IRRATIONAL ASYMPTOTIC BEHAVIOUR OF CASTELNUOVO-MUMFORD REGULARITY

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1. Introduction

Let $\mathcal{I}$ be an ideal sheaf on $\mathbb{P}^r$. The regularity $\mathrm{reg}(\mathcal{I})$ of $\mathcal{I}$ is defined to be the least integer t such that $H^i(\mathbb{P}^r, \mathcal{I}(t-i)) = 0$ for all $i \geq 1$ (c.f. Mumford, Lecture 14 [M]).

This concept can be defined purely algebraically. Let $A = k[X_1, \dots, X_r]$ be a polynomial ring over an arbitrary field k. Let I be a homogeneous ideal in A. The regularity $\mathrm{reg}(I)$ of I is defined to be the maximum degree n for which there is an index j such that $H^j_{\mathfrak{m}}(I)_{n-j} \neq 0$, where $H^j_{\mathfrak{m}}(I)$ denotes the jth local cohomology module of I with respect to the maximal graded ideal $\mathfrak{m}$ of A (c.f Eisenbud and Goto [EG], Bayer and Mumford [BM]).

If $\mathcal{I}$ is the sheafification of a homogeneous ideal I of A, then $\mathrm{reg}(\mathcal{I}) = \mathrm{reg}(\widetilde{I})$, where $\widetilde{I}$ is the saturation of I. The saturation of I is defined by

$$\widetilde{I} = \{f \in A \mid \text{ for each } 0 \leq i \leq n, \text{ there exists } n_i > 0 \text{ such that } X_i^{n_i} f \in I\}$$

The functions $\mathrm{reg}(I^n)$ and $\mathrm{reg}(\mathcal{I}^n)$ are quite interesting. Some references on this and related problems are [BEL], [BPV], [Ch], [CTV], [GGP], [HT], [Hu2], [K], [S], [SS], [ST], [T].

Let I be any homogeneous ideal of A. In Theorem 1.1 of [CHT] it is shown that $\mathrm{reg}(I^n)$ is a linear polynomial for all n large enough.

Because of the examples given in [CS] showing the failure of the Riemann Roch problem to have a good solution in general, it is perhaps to be expected that $\mathrm{reg}(\mathcal{I}^n)$ will also not have polynomial like behaviour. However, there are some significant differences in these problems, and there are special technical difficulties involved in constructing a geometric example calculating the regularity of powers of an ideal sheaf in projective space.

In this paper we construct examples of ideal sheaves $\mathcal{I}$ of nonsingular curves in $\mathbb{P}^3$, showing bizarre behaviour of $\mathrm{reg}(\mathcal{I}^n)$.

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If we take $a = 9$, $b = 1$, $c = 1$ in Theorem 10, and make use of the comments following Lemma 8, we get

$$\operatorname{reg}(\mathcal{I}^r) = [r(9+\sqrt{2})] + 1 + \sigma(r)$$

for $r > 0$, where $[r(9+\sqrt{2})]$ is the greatest integer in $r(9+\sqrt{2})$,

$$\sigma(r) = \begin{cases} 0 & \text{if } r = q_{2n} \text{ for some } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

q_m is defined recursively by $q_0 = 1, q_1 = 2$ and $q_n = 2q_{n-1} + q_{n-2}$. Note that r such that $\sigma(r) = 0$ are quite sparse, as

$$q_{2n} \geq 3^n.$$

In this example,

$$\lim \frac{\operatorname{reg}(\mathcal{I}^n)}{n} = 9 + \sqrt{2}.$$

Example 10 uses the method of [C], which is to find a rational line which intersects the boundary of the cone of effective curves on a projective surface in irrational points.

In [CHT] we give examples showing that $\operatorname{reg}(\mathcal{I}^n)$ is at least not eventually a linear polynomial, or a linear polynomial with periodic coefficients. In particular, using a counter-example to Zariski's Riemann-Roch problem in positive characteristic [CS] we construct an example of a union points in $\mathbb{P}^2$ in positive characteristic such that $\operatorname{reg}(\mathcal{I}^n)$ is not eventually a linear polynomial with periodic coefficients (Example 4.3 [CHT]). In this example,

$$\operatorname{reg}(\mathcal{I}^{5n+1}) = \begin{cases} 29n + 7 & \text{if } n \text{ is not a power of } p \\ 29n + 8 & \text{if } n \text{ is a power of } p. \end{cases}$$

In [CHT] we consider the limits

$$\lim \frac{\operatorname{reg}(I^n)}{n} \tag{1}$$

and

$$\lim \frac{\operatorname{reg}(\mathcal{I}^n)}{n}. \tag{2}$$

It follows from Theorem 1.1 of [CHT] that (1) always has a limit, which must be a natural number. In the examples given in [CHT], (2) has a limit which is a rational number. In the examples of Theorem 10, (2) is an irrational number.

$$\lim \frac{\operatorname{reg}(\mathcal{I}^n)}{n} \notin \mathbb{Q}.$$

Suppose that $\mathcal{I}$ is an ideal sheaf on $\mathbb{P}^n$. Let H be a hyperplane section of $\mathbb{P}^n$. Ein and Lazarsfeld [EL] and Pauletti [P] have defined the Seshadri constant $\epsilon(\mathcal{I})$ of $\mathcal{I}$ to be

$$\epsilon(\mathcal{I}) = \sup\{\eta \in \mathbb{Q} | f^*(H) - \eta E \text{ is ample }\}.$$

Robert Lazarsfeld and Lawrence Ein have pointed out to me that the limit (2) is equal to $\frac{1}{\epsilon(\mathcal{I})}$, the reciprical of the Seshadri constant of $\mathcal{I}$. Our examples thus give examples of smooth space curves which have irrational Seshadri constants. We also see that the limit (2) must always exist.

I would like to thank Keiji Oguiso for suggesting the intersection form $q = 4x^2 - 4y^2 - 4z^2$, and Olivier Piltant and Qi Zhang for helpful discussions on the material of this paper.

2. Preliminaries on K3 surfaces

A K3 surface is a nonsingular complex projective surface F such that $H^1(F, \mathcal{O}_F) = 0$ and the canonical divisor K_F of F is trivial.

For a divisor D on a complex variety V we will denote

$$h^i(D) = \dim_{\mathbb{C}} H^i(V, \mathcal{O}_V(D)). \tag{3}$$

We will write $D \sim E$ if D is linearly equivalent to E.

On a K3 surface F, the Riemann-Roch Theorem (c.f. 2.1.1 [SD]) is

$$\chi(D) = \frac{1}{2}D^2 + 2. \tag{4}$$

If D is ample, we have

$$h^i(D) = 0 \text{ for } i > 0 \tag{5}$$

by the Kodaira Vanishing Theorem (In dimension 2 this follows from Theorem IV.8.6 [BPV] and Serre duality). Let $\mathrm{Num}(F) = \mathrm{Pic}(F)/\equiv$, where $D \equiv 0$ if and only if $(D \cdot C) = 0$ for all curves C on F. $\mathrm{Num}(F) \cong \mathrm{Pic}(F)$, since F is a K3 (c.f. (2.3) [SD]), and $\mathrm{Pic}(F) \cong \mathbb{Z}^n$, where $n \leq 20$ (c.f. Proposition VIII.3.3 [BPV]). $\mathrm{Num}(F)$ has an intersection form q induced by intersection of divisors, $q(D) = (D^2)$. Let

$$\mathrm{N}(F) = \mathrm{Num}(F) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $NE(F)$ be the smallest convex cone in $\mathrm{N}(F)$ containing all effective 1-cycles, $\overline{NE}(F)$ be the closure of $NE(F)$ in the metric topology. We can identify $\mathrm{Pic}(F)$ with the integral points in $\mathrm{N}(F)$. By abuse of notation, we will sometimes identify a divisor with its equivalence class in $\mathrm{N}(F)$.

Nakai's criterion for ampleness (c.f. Theorem 5.1 [H]) is D is ample if and only if $(D^2) > 0$ and $(D \cdot C) > 0$ for all curves C on F.

$[\gamma]$ will denote the greatest integer in a rational number γ.

Theorem 1. *(Theorem 2.9 [Mo]) For $\rho \leq 11$, every even lattice of signature $(1, \rho - 1)$ occurs as the Picard group of a smooth, projective K3 surface.*

3. Construction of a Quartic Surface

By Theorem 1,

$$q = 4x^2 - 4y^2 - 4z^2$$

is the intersection form of a projective K3 surface S.

Lemma 2. *Suppose that C is an integral curve on S. Then $C^2 \geq 0$.*

Proof. Suppose that $(C^2) < 0$. Then $h^0(C) = 0$, and $(C^2) = -2$ (c.f. the note after (2.7.1) [SD]). But $4 \mid (C^2)$ by our choice of q, so $(C^2) \geq 0$.

Theorem 3. *Suppose that D is ample on S. Then*
1. $h^i(D) = 0$ *for* $i > 0$.
2. $h^0(D) = \frac{1}{2}(D^2) + 2$.
3. $\mid D \mid$ *is base point free.*
4. *There exists a nonsingular curve* Γ *on* S *such that* $\Gamma \sim D$.
5. *Let* $g = \frac{1}{2}(D^2) + 1$ *be the genus of a general curve* $\Gamma \in \mid D \mid$. *Let* $\phi : S \to \mathbb{P}^g$ *be the morphism induced by* $\mid D \mid$. *Then either*
   a) ϕ *is an isomorphism onto a surface of degree* $2g - 2$, *or*
   b) ϕ *is a 2-1 morphism onto a rational surface of degree* $g - 1$.

Proof. 1. and 2. are (5) and (4). We will prove 3. By 2. we can assume that D is effective. Suppose that C is an irreducible fixed component of $\mid D \mid$. Then $h^0(C) = 0$ and $(C^2) = -2$ by the note after (2.7.1) [SD]. But this is not possible by Lemma 2. Thus $\mid D \mid$ has no fixed component, and is thus base point free by Corollary 3.2 [SD]. 4. Follows from the fact that $\mid D \mid$ is base point free, $(D^2) > 0$ and by Bertini's Theorem.

We will now prove 5. By (4.1) [SD] and Theorem 6.1 [SD] either b) holds or ϕ is a birational morphism onto a normal surface of degree $2g - 2$. Since S has no curves with negative intersection number, ϕ has no exceptional curves. By Zariski's Main Theorem, ϕ is an isomorphism.

Suppose that $D \in \text{Pic}(S)$ and $D^2 \geq 0$. By Riemann Roch and Serre duality, either D or $-D$ is (linearly equivalent to) an effective divisor.

Suppose that $D \in \text{Pic}(S)$, $D^2 > 0$ and D is effective. Then D is ample. To see this, choose a very ample divisor L on S. Then for $m >> 0$, $mD - L$ is linearly equivalent to an effective divisor. Thus $(C \cdot D) > 0$ for all curves C on S, and D is ample by Nakai's criterion, stated in Section 2.

Let $D = (1, 0, 0) \in \text{Pic}(S)$. Since $D^2 = q(1, 0, 0) = 1 > 0$, either D or $-D$ is ample. Without loss of generality, we may assume that D is ample. $(D^2) = 4$ implies $h^0(D) = 4$. Thus there exists an effective divisor H such that $H = (1, 0, 0)$. We thus have

$$\overline{NE}(S) = \{(x, y, z) \in \mathbb{R}^3 | 4x^2 - 4y^2 - 4z^2 \geq 0, \ x \geq 0\}.$$

By Nakai's criterion, cited in Section 2, a divisor D is ample if and only if D is in the interior of $\overline{NE}(S)$.

Remark 4. *If A nontrivial divisor* $D = (a, b, c) \in \mathbb{Z}^3$ *is linearly equivalent to an effective divisor, then we must have* $a > 0$, *since otherwise* $(D \cdot H) \leq 0$. *Thus we cannot have* $H \sim D_1 + D_2$ *where* D_1 *and* D_2 *are effective divisors, both non trivial.*

Theorem 5. $\mid H \mid$ *embeds* S *as a quartic surface in* $\mathbb{P}^3$.

Proof. Consider the rational map

$$\Phi : S \to \mathbb{P}^3$$

induced by $\mid H \mid$. By Theorem 3, either Φ is an isomorphism onto a quartic surface in $\mathbb{P}^3$, or else Φ is a 2-1 morphism onto a quadric surface in $\mathbb{P}^3$.

If ϕ is 2-1, then Φ is 2-1 onto a quadric surface Q in $\mathbb{P}^3$, which must have, after suitable choice of homogeneous coordinates, one of the forms

$$x_0x_1 - x_2x_3 = 0$$

or

$$x_0x_1 - x_2^2 = 0,$$

and $\Phi^*(\mathcal{O}(1)) \cong \mathcal{O}_S(H)$. In either case there exists a reducible member of $\Gamma(Q, \mathcal{O}_Q(1))$, so that $\mid H \mid$ contains a reducible member, a contradiction to Remark 4.

Choose $(a, b, c) \in \mathbb{Z}^3$ such that $a > 0$, $a^2 - b^2 - c^2 > 0$ and $\sqrt{b^2 + c^2} \notin \mathbb{Q}$. Since (a, b, c) is in the interior of $\overline{NE}(S)$ there exists an ample divisor A such that $A = (a, b, c)$. By Theorem 3, there exists a nonsingular curve C on S such that $C \sim A$. Consider the line $tH - C$, $-\infty < t < \infty$ in $N(S)$. This line intersects the quadric $q = 0$ in 2 irrational points

$$\lambda_2 = a + \sqrt{b^2 + c^2} \text{ and}$$

$$\lambda_1 = a - \sqrt{b^2 + c^2}.$$

Remark 6. *Suppose that $m, r \in \mathbb{N}$. Then*

1. *$mH - rC \in \overline{NE}(S)$ and is ample if $r\lambda_2 < m$.*
2. *$mH - rC \notin \overline{NE}(S)$ and $rC - mH \notin \overline{NE}(S)$ if $r\lambda_1 < m < r\lambda_2$.*
3. *$rC - mH \in \overline{NE}(S)$ and is ample if $m < r\lambda_1$.*

We can choose (a, b, c) so that

$$7 < \lambda_1 < \lambda_2 \tag{6}$$

and

$$\lambda_2 - \lambda_1 > 2. \tag{7}$$

By (4), for all $m, r \in \mathbb{N}$,

$$\chi(mH - rC) = \frac{1}{2}(mH - rC)^2 + 2. \tag{8}$$

Theorem 7. *Suppose that $m, r \in \mathbb{N}$. Then*

$$h^1(mH - rC) = \begin{cases} 0 & \text{if } r\lambda_2 < m \\ -\frac{1}{2}(mH - rC)^2 - 2 & \text{if } r\lambda_1 < m < r\lambda_2 \end{cases}$$

$$h^2(mH - rC) = 0 \text{ if } r\lambda_1 < m$$

Proof. By Remark 6, and (5), we have that $h^i(mH - rC) = 0$ if $r\lambda_2 < m$ and $i > 0$.

For $r\lambda_1 < m < r\lambda_2$ we have by Remark 6 that both $mH - rC \notin \overline{NE(S)}$ and $rC - mH \notin \overline{NE(S)}$. Thus $h^0(mH - rC) = 0$ and (by Serre duality) $h^2(mH - rC) = h^0(rC - mH) = 0$. By (8),

$$h^1(mH - rC) = -\chi(mH - rC) = -\frac{1}{2}(mH - rC)^2 - 2.$$

Lemma 8. *For all $r \in \mathbb{Z}_+$, we must have either*

$$h^1([r\lambda_2]H - rC) > 0 \text{ or}$$
$$h^1([r\lambda_2]H - rC) = 0 \text{ and } h^1(([r\lambda_2] - 1)H - rC) \neq 0$$

Proof. (7) implies that $r\lambda_1 < [r\lambda_2] - 1 < [r\lambda_2] < r\lambda_2$. Suppose that

$$h^1([r\lambda_2]H - rC) = h^1(([r\lambda_2] - 1)H - rC) = 0.$$

Then by Theorem 7,

$$-\frac{1}{2}([r\lambda_2]H - rC)^2 - 2 = -\frac{1}{2}(([r\lambda_2] - 1)H - rC)^2 - 2$$

so that

$$([r\lambda_2]H - rC)^2 = (([r\lambda_2] - 1)H - rC)^2$$

and thus

$$q([r\lambda_2] - ra, -rb, -rc) = q([r\lambda_2] - ra - 1, -rb, -rc).$$

We then have

$$([r\lambda_2] - ra)^2 = ([r\lambda_2] - ra - 1)^2$$

so that $-2([r\lambda_2] - ra) + 1 = 0$, a contradiction since $[r\lambda_2] - ra \in \mathbb{Z}$.

The dependence of the vanishing $h^1([r\lambda_2]H - rC) = 0$ on r is surprisingly subtle. Set $d = b^2 + c^2$.

(9)
$$\begin{aligned} &h^1([r\lambda_2]H - rC) = 0 \text{ iff} \\ &([r\lambda_2] - ra)^2 - r^2b^2 - r^2c^2 = -1 \text{ iff} \\ &[r\lambda_2] = ra + \sqrt{r^2(b^2 + c^2) - 1} \text{ iff} \\ &[r\sqrt{b^2 + c^2}] = \sqrt{r^2(b^2 + c^2) - 1} \text{ iff} \end{aligned}$$

There exists $s \in \mathbb{N}$ such that $r^2(b^2 + c^2) - 1 = s^2$ iff

There exists $s \in \mathbb{N}$ such that $s + r\sqrt{d}$ is a unit in $\mathbb{Q}(\sqrt{d})$ with norm

(10)
$$N(s + r\sqrt{d}) = -1.$$

If $b = c = 1$ so that $d = 2$, we have that the integral solutions to (10) are

$$s + r\sqrt{d} = \pm(1 + \sqrt{2})^{2n+1} \text{ and } s + r\sqrt{d} = \pm(-1 + \sqrt{2})^{2n+1}$$

for $n \in \mathbb{N}$ (c.f. Theorem 244 [HW]). As explained in the proof of Theorem 244 in [HW], if $\frac{p_n}{q_n}$ are the convergents of

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}},$$

then the solutions to (10) are $\pm(p_{2n} + q_{2n}\sqrt{2})$ and $\pm(p_{2n} - q_{2n}\sqrt{2})$ for $n \in \mathbb{N}$. When we impose the condition that $r, s \in \mathbb{N}$, we get that the solutions to (9) are

$$q_{2n}, \ n \in \mathbb{N}$$

when $d = 2$. q_m are defined recursively by

$$q_0 = 1, \ q_1 = 2, \ q_m = 2q_{m-1} + q_{m-2}.$$

4. Construction of the Example

Regard S as a subvariety of $\mathbb{P}^3$, as embedded by $\mid H \mid$. There exists a hyperplane H' of $\mathbb{P}^3$ such that $H' \cdot S = H$. Now let $\pi : X \to \mathbb{P}^3$ be the blow up of C. Let $E = \pi^*(C)$ be the exceptional surface, $\overline{H} = \pi^*(H')$. Let $\overline{S}$ be the strict transform of S. Then $\pi^*(S) = \overline{S} + E$, $S \cong \overline{S}$ and $\overline{S} \cdot E = C$. Let $\mathcal{I}_C$ be the ideal sheaf of C in $\mathbb{P}^3$. Since C is smooth, for $m, r \in \mathbb{N}$,

$$\begin{aligned} &\pi_* \mathcal{O}_X(m\overline{H} - rE) \cong \mathcal{I}_C^r(m) \\ &R^i \pi_* \mathcal{O}_X(m\overline{H} - rE) = 0 \text{ for } i > 0, \end{aligned} \tag{11}$$

(c.f. Proposition 10.2 [Ma]). From the short exact sequence

$$0 \to \mathcal{O}_X(-\overline{S}) \to \mathcal{O}_X \to \mathcal{O}_{\overline{S}} \to 0$$

and the isomorphism $\mathcal{O}_X(-4\overline{H} + E) \cong \mathcal{O}_X(-\overline{S})$ we deduce for all m, r short exact sequences

(12)
$$0 \to \mathcal{O}_X(m\overline{H} - rE) \to \mathcal{O}_X((m+4)\overline{H} - (r+1)E) \to \mathcal{O}_S((m+4)H - (r+1)C) \to 0$$

We have by (11),

$$\begin{aligned} &h^i(\mathcal{O}_X(m\overline{H})) = h^i(\mathcal{O}_{\mathbb{P}^3}(mH')) = 0 \text{ for } m \in \mathbb{N}, \text{ and } 0 < i < 3, \\ &h^3(\mathcal{O}_X(m\overline{H})) = h^3(\mathcal{O}_{\mathbb{P}^3}(mH')) = h^0((-m-4)H') = 0 \text{ for } m \geq 0. \end{aligned}$$

In particular, we have

$$h^i(\mathcal{O}_X(m\overline{H})) = 0 \text{ for } i > 0,\ m \geq 0. \tag{13}$$

Theorem 9. *Suppose that $m, r \in \mathbb{N}$. Then*

$$\begin{aligned} &h^1(m\overline{H} - rE) = \begin{cases} 0 & m > r\lambda_2 \\ h^1(mH - rC) & \textit{if } m = [r\lambda_2] \textit{ or } m = [r\lambda_2] - 1 \end{cases} \\ &h^2(\overline{m}H - rE) = 0 \textit{ if } m > \lambda_1 r \\ &h^3(m\overline{H} - rE) = 0 \textit{ if } m > 4r. \end{aligned}$$

Proof. From (12) we have

(14)
$$0 \to \mathcal{O}_X((n+4(t-1))\overline{H} - (t-1)E) \to \mathcal{O}_X((n+4t)\overline{H} - tE) \to \mathcal{O}_S((n+4t)H - tC) \to 0.$$

From Theorem 7, (13) and induction applied to the long exact cohomology sequence of (14) we deduce that $h^i((4t+n)\overline{H} - tE) = 0$ if $t \geq 0$, $n \geq 0$, $t\lambda_2 < 4t + n$, $i > 0$. Thus $h^i(m\overline{H} - rE) = 0$ if $i > 0$, $m \geq 4r$, $r\lambda_2 < m$. Since $4 < \lambda_2$ by (6), we have

$$h^i(m\overline{H} - rE) = 0 \text{ if } i > 0, r\lambda_2 < m.$$

From Theorem 7, (13) and (14), we also deduce $h^2(m\overline{H} - rE) = 0$ for $r\lambda_1 < m$, since $4 < \lambda_1$ by (6), and $h^3(m\overline{H} - rE) = 0$ for $4r \leq m$.

We now compute $h^1([r\lambda_2]\overline{H} - rE)$. By (6), $\lambda_2 - 4 > 1 > r\lambda_2 - [r\lambda_2]$ implies $[r\lambda_2] - 4 > (r-1)\lambda_2$. We have a short exact sequence

$$0 \to \mathcal{O}_X([r\lambda_2]-4)\overline{H} - (r-1)E) \to \mathcal{O}_X([r\lambda_2]\overline{H} - rE) \to \mathcal{O}_S([r\lambda_2]H - rC) \to 0$$

Since $h^1(([r\lambda_2]-4)\overline{H} - (r-1)E) = h^2(([r\lambda_2]-4)\overline{H} - (r-1)E) = 0$, we have

$$h^1([r\lambda_2]\overline{H} - rE) = h^1([r\lambda_2]H - rC).$$

A similar calculation shows that

$$h^1(([r\lambda_2]-1)\overline{H} - rE) = h^1(([r\lambda_2]-1)H - rC).$$

since (6) implies that $[r\lambda_2] - 5 > (r-1)\lambda_2$.

Define

$$\sigma(r) = \begin{cases} 0 & \text{if } h^1([r\lambda_2]H - rC) = 0 \\ 1 & \text{if } h^1([r\lambda_2]H - rC) \neq 0. \end{cases}$$

Theorem 10.

$$\mathit{reg}(\mathcal{I}_C^r) = [r\lambda_2] + 1 + \sigma(r)$$

for $r > 0$. In particular

$$\lim \frac{\mathrm{reg}(\mathcal{I}^r)}{r} = \lambda_2 \notin \mathbb{Q}.$$

Proof. For $i, m, r \geq 0$ we have, by (11),

$$h^i(\mathcal{I}_C^r(m)) = h^i(m\overline{H} - rE).$$

By Theorem 9 and Lemma 8,

$$h^1(\mathcal{I}_C^r(t-1)) = h^1((t-1)\overline{H} - rE) = \begin{cases} 0 & t \geq [r\lambda_2] + 1 + \sigma(r) \\ \neq 0 & t = [r\lambda_2] + \sigma(r) \end{cases}$$

By (7), $([r\lambda_2]+1) - 2 = [r\lambda_2] - 1 \geq r\lambda_2 - 2 > \lambda_1 r$ and

$$h^2(\mathcal{I}_C^r(t-2)) = h^2((t-2)\overline{H} - rE) = 0 \text{ if } t \geq [r\lambda_2] + 1.$$

By (6), $([r\lambda_2]+1) - 3 = [r\lambda_2] - 2 \geq r\lambda_2 - 3 > 4r$ and

$$h^3(\mathcal{I}_C^r(t-3)) = h^3((t-3)\overline{H} - rE) = 0 \text{ if } t \geq [r\lambda_2] + 1.$$

As a consequence of Theorem 10, we can construct the example stated in the Introduction.

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