ASYMPTOTIC BEHAVIOUR OF THE CASTELNUOVO-MUMFORD REGULARITY

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Dedicated to F. Hirzebruch on the occasion of his seventieth birthday

1. Introduction

Let $A = k[X_1, \dots, X_r]$ be a polynomial ring over an arbitrary field k. Let L be any finitely generated graded A-module. The Castelnuovo-Mumford regularity $\operatorname{reg}(L)$ of L is defined to be the maximum degree n for which there is an index j such that $H^j_{\mathfrak{m}}(L)_{n-j} \neq 0$, where $H^j_{\mathfrak{m}}(L)$ denotes the jth local cohomology module of L with respect to the maximal graded ideal $\mathfrak{m}$ of A. It is also the maximum degree n for which there is an index j such that $\operatorname{Tor}^A_j(k, L)_{n+j} \neq 0$. The Castenuovo-Mumford regularity is an important invariant which measures the complexity of the given module. For instance, if

$$0 \to \cdots \to F_j \to \cdots \to F_1 \to F_0 \to L \to 0$$

is the minimal free resolution of L over A and if a_j is the maximum degree of the generators of F_j, then

$$\operatorname{reg}(L) = \max\{a_j - j \mid j \geq 0\}.$$

See e.g. Eisenbud and Goto [EG], Bayer and Mumford [BM] for more information on this notion.

Let I be any homogeneous ideal of A. Recently, Swanson [S] has proved that there is a number D such that for all $n \geq 1$,

$$\operatorname{reg}(I^n) \leq nD.$$

This result follows from a linear bound on the growth of associated primes of ideals which is closely linked with a version of the uniform Artin-Rees lemma along the line of Huneke's uniform bounds in noetherian rings [Hu2]. However, Swanson could not provide a formula for the number D in general.

A possible candidate for D is $\operatorname{reg}(I)$. In fact, if $\dim A/I = 1$, Geramita, Gimigliano and Pittelloud [GGP] and Chandler [C] showed that $\operatorname{reg}(I^n) \leq n \operatorname{reg}(I)$ for all $n \geq 1$. This result can be easily generalized to the case depth $A/I^n \geq \dim A/I - 1$ for all n. The same bound also holds for a Borel-fixed monomial ideal I by the Eliahou-Kervaire resolution [EK]. See [SS] and [HT] for explicit linear bounds for $\operatorname{reg}(I^n)$ when I is an arbitrary monomial ideal.

The problem of bounding $\operatorname{reg}(I^n)$ is also of interest in algebraic geometry. Given a projective variety $X \subset \mathbb{P}^r$, and let $\mathcal{I}_X$ be the ideal sheaf of the embedding of X. The Castelnuovo-Mumford regularity of

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$\mathcal{I}_X$ is defined to be the the least integer t such that $H^i(\mathbb{P}^r, \mathcal{I}_X(t-i)) = 0$ for all $i \geq 1$. Let d_X denote the minimum of the degrees d such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d. For a *smooth complex* projective variety, Bertram, Ein and Lazarsfeld [BEL] have shown that there is a number e such that

$$H^i(\mathbb{P}^s, \mathcal{I}_X^n(a)) = 0$$

for all $a \geq nd_X + e$, $i \geq 1$. The proof used the Kodaira vanishing theorem. See [B] and [W] for related recent results.

In this paper we will propose a simpler method to estimate $\operatorname{reg}(I^n)$. The main result is the following

Theorem 1.1. *Let I be an arbitrary homogeneous ideal. Let $d(I)$ denote the maximum degree of the homogeneous generators of I. Then*

(i) *There is a number e such that* $\operatorname{reg}(I^n) \leq nd(I) + e$ *for all $n \geq 1$.*

(ii) $\operatorname{reg}(I^n)$ *is a linear function for all n large enough.*

We can estimate the number e (Theorem 2.4) and, if I is generated by forms of the same degree, the place n where $\operatorname{reg}(I^n)$ starts to be a linear function (Proposition 3.7).

We will also show that $d(I^n)$ is a linear function for $n \gg 0$. Since we always have $d(I) \leq \operatorname{reg}(I)$, it follows that

$$\lim \frac{\operatorname{reg}(I^n)}{n} = \lim \frac{d(I^n)}{n}.$$

It is clear that the common limit is a positive number $\leq d(I)$. Therefore, the difference between $\operatorname{reg}(I^n)$ and $n \operatorname{reg}(I)$ can be arbitrarily large if $d(I) < \operatorname{reg}(I)$. This implies that the formula $\operatorname{reg}(I^n) \leq n \operatorname{reg}(I)$ does not hold in general.

Part (i) of the above result implies that for an *arbitrary* projective variety $X \subset \mathbb{P}^r$, there is a number e such that

$$H^i(\mathbb{P}^s, \mathcal{I}_X^n(a)) = 0$$

for all $a \geq nd_X + e$, $i \geq 1$.

However, part (ii) does not have a similar geometric version. In fact, it does not hold if we replace I^n by its saturation $\widetilde{I^n}$, though I^n and $\widetilde{I^n}$ define the same projective scheme. We will give examples of homogeneous ideals of 'fat' points for which $\operatorname{reg}(\widetilde{I^n})$ is not a linear function for large n (Example 4.2). In particular, using a counter-example to Zariski's Riemann-Roch problem in positive characteristic [CS] we can construct an example such that $\operatorname{reg}(I^n)$ is not even a linear polynomial with periodic coefficients (Example 4.3).

We also give an example of a homogeneous ideal in the coordinate ring of an abelian surface such that

$$\lim \frac{\operatorname{reg}(\widetilde{I^n})}{n}$$

is an irrational number (Example 4.4).

Our method is based on a natural bigrading of the Rees algebra $R = \oplus_{n \geq 0} I^n t^n$ given by setting $\deg xt^n = (\deg x, n)$ for all homogeneous element x of I^n. It is not hard to see that

$$H^j_{\mathfrak{m}}(I^n)_a \simeq H^j_M(R)_{(a,n)}$$
$$\operatorname{Tor}^A_j(k, I^n)_a \simeq \operatorname{Tor}^R_j(R/M, R)_{(a,n)}$$

for all numbers a, n, where M is the ideal of R generated by the elements of $\mathfrak{m}$. Therefore, we only need to study the bigraded structure of $H^j_M(R)$ and $\operatorname{Tor}^R_j(R/M, R)$ in order to estimate $\operatorname{reg}(I^n)$.

The proof of Theorem 1.1 (i) and (ii) will be found in Section 2 and Section 3, respectively. We would like to mention that (i) has been also obtained by Lavila-Vidal and Zarzuela by a different method (private communication) and that linear programming is used to prove (ii). The same method can also be applied to give linear bounds for $\operatorname{reg}(\overline{I^n})$, where $\overline{I^n}$ denotes the integral closure of I^n, and for $\operatorname{reg}(I_1^{n_1} \cdots I_m^{n_m})$, where $I_1, \ldots, I_m$ are arbitrary homogeneous ideals. Moreover it can be shown that if the graded algebra $\bigoplus_{n \geq 0} \widetilde{I^n} t^n$ is finitely generated, then there are a finite number of linear functions such that $\operatorname{reg}(\widetilde{I^n})$ varies among these functions for $n \gg 0$ (Theorem 4.3).

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2. Linear bound for the regularity

We begin with some observation on the bigraded structure of local cohomology modules which we shall need in the proof of Theorem 1.1 (i).

Let $R = \oplus_{a,n\ge 0} R_{(a,n)}$ be a noetherian bigraded ring and $E = \oplus_{a,n\in\mathbb{Z}} E_{(m,n)}$ be a bigraded R-module. We may consider R as an $\mathbb{N}$-graded ring with $R_n = \oplus_{a\ge 0} R_{(a,n)}$ and E as a $\mathbb{Z}$-graded module with $E_n = \oplus_{a\ge 0} E_{(a,n)}$. It is clear that R_0 is also an $\mathbb{N}$-graded ring and that E_n is a graded R_0-module.

Let $\mathfrak{m}$ be the maximal graded ideal of R_0. Then the local cohomology module $H^i_{\mathfrak{m}}(E_n)$ is a well-defined graded R_0-module for all $i \ge 0$.

Let M denote the ideal generated by the elements of $\mathfrak{m}$, i.e. $M = \oplus_{n\ge 0}\mathfrak{m}R_n$. We shall see that $H^i_{\mathfrak{m}}(E_n)$ is a $\mathbb{Z}$-graded component of the local cohomology module $H^i_M(E)$.

Lemma 2.1. *$H^i_{\mathfrak{m}}(E_n)_a = H^i_M(E)_{(a,n)}$ for all numbers a, n.*

Proof. We shall use the characterization of local cohomology modules by means of the Koszul complexes (see e.g. [BH], [H1]). Let $x_1,\dots,x_r$ be a family of generating elements for $\mathfrak{m}$. Set $\mathbf{x}^s = x_1^s,\dots,x_r^s$ and denote by $H^i(\mathbf{x}^s,\cdot)$ the ith cohomology of the Koszul complex functor associated with $\mathbf{x}^t$. Then

$$H^i_{\mathfrak{m}}(E_n) = \varinjlim H^i(\mathbf{x}^t, E_n),$$
$$H^i_M(E) = \varinjlim H^i(\mathbf{x}^t, E).$$

Since the elements $x_1,\dots,x_r$ have degree zero in the $\mathbb{N}$-graded ring R, we have

$$H^i(\mathbf{x}^t, E_n) = H^i(\mathbf{x}^t, E)_n.$$

From this it follows that

$$H^i_{\mathfrak{m}}(E_n) = \varinjlim H^i(\mathbf{x}^t, E)_n = H^i_M(E)_n.$$

It is clear that the equation $H^i_{\mathfrak{m}}(E_n) \simeq H^i_M(E)_n$ also reflects the bigraded structure in the sense that $H^i_{\mathfrak{m}}(E_n)_a = H^i_M(E)_{(a,n)}$. □

From now on let $R = \oplus_{n\ge 0} I^n t^n$ be the Rees algebra of a homogeneous ideal I in a polynomial ring $A = k[X_1,\dots,X_r]$. As I is homogeneous, we may view R as a bigraded ring with

$$R_{(a,n)} = (I^n)_a t^n.$$

Let $\mathfrak{m} = (X_1,\dots,X_r)$ be the maximal graded ideal of A. By Lemma 2.1 we have

$$H^i_{\mathfrak{m}}(I^n)_a = H^i_M(R)_{(a,n)}$$

for all numbers a, n. Therefore, we may get information on the graded structure of $H^i_{\mathfrak{m}}(I^n)$ by the bigraded structure of $H^i_M(R)$.

Assume that I is generated by s homogeneous polynomials. Then R may be represented as a factor ring of the bigraded polynomial ring $S = k[X_1,\dots,X_r,Y_1,\dots,Y_s]$. Let N denote the ideal of S generated by $X_1,\dots,X_r$. It is clear that

$$H^i_M(R)_{(a,n)} \simeq H^i_N(R)_{(a.n)}$$

for all numbers a, n. We will use a bigraded minimal free resolution of R over S to study the the bigraded structure of $H^i_N(R)$.

First we have the following description of $H^i_N(S)$.

Lemma 2.2.

$$H^i_N(S) = 0, \quad i \neq r,$$
$$H^r_N(S) = k[X_1^{\alpha_1}\dots X_r^{\alpha_r}Y_1^{\beta_1}\dots Y_s^{\beta_s}\,|\,\alpha_1,\dots,\alpha_r < 0;\ \beta_1,\dots,\beta_s \in \mathbb{N}].$$

Proof. Since S is a direct product of copies of $A = k[X_1,\dots,X_r]$, we have

$$H^i_N(S) = H^i_{\mathfrak{m}}(A) \otimes_A S.$$

It is well-known [H1] that

$$H^i_{\mathfrak{n}}(A) = 0, \quad i \neq r,$$
$$H^r_{\mathfrak{n}}(A) = k[X_1^{\alpha_1}\dots X_r^{\alpha_r}\,|\,\alpha_1,\dots,\alpha_r < 0].$$

Hence the conclusion is immediate. $\square$

Let $d_1,\dots,d_s$ be the degree of the homogeneous generators of I. Then the bigrading of the polynomial ring S is given by

$$\operatorname{bideg} X_i = (1,0), \quad i = 1,\dots,r,$$
$$\operatorname{bideg} Y_j = (d_j,1), \quad j = 1,\dots,s.$$

This can be used to obtain information on the bigraded vanishing of $H^r_N(S)$.

Corollary 2.3. $H^r_N(S)_{(a,n)} = 0$ *for all* $a \geq nd(I) - r + 1$.

Proof. Note that $d(I) = \max\{d_1,\dots,d_s\}$. Since

$$\operatorname{bideg} X_1^{\alpha_1}\dots X_r^{\alpha_r}Y_1^{\beta_1}\dots Y_s^{\beta_s} = (\alpha_1 + \dots + \alpha_r + \beta_1 d_1 + \dots + \beta_s d_s, \beta_1 + \dots + \beta_s),$$

using Lemma 2.2 we get

$$H^r_N(S)_{(a,n)} = k[X_1^{\alpha_1}\dots X_r^{\alpha_r}Y_1^{\beta_1}\dots Y_s^{\beta_s}\,|\,\alpha_1,\dots,\alpha_r < 0;$$
$$\alpha_1 + \dots + \alpha_r + \beta_1 d_1 + \dots + \beta_s d_s = a,\ \beta_1 + \dots + \beta_s = n].$$

If $a \geq nd(I) - r + 1$, then

$$\alpha_1 + \dots + \alpha_r = a - (\beta_1 d_1 + \dots + \beta_s d_s)$$
$$\geq a - (\beta_1 + \dots + \beta_s)d(I) = a - nd(I) \geq 1 - r.$$

Hence at least one of the numbers $\alpha_1,\dots,\alpha_r$ must be non-negative. From this it follows that $H_N(S)_{(a,n)} = 0$. $\square$

The following result gives Theorem 1.1 (i) by setting $E = R$. This result will be used to give a linear bound for $\operatorname{reg}(\overline{I^n})$, too.

Theorem 2.4. *Let* E *be an arbitrary finitely generated bigraded module over the Rees algebra of* I. *Let*

$$0 \to \dots \to \oplus_t S(-a_{tj}, -b_{tj}) \to \dots \to \oplus_t S(-a_{t1}, -b_{t1}) \to \oplus_t S(-a_{t0}, -b_{t0}) \to E \to 0$$

be a bigraded minimal free resolution of E *over* S, *where* S *is defined as above. Put* $c_j = \max_t\{a_{tj} - b_{tj}d(I)\}$ *and* $e = \max\{c_j - j|\ j = 0,\dots,r\}$. *For all* $n \geq 1$ *we have*

$$\operatorname{reg}(E_n) \leq nd(I) + e.$$

Proof. First we will study the graded vanishing of $H^i_N(E)$, $i = 1, \dots, r$. Rewrite the above resolution of E as follows:

$$0 \to \cdots \to F_j \to \cdots \to F_1 \to F_0 \to E \to 0$$

Let K_j denote the image of the map $F_j \to F_{j-1}$ for $j \geq 1$. Then there are the exact sequences

$$0 \to K_j \to F_{j-1} \to K_{j-1} \to 0$$

where $K_0 = E$. Consider the derived exact sequence of local cohomology modules of these exact sequences. For $i < r$, we use Lemma 2.2 to deduce that

$$H^i_N(E) \simeq H^{i+1}_N(K_1) \simeq \ldots \simeq H^{r-1}_N(K_{r-i-1})$$

and that there is an injective map $H^{r-1}_N(K_{r-i-1}) \to H^r_N(K_{r-i})$ and a surjective map $H^r_N(F_{r-i}) \to H^r_N(K_{r-i})$. For $i = r$ we also have a surjective map $H^r_N(F_0) \to H^r_N(E)$. Therefore, for all $i \geq 0$, $H^i_N(E)_{(m,n)} = 0$ if $H_N(F^{r-i})_{(m,n)} = 0$. By Corollary 3.2,

$$H^r_N(S(-a_{tj}, -b_{tj}))_{(m,n)} = 0$$

for $m - a_{tj} \geq (n - b_{tj})d - r + 1$, where $d = d(I)$. Therefore, $H^r_N(F^{r-i})_{(m,n)} = 0$ if $m \geq (n - b_{tr-i})d + a_{tr-i} - r + 1$ for all t. The latter condition is satisfied if $m \geq nd + c_{r-i} - r + 1$. Hence

$$H^i_N(E)_{(m,n)} = 0$$

for all $m \geq nd + c_{r-i} - r + 1$.

By Lemma 2.1 we get

$$H^i_{\mathfrak{m}}(E_n)_{m-i} = H_N(E)_{(m-i,n)} = 0$$

for $m - i \geq nd + c_{r-i} - r + 1$, $i = 1, \dots, r$. Since $nd + e \geq nd + c_{r-i} - r + i$, this vanishing holds if $m \geq nd + e$. Note that $H^0_{\mathfrak{m}}(E_n) = 0$. Then we obtain $\operatorname{reg}(E_n) \leq nd + e$. □

Corollary 2.5. *Let $X \subset \mathbb{P}^r$ be an arbitrary projective variety. Let $\mathcal{I}_X$ be the ideal sheaf of the embedding and d_X the minimum of the degrees d such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d. Then there is a number e such that*

$$H^i(\mathbb{P}^r, \mathcal{I}^n_X(a)) = 0$$

for all $a \geq nd_X + e$, $i \geq 1$.

Proof. Let I be a homogeneous ideal generated by forms of degree at most d_X such that $\mathcal{I}_X$ is the ideal sheaf associated with I. Then $d(I) = d_X$. By Theorem 1.1 (i) there is an integer e such that $H^i_{\mathfrak{m}}(I^n)_a = 0$ for $a \geq nd_X + e$, $i \geq 0$. Therefore the conclusion □

Corollary 2.6. *Let I be a homogeneous ideal generated by s elements. Assume that the Rees algebra of I is Cohen-Macaulay. Then*

$$\operatorname{reg}(I^n) \leq nd(I) + (s-1)[d(I) - 1)]$$

for all $n \geq 1$.

Proof. The assertion follows immediately from the bound $a_{tj} \leq sd(I) - (s-1) + j$, $j \geq 0$, for $E = R$ given by O. Lavila-Vidal [L, Proposition 4.1], in case the Rees algebra of I is Cohen-Macaulay. □

There are several important classes of ideals for which one knows that their Rees algebras are Cohen-Macaulay, see e.g. Eisenbud and Huneke [EH].

Example 2.7. Let I be the ideal generated by the maximal minors of a generic $p \times q$ matrix, $p \leq q$. Then the Rees algebra of I is a Cohen-Macaulay ring [EH]. Therefore,

$$\operatorname{reg}(I^n) \leq np + \left[\binom{q}{p} - 1\right](p-1)$$

for all $n \geq 1$. This is by far the actual value of $\operatorname{reg}(I^n)$. Akin, Buchsbaum, and Weyman [ABW] already gave a linear resolution for I^n from which it follows that $\operatorname{reg}(I^n) = np$ for all $n \geq 1$. We are grateful to A. Conca for this information.

If we set $E = \oplus_{n \geq 0} \overline{I^n} t^n$, where $\overline{I^n}$ denotes the integral closure of I^n, then E is a finitely generated bigraded R-module with $E_n \simeq \overline{I^n}$. Hence from Theorem 2.4 we also obtain a linear bound for $\operatorname{reg}(\overline{I^n})$.

Proposition 2.8. *Let I be an arbitrary homogeneous ideal. Then there is a number e such that $\operatorname{reg}(\overline{I^n}) \leq nd(I) + e$ for all $n \geq 1$.*

3. Asymptotic behaviour of regularity

Let I be a homogeneous ideal in $A = k[X_1, \ldots, X_r]$. In this section we will show that $\operatorname{reg}(I^n)$ is not only bounded by a linear function, but, for $n \gg 0$, is a linear function. The approach will be similar as in the previous section.

For any A-module L each we set

$$\operatorname{reg}_i(L) = \max\{a |\ \operatorname{Tor}_i(k, L)_a \neq 0\} - i.$$

Since $\operatorname{reg}(L) = \max\{\operatorname{reg}_i(L) |\ i \geq 0\}$, Theorem 1.1 (ii) follows from the next result.

Theorem 3.1. *Let I be an arbitrary homogeneous ideal. Then for all $i \geq 0$, the function $\operatorname{reg}_i(I^n)$ is linear for $n \gg 0$.*

Recall that for any homogeneous ideal J, $d(J)$ denotes the maximal degree of the homogeneous generators of J. It is well-known that $d(J)$ is nothing else than $\operatorname{reg}_0(J)$. The next result encodes the fact that the linear functions associated with $\operatorname{reg}_0(I^n)$ and $\operatorname{reg}(I^n)$ have the same slope.

Corollary 3.2. *Let I be an arbitrary homogeneous ideal. Then*

$$\lim \frac{d(I^n)}{n} = \lim \frac{\operatorname{reg}(I^n)}{n},$$

and this common limit is a positive integer $\leq d(I)$.

Proof. Let $\operatorname{reg}(I^n) = an + b$ and $\operatorname{reg}_0(I^n) = cn + d$ for $n \gg 0$. Since $\operatorname{reg}_0(I^n) \leq \operatorname{reg}(I^n)$ for all n, it follows that $c \leq a$. On the other hand, by Theorem 1.1 (i) we have $\operatorname{reg}(I^{mn}) \leq \operatorname{reg}_0(I^n)m + e$ for large n and all $m \geq 0$. This implies that $an \leq \operatorname{reg}_0(I^n) = cn + d$ for all large n. Therefore, $a \leq c$, and so $a = c$. It is clear that c is a positive integer $\leq d(I)$. □

In order to prove Theorem 3.1 we shall consider the Rees algebra $R = \oplus_{n \geq 0} I^n t^n$ as a factor ring of the bigraded polynomial ring $S = k[X_1, \ldots, X_r, Y_1, \ldots, Y_s]$ as in Section 3. Let $\mathfrak{m} = (X_1, \ldots, X_r)$ be the maximal graded ideal of A and $N = \mathfrak{m}S$.

Lemma 3.3. *Let E be a finitely generated bigraded R-module. Put $E_n = \oplus_{a \in \mathbb{Z}} E_{(a,n)}$. Then*

$$\operatorname{Tor}_i^A(k, E_n)_a \simeq \operatorname{Tor}_i^S(S/N, E)_{(a,n)}$$

for all a, n and $i \geq 0$.

Proof. Consider a graded minimal free S-resolution of the R-module E:

$$\mathbb{F}: \quad 0 \to \cdots \to F_j \to \cdots \to F_1 \to F_0 \to E \to 0$$

Taking the n-homogeneous component is an exact functor, so that the sequence

$$\mathbb{F}_n: \quad 0 \to \cdots \to (F_j)_n \to \cdots \to (F_1)_n \to (F_0)_n \to E_n \to 0$$

is exact. Since the modules $(F_i)_n$ are free A-modules, $\mathbb{F}_n$ is a free A-resolution for E_n. We have $\operatorname{Tor}_i^S(S/N, E) = H_i(\mathbb{F}/\mathfrak{m}\mathbb{F})$ so that

$$\operatorname{Tor}_i^S(S/N, E)_n \simeq H_i(\mathbb{F}_n/\mathfrak{m}\mathbb{F}_n)$$

which is isomorphic to $\operatorname{Tor}_i^A(k, E_n)$. Hence $\operatorname{Tor}_i^A(k, E_n)_a \simeq \operatorname{Tor}_i^S(S/N, E)_{(a,n)}$. $\square$

Remark. The above free resolution $\mathbb{F}_n$ of E_n is not minimal in general. For instance, let

$$I = (X_1^2, X_1X_2, X_2^2) \subset A = k[X_1, X_2].$$

Then $R = S/(f_1, f_2, f_3)$ with $f_1 = X_2Y_1 - X_1Y_2$, $f_2 = X_2Y_2 - X_1Y_3$ and $f_3 = Y_2^2 - Y_1Y_3$. One sees easily that (f_1, f_2, f_3) is a height 2 perfect ideal, and hence the Rees algebra R has the S-resolution

$$0 \to S(-5,-2)^2 \to S(-3,-1)^2 \oplus S(-4,-2) \to S \to R \to 0.$$

Thus, if we want to compute a resolution of I^2, we have to take the second component of the above resolution, and get

$$0 \to A(-5)^2 \to A(-5)^6 \oplus A(-4) \to A(-4)^6 \to I^2 \to 0,$$

which, of course, is not minimal.

By Lemma 3.3 we have

$$\operatorname{reg}_i(I^n) = \max\{a | \ \operatorname{Tor}_i^S(S/N, R)_{(a,n)} \neq 0\} - i.$$

Notice that each $\operatorname{Tor}_i(S/N, R)$ is a finitely generated bigraded module over the bigraded polynomial ring $S/N = k[Y_1, \dots, Y_s]$ with $\operatorname{bideg} Y_i = (d_i, 1)$, $i = 1, \dots, s$. Then Theorem 3.1 follows from the following property of such modules.

Theorem 3.4. *Let E be any finitely generated bigraded module over $k[Y_1, \dots, Y_s]$. The function*

$$\rho_E(n) := \max\{a | \ E_{(a,n)} \neq 0\}$$

is linear for $n \gg 0$.

Proof. Put $T = k[Y_1, \dots, Y_s]$. It is clear that for a given exact sequence of bigraded T-modules

$$0 \to E'' \to E \to E' \to 0$$

we have $\rho_E(n) = \max\{\rho_{E''}(n), \rho_{E'}(n)\}$ for all $n \in \mathbb{N}$. Therefore, since there exists a sequence of bigraded submodules

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{i-1} \subset E_i = M$$

of E such that E_j/E_{j-1} is cyclic for $j = 1, \dots, i$, we may assume that E is cyclic.

We represent E as a quotient T/J. Let $<$ be any term order, and denote by $\operatorname{in}(J)$ the initial ideal of J with respect to this term order. It is clear that T/J has a k-basis consisting of the residues classes of all the monomials which do not belong to $\operatorname{in}(J)$, and it is well-known that the residue classes of the same monomials modulo J form a (bigraded) k-basis of T/J. Therefore $\rho_E(n) = \rho_{T/\operatorname{in}(J)}(n)$ for all $n \geq 0$, and we may assume that J itself is a monomial ideal.

Let J be generated by the monomials $Y_1^{c_{i1}} \ldots Y_s^{c_{is}}$ for $i = 1, \dots, p$. For any $\mathbf{a} = (a_1, \dots, a_s) \in \mathbb{N}^s$ let $y^{\mathbf{a}}$ denote the residue class of $Y_1^{a_1} \ldots, Y_s^{a_s}$ in S/J. Let B_n denote the minimal basis of $(S/J)_n$. Then

$$\rho_E(n) = \max\{v(\mathbf{a}) | \ y^{\mathbf{a}} \in B_n\}$$

with $v(\mathbf{a}) = \sum_i a_i d_i$. Note that $y^{\mathbf{a}} \in B_n$ if and only if $\sum_j a_j = n$, and for all $i = 1, \ldots, p$ there exists an integer $1 \le j \le s$ with $a_j < c_{ij}$.

Let L denote the set of maps $\{1, \ldots, p\} \to \{1, \ldots, s\}$, and consider for each $f \in L$ the subset

$$B_{n,f} = \{y^{\mathbf{a}} | \sum_j a_j = n,\ a_{f(i)} < c_{if(i)} \text{ for } i = 1, \ldots, s\}.$$

It is clear that $B_n = \cup_{f \in L} B_{n,f}$. Define $\rho_f(\mathbf{a}) = \max\{v(\mathbf{a}) |\ y^{\mathbf{a}} \in B_{n,f}\}$. Then

$$\rho_E(n) = \max\{\rho_f(n) |\ f \in L\}.$$

Thus it suffices to show that the functions $\rho_f(n)$ are linear for all $f \in F$ and all $n \gg 0$.

Let $\{j_1, \ldots, j_k\}$ be the image of f, and suppose that $j_1 < j_1 \ldots < j_k$. We set $c_{j_t} = \min\{c_{ij(i)} |\ j(i) = j_t\} - 1$ for $t = 1, \ldots, k$. Then

$$B_{n,f} = \{y^{\mathbf{a}} | \sum_j a_j = n \text{ and } a_{j_t} \le c_{j_1},\ \text{for } t = 1, \ldots, k\},$$

and $\rho_f(n)$ is given by the maximum of the linear functional $v(\mathbf{a})$ on the convex bounded set

$$C_n = \{\mathbf{a} | \sum_j a_j = n,\ \text{and } a_{j_t} \le c_{j_1} \text{ for } t = 1, \ldots, k\}.$$

This is a rather trivial example of linear programming. The solution is the following.

Suppose that ℓ is the smallest integer such that $j_t = t$ for $t < \ell$ and $j_\ell > \ell$. In other words, we have $a_1 < c_1, \ldots, a_{\ell-1} < c_{\ell-1}$ and no bound on a_ℓ (except that $\sum_j a_j = n$).

If $\ell = s + 1$, then $\sum_j a_j$ can be at most $\sum_j c_j$, so that for $n \gg 0$, $B_{n,f} = 0$ and hence $\rho_f(n) = 0$.

If $\ell \le s$, let $n \ge c_1 + c_2 + \cdots + c_{\ell-1}$. We claim that v has its maximal value for $\mathbf{a} = (c_1, \ldots, c_{\ell-1}, n - \sum_{j=1}^{\ell-1} c_j, 0, \ldots, 0)$. Then

$$v(\mathbf{a}) = \sum_{j=1}^{\ell-1} d_j c_j + d_\ell (n - \sum_j^{\ell-1} c_j)$$

which is a linear function on n, as we wanted to show.

Indeed, if $\mathbf{a} = (a_1, \ldots, a_s) \in C_n$, and if for some $1 \le i < j \le s$ we have $a_i < c_i$ and $a_j > 0$, then $\mathbf{a}' = (a_1, \ldots, a_i + 1, \ldots, a_j - 1, \ldots, a_s)$ also belongs to C_n and $v(\mathbf{a}') \ge v(\mathbf{a})$ since $d_i \ge d_j$, by assumption. This argument shows that if we fill up the first 'boxes' as much as possible, we must reach the maximal value of v. The resulting $\mathbf{a}$ with maximal value is exactly the one described above. □

Theorem 3.4 also has the following interesting consequence

Corollary 3.5. *Let I be an arbitrary homogeneous ideal. Then $\operatorname{reg}(\overline{I^n})$ is a linear function for $n \gg 0$.*

Proof. Put $E = \oplus_{n \ge 0} \overline{I^n} t^n$. Then E is a finitely generated bigraded module over the Rees algebra of I with $E_n \simeq \overline{I^n}$. By Lemma 3.3 we have

$$\operatorname{reg}_i(E_n) = \rho_{\operatorname{Tor}_i^S(S/N,E)}(n)$$

for all $i \ge 0$. Since $\operatorname{reg}(E_n) = \max\{\operatorname{reg}_i(E_n) |\ i \ge 0\}$, the conclusion follows from Theorem 3.4. □

Remark. With the same method as above one can prove the prove the following modifications of Theorem 3.1: Let $I_1, \ldots, I_m$ be graded ideals in the polynomial ring A. Then there exist integers $a_1, \ldots, a_m$ with $a_j \le d(I_j)$ for $j = 1, \ldots, m$, and an integer b such that

$$\operatorname{reg}(I_1^{n_1} \ldots I_m^{n_m}) = a_1 n_1 + \ldots + a_m n_m + b$$

for all $n_1, \ldots, n_m \gg 0$. For the proof one considers the multi-Rees ring $A[I_1 t_1, \ldots, I_m t_m]$.

Now we will estimate the place where $\operatorname{reg}(I^n)$ starts to be a linear function when I is generated by forms of the same degree. We shall need the following observation.

Lemma 3.6. *Let $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$ be an exact sequence of graded A-modules.*

(i) *If* $\operatorname{reg}(E) > \operatorname{reg}(G) + 1$, *then* $\operatorname{reg}(F) = \operatorname{reg}(E)$.

(ii) *If* $\operatorname{reg}(E) < \operatorname{reg}(G) + 1$, *then* $\operatorname{reg}(F) = \operatorname{reg}(G)$.

Proof. Consider the derived long exact sequence

$$H^{i-1}_{\mathfrak{m}}(G) \to H^i_{\mathfrak{m}}(E) \to H^i_{\mathfrak{m}}(F) \to H^i_{\mathfrak{m}}(G) \to H^{i+1}_{\mathfrak{m}}(E)$$

Put $n = \max\{\operatorname{reg}(E), \operatorname{reg}(G)\}$. It is obvious that $\operatorname{reg}(F) \le n$.

If $\operatorname{reg}(E) > \operatorname{reg}(G)+1$, then $n = \operatorname{reg}(E)$. We choose i such that $H^i_{\mathfrak{m}}(E)_{n-i} \neq 0$. Since $\operatorname{reg}(G) < n-1$, $H^{i-1}_{\mathfrak{m}}(G)_{n-i} = 0$. Hence $H^i_{\mathfrak{m}}(F)_{n-i} \neq 0$. From this it follows that $\operatorname{reg}(F) = n$.

If $\operatorname{reg}(E) < \operatorname{reg}(G) + 1$, then $n = \operatorname{reg}(G)$. We choose i such that $H^i_{\mathfrak{m}}(G)_{n-i} \neq 0$. Since $\operatorname{reg}(E) \le n$, $H^{i+1}_{\mathfrak{m}}(E)_{n-i} = 0$. Hence $H^i_{\mathfrak{m}}(F)_{n-i} \neq 0$. □

Our estimation depends on the minimum number of generators of I and the Castelnuovo Mumford regularity $\operatorname{reg}(R)$ of the Rees algebra $R = \oplus_{n \ge 0} I^n t^n$ as a $\mathbb{N}$-graded ring with the usual grading $\deg xt^n = n$, $x \in I^n$. The regularity $\operatorname{reg}(R)$ can be computed in terms of certain minimal set of generators of I [T]. For instance, if I is generated by a d-sequence [Hu1], then $\operatorname{reg}(R) = 0$.

Recall that the Castelnuovo-Mumford regularity $\operatorname{reg}(E)$ of a graded module E over any $\mathbb{N}$-graded ring B is defined to be the largest integer n for which there exists an index i such that $H^i_{B_+}(E)_{a-i} \neq 0$, where B_+ is the ideal of B generated by the homogeneous elements of positive degree.

If we consider R as a $\mathbb{N}$-graded module over the $\mathbb{N}$-graded polynomial ring S with $\deg X_i = 0$ and $\deg Y_j = 1$, then

$$\operatorname{reg}(R) = \max\{b_{tj} - j |\ j \ge 0\},$$

where b_{tj} are the second coordinates of the bidegree of the generators of the jth term of a minimal bigraded free resolution of R over S.

Proposition 3.7. *Let I be a homogeneous ideal generated by s forms of the same degree d. Put $c = \operatorname{reg}(R) + s + 1$. Then, for $n \ge c$,*

$$\operatorname{reg}(I^n) = (n-c)d + \operatorname{reg}(I^c).$$

Proof. We need to modify the statement as follows. Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_s]$ be a bigraded polynomial ring with $\operatorname{bideg} X_i = (1,0)$ and $\operatorname{bideg} Y_j = (d,1)$, where $d > 0$ is a fixed integer. For any finitely generated bigraded S-module E let $E_n = \oplus_{a \in \mathbb{Z}} E_{(a,n)}$. Then S is an $\mathbb{N}$-graded ring and E an $\mathbb{Z}$-graded S-module. Put $c = \operatorname{reg}(E) + s + 1$. We claim that for $n \ge c$,

$$\operatorname{reg}(E_n) = (n-c)d + \operatorname{reg}(E_c).$$

Since R may be considered as a finitely generated bigraded S-module with $R_n \simeq I^n$, the conclusion clearly follows from this claim.

If $s = 0$, $S_n = 0$ for all $n > 0$. It follows that $\operatorname{reg}(E) = \max\{n |\ E_n \neq 0\}$. Hence $E_n = 0$ for $n \ge \operatorname{reg}(E) + 1$. In this case, $d = 0$.

To prove the claim in the case $s > 0$ we may assume that the base field k is infinite. Then we can find a linear form Y in $Y_1, \ldots, Y_s$ such that $Y \notin P$ for any associated prime $P \not\supseteq (Y_1, \ldots, Y_s)$ of E. In other words, Y is a filter-regular element of E with respect to the ideal $(Y_1, \ldots, Y_s)$. Note that Y is a bihomogeneous form with $\operatorname{bideg} Y = (d,1)$. Put $K = E/0_E : Y$. Consider the exact sequence of graded A-modules:

$$0 \to K_{n-1}(-d) \stackrel{Y}{\to} E_n \to [E/YE]_n \to 0.$$

Note that $\operatorname{reg}(E) \ge \operatorname{reg}(E/YE)$ [T, Lemma 2.1]. By induction on s we may assume that for $n \ge c-1$,

$$\operatorname{reg}([E/YE]_n) = (n-c+1)d + \operatorname{reg}([E/YE]_{c-1}).$$

Moreover, if $n \ge c$, $n - 1 \ge \operatorname{reg}(E) + 1$. Then $[0_E : Y]_{n-1} = 0$ by [T, Proposition 2.2]. In this case we have

$$K_{n-1} = E_{n-1}.$$

We distinguish three cases:

(1) If $\operatorname{reg}(K_{c-2})(-d)) > \operatorname{reg}([E/YE]_{c-1}) + 1$, using Lemma 3.6 we get $\operatorname{reg}(E_{c-1}) = \operatorname{reg}(K_{c-2}(-d))$. From this it follows that

$$\operatorname{reg}(E_{c-1}(-d)) = \operatorname{reg}(K_{c-2}(-d)) + d > \operatorname{reg}([E/YE]_{c-1}) + d + 1 = \operatorname{reg}([E/YE]_c) + 1.$$

By Lemma 3.6 we get

$$\operatorname{reg}(E_c) = \operatorname{reg}(E_{c-1}(-d)) = d + \operatorname{reg}(E_{c-1}).$$

Using the same argument, we will be led to the formula $\operatorname{reg}(E_n) = (n-c+1)d + \operatorname{reg}(E_{c-1})$ for $n \geq c-1$.

(2) If $\operatorname{reg}(K_{c-2}(-d)) < \operatorname{reg}([E/YE]_{c-1}) + 1$, using Lemma 3.6 we get $\operatorname{reg}(E_{c-1}) = \operatorname{reg}([E/YE]_{c-1})$. Therefore,

$$\operatorname{reg}(E_{c-1}(-d)) = \operatorname{reg}([E/YE]_{c-1}) + d = \operatorname{reg}([E/YE]_c).$$

By Lemma 3.6 we get

$$\operatorname{reg}(E_c) = \operatorname{reg}([E/YE]_c) = d + \operatorname{reg}(E_{c-1}).$$

Using Lemma 3.6 again we will be led to the formula $\operatorname{reg}(E_n) = (n-c+1)d + \operatorname{reg}(E_{c-1})$ for $n \geq c-1$.

(3) If $\operatorname{reg}(K_{c-2}(-d)) = \operatorname{reg}([E/YE]_{c-1}) + 1$, then $\operatorname{reg}(E_{c-1}) \leq \operatorname{reg}(K_{c-2}(-d))$. As we have seen in (1), we may assume that $\operatorname{reg}(E_{c-1}) < \operatorname{reg}(K_{c-2}(-d))$. It follows that

$$\operatorname{reg}(E_{c-1}(-d)) < d + \operatorname{reg}([E/YE]_{c-1}) + 1 = \operatorname{reg}([E/YE]_c) + 1.$$

Following (2) we will obtain $\operatorname{reg}(E_n) = (n-c)d + \operatorname{reg}(E_c)$ for $n \geq c$. □

Corollary 3.8. *Let I be an ideal generated by a d-sequence of s forms of the same degree d. For $n \geq s+1$,*

$$\operatorname{reg}(I^n) = (n-s-1)d + \operatorname{reg}(I^{s+1}).$$

4. Regularity of saturations of ideals

In this section we will study the regularity of the saturation $\widetilde{I^n}$ of I^n.

Proposition 4.1. *Let I be an arbitrary homogeneous ideal. There is a number e such that*

$$\operatorname{reg}(\widetilde{I^n}) \leq nd(I) + e$$

for all $n \geq 1$.

Proof. We have

$$H^i_{\mathfrak{m}}(\widetilde{I^n}) \simeq \begin{cases} 0, & i = 0, 1, \\ H^i_{\mathfrak{m}}(I^n), & i \geq 2. \end{cases}$$

Hence the conclusion follows from Theorem 2.4. □

Now we will present examples which show that $\operatorname{reg}(\widetilde{I^n})$ is not a linear polynomial for $n \gg 0$. The ideal I will be the ideal of certain 'fat' points.

Example 4.2. Let $p_1, \ldots, p_s$ be distinct points on a rational normal curve in $\mathbb{P}^r$, $s \geq 2$. Let $\wp_1, \ldots, \wp_s$ denote their defining prime ideals in $A = k[X_0, \ldots, X_r]$, where k is an arbitrary algebraically closed field, and $I = \wp_1 \cap \ldots \cap \wp_s$. Then

$$\widetilde{I^n} = \wp_1^n \cap \ldots \cap \wp_s^n.$$

By [CTV, Proposition 7] we know that

$$\operatorname{reg}(A/\widetilde{I^n}) = \max\left\{2n-1, \left[\frac{ns+r-2}{r}\right]\right\}.$$

Note that $\operatorname{reg}(\widetilde{I^n}) = \operatorname{reg}(A/\widetilde{I^n}) + 1$. If $s \geq 2r$, then

$$\operatorname{reg}(\widetilde{I^n}) = \left[\frac{ns+2r-2}{r}\right].$$

In this case, if s is not divided by r, $\operatorname{reg}(\widetilde{I^n})$ differs from a linear function by a *periodic* function whose values depend on the residue of s modulo r.

A more precise result can be obtained in the following situation

Theorem 4.3. *Let I be a homogeneous ideal. Assume that the graded algebra $\oplus_{n\geq 0}\widetilde{I^n}t^n$ is finitely generated. Then there exists a positive integer r and linear polynomials $f_i(n) = nd_i + e_i$ for $0 \leq i \leq r-1$ such that* $\operatorname{reg}(\widetilde{I^n}) = f_{\sigma(n)}(n)$ *for $n >> 0$, where $\sigma(n) \equiv n$ mod r.*

Proof. Since $\tilde{R} = \oplus\tilde{I^n}t^n$ is finitely generated, it may be written as a factor ring of a bigraded polynomial ring $S = k[X_1,\dots,X_r,Y_1,\dots,Y_s]$ where $\deg X_i = (1,0)$ for $i = 1,\dots,r$, and $\deg Y_j = (d_j,t_j)$ for $j = 1,\dots,s$. The arguments of Lemma 3.3 apply as well to $\tilde{R}$. So we conclude that

$$\operatorname{reg}_i(\tilde{I^n}) = \max\{a \mid \operatorname{Tor}_i^S(S/N,\tilde{R})_{(a,n)} \neq 0\} - i.$$

Thus the conclusion follows if we prove the following analogue of Theorem 3.4: Suppose that E is a finitely generated bigraded module over $T = k[Y_1,\dots,Y_s]$ where $\deg Y_j = (d_j,t_j)$ for $j = 1,\dots,s$. Then there exists an integer k_0 and linear functions ℓ_i, $i = 0,\dots,k_0$, such that for all $n \gg 0$ one has that $\rho_E(n) = \ell_i(n)$ if $n \equiv i \mod k_0$.

Consider the $\mathbb{N}$-grading $T_b = \oplus_a T_{(a,b)}$. Then there exists an integer k_0 such that the k_0th Veronese subring $T^{(k_0)} = \oplus_{i\geq 0}T_{ik_0}$ of T is standard graded in degree 1 (after normalizing the grading). Note that E considered as an $T^{(k_0)}$-module decomposes as $E = \oplus_{i=0}^{k_0-1}T_iE$. Therefore we may apply 3.4, and see that the functions $\rho_{T_iE}(n)$ of the $T^{(k_0)}$-modules T_iE are linear for $n \gg 0$.

Now let n be arbitrary. Then $n = mk_o + i$ with $0 \leq i \leq k_0 - 1$, and $\rho_E^T(n) = \rho_{T_iE}^{T^{(k_0)}}(m)$. Hence the conclusion follows. $\square$

The following example shows that in general $\operatorname{reg}(\widetilde{I^n})$ is not a linear polynomial with periodic coefficients.

Example 4.4. For any $p > 0$ such that p is congruent to 2 mod 3, there exists a field k of characteristic p and an ideal $I \subset k[x,y,z]$ such that the regularity of the saturated powers $\widetilde{I^n}$ is not (eventually) periodic. In fact, $\operatorname{reg}(\widetilde{I^{5n+1}}) = 29n + 7$ if n is not a power of p and $\operatorname{reg}(\widetilde{I^{5n+1}}) = 29n + 8$ if n is a power of p.

In [CS, Section 6] the first author and Srinivas construct a counterexample to Zariski's Riemann-Roch problem in char $p > 0$. There one can find a non singular projective curve C of genus 2 over a field k of characteristic $p \neq 0$ as above with points $\eta, q \in C$ such that

$$h^1(\mathcal{O}_C(n(\eta - q) + q)) = \begin{cases} 0 & \text{if } n \text{ is not a power of } p \\ 1 & \text{if } n \text{ is a power of } p. \end{cases}$$

We will use this curve to construct our example.

Set $D = 6q - \eta$. Then D is a divisor on C such that $\deg(D) = 5 \geq 2g + 1$, where $g = 2$ is the genus of C. Thus D is very ample [H2, Corollary IV.3.2] and $h^0(\mathcal{O}_C(D)) = \deg(D) + 1 - g = 4$ [H2, Example IV.1.3.4 and Theorem IV.1.3]. Hence $H^0(C, \mathcal{O}_C(D))$ gives an embedding of C as a curve of degree 5 in $\mathbb{P}^3$. We can project C onto a degree 5 plane curve γ with only nodes as singularities from a point in $\mathbb{P}^3$ not on C [H2, Theorem IV.3.10]. The arithmetic genus of γ is $p_a(\gamma) = 2 + n$ where n is the number of nodes of γ [H2, Exercise IV.1.8]. Since $d = \deg(\gamma) = 5$, $p_a(\gamma) = \frac{1}{2}(d-1)(d-2) = 6$ [H2, Exercise I.7.2]. Thus γ has $n = 4$ nodes.

Let these singular points be $q_1,\dots,q_4$. Let $\pi_1 : S_1 \to \mathbb{P}^2$ be the blow up of these 4 points. Let F_i be the exceptional curves that map respectively to q_i. Let γ_1 be the strict transform of γ. Then $\gamma_1 \cong C$ since it is nonsingular. Let $H_1 = \pi_1^{-1}(H')$ where H' is a hyperplane on $\mathbb{P}^2$. Since the singular points are nodes,

$$\pi^{-1}(\gamma) = \gamma_1 + 2F_1 + \cdots + 2F_4$$

and

$$F_i \cdot \gamma_1 = q_{i1} + q_{i2}$$

for (distinct) points q_{ij} on γ_1, $1 \leq i \leq 4$, $j = 1, 2$. The divisor

$$5H_1 \cdot \gamma_1 - 2q_{11} - \cdots - 2q_{42} - \eta + 5q$$

has degree 13 since $(H_1 \cdot \gamma_1) = (H' \cdot \gamma) = 5$. Thus it is very ample [H2, Corollary IV.3.2], and there are points $p_1,\dots,p_{13} \in \gamma_1$ such that

$$5H_1 \cdot \gamma_1 - 2q_{11} - \cdots - 2q_{42} - \eta + 5q \;\sim\; p_1 + \cdots + p_{13}$$

where $\sim$ denotes linear equivalence.

Let $\pi_2 : S_2 \to S_1$ be the blowup of the points $p_1, \ldots, p_{13}$, with respective exceptional curves E_i mapping to p_i. Let $\overline{\gamma} \cong C$ be the strict transform of γ_1, $\overline{F}_i$ be the strict transform of F_i for $1 \le i \le 4$. Let $\pi : S_2 \to \mathbb{P}^2$ be the composed map. Let $\overline{H} = \pi^{-1}(H')$. Then

$$5\overline{H} \ \sim \ \pi^{-1}(\gamma) = \overline{\gamma} + E_1 + \ldots + E_{13} + 2\overline{F}_1 + \cdots + 2\overline{F}_4.$$

$$\overline{\gamma} \cdot \overline{\gamma} \ \sim \ (5\overline{H} - E_1 - \cdots - E_{13} - 2\overline{F}_1 - \cdots - 2\overline{F}_4) \cdot \overline{\gamma} \ \sim \ \eta - 5q.$$

By our construction, $\overline{H} \cdot \overline{\gamma} \sim D = 6q - \eta$. Thus

$$\begin{aligned}(5\overline{\gamma} + 4\overline{H}) \cdot \overline{\gamma} \ &\sim \ \eta - q, \\ (\overline{\gamma} + \overline{H}) \cdot \overline{\gamma} \ &\sim \ q.\end{aligned}$$

Set $A = 5\overline{\gamma} + 4\overline{H}$, $B = \overline{\gamma} + \overline{H}$. Observe that $(\overline{\gamma}^2) = -4$ and $(\overline{\gamma} \cdot \overline{H}) = 5$. $H^1(S_2, \mathcal{O}_{S_2}(m\overline{H})) = 0$ for all $m \ge 0$ and $H^1(\overline{\gamma}, \mathcal{O}_{\overline{\gamma}}(m\overline{H} + n\overline{\gamma})) = 0$ if $5m - 4n \ge 3$ since $((m\overline{H} + n\overline{\gamma}) \cdot \overline{\gamma}) = 5m - 4n$ and a divisor on a curve of genus g is nonspecial if its degree is $> 2g - 2$ [H2, Example IV.1.3.4]. Consideration of the cohomology of

$$(*) \qquad 0 \to \mathcal{O}_{S_2}(m\overline{H} + (n-1)\overline{\gamma}) \to \mathcal{O}_{S_2}(m\overline{H} + n\overline{\gamma}) \to \mathcal{O}_{\overline{\gamma}}(m\overline{H} + n\overline{\gamma}) \to 0$$

and induction imply $H^1(S_2, \mathcal{O}_{S_2}(m\overline{H} + n\overline{\gamma})) = 0$ if $5m - 4n \ge 3$. The relations $H^2(S_2, \mathcal{O}_{S_2}(m\overline{H})) = 0$ for all $m \ge 0$ and $H^2(\overline{\gamma}, \mathcal{O}_{\overline{\gamma}}(m\overline{H} + n\overline{\gamma})) = 0$ for all m, n imply $H^2(S_2, \mathcal{O}_{S_2}(m\overline{H} + n\overline{\gamma})) = 0$ for all $m, n > 0$.

For all $n \ge 0$ we have

$$0 \to \mathcal{O}_{S_2}(nA + \overline{H}) \to \mathcal{O}_{S_2}(nA + B) \to \mathcal{O}_{\overline{\gamma}}(nA + B) \to 0.$$

By the above, $H^1(S_2, \mathcal{O}_{S_2}(nA + \overline{H})) = H^2(S_2, \mathcal{O}_{S_2}(nA + \overline{H})) = 0$ for all $n \ge 0$. From (*) we see that

$$h^1(\mathcal{O}_{S_2}(nA + B)) = h^1(\mathcal{O}_C(n(\eta - q) + q)) = \begin{cases} 0 & \text{if } n \text{ is not a power of } p \\ 1 & \text{if } n \text{ is a power of } p. \end{cases}$$

By (*), $H^1(S_2, \mathcal{O}_{S_2}(4n\overline{H} + (5n-1)\overline{\gamma})) = 0$ for all $n > 0$. Then by the Riemann-Roch Theorem on $\overline{\gamma}$ and (*),

$$h^1(\mathcal{O}_{S_2}(4n\overline{H} + 5n\overline{\gamma})) = h^1(\mathcal{O}_{\overline{\gamma}}(4n\overline{H} + 5n\overline{\gamma})) = 1$$

for $n > 0$ since $((4n\overline{H} + 5n\overline{\gamma}) \cdot \overline{\gamma}) = 0$ and by Riemann-Roch. $((4n\overline{H} + (5n+1)\overline{\gamma}) \cdot \overline{\gamma}) = -4$. Thus $h^0(\mathcal{O}_{\overline{\gamma}}(4n\overline{H} + (5n+1)\overline{\gamma})) = 0$ and $h^1(\mathcal{O}_{\overline{\gamma}}(4n\overline{H} + (5n+1)\overline{\gamma})) = 5$ by Riemann-Roch. By (*) we have $h^1(\mathcal{O}_{S_2}(4n\overline{H} + (5n+1)\overline{\gamma}) = 4$.

The formulas $((nA + B + m\overline{H}) \cdot E_i) > 0$ and $((nA + B + m\overline{H}) \cdot \overline{F}_i) > 0$ for all $m, n \ge 0$ imply that $R^i\pi_*\mathcal{O}_{S_2}(nA+B+m\overline{H}) = 0$ for $m, n \ge 0$, and $H^1(S_2, \mathcal{O}_{S_2}(nA+B+m\overline{H})) = H^1(S, \pi_*\mathcal{O}_{S_2}(nA+B+m\overline{H}))$. The relation

$$nA + B + m\overline{H} \sim (29n + 6 + m)\overline{H} - (5n+1)(E_1 + \cdots + E_{13} + 2\overline{F}_1 + \cdots + 2\overline{F}_4)$$

implies

$$\pi_*\mathcal{O}(nA + B + m\overline{H}) \cong (\mathcal{I}_1^{5n+1} \cap \cdots \cap \mathcal{I}_{13}^{5n+1} \cap \mathcal{J}_1^{10n+2} \cap \cdots \cap \mathcal{J}_4^{10n+2}) \otimes \mathcal{O}(29n + 6 + m)$$

where $\mathcal{I}_i$ are the ideal sheaves of the points p_i and $\mathcal{J}_j$ are the ideal sheaves of the points q_j in $\mathbb{P}^2$.

Let $\wp_1, \ldots, \wp_{13}$ and $\wp_{14}, \ldots, \wp_{17}$ be the homogeneous primes in $k[x, y, z]$ which sheafify to $\mathcal{I}_1, \ldots, \mathcal{I}_{13}$ and $\mathcal{J}_1, \ldots, \mathcal{J}_4$, respectively. Set

$$I = \wp_1 \cap \cdots \cap \wp_{13} \cap \wp_{14}^2 \cap \cdots \cap \wp_{17}^2.$$

Let $\mathfrak{m} = (x, y, z)$. Then

$$H^0_{\mathfrak{m}}(\widetilde{I^n}) = H^1_{\mathfrak{m}}(\widetilde{I^n}) = 0,$$
$$H^2_{\mathfrak{m}}(\widetilde{I^n}) = \oplus_{a\in\mathbb{Z}} H^1(\mathbb{P}^2, \mathcal{I}^n(a)),$$
$$H^3_{\mathfrak{m}}(\widetilde{I^n}) = \oplus_{a\in\mathbb{Z}} H^2(\mathbb{P}^2, \mathcal{I}^n(a)),$$

where $\mathcal{I}$ is the ideal sheaf of I. Putting everything together, we obtain

$$\dim_k H^2_{\mathfrak{m}}(\widetilde{I^{5n+1}})_{(s-2)} = \begin{cases} 0 & \text{if } s > 29n+8 \\ 0 & \text{if } s = 29n+8 \text{ and } n \text{ is not a power of } p \\ 1 & \text{if } s = 29n+8 \text{ and } n \text{ is a power of } p. \\ 4 & \text{if } s = 29n+7. \end{cases}$$
$$H^3_{\mathfrak{m}}(\widetilde{I^{5(n+1)}})_{(s-3)} = 0 \quad \text{if } s \geq 29n+7.$$

By Theorem 4.3 we know that $\oplus_{n\geq 0}\widetilde{I^n}$ is not a finitely generated k-algebra. We can verify this directly.

If $\oplus_{n\geq 0}\widetilde{I^n}$ were finitely generated, there would be a surjection of a bigraded polynomial ring onto $\oplus_{n\geq 0}\widetilde{I^n}$. Then the subalgebra

$$R = \oplus_{n\geq 0}\left(\widetilde{I^{5n}}\right)_{29n}$$

would be finitely generated. We will show that R is not finitely generated.

$$R \cong \oplus_{n\geq 0} H^0(S_2, \mathcal{O}_{S_2}(nA)).$$

¿From (*), and our calculation $H^1(S_2, \mathcal{O}_{S_2}(m\overline{H}+n\overline{\gamma})) = 0$ if $5m-4n \geq 3$, we see that we have surjections

$$H^0(S_2, \mathcal{O}_{S_2}(nA)) \to H^0(\overline{\gamma}, \mathcal{O}_{\overline{\gamma}}(nA)) \cong H^0(\overline{\gamma}, \mathcal{O}_{\overline{\gamma}}(n(\eta - q))) = 0$$

since $\eta - q$ must have infinite order in the Jacobian of $\overline{\gamma}$, and

$$H^0(S_2, \mathcal{O}_{S_2}(nA - \overline{\gamma})) \to H^0(\overline{\gamma}, \mathcal{O}_{\overline{\gamma}}(nA - \overline{\gamma})) \neq 0$$

since

$$(\overline{\gamma} \cdot (nA - \overline{\gamma})) = -(\overline{\gamma}\cdot\overline{\gamma}) = 4 \geq 2g$$

and by [H2, Corollary IV.3.2]. Thus the fixed locus (counting multiplicity) of the complete linear system $\mid nA \mid$ is $\overline{\gamma}$ for all $n > 0$. Since this multiplicity is nonzero and bounded for all $n > 0$, R is not finitely generated (c.f. [[Z], part I, section 2]).

The following example shows interesting asymptotic behaviour for an ideal in the coordinate ring of an abelian surface. In this example, $\lim \frac{\text{reg}(\widetilde{I^n})}{n}$ is an irrational number. The construction is based on an example in [Cu].

Example 4.4. Let k be a an algebraically closed field of arbitrary characteristic. Let C be an elliptic curve over k and let $S = C \times C$. Let $\Delta \subset S$ be the diagonal, $P \in S$ a closed point and $A = \pi_1^{-1}(p)$, $B = \pi_2^{-1}(P)$, where $\pi_i : S \to C$, $i = 1, 2$ are the projections. Let $\text{NS}(S)$ be the Neron-Severi group of S and $\overline{\text{NE}}(S)$ be the closure in the metric topology on $\text{NS}(S)\otimes_{\mathbf{Z}}\mathbf{R}$ of the cone generated by the curves on S. Let $\mathcal{V} \subset \text{NS}(S)\otimes_{\mathbf{Z}}\mathbf{R}$ be the real vector space with basis $\{A, B, \Delta\}$. Observe that $(\Delta^2) = (A^2) = (B^2) = 0$, $(A\cdot B) = (A\cdot\Delta) = (B\cdot\Delta) = 1$. Let

$$U = \{(x, y, z) \mid (xA + yB + z\Delta)^2 > 0\} = \{(x, y, z) \mid (xy + xz + yz) > 0\}.$$

U consists of two disjoint, connected cones. Let G be the connected component containing $L = A+B+\Delta$. By the index Theorem $(E\cdot L) > 0$ for any rational $E \in G$. Hence the effective classes in $\mathcal{V}$ are contained in the closure $\overline{G}$ of G. If E is a rational class in G, then E is ample by the Riemann-Roch Theorem, and the fact that any effective divisor on an abelian surface with a positive intersection number is ample. Hence $\overline{G} = \mathcal{V} \cap \overline{\mathrm{NE}}(S)$. Let $H = 3A + 6B + 9\Delta$, $D = A + B + \Delta$.

$$(sH - D)^2 = 198s^2 - 72s + 6 = 0$$

has the roots

$$s_1 = \frac{1}{33}(6 - \sqrt{3}), s_2 = \frac{1}{33}(6 + \sqrt{3}).$$

If $s > s_2$ then $sH - D$ is in the ample cone. If $s_1 < s < s_2$ then $sH - D$ is not in the effective cone, and $D - sH$ is not in the effective cone.

By Mumford's Vanishing Theorem (section 16 of [Mu]), if m and r are nonnegative integers,

$$H^1(S, \mathcal{O}_S(mH - rD)) = 0 \text{ if } m > rs_2 \text{ and } H^2(S, \mathcal{O}_S(mH - rD)) = 0 \text{ if } m > rs_2.$$

Suppose that m, r are nonnegative integers such that $s_1 r < m < s_2 r$. Then

$$H^0(S, \mathcal{O}_S(mH - rD)) = 0 \text{ and } H^2(S, \mathcal{O}_S(mH - rD)) = H^0(S, \mathcal{O}_S(rD - mH)) = 0.$$

By the Riemann-Roch Theorem of section 16 [Mu],

$$\chi(mH - rD) = \frac{(mH - rD)^2}{2}.$$

Thus if $s_1 r < m < s_2 r$ we have

$$h^1(mH - rD) = -\frac{(mH - rD)^2}{2} > 0.$$

H is very ample on S by the Lefschetz Theorem (section 17 of [Mu]). Set $R = \oplus_{n \geq 0} H^0(S, \mathcal{O}_S(nH))$, with graded maximal ideal m. Let I_1 be the homogeneous ideal of A, I_2 be the homogeneous ideal of B, I_3 the homogeneous ideal of Δ. Let $I = I_1 \cap I_2 \cap I_3$. Let $\mathcal{I}$ be the sheafification of I. Since $H^2_m(\widetilde{I^r})_{n-2} \cong H^1(S, \mathcal{O}_S(n-2)H - rD))$ and $H^3_m(\widetilde{I^r})_{n-3} \cong H^2(S, \mathcal{O}_S(n-3)H - rD))$, we have that the "regularity" of $\widetilde{I^r}$ is $[s_2 r] + 2 = \left[\frac{r}{33}(6 + \sqrt{3})\right] + 2$.

The ring $\oplus_{n \geq 0} \widetilde{I^n}$ of Example 4.4 is not finitely generated. This follows since

$$\left(\widetilde{I^r}\right)_m = H^0(S, \mathcal{O}_S(mH - rD)) = \begin{cases} 0 & \text{if } m < s_2 r \\ \neq 0 & \text{if } m > s_2 r. \end{cases}$$

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