominates R. Write $d = \frac{a}{b}$ where $a, b \in R$. Setting $\overline{z} = b^p\overline{z}'$, we obtain that $K^* = K[\overline{z}]$ and the minimal polynomial of $\overline{z}$ over K has the form

$$g(z) = z^p + h^{p-1}z + f \tag{6}$$

where $h, f \in R$. The domain $S = R[\overline{z}] \cong R[z]/(g(z))$ is integral over R and has the quotient field K^*. Thus ω^* is nonnegative on S. Let T be the integral closure of R in K^*. Since ω does not split in K^*, T is a local ring. The center of ω^* on S is the maximal ideal $m_T \cap S$.

Let m_R be the maximal ideal of R, and let $N = m_R R[z] + zR[z]$, a maximal ideal of $R[z]$. For $h(z) \in R[z]$, define $\mathrm{ord}(h) = \max\{n \mid h \in N^n\}$. For $e \in R$, we define $\mathrm{ord}(e) = \max\{n \mid e \in m_R^n\}$.

We now make a fundamental observation.

Lemma 0.2. *Suppose that x, y are a regular system of parameters in R, so that the polynomial ring $k[x, y]$ is a subset of R. Suppose that $\mathrm{ord}(g(z)) > 0$. Then the center of ω^* on S is the maximal ideal $(x, y, \overline{z})$.*

Proof. The assumption that $\mathrm{ord}(g(z)) > 0$ implies that the ideal $m_1 = (x, y, \overline{z})$ is a maximal ideal of S which contracts to m_R. But m_1 is then the unique maximal ideal which dominates m_R, since ω does not split in K^*. ☐

We point out that our assumption of nonsplitting implies that $\mathrm{ord}(h) > 0$ in (6). Otherwise, the residue of $g(z)$ in $R/m_R[z] \cong k[z]$ would be an Artin Schreier polynomial, and there would be p distinct maximal ideals in S which contract to m_R, which we know cannot happen (it would contradict the assumption that ω does not split in K^*).

We will perform 3 types of operations on the polynomial ring $R[z]$, which induce birational transformations of S. Suppose that x, y are regular parameters in R. Then there is a natural inclusion of the polynomial ring $k[x, y]$ into R.

The first and simplest operation is to "clean" the coefficients of $g(z)$. Suppose that $A(x, y) \in k[x, y]$. We can make a change of variables in $R[z]$, replacing z with $z' = z - A(x, y)$. We then set $g'(z') = g(z' + A(x, y)) \in R[z']$. Set $\overline{z}' = \overline{z} - A(x, y)$. We then have that $S = R[\overline{z}'] \cong R[z']/(g'(z'))$.

The most basic case of this transformation is to make $\mathrm{ord}(g'(z')) > 0$. There exists $\alpha \in k$ such that $f - \alpha \in m_R$. Set $z' = z - \sqrt[p]{\alpha}$. Then, since $\mathrm{ord}(h) > 0$, we have that $\mathrm{ord}(g'(z')) > 0$. More generally, we can view f as an element of the completion $\hat{R} \cong k[[x,y]]$ of R, and "clean" to remove p-th powers from f be making substitutions $z' = z - A(x,y)$.

The second type of operation is to perform a quadratic transformation $R \rightarrow R_1$ along ω. The regular local ring R_1 has a regular system of parameters x_1, y_1 defined by (1) or (2). We view $g(z)$ as an (irreducible) element of the polynomial ring $R_1[z]$. Set $S_1 = R_1[\overline{z}] \cong R_1[z]/(g(z))$. S_1 is a birational extension of S, such that ω^* is nonnegative on S_1.

Using quadratic transformations of R, we can make h a monomial. By "embedded principalization of ideals" in R, we can construct a sequence $R \rightarrow R_i$ of quadratic transformations (3) along ω, such that in $R_i[z]$, we have

$$g(z) = z^p + (x_i^{a_i} y_i^{b_i})^{(p-1)} \delta_i z + f_i \tag{7}$$

where x_i, y_i are a regular system of parameters in R_i, $\delta_i, f_i \in R_i$ and δ_i is a unit in R_i. We may thus assume that this forms holds in R, so that

$$g(z) = z^p + (x^a y^b)^{(p-1)} \delta z + f. \tag{8}$$

If (8) holds and $R \rightarrow R_1$ is a quadratic transformation along ω, then

$$a_1 = a + b, b_1 = b$$

if (1) holds with $\alpha_1 = 0$,

$$a_1 = a + b, b_1 = 0$$

if (1) holds with $\alpha_1 \neq 0$, and

$$a_1 = a, b_1 = a + b$$

if (2) holds.

The third type of operation is to make a monomial substitution for z. Suppose that $s, t \in \mathbf{N}$ are such that $x^{sp} y^{tp}$ divides f in R, with $s \leq a$, $t \leq b$. Define z_1 by

$$z = x^s y^t z_1.$$

Define

$$g_1(z_1) = \frac{g(x^s y^t z_1)}{x^{sp} y^{tp}}.$$

The element $g_1(z_1)$ is in the polynomial ring $R[z_1]$. Substituting into (8), we see that

$$g_1(z_1) = z_1^p + (x^{a_1} y^{b_1})^{(p-1)} \delta z_1 + f_1,$$

where $a_1 = a - s$, $b_1 = b - t$ and

$$f_1 = \frac{f}{x^{sp} y^{tp}} \in R.$$

Define $\overline{z}_1 \in K^*$ by

$$\overline{z} = x^s y^t \overline{z}_1.$$

Let $S_1 = R[\overline{z}_1] \cong R[z_1]/(g_1(z_1))$. S_1 is a birational extension of S. The valuation ω^* is nonnegative on S_1 since S_1 is integral over R.

We will construct sequences of operations of these three types. Composing the operations will give us the data of a birational extension of regular local rings $R \rightarrow R_1$, with a regular system of parameters x_1, y_1 in R_1, a polynomial ring $R_1[z_1]$, an irreducible polynomial $g_1(z_1) \in R_1[z_1]$ which has the form

$$g_1(z_1) = z_1^p + (x_1^{a_1} y_1^{b_1})^{(p-1)} \delta_1 z_1 + f_1 \tag{9}$$

where $\delta_1, f_1 \in R_1$ and δ_1 is a unit. We further have a birational extension $S \rightarrow S_1 = R_1[\overline{z}_1] \cong R_1[z_1]/(g_1(z_1))$, where ω^* is nonnegative on S_1.

Our choice of regular parameters x, y in R gives us an identification of $\hat{R}$ with the power series ring $k[[x,y]]$. We then have an expansion

$$f = \sum_{i,j \in \mathbf{N}} f_{i,j} x^i y^j$$

with $f_{i,j} \in k$. We also have an associated series $f_1 \in k[[x_1, y_1]]$, with coefficients $(f_1)_{i,j} \in k$.

We will summarize the above data by saying that (R, g), (R_1, g_1) are states (with associated equations (8) and (9)), and call such a sequence of operations a transformation from (R, g) to (R_1, g_1). We will also refer to states such as (R', g'), where it is understood that the complete set of data will be written as $S', x', y', z', f', a', b'$, etc. We will also find it convenient at one point to interchange the variables x and y in R, and then make the obvious change of notation in the state (R, g).

We will say that a state (R, g) is resolved if $0 < \mathrm{ord}(g) < p$.

The following two lemmas, Lemma 0.3 and Lemma 0.4, are completely worked on in (5.1), (5.2) and (5.3) of [2]. The proofs are straightforward, but somewhat technical.

Suppose that $i, j \in \mathbf{Z}$. We will write $(i,j) \equiv 0\,(p)$ if p divides both i and j.

Lemma 0.3. *Suppose that (R, g) is a state. Then there exists $A(x,y) \in k[x,y]$ and a transformation $(R, g) \rightarrow (R_1, g_1)$ obtained by setting $z_1 = z - A(x,y)$ such that $g_1(z_1) = g(z_1 + A(x,y))$ has the form (9), with $(f_1)_{i,j} = 0$ for all (i,j) such that $(i,j) \equiv 0$ (p) and $i + j \leq p \max\{a, b\}$.*

This is (5.3) [2].

Lemma 0.4. *Suppose that $n \in \mathbf{Z}_+$ and (R, g) is a state such that $a > 0$ (in (8)) and there exist $l, m \in \mathbf{N}$ with $l < p$, $(l,m) \not\equiv 0\,(p)$, $f_{l,m} \neq 0$, and $f_{i,j} = 0$ for all $i < l$. Then there exists $A(x,y) \in k[x,y]$ and a transformation $(R, g) \rightarrow (R_1, g_1)$ obtained by setting $z_1 = z - A(x,y)$ such that $g_1(z_1) = g(z_1 + A(x,y))$ has the form (9), with $(f_1)_{l,m} \neq 0$, $(f_1)_{i,j} = 0$ for all $i < l$ and $(f_1)_{i,j} = 0$ for all (i,j) such that $(i,j) \equiv 0$ (p) and $i + j \leq p \max\{a, b\}$.*

This is (5.2) of [2].

Lemma 0.5. *Suppose that $n \in \mathbf{Z}_+$ and (R, g) is a state such that $\max\{a, b\} = n$ in (8). (The number n is necessarily ≥ 1, as remarked after Lemma 0.2). Then there exists a transformation of states $(R, g) \rightarrow (R_1, g_1)$ such that one of the following holds in (9):*

1. *$\max\{a_1, b_1\} < n$ or*
2. *$\max\{a_1, b_1\} = n$ and there exists $l, m \in \mathbf{N}$ with $l + m < np$ such that $(f_1)_{l,m} \neq 0$, and $(f_1)_{i,j} = 0$ whenever $(i,j) \equiv 0\,(p)$ with $i + j \leq l + m$.*

Proof. By Lemma 0.3, we can make a a change of variables (a transformation of the first type) in z, to achieve that $f_{i,j} = 0$ for all (i,j) such that $(i,j) \equiv 0\,(p)$ with $i + j \leq pn$. Suppose that 2 does not hold. Then we have that $\mathrm{ord}(f) \geq np$. Perform the quadratic transformation $R \rightarrow R_1$ along ω. Let x_1, y_1 be the regular system of parameters in R_1 determined by this quadratic transformation. If x_1, y_1 are of of the type (1), then define $z = x_1^n z_1$. If x_1, y_1 are of type (2), then define $z = y_1^n z_1$. Since $\mathrm{ord}(f) \geq np$ and $a + b \geq n$, this defines a transformation $(R, g) \rightarrow (R, g_1)$.

In the case that $x = x_1, y = x_1(y_1 + \alpha_1), z = x_1^n z_1$, we have that

$$g_1(z_1) = z_1^p + (x_1^{(a+b)-n}(y_1 + \alpha_1)^b)^{p-1} \delta z_1 + \frac{f}{x_1^{np}},$$

with a similar expression if $x = x_1y_1, y = y_1, z = y_1^n z_1$. We see that

$$(a_1, b_1) = \begin{cases} (a+b-n, 0) & \text{if } x = x_1, y = x_1(y_1 + \alpha_1) \text{ with } \alpha_1 \neq 0 \\ (a+b-n, b) & \text{if } x = x_1, y = x_1y_1 \\ (a, a+b-n) & \text{if } x = x_1y_1, y = y_1. \end{cases}$$

We thus have $\max\{a_1, b_1\} \leq \max\{a, b\} = n$.

If the conclusions of the theorem do not hold for (R_1, g_1), then we may repeat the above process. Assume that after a finite number of iterations of this process we do not achieve the conclusions of the theorem. Since ω is nondiscrete rational, we must eventually perform a quadratic transformation of the type (4) with $\alpha_{i+1} \neq 0$. Then we have $(a_{i+1}, b_{i+1}) = (n, 0)$. Since we do not achieve a reduction of $\max\{a_{i+1}, b_{i+1}\}$ in the next iteration, we must perform a quadratic transformation of the type of (5), and we have $(a_{i+2}, b_{i+2}) = (n, 0)$. Thus all quadratic transformations that we perform must be of the type (5) from then on, which is impossible since ω is rational and nondiscrete. □

Lemma 0.6. Suppose that $n \in \mathbf{Z}_+$ and (R, g) is a state such that $\max\{a, b\} = n$ in (8) and there exists $l, m \in \mathbf{N}$ with $l+m < np$ such that $f_{l,m} \neq 0$, and $f_{i,j} = 0$ whenever $(i, j) \equiv 0\,(p)$ with $i+j \leq l+m$. Then there exists a transformation of states $(R, g) \to (R_1, g_1)$ such that $a_1 > 0$ and there exist $l_1, m_1 \in \mathbf{N}$ with $l_1 < p$, $m_1 < np$, $(l_1, m_1) \not\equiv 0\,(p)$, $(f_1)_{l_1,m_1} \neq 0$, and $(f_1)_{i,j} = 0$ for all $i < l_1$.

Proof. Let $d = \operatorname{ord}(f)$. We have $d \leq l+m$. Let q be the greatest integer such that $qp \leq d$. By our assumptions, $q \leq n-1$. After possibly interchanging x and y (the assumptions of Lemma 0.6 are symmetric in x and y), we may assume that $\omega(y) \geq \omega(x)$.

Set $l_1 = d - qp < p$. Let

$$m_1 = \max\{j \mid f_{d-j,j} \neq 0\}.$$

By our assumptions, $(d - m_1, m_1) \not\equiv 0\,(p)$. Thus $(l_1, m_1) \not\equiv 0\,(p)$. We further have $m_1 \leq m < np$.

We now perform the quadratic transformation $R \to R_1$ along ω. R_1 has regular parameters x_1, y_1 defined by

$$x = x_1, y = x_1(y_1 + \alpha_1)$$

for some $\alpha_1 \in k$. We have that $qp \leq d$ and x_1^d divides f in R_1. We now make the substitution $z = x_1^q z_1$ to construct a transformation of states $(R, g) \to (R_1, g_1)$. We have

$$g_1(z_1) = z_1^p + (x_1^{a+b-q}(y_1 + \alpha_1)^b)^{p-1}\delta z_1 + f_1$$

where

$$f_1 = \frac{f(x_1, x_1(y_1 + \alpha_1))}{x_1^{qp}}.$$

We have that

$$a_1 = a + b - q \geq \max\{a, b\} - (n-1) \geq 1.$$

Further,

$$\begin{array}{rcl} f_1 & = & x_1^{d-qp}(\sum_{i+j=d} f_{i,j}(y_1 + \alpha_1)^j + x_1\Omega) \\ & = & x_1^{l_1}(f_{d-m_1,m_1}y_1^{m_1} + \text{ lower order terms in } y_1 + x_1\Omega). \end{array}$$

Thus the conclusions of the lemma hold.

□

Remark 0.7. We could have an increase $\max\{a_1, b_1\} > n$ in the state (R_1, g_1) of the conclusions of Lemma 0.6.

Lemma 0.8. Suppose that $n \in \mathbf{Z}_+$ and (R, g) is a state such that $a > 0$ in (8), and there exist $l, m \in \mathbf{N}$ with $l < p$, $m < np$, $(l, m) \not\equiv 0\,(p)$, $f_{l,m} \neq 0$, and $f_{i,j} = 0$ for all $i < l$. Then there exists a transformation of states $(R, g) \to (R_1, g_1)$ such that one of the following holds:

1. (R_1, g_1) is resolved, or
2. $\max\{a_1, b_1\} < n$ (in (9)) or
3. The assumptions of this lemma hold, with a reduction in m; that is, $a_1 > 0$ and there exist $l_1, m_1 \in \mathbf{N}$ with $l_1 < p$, $m_1 < m < np$, $(l_1, m_1) \not\equiv 0\,(p)$, $(f_1)_{l_1,m_1} \neq 0$, and $f_{i,j} = 0$ for all $i < l_1$.

Proof. By Lemma 0.4, we can make a a change of variables (a transformation of the first type) in z, to achieve that $f_{l,m} \neq 0$, $f_{i,j} = 0$ for all $i < l$ and $f_{i,j} = 0$ for all (i, j) such that $(i, j) \equiv 0\,(p)$ with $i + j \leq p \max\{a, b\}$. Assume that $\max\{a, b\} \geq n$ and (R, g) is not resolved. Then $\operatorname{ord}(f) \geq p$. We must have that $m > 0$, since $l < p$ and $\operatorname{ord}(f) \geq p$.

Perform a quadratic transformation $R \to R_1$ along ω. Let x_1, y_1 be the regular system of parameters in R_1 determined by this transformation.

Case I. Suppose that $\omega(y) < \omega(x)$, so that R_1 has regular parameters x_1, y_1 defined by $x = x_1 y_1, y = y_1$. We have that y_1^p divides f in R_1 since $\operatorname{ord}(f) \geq p$. We also have that $a + b \geq 1$. We may thus define a transformation $(R, g) \to (R_1, g_1)$ by the substitution $z = y_1 z_1$. We have that

$$g_1(z_1) = z_1^p + (x_1^a y_1^{a+b-1})^{p-1}\delta z_1 + f_1$$

where

$$f_1 = \frac{f(x_1 y_1, y_1)}{y_1^p}.$$

Let $l_1 = l$ and $m_1 = l + m - p < m$. Since $(f_1)_{i,j} \neq 0$ if and only if $f_{i,j-i+p} \neq 0$, we have that (R_1, g_1) is resolved, or $\max\{a_1, b_1\} < n$ or the conclusion 3 of Lemma 0.8 holds for (R_1, g_1) for l_1, m_1, with $m_1 < m$.

Case II. Suppose that $\omega(y) \geq \omega(x)$, so that R_1 has regular parameters x_1, y_1 defined by $x = x_1, y = x_1(y_1 + \alpha_1)$ for some $\alpha_1 \in k$. Let $d = \operatorname{ord}(f)$. Let q be the greatest integer such that $qp \leq d$. We have that $qp \leq l + m < (n + 1)p$, so that

$$q \leq n.$$

Suppose that i, j are such that $i + j \leq l + m$ and $(i, j) \equiv 0\,(p)$. Since $l + m < (n + 1)p$, we have that $i + j \leq np \leq p \max\{a, b\}$, so that $f_{i,j} = 0$ by our assumptions.

We have that x_1^{qp} divides f in R_1 and $a+b \geq n \geq q$. We may thus define a transformation $(R, g) \to (R_1, g_1)$ by the substitution $z = x_1^q z_1$. We have that

$$g_1(z_1) = z_1^p + (x_1^{a+b-q}(y_1 + \alpha_1)^b)^{p-1}\delta z_1 + f_1$$

where

$$f_1 = \frac{f(x_1, x_1(y_1 + \alpha_1))}{x_1^{qp}}.$$

We have that $a_1 = a + b - q$ and $b_1 = b$ if $\alpha_1 = 0$, $b_1 = 0$ if $\alpha_1 \neq 0$. Let $x^{\bar{l}} y^{\overline{m}}$ be the term (with nonzero coefficient) of the expansion of f in $k[[x, y]]$ with $\bar{l} + \overline{m} = d$ with largest value of $\overline{m}$. By our assumptions, we must have that $\overline{m} \leq m$, $\bar{l} \geq l$, and $(\bar{l}, \overline{m}) \not\equiv 0\,(p)$.

The monomial $x_1^{d-qp}y_1^{\overline{m}}$ thus appears (with nonzero coefficient) in f_1. Set $l_1 = d - qp$ and $m_1 = \overline{m}$.

If $\max\{a_1, b_1\} \geq n$, then we must have $a_1 > 0$. The assumptions of Lemma 0.8 then hold for (R_1, g_1), with $m_1 \leq m$. If $m_1 < m$, then case 3 of the conclusions of the lemma hold, and we are done.

If $m_1 = m$ (and (R_1, g_1) is not resolved and $\max\{a_1, b_1\} \geq n$), then we repeat the algorithm of this lemma, applied to (R_1, g_1). If we continue to iterate and not reach the conclusions of the lemma, then we must eventually reach the Case I, since ω is not discrete. This is then necessarily the last iteration of the algorithm, and the conclusions of the lemma are reached.

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Lemma 0.9. Suppose that $n \in \mathbf{Z}_+$ and (R, g) is a state such that $a > 0$ in (8), and there exist $l, m \in \mathbf{N}$ with $l < p$, $m < np$, $(l, m) \not\equiv 0\,(p)$, $f_{l,m} \neq 0$, and $f_{i,j} = 0$ for all $i < l$. Then there exists a transformation of states $(R, g) \to (R_1, g_1)$ such that either (R_1, g_1) is resolved, or $\max\{a_1, b_1\} < n$ (in (9)).

Proof. Lemma 0.9 follows from descending induction on l in Lemma 0.8. ☐

Proposition 0.10. Suppose that $n \in \mathbf{Z}_+$ and (R, g) is a state such that $\max\{a, b\} = n$ in (8). Then there exists a transformation $(R, g) \to (R_1, g_1)$ such that either (R_1, g_1) is resolved or $\max\{a_1, b_1\} < n$ in (9).

Proof. The proposition follows from successive application (as necessary) of Lemmas 0.5, 0.6 and 0.9. ☐

We now can easily finish the proof of Theorem 0.1. We start with a state (R, g). By descending induction on n in Proposition 0.10, we can constuct a transformation of states $(R, g) \to (R_1, g_1)$ such that (R_1, g_1) is resolved. let $A = (S_1)_{(x_1, y_1, \overline{z}_1)} \cong (R_1[z_1]/(g_1))_{(x_1, y_1, z_1)}$ be the associated algebraic local ring of K^* which is dominated by ω^*.

We have that $0 < \mathrm{ord}(g_1) < p$, so that A is a hypersurface singularity of multiplicity less than p. We may now construct a birational extension $A \to B$ where B is a regular algebraic local ring of K^* dominated by ω^* using characteristic zero techniques. For instance, we can make a Tschirnhausen transformation to find a hypersurface of maximal contact.

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