SIMULTANEOUS RESOLUTION OF SINGULARITIES (TO APPEAR IN PROC. AMS)

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Abstract. We prove a local theorem on simultaneous resolution of singularities, which is valid in all dimensions. This theorem is proven in dimension 2 (and in all characteristics) by Abhyankar in his book "Ramification theoretic methods in algebraic geometry" [2].

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1. Introduction

Suppose that $f : V \to W$ is a generically finite morphism of normal varieties. If the characteristic of the ground field is zero, or if we are in char p and dimension ≤ 3, we know from resolution of singularities and resolution of indeterminancy of morphisms ([8], [3]) that there are sequences of blowups of nonsingular subvarieties (monoidal transforms) $V_1 \to V$ and $W_1 \to W$ such that V_1 and W_1 are nonsingular, and $g : V_1 \to W_1$ is a generically finite morphism.

A simultaneous resolution of V and W is a pair of nonsingular V_1 and W_1 as above, such that $V_1 \to W_1$ is a finite morphism. Even if V and W have dimension two, a simultaneous resolution may not exist, even locally along a valuation, as shown by an example of Abhyankar (Theorem 12 [4]).

If we only require that W_1 be nonsingular, but allow V_1 to be normal, we can take V_1 to be the normalization of a resolution of singularities W_1 of W in the function field of V, and obtain a finite map $V_1 \to W_1$.

However, the essential problem is to obtain a finite map such that V_1 is nonsingular.

In Theorem 1.1 we prove that given a fixed valuation ν of the characteristic zero function field K of V, we can find a nonsingular variety V_1, birationally dominating V, and a normal variety W_1 birationally dominating W, such that $g : V_1 \to W_1$ is finite in a neighborhood of the center of ν. That is, if p is the center of ν on V_1 and $q = g(p)$ is the center of ν on W_1, then $\mathcal{O}_{\mathcal{W}_\infty,\mathrm{II}}$ lies below $\mathcal{O}_{\mathcal{V}_\infty,\surd}$.

Theorem 1.1. *Let k be a field of characteristic zero, L/k an algebraic function field, K a finite algebraic extension of L, ν a valuation of K/k, and (R, M) a regular local*

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ring with quotient field K, essentially of finite type over k, such that ν dominates R. Then for some sequence of monodial transforms $R \to R^*$ along ν, there exists a normal local ring S^* with quotient field L, essentially of finite type over k, such that R^* is the localization of the integral closure T of S^* in K at a maximal ideal of T.

We review notation from [2]. A local domain (R, M) is algebraic with ground field k if R is essentially of finite type over k. Suppose that k, L, K are as in the statement of Theorem 1.1. Suppose that $(S^*, N^)$ and (R^*, M^*) are normal algebraic local domains with ground field k such that the quotient field of S^* is L and the quotient field of R^* is K. We say that S^* lies below R^* if R^* is a localization of the integral closure T of S^* in K at a maximal ideal of T. Suppose that K is Galois over L and S^* lies below R^*. The splitting field $F^s(R^*/S^*)$ is the smallest field K' between L and K such that R^* is the only local ring in K lying above $R^* \cap K'$. If ν is a valuation of L with valuation ring R_ν, and ν^* is an extension of ν to K with valuation ring R_{ν^*}, then $F^s(\nu^*/\nu)$ is defined to be $F^s(R_{\nu^*}/R_\nu)$.

The following Theorem 1.2 follows immediately from our Theorem 1.1.

Theorem 1.2. *Let k be a field of characteristic zero, L/k an n-dimensional algebraic function field, K a finite algebraic extension of L, ν a zero dimensional valuation of K/k, and (R, M) a regular algebraic local domain with quotient field K and ground field k such that ν has center M in R. Then for some sequence of monoidal transforms $R \to R^*$ along ν, there exists an algebraic local domain S^* with quotient field L and ground field k lying below R^*.*

This Theorem is proved in dimension two (and in any characteristic) by Abhyankar (Theorem 4.8 [2]). Our Theorem 2 is stated in the notation of Theorem 4.8 [2]. In the special case when the valuation ν has maximal rational rank n, Fu [7] has shown that the conclusions of Theorem 2 hold.

In Theorem 3 we prove simultaneous resolution in a setting which is useful in the theory of resolution of singularities.

Theorem 1.3. *Let K be an n-dimensional algebraic function field with ground field k of characteristic zero. Let ν be a zero dimensional valuation of K/k. Suppose that K^* is a Galois extension of K, and ν^* is an extension of ν to K^*, (R^*, M^*) and (R, M) are n-dimensional normal algebraic local domains with ground field k and quotient field K^* and K respectively, such that ν^* has center M^* in R^* and $R = R^* \cap K$. Suppose that $F^s(\nu^*/\nu) = F^s(R^*/R)$ and $R^s = R^* \cap F^s(\nu^*/\nu)$ is a regular algebraic domain. Then there exists a sequence of monoidal transforms $R^s \to \overline{R}^s$ along ν^* such that there exists a regular algebraic domain $\overline{R}$ with ground field k and quotient field K lying below $\overline{R}^s$.*

In Theorem 4.9 [2], Abhyankar proves local uniformization along a valuation in dimension 2, when the ground field k is algebraically closed of characteristic zero. The proof in fact shows more. It proves that (over an algebraically closed ground field of characteristic zero), "embedded local uniformization in algebraic local domains of dimension n" + "the conclusions of our Theorem 2 (or Theorem 3) in dimension n" implies "local uniformization in dimension n".

Our theorems are an application of our theorem on monomialization of generically finite morphisms along a valuation (Theorem A [6]) and the general theory developed in [2]. We thank Professor Abhyankar for pointing out to us that simultaneous resolution is related to our monomialization theorem. Here we state the monomialization theorem from our paper [6], which we use in this paper.

Theorem 1.4. *(Theorem A [6] - Monomialization) Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, such that the quotient field K of S is a finite extension of the quotient field J of R.*

Let V be a valuation ring of K which dominates S. Then there exist sequences of monoidal transforms (blow ups of regular primes) $R \rightarrow R'$ and $S \rightarrow S'$ such that V dominates S', S' dominates R' and there are regular parameters $(x_1, ..., x_n)$ in R', $(y_1, ..., y_n)$ in S', units $\delta_1, \dots, \delta_n \in S'$ and a matrix (a_{ij}) of nonnegative integers such that $Det(a_{ij}) \neq 0$ and

$$\begin{aligned} x_1 &= y_1^{a_{11}} .....y_n^{a_{1n}} \delta_1 \\ &\vdots \\ x_n &= y_1^{a_{n1}} .....y_n^{a_{nn}} \delta_n. \end{aligned}$$

Theorem A is used in [6] to prove a factorization theorem for birational morphisms. This theorem is proved in dimension 3 in our paper [5].

2. Simultaneous resolution

2.1. **Resolution of Singularities.** Hironaka's theorems on resolution, the "fundamental theorems" and "main theorems" stated in chapter I of [8], apply to "algebraic schemes". An algebraic scheme is defined (on page 162 of [8]) to be a separated scheme of finite type over spec(S), with S a local ring in β. The class β of local rings is defined on page 161 of [8]. $S \in \beta$ if

(1) S is a (Noetherian) local ring with residue field of characteristic 0.
(2) If $\hat{S}$ denotes the completion of the local ring S, then for every S-algebra of finite type A, the singular locus of spec$(A \otimes_S \hat{S})$ is the preimage of that of spec(A) under the canonical map of the first spectrum into the second.

Theorem 2.1. *Suppose that S is an excellent local ring containing a field of characteristic 0. Then $S \in \beta$. Thus spec(S) is an "algebraic scheme".*

Proof. For a Noetherian ring B, let reg(B) denote the subset of spec(B) of primes p such that B_p is a regular local ring, sing(B) denote the subset of spec(B) of primes p such that B_p is not a regular local ring.

Suppose that A is an S-algebra of finite type. $S \rightarrow \hat{S}$ is faithfully flat and regular, since S is a local G-ring. The natural map $\varphi : A \rightarrow A \otimes_S \hat{S}$ is then faithfully flat and regular by Lemma 4 (33.E) [9]. $(\varphi^*)^{-1}(\text{reg}(A)) = \text{reg}(A \otimes_S \hat{S})$ by Theorem 51 (21.D) [9]. Thus $(\varphi^*)^{-1}(\text{sing}(A)) = \text{sing}(A \otimes_S \hat{S})$. □

Suppose that (R, m) is a local domain, with maximal ideal m, and that $P \subset R$ is a prime ideal, such that R/P is regular. Suppose that $0 \neq f \in P$, and m_1 is a prime ideal in $R[\frac{P}{f}]$ such that $m_1 \cap R = m$. Set $R_1 = (R[\frac{P}{f}])_{m_1}$. R_1 (or $R \rightarrow R_1$) is called a monoidal transform of R. If $P = m$, then R_1 is called a quadratic transform.

We will say that a valuation ν of the quotient field of R dominates R if ν has nonnegative value on R, and $m = \{f \in R | \nu(f) > 0\}$. A monoidal transform $R \rightarrow R_1$ is along ν if ν dominates R and R_1.

Theorem 2.2. *Suppose that R, S are excellent local domains containing a field k of characteristic zero such that S dominates R and S is regular. Let ν be a valuation of the quotient field K of S that dominates S, $R \rightarrow R_1$ a monoidal transform such that ν dominates R_1. R_1 is a local ring on $X = Proj(\bigoplus_{n \geq 0} p^n)$ for some prime $p \subset R$. Let*

$$U = \{Q \in spec(S) : pS_Q \text{ is invertible } \}$$

an open subset of $spec(S)$. Then there exists a projective morphism $f : Y \rightarrow spec(S)$ which is a product of monoidal transforms such that if S_1 is the local ring of Y dominated by ν, then S_1 dominates R_1, and $(f)^{-1}(U) \rightarrow U$ is an isomorphism.

Proof. Since S is a UFD, we can write $pS = gI$, where $g \in S$, $I \subset S$ has height ≥ 2. Then $U = \text{spec}(S) - V(I)$. By Main Theorem II(N) [8], there exists a sequence of monoidal transforms $\pi : Y \rightarrow \text{spec}(S)$ such that $I\mathcal{O}_{\mathcal{Y}}$ is invertible, and $\pi^{-1}(U) \rightarrow U$ is an isomorphism. Let S_1 be the local ring of the center of ν on Y. We have $pS_1 = hS_1$ for some $h \in p$. Hence $R[\frac{p}{h}] \subset S_1$, and since ν dominates S_1, R_1 is the localization of $R[\frac{p}{h}]$ which is dominated by S_1. □

Theorem 2.3. *Suppose that R is an excellent regular local domain containing a field of characteristic zero, with quotient field K. Let ν be a valuation of K dominating R. Suppose that $f \in K$ is such that $\nu(f) \geq 0$. Then there exists a sequence of monoidal transforms along ν*

$$R \rightarrow R_1 \rightarrow \cdots \rightarrow R_n$$

such that $f \in R_n$.

Proof. Write $f = \frac{a}{b}$ with $a, b \in R$. By Main Theorem II(N) [8] applied to the ideal $I = (a, b)$ in R, there exists a sequence of monoidal transforms along ν, $R \rightarrow R_n$ such that $IR_n = \alpha R_n$ is a principal ideal. There exist c, d, u_1, u_2 in R_n such that $a = c\alpha, b = d\alpha, \alpha = u_1 a + u_2 b$. Then $u_1 c + u_2 d = 1$, so that $cR_n + dR_n = R_n$, and one of c or d is a unit in R_n. If c is a unit, then $0 \leq \nu(f) = \nu(\frac{c}{d}) = \nu(c) - \nu(d)$ implies $\nu(d) = 0$, and since ν dominates R_n, d is a unit and $f \in R_n$. □

Suppose that Y is an algebraic scheme, X, D are subschemes of Y. Suppose that $g : Y' \rightarrow Y$, $f : X' \rightarrow X$ are the monoidal transforms of Y and X with center D and $D \cap X$ respectively. Then there exists a unique isomorphism of X' to a subscheme X'' of Y' such that g induces f (cf. chapter 0, section 2 [8]). X'' is called the strict transform of X by the monoidal transform g.

Theorem 2.4. *Let R be an excellent regular local ring, containing a field of characteristic zero. Let $W \subset spec(R)$ be an integral subscheme, $V \subset spec(R)$ be the singular locus of W. Then there exists a sequence of monoidal transforms $f : X \rightarrow spec(R)$ such that the strict transform of W is nonsingular in X, and f is an isomorphism over $spec(R) - V$.*

Proof. This is immediate from Theorem $I_2^{N,n}$ [8]. □

2.2. **Proof of Theorem 1.1.** Let V be the valuation ring of ν with maximal ideal m_ν.

We will first prove the theorem with the assumption that $\text{trdeg}_k V/m_\nu = 0$. Let $q_1, \dots, q_n$ be a transcendence basis of L/k. After replacing q_i by $\frac{1}{q_i}$ if necessary, we may assume that $\nu(q_i) \geq 0$ for all i. By Theorem 2.3, after possibly replacing R by a sequence of monoidal tranforms, we may assume that $q_1, \dots, q_n \in R$.

Let T be the integral closure of $k[q_1, \dots, q_n]$ in L. $T \subset R$. Set $N' = M \cap T$, $S = T_{N'}$, $N = N'T_{N'}$. $\text{trdeg}_k S/N \leq \text{trdeg}_k R/M \leq \text{trdeg}_k V/m_\nu = 0$. Thus $\dim S = \text{trdeg}_k L - \text{trdeg}_k S/N = n$.

By Theorems 2.4 and 2.2, we can perform a sequence of monoidal transforms $S \to S_1$ and $R \to R_1$ so that ν dominates R_1, R_1 dominates S_1, R_1 and S_1 are regular, and $\dim S_1 = \dim R_1 = n$.

By Theorem 1.4, we can perform sequences of monoidal transforms $R_1 \to R^*$ and $S_1 \to S^*$ so that ν dominates R^*, R^* dominates S^*, R^* has regular parameters $(y_1, \dots, y_n)$ and S^* has regular parameters $(x_1, \dots, x_n)$, such that there are units $\delta_1, \dots, \delta_n \in R^*$ and a matrix (a_{ij}) of nonnegative integers such that $\text{Det}(a_{ij}) \neq 0$ and

$$\begin{aligned} x_1 &= y_1^{a_{11}} \cdots\cdots y_n^{a_{1n}} \delta_1 \\ &\vdots \\ x_n &= y_1^{a_{n1}} \cdots\cdots y_n^{a_{nn}} \delta_n. \end{aligned}$$

By permuting variables, we may asume that $D = \text{Det}(a_{ij}) > 0$. Let (b_{ij}) be the adjoint matrix of (a_{ij}). We have that

$$x_1^{b_{j1}} \cdots x_n^{b_{jn}} = y_j^D \varepsilon_j$$

for $1 \leq j \leq n$, where ε_j are units in R^*. Since these elements are in $R^* \cap L$, we have the existence of a normal algebraic local domain S^* with ground field k and quotient field L lying below R^* by Proposition 3.12 [2].

Now we will prove the theorem for $\text{trdeg}_k V/m_\nu$ arbitrary. Let ν' be the restriction of ν to L. The valuation ring of ν is $V' = V \cap L$, with maximal ideal $m_{\nu'} = m_\nu \cap L$.

$$\text{trdeg}_k V/m_\nu = \text{trdeg}_k V'/m_{\nu'} \leq \text{trdeg}_k L < \infty$$

by Proposition 2.46 [2]. We can lift a transcendence basis of $V'/m_{\nu'}$ over k to $f_1, \dots f_m \in L$. $f_1, \dots, f_m$ are algebraically independent over k. Set $k' = k(f_1, \dots, f_m)$. By Theorem 2.3 we can perform a sequence of monoidal transforms $R \to R'$ along ν so that $f_1, \dots, f_m \in R'$. $k' \subset R'$ since $f_1, \dots, f_m$ are algebraically independent over k and ν dominates R'.

We have that L is a field of algebraic functions over k', R' is essentially of finite type over k' and V is a valuation of K/k' such that $\text{trdeg}_{k'} V/m_\nu = 0$. By the first part of the proof there exists a sequence of monodial transforms $R' \to R^*$ along ν, a normal local ring S^* with quotient field L, essentially of finite type over k', such that R^* is the localization of the integral closure T of S^* in K at a maximal ideal of T. S^* is essentially of finite type over k since k' is essentially of finite type over k.

2.3. **Proof of Theorem 1.3.** By Theorem 1.2, there exists a monoidal transform sequence $R^s \to \overline{R}^s$ along ν^* such that there exists a normal algebraic domain $\overline{R}$ in K lying below $\overline{R}^s$. Let $\overline{M}$, $\overline{M}^s$ be the respective maximal ideals in $\overline{R}$ and and $\overline{R}^s$. Let $(\overline{R}^*, \overline{M}^*)$ be the local ring in K^* lying above $\overline{R}^s$ such that ν^* has center $\overline{M}^*$ in $\overline{R}^*$.

$$F^s(\nu^*/\nu) = F^s(R^*/R) \subset F^s(\overline{R}^*/\overline{R}) \subset F^s(\nu^*/\nu)$$

implies $F^s(\overline{R}^*/\overline{R}) = F^s(\nu^*/\nu)$. Thus $\overline{R}^s/\overline{M}^s = \overline{R}/\overline{M}$, $\overline{M}\,\overline{R}^s = \overline{M}^s$ and $\overline{R}$ is regular by Theorem 1.47 [2] and Proposition 3.18B [2].

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