LOCAL MONOMIALIZATION OF TRANSCENDENTAL EXTENSIONS

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1. Introduction

Suppose that we are given a system of polynomial equations

$$
\begin{array}{ll}
y_1 & = f_1(x_1,\ldots,x_n)\\
& \vdots\\
y_m & = f_m(x_1,\ldots,x_n)
\end{array}
\tag{1}
$$

such that $m \le n$ and some $m \times m$ minor of the Jacobian matrix

$$\frac{\partial(y_1,\ldots,y_m)}{\partial(x_1,\ldots,x_n)}$$

is not identically zero.

In this paper we show that there exists a finite resolving system of (1). That is, we show that there are finitely many pairs of charts $(U_1,V_1),\ldots,(U_r,V_r)$ such that all solutions to (1) are transformed to a monomial solution in a pair (U_i,V_i) of the form

$$
\begin{array}{ll}
y_1(1) & = x_1(1)^{a_{11}}\cdots x_n^{a_{1n}}\\
& \vdots\\
y_m(1) & = x_1(1)^{a_{m1}}\cdots x_n(1)^{a_{mn}}.
\end{array}
\tag{2}
$$

The x and y variables are related to the $x(1)$ and $y(1)$ variables (in U_i and V_i respectively) by a sequence of monomial transforms. That is, they are related by a sequence of changes of variables and transformations of the form

$$z_j = \begin{cases} z_1'z_2' & \text{if } j=1\\ z_j' & \text{if } j\ne 1.\end{cases}$$

In our paper [14] we have proven the above result in the case when $m = n$. Our method of proof is through valuation theory and the development of other methods which properly belong to resolution of singularities. There are series obstacles to be overcome in generalizing the result of [14] to the main result of this paper.

1.1. Monomialization and toroidalization of morphisms of varieties.

We discuss an application of our Theorem 1.4 to proper morphisms of varieties, and the problem of monomialization of morphisms of varieties.

Definition 1.1. *Suppose that $\Phi : X \to Y$ is a dominant morphism of nonsingular integral finite type k schemes. Φ is monomial if for every $p \in X$ there exist regular*

partially supported by NSF.

parameters $(y_1, \ldots, y_m)$ *in* $\mathcal{O}_{Y,\Phi(p)}$*, and an étale cover* U *of an affine neighborhood of* p*, uniformizing parameters* $(x_1, \ldots, x_n)$ *on* U *and a matrix* a_{ij} *such that*

$$\begin{array}{rl} y_1 = & x_1^{a_{11}} \cdots x_n^{a_{1n}} \\ & \vdots \\ y_m = & x_1^{a_{m1}} \cdots x_n^{a_{mn}}. \end{array}$$

We do not assume that X and Y are separated in the above definition. Since Φ is dominant, the matrix (a_{ij}) must have maximal rank m.

A quasi-complete variety over a field k is an integral finite type k-scheme which satisfies the existence part of the valuative criterion for properness (Hironaka, Chapter 0, Section 6 of [26] and Chapter 8 of [14]).

The construction of a monomialization by quasi-complete varieties follows from Theorem 1.4.

Theorem 1.2. *Let* k *be a field of characteristic zero,* $\Phi : X \to Y$ *a dominant morphism of proper* k*-varieties. Then there are birational morphisms of nonsingular quasi-complete* k*-varieties* $\alpha : X_1 \to X$ *and* $\beta : Y_1 \to Y$*, and a monomial morphism* $\Psi : X_1 \to Y_1$ *such that the diagram*

$$\begin{array}{ccc} X_1 & \stackrel{\Psi}{\to} & Y_1 \\ \downarrow & & \downarrow \\ X & \stackrel{\Phi}{\to} & Y \end{array}$$

commutes and α *and* β *are locally products of blow ups of nonsingular subvarieties. That is, for every* $z \in X_1$*, there exist affine neighborhoods* V_1 *of* z*,* V *of* $x = \alpha(z)$*, such that* $\alpha : V_1 \to V$ *is a finite product of monoidal transforms, and there exist affine neighborhoods* W_1 *of* $\Psi(z)$*,* W *of* $y = \beta(\Psi(z))$*, such that* $\beta : W_1 \to W$ *is a finite product of monoidal transforms.*

Theorem 1.2 proves a local version of the toroidalization conjecture stated on page 568 of [9].

A monoidal transform of a nonsingular k-scheme S is the map $T \to S$ induced by an open subset T of $\mathrm{Proj}(\oplus \mathcal{I}^n)$, where $\mathcal{I}$ is the ideal sheaf of a nonsingular subvariety of S.

The case of Theorem 1.2 when $X \to Y$ is generically finite is proven in Theorem 1.2 of our paper [14].

The proof of Theorem 1.2 in general follows from Theorem 1.4, by patching a finite number of local solutions, as in the proof of Theorem 1.2 [14]. The resulting schemes may not be separated.

The strongest known result on monomialization is our theorem below.

Theorem 1.3. *(Theorem 18.21 and Theorem 19.11* [17]*) Suppose that* $\Phi : X \to S$ *is a dominant morphism from a 3 fold* X *to a surface* S *(over an algebraically closed field* k *of characteristic zero). Then there exist sequences of blow ups of nonsingular subvarieties* $X_1 \to X$ *and* $S_1 \to S$ *such that the induced map* $\Phi_1 : X_1 \to S_1$ *is a monomial (and toroidal) morphism.*

Theorem 1.3 proves the toroidalization conjecture of page 568 [9] for morphisms from 3 folds to surfaces.

A generalization of this result to prove monomialization (and toroidalization) of strongly prepared morphisms from N-folds to surfaces appears in the paper [19] with Olga Kashcheyeva.

1.2. **Local Monomialization.** Suppose that $R \subset S$ is a local homomorphism of local rings essentially of finite type over a field k and that V is a valuation ring of the quotient field K of S, such that V dominates S. Then we can ask if there are sequences of monoidal transforms $R \rightarrow R'$ and $S \rightarrow S'$ along V such that V dominates S', S' dominates R', and $R \rightarrow R'$ is a "monomial mapping",

$$\begin{array}{ccc} R' & \rightarrow & S' \subset V \\ \uparrow & & \uparrow \\ R & \rightarrow & S. \end{array}$$

We completely answer this question in the affirmative when k has characteristic 0 in Theorem 1.4. Notations are as in Section 2

Theorem 1.4. *Suppose that k is a field of characteristic zero, $K \rightarrow K^*$ is a (possibly transcendental) extension of algebraic function fields over k, and that ν^* is a valuation of K^* which is trivial on k. Further suppose that R is an algebraic local ring of K and S is an algebraic local ring of K^* such that S dominates R and ν^* dominates S. Then there exist sequences of monoidal transforms $R \rightarrow R'$ and $S \rightarrow S'$ along ν^* such that R' and S' are regular local rings, S' dominates R', there exist regular parameters $(y_1, \ldots, y_n)$ in S', $(x_1, \ldots, x_m)$ in R', units $\delta_1, \ldots, \delta_m \in S'$ and an $m \times n$ matrix (c_{ij}) of nonnegative integers such that (c_{ij}) has rank m, and*

$$x_i = \prod_{j=1}^{n} y_j^{c_{ij}} \delta_i \tag{3}$$

for $1 \leq i \leq m$.

When $K = k$, so that R is just the field k, Theorem 1.4 is Zariski's classical Local Uniformization Theorem, proven in [37]. In this case, the condition (3) is vacuous. Our Theorem 1.4, which is the most general possible relative Local Uniformization Theorem for mappings, is a substantial generalization of Zariski's theorem.

The case when the field extension $K \rightarrow K^*$ is finite is solved in Theorem 1.1 of our paper [14]. When $K = K^*$, Theorem 1.4 (or Theorem 1.1 [14]) implies local "weak factorization" of birational mappings (Theorem 1.6 [14]). The global version of this "weak factorization" conjecture has since been proven in [9]. As a corollary of our local monomialization theorem, we prove the stronger "local strong factorization" conjecture, which was conjectured by Abhyankar (page 237 [8], [11]). This is proven in Theorem A [13] (dimension 3), and in Theorem 1.6 [14] (general dimension), which reduces the proof to the case of toric varieties and toric valuations. "Local strong factorization" of morphisms of toric varieties along a toric valuation is proven in dimension 3 by Christensen [11], and in general by Karu [28]. A proof in the spirit of [11] using only elementary properties of determinants is given in [23].

In [15] and [21] we use Theorem 1.1 [14], which is the finite field extension $K \rightarrow K^*$ case of Theorem 1.4 to prove very strong results in the ramification theory of general valuations on characteristic zero algebraic function fields, such as Abhyankar's " Weak local simultaneous resolution conjecture" (this is conjectured in [3] and on page 144 [6]). It is expected that Theorem 1.4 can be used to extend this ramification theory to arbitrary extensions of characteristic zero algebraic function fields.

The standard theorems on resolution of singularities allow one to easily find R' and S' such that (3) holds, but, in general, the essential condition that (a_{ij}) has maximal rank m will not hold. It is for this reason that we must construct the sequence of monoidal transforms $R \rightarrow R'$, even if R is regular. The difficulty of the proof of the Theorem is to achieve this condition.

It is an interesting open problem to prove Theorem 1.4 in positive characteristic, even in dimension 2 ([20], [21]).

We will make a few comments here about the proof of Theorem 1.4. Our starting point is the proof for finite extensions $K \to K^*$ of our paper [14]. An overview of the proof (in the finite field extension case) can be found in Section 1.3 of [14].

Some parts of this proof generalize readily to the case when K^* is transcendental over K. For these parts, we give here the modified statements, and indicate the changes which must be made in the original proofs. However, there are some parts of the proof which are quite different. The really new ingrediants in the proof are given in the critical sections 6, 7 and 8 of this paper. As in the proof for the case when $K \to K^*$ is finite, we reduce to the case when V^* has rank 1. Since $V = V^* \cap K$ then has rank ≤ 1, and we can assume that V is nontrivial, we are reduced to the case when V has rank 1 also. Two new complexities arise in the case when K^* is transcendental over K. The rational rank of a valuation ν is the dimension of the $\mathbf{Q}$ vector space $\Gamma_\nu \otimes \mathbf{Q}$, where Γ_ν is the valuation group of ν. We have an inequality $\overline{r} = \text{ ratrank } (\nu) \le \overline{s} = \text{ ratrank } (\nu^*)$. If K^* is finite over K this is an equality. The case when $\overline{r} = \text{ ratrank } (\nu) < \overline{s} = \text{ ratrank } (\nu^*)$ is significantly more difficult. It is addressed in Section 8. The second major new complexity lies in the extension of residue fields of valuations. If $k(V^*)$ is the residue field of V^* and $k(V)$ is the residue field of V, then we have $\text{trdeg }_{k(V)} k(V^*) \le \text{ trdeg }_K K^*$. Thus $k(V^*)$ is algebraic (though not generally finite) over $k(V)$ if K^* is finite over K. The new arguments which are required to handle the case when $k(V^*)$ is transcendental over $k(V)$ are in Sections 6, 7 and 8.

We give a quick outline of the proof of Theorem 1.4 in the essential case when V^* has rank 1.

We easily reduce to the case when $k(V)$ is algebraic over k. Let $m = \text{trdeg}_k K$, $n = \text{trdeg}_k K^* - \text{trdeg}_k k(V^*)$.

It is straightforward to reduce to the case when R has regular parameters $x_1, \ldots, x_m$, S has regular parameters $y_1, \ldots, y_n$ such that

$$\begin{aligned} x_1 &= y_1^{c_{11}} \cdots y_{\overline{s}}^{c_{1s}} \phi_1 \\ &\vdots \\ x_{\overline{r}} &= y_1^{c_{\overline{r}1}} \cdots y_{\overline{s}}^{c_{\overline{rs}}} \phi_{\overline{r}} \end{aligned}$$

where $\phi_1, \ldots, \phi_{\overline{r}}$ are units in S. Necessarily, $\text{rank}(c_{ij}) = \overline{r}$.

The proof now proceeds by induction on l with $0 \le l < m - \overline{r}$. We assume that we have an expression

$$\begin{aligned} x_1 &= y_1^{c_{11}} \cdots y_{\overline{s}}^{c_{1s}} \phi_1 \\ &\vdots \\ x_{\overline{r}} &= y_1^{c_{\overline{r}1}} \cdots y_{\overline{s}}^{c_{\overline{rs}}} \phi_{\overline{r}} \\ x_{\overline{r}+1} &= y_{\overline{s}+1} \\ &\vdots \\ x_{\overline{r}+l} &= y_{\overline{s}+l} \end{aligned} \tag{4}$$

and perform monoidal transforms along V and V^* to achieve an improvement

$$\begin{aligned} x_1 &= y_1^{c_{11}} \cdots y_{\overline{s}}^{c_{1s}} \phi_1 \\ &\vdots \\ x_{\overline{r}} &= y_1^{c_{\overline{r}1}} \cdots y_{\overline{s}}^{c_{\overline{r}s}} \phi_{\overline{r}} \\ x_{\overline{r}+1} &= y_{\overline{s}+1} \\ &\vdots \\ x_{\overline{r}+l+1} &= y_{\overline{s}+l+1}. \end{aligned}$$

The proof of the induction step is by first applying Lemma 7.5 to $f = x_{\overline{r}+l+1}$ to see that f depends on some variable y_i with $i > \overline{s} + l$. The proof proceeds by performing a series of monoidal transforms which preserve the form (4), to simplify the form of f. Theorems 8.1 and 8.2 (which are formal) transform f into the expression given at the end of the statement of Theorem 8.2. Algebraization is performed in Theorems 9.1, 9.2 and Theorem 9.3. Finally, Theorem 9.4 (and Lemmas 5.3 and 5.4) conclude the proof.

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2. Notations

We will denote the maximal ideal of a local ring R by $m(R)$. If R contains a field k, we will denote its residue field by $k(R)$. We will denote the quotient field of a domain R by $Q(R)$. Suppose that $R \subset S$ is an inclusion of local rings. We will say that R dominates S if $m(S) \cap R = m(R)$. Suppose that K is an algebraic function field over a field k. We will say that a local ring R with quotient field K is an algebraic local ring of K if R is essentially of finite type over k. If R is a local ring, $\hat{R}$ will denote the completion of R at its maximal ideal.

If L_1 and L_2 are 2 subfields of a field M, then $L_1 * L_2$ will denote the subfield of M generated by L_1 and L_2.

Good introductions to the valuation theory which we require in this paper can be found in Chapter VI of [39] and in [4]. A valuation ν of K will be called a k-valuation if $\nu(k) = 0$. We will denote by V_ν the associated valuation ring, which necessarily contains k. A valuation ring V of K will be called a k-valuation ring if $k \subset V$. The value group of a valuation ν with valuation ring V will be denoted by Γ_ν or Γ_V. We will abuse notation by denoting the valuation ν, which is a homorphism of the group of units of K, as a function on K. If R is a subring of V_ν then the center of ν (the center of V_ν) on R is the prime ideal $R \cap m(V_\nu)$. If R is a Noetherian subring of V_ν and $I \subset R$ is an ideal, we will write $\nu(I) = \rho$ if $\rho = \min \{\nu(f) \mid f \in I\}$.

We will review the concept of composite valuations. For details, we refer to Section 10 of Chapter II of [4] and Section 10, Chapter VI [39]. If ν is a valuation of rank greater than 1, then ν is a composite valuation. That is, there are valuations w and $\overline{\nu}$ where w is a valuation of K and $\overline{\nu}$ is a valuation of the residue field of V_w such that if $\pi : V_w \rightarrow k(V_w)$ is the residue map, then $V_v = \pi^{-1}(V_{\overline{\nu}})$. For $f \in V_w$ such that $\pi(f) \neq 0$ we have $\nu(f) = \overline{\nu}(\pi(f))$. This gives us an inclusion of value groups $\Gamma_{\overline{\nu}} \subset \Gamma_\nu$. $\Gamma_{\overline{\nu}}$ is an isolated subgroup of Γ_ν. There exists a prime ideal p in V_ν such that $V_w = (V_\nu)_p$. For $f \in K$, $w(f)$ is the residue of $\nu(f)$ in $\Gamma_w = \Gamma_\nu / \Gamma_{\overline{\nu}}$. We say that ν is the composite of w and $\overline{\nu}$ and write $\nu = w \circ \overline{\nu}$.

Suppose that R is a local domain. A monoidal transform $R \rightarrow R_1$ is a birational extension of local domains such that $R_1 = R[\frac{P}{x}]_m$ where P is a regular prime ideal of R, $0 \neq x \in P$ and m is a prime ideal of $R[\frac{P}{x}]$ such that $m \cap R = m(R)$. $R \rightarrow R_1$ is called a quadratic transform if $P = m(R)$.

If R is regular, and $R \to R_1$ is a monodial transform, then there exists a regular sustem of parameters $(x_1, \dots, x_n)$ in R and $r \le n$ such that

$$R_1 = R\left[\frac{x_2}{x_1}, \dots, \frac{x_r}{x_1}\right]_m.$$

Suppose that ν is a valuation of the quotient field R with valuation ring V_ν which dominates R. Then $R \to R_1$ is a monoidal transform along ν (along V_ν) if ν dominates R_1.

3. Valuations

Lemma 3.1. *Suppose that K is a field containing a subfield k, $t_1, \dots, t_\alpha$ are algebraically independent over K and ν is a k-valuation of K with valuation ring V. Then there exists a unique extension $\overline{\nu}$ of ν to $K(t_1, \dots, t_\alpha)$, such that $\overline{\nu}(f) = \nu(f)$ for $f \in K$, $\overline{\nu}(t_i) = 0$, and if $\overline{V}$ is the valuation ring of $\overline{\nu}$, then the images of $t_1, \dots, t_\alpha$ in $k(\overline{V})$ are algebraically independent over $k(V)$.*

Proof. For $I = (i_1, \dots, i_\alpha) \in \mathbf{N}^\alpha$, let $t^I = t_1^{i_1} \dots t_\alpha^{i_\alpha}$. If

$$0 \ne h = \sum_{I \in \mathbf{N}^\alpha} f_I t^I \in K[t_1, \dots, t_\alpha]$$

with $f_I \in K$, define

$$\overline{\nu}(f) = \min\{\nu(f_I) \mid f_I \ne 0\}.$$

This induces an extension of ν as desired. We will verify that $\overline{\nu}(fg) = \overline{\nu}(f) + \overline{\nu}(g)$ for

$$f = \sum_I f_I t^I, g = \sum_J g_j t^J \in K[t_1, \dots, t_\alpha].$$

$$fg = \sum_A \left(\sum_{I+J=A} f_I h_J\right) t^A.$$

For each A,

$$\nu\left(\sum_{I+J=A} f_I g_J\right) \ge \min\{\nu(f_I) + \nu(g_J) \mid I + J = A\} \ge \overline{\nu}(f) + \overline{\nu}(g).$$

Let I_0 be such that $\nu(f_{I_0}) > \nu(f_I)$ if $I < I_0$ (in the Lex order) and $\nu(f_I) \ge \nu(f_{I_0}))$ if $I > I_0$. Similarily, let J_0 be such that $\nu(g_{J_0}) > \nu(g_J)$ if $J < J_0$ and $\nu(g_J) \ge \nu(g_{J_0}))$ if $J > J_0$. Let $A_0 = I_0 + J_0$. Then

$$\nu(\sum_{I+J=A_0} f_I g_J) = \nu(f_{I_0} g_{J_0}) = \overline{\nu}(f) + \overline{\nu}(g).$$

Thus $\overline{\nu}(fg) = \overline{\nu}(f) + \overline{\nu}(g)$.

Suppose that $\overline{\nu}$ is an extension of ν with the desired properties. If $h = \sum f_I t^I \in K[t_1, \dots, t_\alpha]$ with $f_I \in K$, let f_J be such that $\nu(f_J) = \min\{\nu(f_I)\}$. If $\overline{\nu}(h) > \nu(f_J)$, then $\overline{\nu}(\sum_I \frac{f_I}{f_J} t^I) > 0$ so that

$$[\sum \frac{f_I}{f_J} t^J] = 0 \text{ in } k(\overline{V}),$$

where $[\beta]$ denotes the class of $\beta \in \overline{V}$ in $k(\overline{V})$. But by assumption, $[t_1], \dots, [t_\alpha]$ are algebraically independent in $k(\overline{V})$ over $k(V)$. This is a contradiction. $\square$

Lemma 3.2. *Suppose that K is a field containing a subfield k, $t_1, \ldots, t_\alpha$ are analytically independent over K and ν is a k-valuation of K with valuation ring V. Suppose that R is a noetherian local domain with quotient field K such that V dominates R. Then there exists a unique extension $\overline{\nu}$ of ν to $Q(R[[t_1 \ldots, t_\alpha]])$ such that $\overline{\nu}(f) = \nu(f)$ for $f \in K$, $\overline{\nu}(t_i) = 0$ for $1 \le i \le \alpha$, and if $\overline{V}$ is the valuation ring of $\overline{\nu}$, then the images of $t_1, \ldots, t_\alpha \in k(\overline{V})$ are analytically independent over $k(V)$.*

Proof. For a series

$$f = \sum a_I t^I \in R[[t_1, \ldots, t_\alpha]]$$

with $a_I \in R$, we define

$$\overline{\nu}(f) = \min \{\nu(f_I) \mid f_I \in R\}.$$

We first verify that $\overline{\nu}$ is well defined. Suppose that

$$\min \{\nu(f) \mid f_I \in R\}$$

does not exist. Then there exists an infinite descending chain of values

$$a_1 > a_2 > a_3 > \cdots > 0$$

and $f_i \in R$ such that $\nu(f_i) = a_i$ for all positive integers i. Let I_i be the R ideal

$$I_i = \{g \in R \mid \nu(g) \ge a_i\}.$$

Then we have an infinite strictly ascending chain of ideals in R,

$$I_1 \subset I_2 \subset \cdots,$$

a contradiction to the assumption that R is Noetherian.

As in the proof of Lemma 3.1, $\overline{\nu}$ induces an extension of ν as desired. As in the proof of Lemma 3.1, $\overline{\nu}$ is unique. □

4. Rational Rank 1 Valuations

Suppose that k is a field of characteristic 0 and $K \to K^*$ is an extension of algebraic function fields over k. Suppose that ν^* is a rank 1 k-valuation of K^* with valuation ring V^*. Let $\nu = \nu^* \mid K$ with valuation ring $V = V^* \cap K$. Necessarily, ν has rank ≤ 1 (c.f. Lemma 10.3 and the discussion following Lemma 10.3). Assume that ν has rank 1 and that $k(V)$ is algebraic over k.

Let $\overline{r} = \text{ratrank}\ \nu$, $\overline{s} = \text{ratrank}\ \nu^*$ be the respective rational ranks. Let $m = \text{trdeg}_k K$, $n = \text{trdeg}_k K^* - \text{trdeg}_k k(V^*)$. We necessarily have $m \le n$, $\overline{r} \le m$, $\overline{s} \le n$ and $\overline{r} \le \overline{s}$.

Suppose that R is an algebraic local ring of K, S is an algebraic local ring of K^* such that R and S are regular, S dominates R, and V^* dominates S (so that V dominates R).

Suppose that $\text{trdeg}_k k(S) = \text{trdeg}_k k(V^*)$. Further suppose that $t_1, \ldots, t_{\overline{\alpha}} \in S$ are such that their residues in $k(V^*)$ are a transcendence basis of $k(V^*)$ over $k(V)$. We then have that the residues of $t_1, \ldots t_{\overline{\alpha}}$ in $k(S)$ are also a transcendence basis of $k(S)$ over $k(R)$.

We define a monoidal transform sequence (MTS) as in Definition 3.1 of [14] and define a uniformizing transform sequence (UTS), a rational uniformizing transform sequence (RUTS) and a UTS along a valuation as in Definition 3.2 [14].

We also define, for our R with quotient field K and extension ring S with quotient field K^* a compatable UTS (CUTS), a compatable RUTS (CRUTS) and a CUTS along ν^* as on page 29 of [14]. Of course, in a CUTS $(R, \overline{R}_n'', \overline{T}_n)$ and $(S, \overline{U}_n'', \overline{U}_n)$, we now have that the quotient field of $\overline{U}_i''$ is a finitely generated extension field of the quotient field of $\overline{T}''_i$ for all i, as opposed to the much stronger condition of being a finite extension, which holds in [14].

Lemma 3.3 of page 29 of [14] on the compatability of a CRUTS and its associated MTS is valid in our extended setting. The same proof holds.

Suppose that $(R,\overline{T}'',\overline{T})$ and $(S,\overline{U}'',\overline{U})$ is a CUTS along ν^*. When there is no danger of confusion, we will denote by ν^* our extension of ν^* to the quotient field of $\overline{U}''$ which dominates $\overline{U}''$, ν our extension of ν to the quotient field of $\overline{T}''$ which dominates $\overline{T}''$, $\tilde\nu^*$ our extension of ν^* to the quotient field of $\overline{U}$ which dominates $\overline{U}$, and $\tilde\nu$ our extension of ν to the quotient field of $\overline{T}$ which dominates $\overline{T}$.

For $f\in\overline{U}$, we will write $\nu^*(f)<\infty$ to mean $\tilde\nu^*(f)\in\Gamma_{\nu^*}$. For $f\in\overline{T}$, $\nu(f)<\infty$ will mean $\tilde\nu^*(f)\in\Gamma_\nu$.

Let $p_{\overline{U}}=\{f\in\overline{U}\mid \nu^*(f)=\infty\}$, $p_{\overline{T}}=\{f\in\overline{T}\mid \nu(f)=\infty\}$. Our extension of ν^* to $Q(\overline{U}/p_{\overline{U}})$ and of ν to $Q(\overline{T}/p_{\overline{T}})$ are canonical and have value groups Γ_{ν^*} and Γ_ν respectively. Note that we have natural embeddings $\overline{T}''\subset\overline{T}/p_{\overline{T}}$ and $\overline{U}''\subset\overline{U}/p_{\overline{U}}$. We will in general not be concerned with precise values of elements in $Q(\overline{U})$ and $Q(\overline{T})$ which have infinite value.

5. Perron Transforms

In this section, assumptions and notations will be as in Section 4.

We define a UTS $\overline{T}\rightarrow\overline{T}(1)$ of type I and a UTS $\overline{T}\rightarrow\overline{T}(1)$ of type II_r along ν, using the "Algorithm of Perron" [37] as in section 4.1 of [14]. Since our notations are a little different, we summarize the final forms of the transformations here. We assume (as in section 4.1 of [14]) that $\overline{T}''$ has regular parameters $(\overline{x}_1,\ldots,\overline{x}_m)$ such that $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_{\overline{r}})$ is a rational basis of $\Gamma_\nu\otimes\mathbf{Q}$.

We first state the equations defining a UTS $\overline{T}\rightarrow\overline{T}(1)$ of type I. $\overline{T}''(1)=\overline{T}'(1)$ has regular parameters $(\overline{x}_1(1),\ldots,\overline{x}_m(1))$ such that

$$\begin{aligned}\overline{x}_1=&\ \overline{x}_1(1)^{a_{11}}\cdots\overline{x}_{\overline{r}}(1)^{a_{1\overline{r}}}\\ &\ \vdots\\ \overline{x}_{\overline{r}}=&\ \overline{x}_1(1)^{a_{\overline{r}1}}\cdots\overline{x}_{\overline{r}}(1)^{a_{\overline{r}\overline{r}}}\end{aligned}$$

and $\overline{x}_i=\overline{x}_i(1)$ for $\overline{r}<i\le m$. The matrix $A=(a_{ij})$ of natural numbers is computed using Perron's algorithm. We have $\mathrm{Det}(A)=\pm1$, and $\nu(\overline{x}_1(1)),\ldots,\nu(\overline{x}_{\overline{r}}(1))$ are a rational basis of $\Gamma_\nu\otimes\mathbf{Q}$.

We now state the equations defining a UTS $\overline{T}\rightarrow\overline{T}(1)$ of type II_r with $0<r\le m-\overline{r}$. $\overline{T}''(1)$ has regular parameters $(\overline{x}_1(1),\ldots,\overline{x}_m(1))$ such that

$$\begin{aligned}\overline{x}_1=&\ \overline{x}_1(1)^{a_{11}}\cdots\overline{x}_{\overline{r}}(1)^{a_{1\overline{r}}}c^{a_{1,\overline{r}+1}}\\ &\ \vdots\\ \overline{x}_{\overline{r}}=&\ \overline{x}_1(1)^{a_{\overline{r}1}}\cdots\overline{x}_{\overline{r}}(1)^{a_{\overline{r}\overline{r}}}c^{a_{\overline{r},\overline{r}+1}}\\ \overline{x}_{\overline{r}+r}=&\ \overline{x}_1(1)^{a_{\overline{r}+1,1}}\cdots\overline{x}_{\overline{r}}(1)^{a_{\overline{r}+1,\overline{r}}}(\overline{x}_{\overline{r}+r}(1)+1)c^{a_{\overline{r}+1,\overline{r}+1}}\end{aligned}$$

and $\overline{x}_i=\overline{x}_i(1)$ for $\overline{r}<i\le m$ and $i\ne\overline{r}+r$. We have that $c\in k(\overline{T}(1))$ and $A=(a_{ij})$ is a matrix of natural numbers such that $\mathrm{Det}(A)=\pm1$. $\nu(\overline{x}_1(1)),\ldots,\nu(\overline{x}_{\overline{r}}(1))$ are a rational basis of $\Gamma_\nu\otimes\mathbf{Q}$.

We define UTSs $\overline{U}\rightarrow\overline{U}(1)$ along ν^* in a similar way. Starting with regular parameters $(\overline{y}_1,\ldots,\overline{y}_n)$ in $\overline{U}''$ such that $\nu^*(\overline{y}_1)\ldots,\nu^*(\overline{y}_{\overline{s}})$ are a rational basis of $\Gamma_{\nu^*}\otimes\mathbf{Q}$, we define a UTS $\overline{U}\rightarrow\overline{U}(1)$ of type I so that $\overline{U}''(1)=\overline{U}'(1)$ has regular parameters $(\overline{y}_1(1),\ldots,\overline{y}_n(1))$ such that

$$\begin{aligned}\overline{y}_1 &= \overline{y}_1(1)^{b_{11}}\cdots\overline{y}_{\overline{s}}(1)^{b_{1\overline{s}}}\\ &\vdots\\ \overline{y}_{\overline{s}} &= \overline{y}_1(1)^{b_{\overline{s}1}}\cdots\overline{y}_{\overline{s}}(1)^{b_{\overline{s}\overline{s}}}\end{aligned}$$

and $\overline{y}_i=\overline{y}_i(1)$ for $\overline{s}<i\le n$. We have that $B=(b_{ij})$ is a matrix of natural numbers such that $\mathrm{Det}(B)=\pm1$ and $\nu^*(\overline{y}_1(1)),\ldots,\nu^*(\overline{y}_{\overline{s}}(1))$ are a rational basis of $\Gamma_{\nu^*}\otimes\mathbf{Q}$.

We define a UTS $\overline{U}\rightarrow\overline{U}(1)$ of type II_r with $0<r\le n-\overline{s}$ so that $\overline{U}''(1)$ has regular parameters $(\overline{y}_1(1),\ldots,\overline{y}_n(1))$ such that

$$\begin{aligned}\overline{y}_1 =&\ \overline{y}_1(1)^{b_{11}}\cdots\overline{y}_{\overline{s}}(1)^{b_{1\overline{s}}}d^{b_{1,\overline{s}+1}}\\ &\vdots\\ \overline{y}_{\overline{s}} =&\ \overline{y}_1(1)^{b_{\overline{s}1}}\cdots\overline{y}_{\overline{s}}(1)^{b_{\overline{s}\overline{s}}}d^{b_{\overline{s},\overline{s}+1}}\\ \overline{y}_{\overline{s}+r} =&\ \overline{y}_1(1)^{b_{\overline{s}+1,1}}\cdots\overline{y}_{\overline{s}}(1)^{b_{\overline{s}+1,\overline{s}}}(\overline{y}_{\overline{s}+r}(1)+d)d^{b_{\overline{s}+1,\overline{s}+1}}\end{aligned}$$

and $\overline{y}_i=\overline{y}_i(1)$ for $\overline{s}<i\le n$ and $i\ne\overline{s}+r$. We have $d\in k(\overline{U}(1))$ and $B=(b_{ij})$ is a matrix of natural numbers such that $\mathrm{Det}(B)=\pm1$ and $\nu^*(\overline{y}_1(1)),\ldots,\nu^*(\overline{y}_{\overline{s}}(1))$ are a rational basis of $\Gamma_{\nu^*}\otimes\mathbf{Q}$.

Lemma 5.1. *Suppose that $(R,\overline{T}'',\overline{T})$ and $(S,\overline{U}'',\overline{U})$ is a CUTS along ν^*, $\overline{T}''$ has regular parameters $(\overline{x}_1,\ldots,\overline{x}_m)$ and $\overline{U}''$ has regular parameters $(\overline{y}_1,\ldots,\overline{y}_n)$, related by*

$$\begin{aligned}\overline{x}_1 &= \overline{y}_1^{c_{11}}\cdots\overline{y}_{\overline{s}}^{c_{1\overline{s}}}\alpha_1\\ &\vdots\\ \overline{x}_{\overline{r}} &= \overline{y}_1^{c_{\overline{r}1}}\cdots\overline{z}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\alpha_{\overline{r}}\end{aligned}$$

such that $\alpha_1,\ldots,\alpha_{\overline{r}}\in k(\overline{U})$, $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_{\overline{r}})$ are rationally independent, $\nu^*(\overline{y}_1),\ldots,\nu^*(\overline{y}_{\overline{s}})$ are rationally independent and (c_{ij}) has rank $\overline{r}$. Suppose that $\overline{T}\rightarrow\overline{T}(1)$ is a UTS of type I along ν, such that $\overline{T}'(1)=\overline{T}''(1)$ has regular parameters $(\overline{x}_1(1),\ldots,\overline{x}_m(1))$ with

$$\begin{aligned}\overline{x}_1 &= \overline{x}_1(1)^{a_{11}}\cdots\overline{x}_{\overline{r}}(1)^{a_{1\overline{r}}}\\ &\vdots\\ \overline{x}_{\overline{r}} &= \overline{x}_1(1)^{a_{\overline{r}1}}\cdots\overline{x}_{\overline{r}}(1)^{a_{\overline{r},\overline{r}}}.\end{aligned}$$

Then there exists a UTS of type I along ν^*, $\overline{U}\rightarrow\overline{U}(1)$ such that $(R,\overline{T}''(1),\overline{T}(1))$ and $(S,\overline{U}''(1),\overline{U}(1))$ is a CUTS along ν^* and $\overline{U}'(1)=\overline{U}''(1)$ has regular parameters $(\overline{y}_1(1),\ldots,\overline{y}_n(1))$ with

$$\begin{aligned}\overline{y}_1 &= \overline{y}_1(1)^{b_{11}}\cdots\overline{y}_{\overline{s}}(1)^{b_{1\overline{s}}}\\ &\vdots\\ \overline{y}_{\overline{s}} &= \overline{y}_1(1)^{b_{\overline{s}1}}\cdots\overline{y}_{\overline{s}}(1)^{b_{\overline{s}\overline{s}}}\end{aligned}$$

and

$$\begin{aligned}\overline{x}_1(1) &= \overline{y}_1(1)^{c_{11}(1)}\cdots\overline{y}_{\overline{s}}(1)^{c_{1\overline{s}}(1)}\alpha_1(1)\\ &\vdots\\ \overline{x}_{\overline{r}}(1) &= \overline{y}_1(1)^{c_{\overline{r}1}(1)}\cdots\overline{y}_{\overline{s}}(1)^{c_{\overline{r}\overline{s}}(1)}\alpha_{\overline{r}}(1)\end{aligned}$$

where $\alpha_1(1),\ldots,\alpha_{\overline{r}}(1)\in k(\overline{U}(1))$ are products of integral powers of the α_i, $\nu(\overline{x}_1(1)),\ldots,\nu(\overline{x}_{\overline{r}}(1))$ are rationally independent, $\nu^*(\overline{y}_1(1)),\ldots,\nu^*(\overline{y}_{\overline{s}}(1))$ are rationally independent and $(c_{ij}(1))$ has rank $\overline{r}$.

Lemma 5.1 is a minor extension of Lemma 4.3 [14]. The same proof is valid, after replacing s in Lemma 4.3 with $\overline{r}$ and $\overline{s}$ as necessary.

Lemma 5.2. *Suppose that $(R,\overline{T}'',\overline{T})$ and $(S,\overline{U}'',\overline{U})$ is a CUTS along ν^*, $\overline{T}''$ has regular parameters $(\overline{x}_1,\ldots,\overline{x}_m)$ and $\overline{U}''$ has regular parameters $(\overline{y}_1,\ldots,\overline{y}_n)$ with*

$$
\begin{array}{ll}
\overline{x}_1 & = \overline{y}_1^{c_{11}}\cdots\overline{y}_{\overline{s}}^{c_{1\overline{s}}}\alpha_1\\
 & \vdots\\
\overline{x}_{\overline{r}} & = \overline{y}_1^{c_{\overline{r}1}}\cdots\overline{z}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\alpha_{\overline{r}}\\
\overline{x}_{\overline{r}+1} & = \overline{y}_{\overline{s}+1}\\
 & \vdots\\
\overline{x}_{\overline{r}+l} & = \overline{y}_{\overline{s}+l}
\end{array}
\tag{5}
$$

such that $\alpha_1,\ldots,\alpha_{\overline{r}}\in k(\overline{U})$, $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_{\overline{r}})$ are rationally independent, $\nu^*(\overline{y}_1),\ldots,\nu^*(\overline{y}_{\overline{s}})$ are rationally independent and (c_{ij}) has rank $\overline{r}$.

Suppose that $\overline{T}\rightarrow\overline{T}(1)$ is a UTS of type II_r along ν, with $r\le l$, such that $\overline{T}''(1)$ has regular parameters $(\overline{x}_1(1),\ldots,\overline{x}_m(1))$ with

$$
\begin{array}{ll}
\overline{x}_1 & = \overline{x}_1(1)^{a_{11}}\cdots\overline{x}_{\overline{r}}(1)^{a_{1\overline{r}}}c^{a_{1,\overline{r}+1}}\\
 & \vdots\\
\overline{x}_{\overline{r}} & = \overline{x}_1(1)^{a_{\overline{r}1}}\cdots\overline{x}_{\overline{r}}(1)^{a_{\overline{r},\overline{r}}}c^{a_{\overline{r},\overline{r}+1}}\\
\overline{x}_{\overline{r}+r} & = \overline{x}_1(1)^{a_{\overline{r}+1,1}}\cdots\overline{x}_{\overline{r}}(1)^{a_{\overline{r}+1,\overline{r}}}(\overline{x}_{\overline{r}+r}(1)+1)c^{a_{\overline{r}+1,\overline{r}+1}}.
\end{array}
$$

Then there exists a UTS of type II_r (followed by a UTS of type I) along ν^*, $\overline{U}\rightarrow\overline{U}(1)$, such that $(R,\overline{T}''(1),\overline{T}(1))$ and $(S,\overline{U}''(1),\overline{U}(1))$ is a CUTS along ν^* and $\overline{U}''(1)$ has regular parameters $(\overline{y}_1(1),\ldots,\overline{y}_n(1))$ with

$$
\begin{array}{ll}
\overline{y}_1 & = \overline{y}_1(1)^{b_{11}}\cdots\overline{y}_{\overline{s}}(1)^{b_{1\overline{s}}}d^{b_{1\overline{s}+1}}\\
 & \vdots\\
\overline{y}_{\overline{s}} & = \overline{y}_1(1)^{b_{\overline{s}1}}\cdots\overline{y}_{\overline{s}}(1)^{b_{\overline{s}\overline{s}}}d^{b_{\overline{s},\overline{s}+1}}\\
\overline{y}_{\overline{s}+r} & = \overline{y}_1(1)^{b_{\overline{s}+1,1}}\cdots\overline{y}_{\overline{s}}(1)^{b_{\overline{s}+1,\overline{s}}}(\overline{y}_{\overline{s}+r}(1)+1)d^{b_{\overline{s}+1,\overline{s}+1}}
\end{array}
$$

and

$$
\begin{array}{ll}
\overline{x}_1(1) & = \overline{y}_1(1)^{c_{11}(1)}\cdots\overline{y}_{\overline{s}}(1)^{c_{1\overline{s}}(1)}\alpha_1(1)\\
 & \vdots\\
\overline{x}_{\overline{r}}(1) & = \overline{y}_1(1)^{c_{\overline{r}1}(1)}\cdots\overline{y}_{\overline{s}}(1)^{c_{\overline{r}\overline{s}}(1)}\alpha_{\overline{r}}(1)\\
\overline{x}_{\overline{r}+1}(1) & = \overline{y}_{\overline{s}+1}\\
 & \vdots\\
\overline{x}_{\overline{r}+l} & = \overline{y}_{\overline{s}+l}
\end{array}
$$

where $\alpha_1(1),\ldots,\alpha_{\overline{r}}(1)\in k(\overline{U}(1))$, $\nu(\overline{x}_1(1)),\ldots,\nu(\overline{x}_{\overline{r}}(1))$ are rationally independent, $\nu^*(\overline{y}_1(1)),\ldots,\nu^*(\overline{y}_{\overline{s}}(1))$ are rationally independent and $(c_{ij}(1))$ has rank $\overline{r}$.

Lemma 5.2 is a simple variation of Lemma 4.4 [14].

Lemma 5.3. *Suppose that $(R,\overline{T}'',\overline{T})$ and $(S,\overline{U}'',\overline{U})$ is a CUTS along ν^*, $\overline{T}''$ has regular parameters $(\overline{x}_1,\ldots,\overline{x}_m)$ and $\overline{U}''$ has regular parameters $(\overline{y}_1,\ldots,\overline{y}_n)$ such that*

$$
\begin{aligned}
\overline{x}_1 &= \overline{y}_1^{c_{11}} \cdots \overline{y}_{\overline{s}}^{c_{1\overline{s}}} \alpha_1\\
&\vdots\\
\overline{x}_{\overline{r}} &= \overline{y}_1^{c_{\overline{r}1}} \cdots \overline{z}_{\overline{s}}^{c_{\overline{r}\overline{s}}} \alpha_{\overline{r}}\\
\overline{x}_{\overline{r}+1} &= \overline{y}_{\overline{s}+1}\\
&\vdots\\
\overline{x}_{\overline{r}+l} &= \overline{y}_{\overline{s}+l}\\
\overline{x}_{\overline{r}+l+1} &= \overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}} \overline{y}_{\overline{s}+l+1}
\end{aligned} \tag{6}
$$

such that $\alpha_1, \ldots, \alpha_{\overline{r}} \in k(\overline{U})$, $\nu(\overline{x}_1), \ldots, \nu(\overline{x}_{\overline{r}})$ *are rationally independent,* $\nu^*(\overline{y}_1), \ldots, \nu^*(\overline{y}_{\overline{s}})$ *are rationally independent and* (c_{ij}) *has rank* $\overline{r}$*.*

Suppose that $\overline{T} \to \overline{T}(1)$ *is a UTS of type* II_{l+1} *along* ν*, such that* $\overline{T}''(1)$ *has regular parameters* $(\overline{x}_1(1), \ldots, \overline{x}_m(1))$ *with*

$$
\begin{aligned}
\overline{x}_1 &= \overline{x}_1(1)^{a_{11}} \cdots \overline{x}_{\overline{r}}(1)^{a_{1\overline{r}}} c^{a_{1,\overline{r}+1}}\\
&\vdots\\
\overline{x}_{\overline{r}} &= \overline{x}_1(1)^{a_{\overline{r}1}} \cdots \overline{x}_{\overline{r}}(1)^{a_{\overline{r},\overline{r}}} c^{a_{\overline{r},\overline{r}+1}}\\
\overline{x}_{\overline{r}+l+1} &= \overline{x}_1(1)^{a_{\overline{r}+1,1}} \cdots \overline{x}_{\overline{r}}(1)^{a_{\overline{r},\overline{r}+1}} (\overline{x}_{\overline{r}+l+1}(1)+1) c^{a_{\overline{r}+1,\overline{r}+1}}
\end{aligned}
$$

Then there exists a UTS of type II_{l+1} *(followed by a UTS of type* I*) along* ν^* $\overline{U} \to \overline{U}(1)$ *such that* $(R, \overline{T}''(1), \overline{T}(1))$ *and* $(S, \overline{U}''(1), \overline{U}(1))$ *is a CUTS along* ν^* *and* $\overline{U}''(1)$ *has regular parameters* $(\overline{y}_1(1), \ldots, \overline{y}_n(1))$ *with*

$$
\begin{aligned}
\overline{y}_1 &= \overline{y}_1(1)^{b_{11}} \cdots \overline{y}_{\overline{s}}(1)^{b_{1\overline{s}}} d^{b_{1\overline{s}+1}}\\
&\vdots\\
\overline{y}_{\overline{s}} &= \overline{y}_1(1)^{b_{\overline{s}1}} \cdots \overline{y}_{\overline{s}}(1)^{b_{\overline{s}\overline{s}}} d^{b_{\overline{s},\overline{s}+1}}\\
\overline{y}_{\overline{s}+l+1} &= \overline{y}_1(1)^{b_{\overline{s}+1,1}} \cdots \overline{y}_{\overline{s}}(1)^{b_{\overline{s}+1,\overline{s}}} (\overline{y}_{\overline{s}+l+1}(1)+1) d^{b_{\overline{s}+1,\overline{s}+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{x}_1(1) &= \overline{y}_1(1)^{c_{11}(1)} \cdots \overline{y}_{\overline{s}}(1)^{c_{1\overline{s}}(1)} \alpha_1(1)\\
&\vdots\\
\overline{x}_{\overline{r}}(1) &= \overline{y}_1(1)^{c_{\overline{r}1}(1)} \cdots \overline{y}_{\overline{s}}(1)^{c_{\overline{r}\overline{s}}(1)} \alpha_{\overline{r}}(1)\\
\overline{x}_{\overline{r}+1}(1) &= \overline{y}_{\overline{s}+1}\\
&\vdots\\
\overline{x}_{\overline{r}+l} &= \overline{y}_{\overline{s}+l}\\
\overline{x}_{\overline{r}+l+1} &= \overline{y}_{\overline{s}+l+1}
\end{aligned}
$$

where $\alpha_1(1), \ldots, \alpha_{\overline{r}}(1) \in k(\overline{U}(1))$, $\nu(\overline{x}_1(1)), \ldots, \nu(\overline{x}_{\overline{r}}(1))$ *are rationally independent,* $\nu^*(\overline{y}_1(1)), \ldots, \nu^*(\overline{y}_{\overline{s}}(1))$ *are rationally independent and* $(c_{ij}(1))$ *has rank* $\overline{r}$*.*

Lemma 5.3 is a simple variation of Lemma 4.5 [14].

Lemma 5.4. *Suppose that $(R,\overline{T}'',\overline{T})$ and $(S,\overline{U}'',\overline{U})$ is a CUTS along ν^*, $\overline{T}''$ has regular parameters $(\overline{x}_1,\dots,\overline{x}_m)$ and $\overline{U}''$ has regular parameters $(\overline{y}_1,\dots,\overline{y}_n)$ such that*

$$\begin{array}{ll} \overline{x}_1 & = \overline{y}_1^{c_{11}}\cdots\overline{y}_{\overline{s}}^{c_{1\overline{s}}}\alpha_1\\ & \vdots\\ \overline{x}_{\overline{r}} & = \overline{y}_1^{c_{\overline{r}1}}\cdots\overline{y}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\alpha_{\overline{r}}\\ \overline{x}_{\overline{r}+1} & = \overline{y}_{\overline{s}+1}\\ & \vdots\\ \overline{x}_{\overline{r}+l} & = \overline{y}_{\overline{s}+l}\\ \overline{x}_{\overline{r}+l+1} & = \overline{y}_1^{d_1}\cdots\overline{y}_{\overline{s}}^{d_{\overline{s}}}\delta \end{array}$$

such that $\delta\in\overline{U}''$ is a unit, $\alpha_1,\dots,\alpha_{\overline{r}}\in k(\overline{U})$, $\nu(\overline{x}_1),\dots,\nu(\overline{x}_{\overline{r}})$ are rationally independent, $\nu^*(\overline{y}_1),\dots,\nu^*(\overline{y}_{\overline{s}})$ are rationally independent and (c_{ij}) has rank $\overline{r}$.

Suppose that $\overline{T}\rightarrow\overline{T}(1)$ is a UTS of type II_{l+1} along ν, such that $\overline{T}''(1)$ has regular parameters $(\overline{x}_1(1),\dots,\overline{x}_m(1))$ with

$$\begin{array}{ll} \overline{x}_1 & = \overline{x}_1(1)^{a_{11}}\cdots\overline{x}_{\overline{r}}(1)^{a_{1\overline{r}}}c^{a_{1,\overline{r}+1}}\\ & \vdots\\ \overline{x}_{\overline{r}} & = \overline{x}_1(1)^{a_{\overline{r}1}}\cdots\overline{x}_{\overline{r}}(1)^{a_{\overline{r},\overline{r}}}c^{a_{\overline{r},\overline{r}+1}}\\ \overline{x}_{\overline{r}+l+1} & = \overline{x}_1(1)^{a_{\overline{r}+1,1}}\cdots\overline{x}_{\overline{r}}(1)^{a_{\overline{r},\overline{r}+1}}(\overline{x}_{\overline{r}+l+1}(1)+1)c^{a_{\overline{r}+1,\overline{r}+1}}. \end{array}$$

Then there exists a UTS of type I along ν^* $\overline{U}\rightarrow\overline{U}(1)$ such that $(R,\overline{T}''(1),\overline{T}(1))$ and $(S,\overline{U}''(1),\overline{U}(1))$ is a CUTS along ν^* and $\overline{U}'(1)$ has regular parameters $(\hat{y}_1(1),\dots,\hat{y}_n(1))$ with

$$\begin{array}{ll} \overline{y}_1 & = \hat{y}_1(1)^{b_{11}}\cdots\hat{y}_{\overline{s}}(1)^{b_{1\overline{s}}}\\ & \vdots\\ \overline{y}_{\overline{s}} & = \hat{y}_1(1)^{b_{\overline{s}1}}\cdots\hat{y}_{\overline{s}}(1)^{b_{\overline{s}\overline{s}}} \end{array}$$

and $\overline{U}''(1)$ has regular parameters $(\overline{y}_1(1),\dots,\overline{y}_n(1))$ such that $\overline{y}_i(1)=\epsilon_i\hat{y}_i(1)$ for $1\le i\le\overline{s}$ for some units $\epsilon_i\in\overline{U}''(1)$,

$$\begin{array}{ll} \overline{x}_1(1) & = \overline{y}_1(1)^{c_{11}(1)}\cdots\overline{y}_{\overline{s}}(1)^{c_{1\overline{s}}(1)}\alpha_1(1)\\ & \vdots\\ \overline{x}_{\overline{r}}(1) & = \overline{y}_1(1)^{c_{\overline{r}1}(1)}\cdots\overline{y}_{\overline{s}}(1)^{c_{\overline{r}\overline{s}}(1)}\alpha_{\overline{r}}(1)\\ \overline{x}_{\overline{r}+1}(1) & = \overline{y}_{\overline{s}+1}\\ & \vdots\\ \overline{x}_{\overline{r}+l} & = \overline{y}_{\overline{s}+l} \end{array}$$

where $\alpha_1(1),\dots,\alpha_{\overline{r}}(1)\in k(\overline{U}(1))$, $\nu(\overline{x}_1(1)),\dots,\nu(\overline{x}_{\overline{r}}(1))$ are rationally independent, $\nu^*(\overline{y}_1(1)),\dots,\nu^*(\overline{y}_{\overline{s}}(1))$ are rationally independent and $(c_{ij}(1))$ has rank $\overline{r}$.

Lemma 5.4 is a simple variation of Lemma 4.6 [14]. The last two lines of page 45 [14] must be replaced with the following lines:

After possibly interchanging $\hat{y}_1(1),\dots,\hat{y}_{\overline{s}}(1)$, we may assume that if

$$\tilde{C}=\begin{pmatrix} c_{11}(1) & \cdots & c_{1\overline{r}}(1)\\ \vdots & & \vdots\\ c_{\overline{r}1}(1) & \cdots & c_{\overline{r}\overline{r}}(1)\end{pmatrix}$$

then $\mathrm{Det}(\tilde{C}) \neq 0$. Set $(e_{ij}) = \tilde{C}^{-1}$, and set

$$\epsilon_i = \begin{cases} \delta_1^{e_{i1}} \cdots \delta_{\overline{r}}^{e_{i\overline{r}}} (\frac{\delta_{l+1}}{c})^{-\gamma_1 e_{i1} - \cdots - \gamma_{\overline{r}} e_{i\overline{r}}} & \text{for } 1 \le i \le \overline{r}, \\ 1 & \text{for } \overline{r} < i \le \overline{s} \end{cases}$$

6. UTSs of form $\overline{m}$

In this section, assumptions and notations will be as in Section 4.

Definition 6.1. *Suppose that $(R, \overline{T}'', \overline{T})$ is a UTS along ν, such that $\overline{T}''$ contains a subfield isomorphic to $k(c_0)$ for some $c_0 \in \overline{T}''$ and $\overline{T}''$ has regular parameters $(\overline{z}_1, \ldots, \overline{z}_m)$*

Suppose that $0 \le \overline{m} \le m - \overline{r}$. We will say that a UTS along ν

$$\overline{T} \to \overline{T}(1) \to \cdots \to \overline{T}(t) \tag{7}$$

is of form $\overline{m}$ if for $0 \le \alpha \le t$, $\overline{T}''(\alpha)$ has regular parameters

$$(\overline{z}_1(\alpha), \ldots, \overline{z}_m(\alpha)) \text{ and } (\tilde{\overline{z}}_1'(\alpha), \ldots, \tilde{\overline{z}}_m'(\alpha)),$$

where $\overline{z}_i(0) = \overline{z}_i$ for $1 \le i \le m$, $\overline{T}''(\alpha)$ contains a subfield isomorphic to $k(c_0, \ldots, c_\alpha)$ and there are polynomials

$$P_{i,\alpha} \in k(c_0, \ldots, c_\alpha)[\overline{z}_1(\alpha), \ldots, \overline{z}_{i-1}(\alpha)] \text{ for } \overline{r}+1 \le i \le \overline{r} + \overline{m}$$

such that

$$\tilde{\overline{z}}_i'(\alpha) = \begin{cases} \overline{z}_i(\alpha) - P_{i,\alpha} & \text{if } \overline{r}+1 \le i \le \overline{r} + \overline{m} \\ \overline{z}_i(\alpha) & \text{otherwise} \end{cases}$$

Each $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ will either be of type I or of type II_r with $1 \le r \le \overline{r} + \overline{m}$.

In a transformation $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ of type I, $\overline{T}''(\alpha+1)$ will have regular parameters $(\overline{z}_1(\alpha+1), \ldots, \overline{z}_m(\alpha+1))$ defined by

$$\begin{aligned} \tilde{\overline{z}}_1'(\alpha) &= \overline{z}_1(\alpha+1)^{a_{11}(\alpha+1)} \cdots \overline{z}_{\overline{r}}(\alpha+1)^{a_{1\overline{r}}(\alpha+1)} \\ &\vdots \\ \tilde{\overline{z}}_{\overline{r}}'(\alpha) &= \overline{z}_1(\alpha+1)^{a_{\overline{r}1}(\alpha+1)} \cdots \overline{z}_{\overline{r}}(\alpha+1)^{a_{\overline{r}\overline{r}}(\alpha+1)}. \end{aligned}$$

and $c_{\alpha+1}$ is defined to be 1. In a transformation $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ of type II_r ($1 \le r \le \overline{r} + \overline{m}$) $\overline{T}''(\alpha+1)$ will have regular parameters $(\overline{z}_1(\alpha+1), \ldots, \overline{z}_m(\alpha+1))$ defined by

$$\begin{aligned} \tilde{\overline{z}}_1'(\alpha) &= \overline{z}_1(\alpha+1)^{a_{11}(\alpha+1)} \cdots \overline{z}_{\overline{r}}(\alpha+1)^{a_{1\overline{r}}(\alpha+1)} c_{\alpha+1}^{a_{1\overline{r}+1}(\alpha+1)} \\ &\vdots \\ \tilde{\overline{z}}_{\overline{r}}'(\alpha) &= \overline{z}_1(\alpha+1)^{a_{\overline{r}1}(\alpha+1)} \cdots \overline{z}_{\overline{r}}(\alpha+1)^{a_{\overline{r}\overline{r}}(\alpha+1)} c_{\alpha+1}^{a_{\overline{r},\overline{r}+1}(\alpha+1)} \\ \tilde{\overline{z}}_{\overline{s}+r}'(\alpha) &= \overline{z}_1(\alpha+1)^{a_{\overline{r}+1,1}(\alpha+1)} \cdots \overline{z}_{\overline{r}}(\alpha+1)^{a_{\overline{r}+1,\overline{r}}(\alpha+1)} \\ &\quad \cdot (\overline{z}_{\overline{r}+r}(\alpha+1)+1) c_{\alpha+1}^{a_{\overline{r}+1,\overline{r}+1}(\alpha+1)} \end{aligned}$$

Theorem 6.2. *Suppose that $(R, \overline{T}'', \overline{T})$ is a UTS along ν, such that $\overline{T}''$ contains a subfield isomorphic to $k(c_0)$ for some $c_0 \in \overline{T}''$ and $\overline{T}''$ has regular parameters $(\overline{z}_1, \ldots, \overline{z}_m)$.*

(A1): *Suppose that $0 \le \overline{m} \le m - \overline{r}$. Then there exists a UTS (7) of form $\overline{m}$ such that*

$$p_{\overline{m}}(i) = \{f \in k(\overline{T}(i))[[\overline{z}_1(i), \ldots, \overline{z}_{\overline{r}+\overline{m}}(i)]] \mid \nu(f) = \infty\}$$

has the form

$$p_{\overline{m}}(t)=(\overline{z}_{r(1)}(t)-Q_{r(1)}(\overline{z}_1(t),\ldots,\overline{z}_{r(1)-1}),\ldots,\overline{z}_{r(\tilde{m})}(t)-Q_{r(\tilde{m})}(\overline{z}_1(t),\ldots,\overline{z}_{r(\tilde{m})-1})) \tag{8}$$

for some $0\le\tilde{m}\le\overline{m}$ *and* $\overline{r}<r(1)<r(2)<\cdots<r(\tilde{m})\le\overline{r}+\overline{m}$*, where* $Q_{r(i)}$ *are power series with coefficients in* $k(c_0,\ldots,c_t)$

(A2): *Suppose that* L *is a finite extension field of* $k(c_0)(t_1,\ldots,t_\beta)$*, with* $0\le\beta\le\overline{\alpha}$ *(For the definition of* $t_1,\ldots t_{\overline{\alpha}}$ *see section 4). Suppose that* ν' *is an extension of* $\tilde{\nu}$ *to* $Q(L[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]])$ *such that* ν' *dominates* $L[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]]$*.* $h\in L[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]]$ *for some* $0\le\overline{m}$ *with* $\overline{r}+\overline{m}\le m$ *and* $\nu(h)<\infty$*. Then there exists a UTS (7), of form* $\overline{m}$*, such that (8) holds in* $\overline{T}(t)$ *and*

$$h=\overline{z}_1(t)^{d_1}\cdots\overline{z}_{\overline{r}}(t)^{d_{\overline{r}}}u$$

where $u\in L(c_1,\ldots,c_t)[[\overline{z}_1(t),\ldots,\overline{z}_{\overline{r}+\overline{m}}(t)]]$ *is a unit power series.*

If $h\in L[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]$*, then* $u\in L(c_1,\ldots,c_t)[\overline{z}_1(t),\ldots,\overline{z}_{\overline{r}+\overline{m}}(t)]$*.*

(A3): *Suppose that* L *is a finite extension field of* $k(c_0)(t_1,\ldots,t_\beta)$ *with* $0\le\beta\le\overline{\alpha}$ *(For the definition of* $t_1,\ldots t_{\overline{\alpha}}$ *see section 4). Suppose that* ν' *is an extension of* $\tilde{\nu}$ *to* $Q(L[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]])$ *such that* ν' *dominates* $L[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]]$*.* $h\in L[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]]$ *for some* $0<\overline{m}$ *with* $\overline{r}+\overline{m}\le m$*,* $\nu(h)=\infty$ *and* $A>0$ *is given. Then there exists a UTS (7), of form* $\overline{m}$*, such that (8) holds in* $\overline{T}(t)$ *and*

$$h=\overline{z}_1(t)^{d_1}\cdots\overline{z}_{\overline{r}}(t)^{d_{\overline{r}}}\Sigma$$

where $\nu(\overline{z}_1(t)^{d_1}\cdots\overline{z}_{\overline{r}}(t)^{d_{\overline{r}}})>A$ *and* $\Sigma\in L(c_1,\ldots,c_t)[[\overline{z}_1(t),\ldots,\overline{z}_{\overline{r}+\overline{m}}]]$*.*

If $h\in L[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]$*, then* $\Sigma\in L(c_1,\ldots,c_t)[\overline{z}_1(t),\ldots,\overline{z}_{\overline{r}+\overline{m}}(t)]$*.*

Proof. (A1), (A2), (A3) replace (53), (54) and (55) of the proof of Theorem 4.7 [14]. We observe that (A1) is trivial for $\overline{m}=0$ since $p_0=(0)$. The proof of (A2) when $\overline{m}=0$ follows from the "Proof of (54) for $m=s$" on page 50 of the proof of Theorem 4.7 [14].

We will now establish (A1), (A2) and (A3) by proving the following inductive statements.

$A(m')$: (A1), (A2) and (A3) for $\overline{m}\le m'$ imply (A1) for $\overline{m}=m'$.

$B(m')$: (A2), (A3) for $\overline{m}<m'$ and (A1) for $\overline{m}=m'$ imply (A2) and (A3) for $\overline{m}=m'$.

The "Proof of $A(\overline{m})$ $(s<\overline{m})$" on pages 51-55 of [14] proves $A(m')$ for $0<m'$.

We now give the proof of $B(m')$ for $m'>0$. By assumption, there exists a UTS $\overline{T}\to\overline{T}(t)$ satisfying (A1) for $\overline{m}=m'$. After replacing $\overline{T}''$ with $\overline{T}''(t)$ and replacing c_0 with a primitive element of $k(c_0,\ldots,c_t)$ over k, we may assume that

$$p_{m'}=(\overline{z}_{r(1)}-Q_{r(1)}(\overline{z}_1,\ldots,\overline{z}_{r(1)-1}),\ldots,\overline{z}_{r(\tilde{m}')}-Q_{r(\tilde{m}')}(\overline{z}_1,\ldots,\overline{z}_{r(\tilde{m}')-1}))$$

where the $Q_{r(i)}$ are power series with coefficients in $k(c_0)$.

Let $\overline{R}=L[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+m'}]]$. Let

$$\overline{p}=\{f\in\overline{R}\mid\nu(f)=\infty\}.$$

Let

$$\overline{p}_{m'}=p_{m'}\cap k(c_0)[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+m'}]].$$

We first establish the following formula:

$$\overline{p}=\overline{p}_{m'}\overline{R}. \tag{9}$$

We first prove the identity (9) when $L = k(c_0)(t_1, \ldots, t_\beta)$ for β with $0 \le \beta \le \overline{\alpha}$. Let

$$\tilde{R} = k(c_0)[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]]/\overline{p}_{m'}.$$

Let $a \in \overline{p}$. Suppose that $N > 0$ is given. Chevalley's Theorem (Theorem 13, Section 5, Chapter VIII [39]) implies there exists M such that $g \in \tilde{R}$ and $\nu(g) > M$ implies $g \in m(\tilde{R})^N$. There also exists $N_0 \ge N$ such that $\Omega \in m(\overline{R})^{N_0}$ implies $\nu(\Omega) > M$. Write $a = H + \Omega$ with $H \in k(c_0)(t_1, \ldots, t_\beta)[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]$ and $\Omega \in m(\overline{R})^{N_0}$. Then $\nu(H) = \nu(\Omega) > M$. There exists $0 \ne h \in k(c_0)[t_1, \ldots, t_\beta]$ such that

$$hH = \sum_{I=(i_1,\ldots,i_\beta)} \alpha_I t_1^{i_1} \cdots t_\beta^{i_\beta}$$

is a polynomial with all $\alpha_I \in k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]$. Thus $\nu(\alpha_I) > M$ for all I (by Lemma 3.1). Let $\tilde{\alpha}_I$ be the residue of α_I in $\tilde{R}$. $\tilde{\alpha}_I \in m(\tilde{R})^N$ for all I implies

$$\alpha_I \in \overline{p}_{m'} + m(k(c_0)[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}'_m}]])^N$$

for all I so that $a \in \overline{p}_{m'}\overline{R} + m(\overline{R})^N$. Since this is true for all N,

$$a \in \cap_{N>0} \left(\overline{p}_{m'}\overline{R} + m(\overline{R})^N\right) = \overline{p}_{m'}\overline{R}.$$

(9) is thus established when $L = k(c_0)(t_1, \ldots, t_\beta)$.

Now suppose that L is a finite extension of $k(c_0)(t_1, \ldots, t_\beta)$. Let $\overline{M} = k(c_0)(t_1, \ldots, t_\beta)$ and let M' be a Galois closure of L over $\overline{M}$. Let G be the Galois group of M' over $\overline{M}$. Suppose that $a \in \overline{p}$. Set

$$g = \prod_{\sigma \in G} \sigma(a) \in \overline{M}[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]].$$

$\nu(g) = \infty$ since $a \mid g$ in $L[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]]$. Thus $g \in \overline{p}_{m'}M'[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]]$ which is a prime ideal invariant under G. Thus there exists $\sigma \in G$ such that $\sigma(a) \in \overline{p}_{m'}M'[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]]$. Necessarily we then have that

$$a \in (p_{m'}M'[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]]) \cap L[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]] = \overline{p}_{m'}L[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]].$$

(9) is now established.

Now suppose that $h \in L[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]]$ and $\nu(h) < \infty$. Let $\overline{M} = k(c_0)(t_1, \ldots, t_\beta)$ and let M' be a Galois closure of L over $\overline{M}$. Let G be the Galois group of M' over $\overline{M}$. Set

$$g = \prod_{\sigma \in G} \sigma(h) \in \overline{M}[[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}]].$$

It follows from (9) that $\nu(g) < \infty$. We will construct a UTS (7) so that

$$g = u\overline{z}_1(t)^{e_1} \cdots \overline{z}_{\overline{r}}^{e_{\overline{r}}}(t)$$

where u is a unit power series in $k(c_0, \ldots, c_t, t_1, \ldots, t_\beta)[[\overline{z}_1(t), \ldots, \overline{z}_{\overline{r}+m'}(t)]]$ and $h \in L(c_1, \ldots, c_t)[[\overline{z}_1(t), \ldots, \overline{z}_{\overline{r}+m'}(t)]]$. Since $h \mid g$ in $L(c_1, \ldots, c_t)[[\overline{z}_1(t), \ldots, \overline{z}_{\overline{r}+m'}(t)]]$, we will have that h has the desired form in $L(c_1, \ldots, c_t)[[\overline{z}_1(t), \ldots, \overline{z}_{\overline{r}+m'}(t)]]$.

We now follow the argument of the proof of Theorem 4.7 [14] as modified to fit our notation, from "Set $g = \overline{z}_1^{d_1} \cdots \overline{z}_{\overline{r}}^{d_{\overline{r}}} g_0$" on the 17th line of the "Proof of $B(\overline{m})$" of page 55 of [14] to the fourth line from the last on page 56, ending with "$\nu(\sigma_{d-1}) = \nu(\overline{z}_{\overline{r}+m'})$". We must substitute $k(c_0, \ldots, c_\alpha, t_1, \ldots, t_\beta)$ for $k(c_0, \ldots, c_\alpha)$, $\overline{r} + m'$ for $\overline{m}$ and $\overline{r}$ for s in the proof, and for "(54), (55) for $m < \overline{m}$" on line 5 of page 56 [14] we must substitute "(A2), (A3) for $m < m'$".

We now argue as follows. σ_d is a unit. Let

$$\overline{R} = k(c_0, \ldots, c_{\alpha+2})(t_1, \ldots, t_\beta)[[\overline{z}_1(\alpha+2), \ldots, \overline{z}_{r+m'}(\alpha+2)]].$$

Let $\tilde{\alpha}_d$ be the residue of $\sigma_d = \overline{u}_d$ in $k(c_0)(t_1, \ldots, t_\beta)$, $\tilde{\alpha}_{d-1}$ be the residue of $\overline{u}_{d-1}$ in $k(c_0, \ldots, c_{\alpha+2})(t_1, \ldots, t_\beta)$. After replacing g_0 with $\frac{1}{\tilde{\alpha}_d} g_0$, we may assume that $\tilde{\alpha}_d = 1$. We have

$$\lambda_d r \tilde{\alpha}_d + \lambda_{d-1}\tilde{\alpha}_{d-1} = 0, \tag{10}$$

by the argument of page 53 of [14], where $\lambda_d, \lambda_{d-1} \in k(c_0, \ldots, c_{\alpha+2})$. Set $\tau = \nu(\sigma_{d-1}) = \nu(\overline{z}_{\overline{r}+m'}) < \infty$. Write

$$\sigma_{d-1} = \sum_{\nu(\overline{z}_1^{i_1}\cdots \overline{z}_{\overline{r}+m'-1}^{i_{\overline{r}+m'-1}})\le \tau} g_I \overline{z}_1^{i_1}\cdots \overline{z}_{\overline{r}+m'-1}^{i_{\overline{r}+m'-1}} + \sum_{\nu(\overline{z}_1^{i_1}\cdots \overline{z}_{\overline{r}+m'-1}^{i_{\overline{r}+m'-1}})> \tau} g_I \overline{z}_1^{i_1}\cdots \overline{z}_{\overline{r}+m'-1}^{i_{\overline{r}+m'-1}}$$

with $g_I \in k(c_0)(t_1, \ldots, t_\beta)$ for all $I = (i_1, \ldots, i_{\overline{r}+m'-1})$. Set

$$\Omega = \sum_{\nu(\overline{z}_1^{i_1}\cdots \overline{z}_{\overline{r}+m'-1}^{i_{\overline{r}+m'-1}})\le \tau} g_I \overline{z}_1^{i_1}\cdots \overline{z}_{\overline{r}+m'-1}^{i_{\overline{r}+m'-1}}.$$

We have $\nu(\Omega) = \nu(\sigma_{d-1})$. There exists $0 \ne \overline{h} \in k(c_0)(t_1, \ldots, t_\beta)$ such that $\overline{h} g_I \in k(c_0)[t_1, \ldots, t_\beta]$ for all g_I in the finite sum Ω. Thus we have

$$\overline{h}\Omega = \sum_{J=(j_1,\ldots,j_\beta)} \Psi_J t_1^{j_1}\cdots t_\beta^{j_\beta}$$

with all $\Psi_J \in k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'-1}]$. By Lemma 3.1,

$$\nu(\Omega) = \min\{\nu(\Psi_J)\}.$$

$\frac{\sigma_{d-1}}{\overline{z}_{\overline{r}+m'}}$ has residue $-r\tilde{\alpha}_d = -r$ in $k(V)(t_1, \ldots, t_\beta) \subset k(V^*)$ (by the argument of page 53 of [14]). Let

$$\left[\frac{\Psi_J}{\overline{z}_{\overline{r}+m'}}\right]$$

be the residue of $\frac{\Psi_J}{\overline{z}_{\overline{r}+m'}}$ in $k(V)$.

In $k(V(t_1, \ldots, t_\beta))$, we have

$$-r\overline{h} = \sum_J \left[\frac{\Psi_J}{\overline{z}_{\overline{r}+m'}}\right] t_1^{j_1}\cdots t_\beta^{j_\beta}.$$

Since $0 \ne \overline{h} \in k(c_0)(t_1, \ldots, t_\beta)$ and the $t_1^{j_1}\cdots t_\beta^{j_\beta}$ are linearly independent over $k(V)$, we have that

$$\left[\frac{\Psi_J}{\overline{z}_{\overline{r}+m'}}\right] \in k(c_0) \text{ for all } J$$

and

$$\left[\frac{\Psi_{J_0}}{\overline{z}_{\overline{r}+m'}}\right] \ne 0 \text{ for some } J_0.$$

Let $c = \left[\frac{\overline{z}_{\overline{r}+m'}}{\Psi_{J_0}}\right]$. Set

$$\overline{z}'_{\overline{r}+m'} = \overline{z}_{\overline{r}+m'} - c\Psi_{J_0} \in k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'}].$$

We have

$$\nu(\overline{z}'_{\overline{r}+m'}) > \nu(\overline{z}_{\overline{r}+m'}).$$

We further have

$$\nu(g_0) \ge \nu(\overline{z}^r_{\overline{r}+m'}) \ge \nu(\overline{z}_{\overline{r}+m'})$$

since $\overline{z}^r_{\overline{r}+m'}$ is a minimal value term of g_0.

Now we finish the proof as on lines 1 - 16 of page 57 of [14]. On line 7 of page 57 we must replace "$P_{\overline{m},0} \in k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{m}-1}]$" with "$P_{m',0} \in k(c_0)[\overline{z}_1, \ldots, \overline{z}_{\overline{r}+m'-1}]$".

on line 11 we must replace "$\overline{u}$ is a unit power series with coefficients in $k(c_0,\ldots,c_t)$" with "$\overline{u}$ is a unit power series with coefficients in $k(c_0,\ldots,c_t)(t_1,\ldots,t_\beta)$". (53) on line 12 must be replaced with (A1). This concludes the proof of Case 1, $\nu(h)<\infty$ of the proof of $B(m')$.

The proof of $B(m')$, when $\nu(h)=\infty$ is only a slight modification of the proof of case 2 on page 57 of [14]. We must replace (53) on line 17 with (A1) and replace "$\sigma_i\in k(\overline{T})[[\overline{z}_1,\ldots,\overline{z}_m]]$" with "$\sigma_i\in L[[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+m'}]]$" on line 21 of page 57.

The final statements that $h\in L[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}]$ imply $h\in L(c_1,\ldots,c_t)[\overline{z}_1(t),\ldots,\overline{z}_{\overline{r}+\overline{m}}(t)]$ follow since $\overline{z}_1,\ldots,\overline{z}_{\overline{r}+\overline{m}}\in k(c_1,\ldots,c_t)[\overline{z}_1(t),\ldots,\overline{z}_{\overline{r}+\overline{m}}(t)]$ by the definition of a UTS of form $\overline{m}$. □

Lemma 6.3. *Suppose that*

$$\overline{T}\to\overline{T}(1)\to\cdots\to\overline{T}(t)$$

is a UTS of form $\overline{m}$ as in (7) of Definition 6.1. Let

$$A_i=k(\overline{T}(i))[[\overline{z}_1(i),\ldots,\overline{z}_{\overline{r}+\overline{m}}(i)]],$$

and let

$$\sigma(i)=\ dim\ A_i/p_{\overline{m}}(i)$$

where $p_{\overline{m}}(i)$ is defined by (A1) of Theorem 6.2. Then

$$\sigma(i+1)\le\sigma(i)$$

for $0\le i\le t-1$.

Proof. There exists an ideal $q\subset A_i$, $0\ne\lambda\in q$ and a maximal ideal n in $A_i[\frac{q}{n}]$ such that

$$A_{i+1}=(A_i[\frac{q}{\lambda}]_n)\hat{}.$$

Let

$$\tilde{p}=\cup_{j=1}^{\infty}\left(p_{\overline{m}}(i)A_i[\frac{q}{\lambda}]_n:q^jA_i[\frac{q}{\lambda}]_n\right),$$

the strict transform of $p_{\overline{m}}(i)$ in $A_i[\frac{q}{\lambda}]_n$. $q\not\subset p_{\overline{m}}(i)$ implies $\nu(f)=\infty$ if $f\in\tilde{p}$. Thus $\tilde{p}\subset p_{\overline{m}}(i+1)$.

$$A_i/p_{\overline{m}}(i)\to A_i[\frac{q}{\lambda}]_n/\tilde{p}$$

is birational (c.f. [26]) and the residue field extension is finite, so by the dimension formula (c.f. Theorem 15.6 [31]) $\sigma(i)=\ \dim\ A_i[\frac{q}{\lambda}]_n/\tilde{p}$. Thus $\sigma(i)=\ \dim\ A_{i+1}/\tilde{p}A_{i+1}$ since completion is flat (c.f. Theorem 8.14 [31]) and by Theorem 15.1 [31]. We thus have

$$\sigma(i)\ge\ \dim\ A_{i+1}/p_{\overline{m}}(i+1)=\sigma(i+1).$$

□

7. Expansions of Power Series

Let assumptions and notations be as in Section 4 throughout this section. Suppose that $(R,\overline{T}'',\overline{T})$ and $(S,\overline{U}'',\overline{U})$ is a CUTS along ν^*, $\overline{T}''$ has regular parameters $(\overline{x}_1,\ldots,\overline{x}_m)$ and $\overline{U}''$ has regular parameters $(\overline{y}_1,\ldots,\overline{y}_n)$ with

$$\begin{aligned}\overline{x}_1&=\overline{y}_1^{c_{11}}\cdots\overline{y}_{\overline{s}}^{c_{1\overline{s}}}\phi_1\\&\vdots\\\overline{x}_{\overline{r}}&=\overline{y}_1^{c_{\overline{r}1}}\cdots\overline{y}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\phi_{\overline{r}}\\\overline{x}_{\overline{r}+1}&=\overline{y}_{\overline{s}+1}\\&\vdots\\\overline{x}_{\overline{r}+l}&=\overline{y}_{\overline{s}+l}\end{aligned}\tag{11}$$

such that $\phi_1,\ldots,\phi_{\overline{r}} \in k(\overline{U})$, $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_{\overline{r}})$ are rationally independent, $\nu^*(\overline{y}_1),\ldots,\nu^*(\overline{y}_{\overline{s}})$ are rationally independent and (c_{ij}) has rank $\overline{r}$.

Let $C=(C_1,\ldots,C_{\overline{s}})$ be the $\overline{r}\times\overline{s}$ matrix (c_{ij}) of (11). Multiplication by C defines a linear map $\Phi:\mathbf{Q}^{\overline{r}}\rightarrow\mathbf{Q}^{\overline{s}}$, $\Phi(v)=vC$. Φ is 1-1 since C has rank $\overline{r}$.

Suppose that we have a CUTS as in (11), and that $f\in k(\overline{U})[[\overline{y}_1,\ldots,\overline{y}_{\overline{s}+l}]]$.

For $\Lambda\in\mathbf{Z}^{\overline{s}}$, let $[\Lambda]$ denote the class of Λ in $\mathbf{Z}^{\overline{s}}/(\mathbf{Q}^{\overline{r}}C)\cap\mathbf{Z}^{\overline{s}}$. f has a unique expression

$$f=\sum_{[\Lambda]\in\mathbf{Z}^{\overline{s}}/(\mathbf{Q}^{\overline{r}}C)\cap\mathbf{Z}^{\overline{s}}}h_{[\Lambda]} \tag{12}$$

where

$$h_{[\Lambda]}=\sum_{\alpha\in\mathbf{N}^{\overline{s}}\mid[\alpha]=[\Lambda]}g_\alpha\overline{y}_1^{\alpha_1}\cdots\overline{y}_{\overline{s}}^{\alpha_{\overline{s}}} \tag{13}$$

and $g_\alpha\in k(\overline{U})[[\overline{y}_{\overline{s}+1},\ldots,\overline{y}_{\overline{s}+l}]]$.

Set $G=\Phi^{-1}(\mathbf{Z}^{\overline{s}})$. For $\Lambda=(\lambda_1,\ldots,\lambda_{\overline{s}})\in\mathbf{N}^{\overline{s}}$, define

$$P_\Lambda=\{v\in\mathbf{Q}^{\overline{r}}\mid vC_i+\lambda_i\ge 0\text{ for }1\le i\le\overline{s}\}.$$

For $\Lambda\in\mathbf{N}^{\overline{s}}$, we have

$$h_{[\Lambda]}=\overline{y}_1^{\lambda_1}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}}\left[\sum_{v=(v_1,\ldots,v_{\overline{r}})\in G\cap P_\Lambda}\phi_1^{-v_1}\cdots\phi_{\overline{r}}^{-v_{\overline{r}}}\overline{x}_1^{v_1}\cdots\overline{x}_{\overline{r}}^{v_{\overline{r}}}g_v\right] \tag{14}$$

where each $g_v\in k(\overline{U})[[\overline{x}_{\overline{r}+1},\ldots,\overline{x}_{\overline{r}+l}]]$. Here we have reindexed the $g_\alpha=g_{\Phi(v)+\Lambda}$ in (13) as g_v. Let

$$H=\{v\in\mathbf{Z}^{\overline{r}}\mid vC_i\ge 0\text{ for }1\le i\le\overline{s}\},$$
$$I=\{v\in G\mid vC_i\ge 0\text{ for }1\le i\le\overline{s}\}$$

and for $\Lambda=(\lambda_1,\ldots,\lambda_{\overline{s}})\in\mathbf{N}^{\overline{s}}$,

$$M_\Lambda=\{v\in G\mid vC_i+\lambda_i\ge 0\text{ for }1\le i\le\overline{s}\}.$$

P_Λ is a rational polyhedral set in $\mathbf{Q}^{\overline{r}}$ whose associated cone is

$$\sigma=\{v\in\mathbf{Q}^{\overline{r}}\mid vC_i=0\text{ for }1\le i\le\overline{s}\}=\{0\}.$$

Let $W=\mathbf{Q}^{\overline{r}}$. G is a lattice in W. Thus P_Λ is strongly convex and $M_\Lambda=P_\Lambda\cap G$ is a finitely generated module over the semigroup I (c.f. Theorem 7.1 [18]). Let $\overline{n}=[G:\mathbf{Z}^{\overline{r}}]$. We have $\overline{n}x\in H$ for all $x\in I$. Gordon's Lemma (c.f. proposition 1, page 12 [24]) implies that H and I are finitely generated semigroups. There exist $w_1,\ldots,w_{\overline{m}}\in I$ which generate I as a semi-group. Then the finite set

$$\{a_1w_1+\cdots+a_{\overline{m}}w_{\overline{m}}\mid a_i\in\mathbf{N}\text{ and }0\le a_i<\overline{n}\text{ for }1\le i\le\overline{m}\}$$

generate I as an H module. We have then that M_Λ is a finitely generated module over the semigroup H. Thus there exist $\overline{v}_1,\ldots.\overline{v}_{\overline{a}}\in H$ and $\overline{u}_1,\ldots,\overline{u}_{\overline{b}}\in M_\Lambda$ such that if $v=(v_1,\ldots,v_{\overline{r}})\in M_\Lambda=G\cap P_\Lambda$, then

$$v=\overline{u}_i+\sum_{j=1}^{\overline{a}}n_j\overline{v}_j$$

for some $1\le i\le\overline{b}$ and $n_1,\ldots,n_{\overline{a}}\in\mathbf{N}$. Thus,

$$\overline{x}_1^{v_1}\cdots\overline{x}_{\overline{r}}^{v_{\overline{r}}}=\overline{x}_1^{\overline{u}_{i,1}}\cdots\overline{x}_{\overline{r}}^{\overline{u}_{i,\overline{r}}}\prod_{j=1}^{\overline{a}}(\overline{x}_1^{\overline{v}_{j,1}}\cdots\overline{x}_{\overline{r}}^{\overline{v}_{j,\overline{r}}})^{n_j}$$

where $\overline{u}_i = (\overline{u}_{i,1}, \ldots, \overline{u}_{i,\overline{r}})$ and $\overline{v}_j = (\overline{v}_{j,1}, \ldots, \overline{v}_{j,\overline{r}})$ for $1 \le j \le \overline{a}$. Since each $\overline{v}_j \in H$, $\nu(\overline{x}_1^{\overline{v}_{j,1}} \cdots \overline{x}_{\overline{r}}^{\overline{v}_{j,\overline{r}}}) \ge 0$. Thus there exists by Lemma 4.2 [14] and Lemma 5.1 a CUTS of type I along ν^*

$$\begin{array}{ccc} \overline{U} & \to & \overline{U}(1) \\ \uparrow & & \uparrow \\ \overline{T} & \to & \overline{T}(1) \end{array} \tag{15}$$

such that

$$\overline{x}_1^{\overline{v}_{j,1}} \cdots \overline{x}_{\overline{r}}^{\overline{v}_{j,\overline{r}}} = \overline{x}_1(1)^{\overline{v}(1)_{j,1}} \cdots \overline{x}_{\overline{r}}(1)^{\overline{v}(1)_{j,\overline{r}}}$$

with $\overline{v}_j(1) = (\overline{v}(1)_{j,1}, \ldots, \overline{v}(1)_{j,\overline{r}}) \in \mathbf{N}^{\overline{r}}$ for $1 \le j \le \overline{a}$.

We then have expressions for all $\Lambda = (\lambda_1, \ldots, \lambda_{\overline{s}}) \in \mathbf{N}^{\overline{s}}$, where $\overline{u}_1, \ldots, \overline{u}_{\overline{b}} \in \mathbf{Q}^{\overline{r}}$ depend on Λ,

$$h_{[\Lambda]} = \overline{y}_1(1)^{\lambda_1(1)} \cdots \overline{y}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)} \left[\sum_{i=1}^{\overline{b}} \phi_1^{-\overline{u}_{i,1}} \cdots \phi_{\overline{r}}^{-\overline{u}_{i,\overline{r}}} \overline{x}_1(1)^{\overline{u}_{i,1}(1)} \cdots \overline{x}_{\overline{r}}(1)^{\overline{u}_{i,\overline{r}}(1)} g_i\right] \tag{16}$$

with $g_i \in k(\overline{U})[[\overline{x}_1(1), \ldots, \overline{x}_{\overline{r}+l}(1)]]$,

$$\overline{y}_1(1)^{\lambda_1(1)} \cdots \overline{y}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)} = \overline{y}_1^{\lambda_1} \cdots \overline{y}_{\overline{s}}^{\lambda_{\overline{s}}}$$

and

$$\overline{x}_1(1)^{\overline{u}_{i,1}(1)} \cdots \overline{x}_{\overline{r}}(1)^{\overline{u}_{i,\overline{r}}(1)} = \overline{x}_1^{\overline{u}_{i,1}} \cdots \overline{x}_{\overline{r}}^{\overline{u}_{i,\overline{r}}}$$

for $1 \le i \le \overline{b}$.

Remark 7.1. *If $\Lambda = 0$, we have $\nu(\overline{x}_1^{\overline{u}_{i,1}} \cdots \overline{x}_{\overline{r}}^{\overline{u}_{i,\overline{r}}}) \ge 0$ for $1 \le i \le \overline{b}$, so we can construct our CUTS (15) so that $\overline{u}_i(1) \in \mathbf{Q}_+^{\overline{r}}$ for $1 \le i \le \overline{b}$. Thus if d is a common denominator of the coefficients of the $\overline{u}_i$, then $\overline{u}_i(1) \in \frac{1}{d}\mathbf{N}^{\overline{r}}$ for $1 \le i \le \overline{b}$, and we have*

$$h_{[\Lambda]} = h_0 \in k(\overline{U})[\phi_1^{\frac{1}{d}}, \ldots \phi_{\overline{r}}^{\frac{1}{d}}][[\overline{x}_1(1)^{\frac{1}{d}}, \ldots, \overline{x}_{\overline{r}}(1)^{\frac{1}{d}}, \overline{x}_{\overline{r}+1}(1), \ldots, \overline{x}_{\overline{r}+l}(1)]]. \tag{17}$$

Lemma 7.2. *Suppose that $h \in k(\overline{U})[[\overline{x}_1, \ldots, \overline{x}_m]] \subset \overline{U}$ is such that $\nu^*(h) < \infty$. Then $\nu^*(h) \in \Gamma_\nu \otimes \mathbf{Q}$.*

Proof. Recall that $t_1, \ldots, t_{\overline{\alpha}}$ is a transcendence basis of $k(\overline{U})$ over $k(\overline{T})$. Let $A = k(\overline{U})[[\overline{x}_1, \ldots, \overline{x}_m]]$, $B = k(\overline{T})(t_1, \ldots, t_{\overline{\alpha}})[[\overline{x}_1, \ldots, \overline{x}_m]]$. We will first assume that $h \in B$. Since $\nu^*(h) < \infty$, there exists $s > 0$ such that $\nu^*(m(B)^s) > \nu^*(h)$. We have

$$h = f + g$$

with $f \in k(\overline{T})(t_1, \ldots, t_{\overline{\alpha}})[\overline{x}_1, \ldots, \overline{x}_m]$ and $g \in m(B)^s$. $\nu^*(h) = \nu^*(f)$. There exists $0 \ne a \in k(\overline{T})[t_1, \ldots, t_{\overline{\alpha}}]$ such that we have a finite expansion

$$af = \sum_{I=(i_1,\ldots,i_{\overline{\alpha}}) \in \mathbf{N}^{\overline{\alpha}}} a_I t_1^{i_1} \cdots t_{\overline{\alpha}}^{i_{\overline{\alpha}}}$$

with $a_I \in k(\overline{T})[\overline{x}_1, \ldots, \overline{x}_{\overline{r}+l}]$. By Lemma 3.1,

$$\nu^*(h) = \nu^*(f) = \min \{\nu(a_I) \mid a_I \ne 0\} \in \Gamma_\nu.$$

Since A is a finite extension of B, we have $\nu^*(h) \in \Gamma_\nu \otimes \mathbf{Q}$ if $h \in B$, by the Corollary to Lemma 3, Section 11, Chapter VI [39].

□

Lemma 7.3. (1) *Suppose that* $\Lambda \in \mathbf{N}^{\overline{s}}$ *and* $\nu^*(h_{[\Lambda]}) < \infty$. *Then we have*

$$\nu^*\left(\frac{h_{[\Lambda]}}{\overline{y}_1^{\lambda_1}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}}}\right) \in \Gamma_\nu \otimes \mathbf{Q}.$$

In particular, $\nu^*(h_{[\Lambda]}) \in \Gamma_\nu \otimes \mathbf{Q}$ *implies* $[\Lambda] = 0$.

(2) *In the expansion (12), for* $\Lambda_1, \Lambda_2 \in \mathbf{N}^{\overline{s}}$, *suppose that*

$$\nu^*(h_{[\Lambda_1]}) = \nu^*(h_{[\Lambda_2]}) < \infty.$$

Then $[\Lambda_1] = [\Lambda_2]$.

Proof. For $\Lambda \in \mathbf{N}^{\overline{s}}$ such that $\nu^*(h_{\Lambda]}) < \infty$, consider the expansion (16) of $h_{[\Lambda]}$.

There exists $w = (w_1, \ldots, w_{\overline{r}}) \in \mathbf{N}^{\overline{r}}$ such that $w + \overline{u}_i \in \mathbf{Q}_+^{\overline{r}}$ for $1 \le i \le \overline{b}$.

Let d be a common denominator of the coefficients $\overline{u}_i$ for $1 \le i \le \overline{b}$. Let

$$A = k(\overline{U})[\phi_1^{\frac{1}{d}}, \ldots, \phi_{\overline{r}}^{\frac{1}{d}}][[\overline{x}_1(1)^{\frac{1}{d}}, \ldots, \overline{x}_{\overline{r}}(1)^{\frac{1}{d}}, \overline{x}_{\overline{r}+1}(1), \ldots, \overline{x}_{\overline{r}+l}(1)]].$$

Set

$$f = \frac{h_{[\Lambda]}}{\overline{y}_1^{\lambda_1}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}}}\overline{x}_1^{w_1}\cdots\overline{x}_{\overline{r}}^{w_{\overline{r}}} \in A.$$

If we restrict $\tilde{\nu}^*$ to $Q(k(\overline{U})[[\overline{y}_1(1), \ldots, \overline{y}_{\overline{s}+l}(1)]])$, extend it to the finite extension $Q(k(\overline{U})[\phi_1^{\frac{1}{d}}, \ldots, \phi_{\overline{r}}^{\frac{1}{d}}][\overline{y}_1(1)^{\frac{1}{d}}, \ldots, \overline{y}_{\overline{s}}(1)^{\frac{1}{d}}, \overline{y}_{\overline{s}+1}(1), \ldots, \overline{y}_{\overline{s}+l}(1)]])$ so that it dominates $k(\overline{U})[\phi_1^{\frac{1}{d}}, \ldots, \phi_{\overline{r}}^{\frac{1}{d}}][\overline{y}_1(1)^{\frac{1}{d}}, \ldots, \overline{y}_{\overline{s}}(1)^{\frac{1}{d}}, \overline{y}_{\overline{s}+1}(1), \ldots, \overline{y}_{\overline{s}+l}(1)]]$ and restrict to A, we get a valuation $\overline{\nu}'$ on $Q(A)$ which extends $\tilde{\nu}$ restricted to $Q(k(\overline{T})[[\overline{x}_1(1), \ldots, \overline{x}_{\overline{r}+l}(1)]])$. By Lemma 7.2 and the Corollary to Lemma 3, Section 11, Chapter VI [39], we have $\overline{\nu}'(f) \in \Gamma_\nu \otimes \mathbf{Q}$. Thus, we have

$$\nu^*(\frac{h_{[\Lambda]}}{\overline{y}_1^{\lambda_1}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}}}) \in \Gamma_\nu \otimes \mathbf{Q}.$$

We now compare $\nu^*(h_{[\Lambda_1]})$ and $\nu^*(h_{[\Lambda_2]})$.

$$\nu^*\left(\frac{h_{[\Lambda_1]}}{h_{[\Lambda_2]}}\right) = \nu^*\left(\frac{h_{[\Lambda_1]}}{\overline{y}_1^{\lambda_1^1}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}^1}}\right) - \nu^*\left(\frac{h_{[\Lambda_2]}}{\overline{y}_1^{\lambda_1^2}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}^2}}\right) + \nu^*\left(\frac{\overline{y}_1^{\lambda_1^1}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}^1}}{\overline{y}_1^{\lambda_1^2}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}^2}}\right)$$

where $\Lambda_1 = (\lambda_1^1, \ldots, \lambda_{\overline{s}}^1)$, $\Lambda_2 = (\lambda_1^2, \ldots, \lambda_{\overline{s}}^2)$. Thus $\nu^*(h_{[\Lambda_1]}) = \nu^*(h_{[\Lambda_2]}) < \infty$ implies

$$\nu^*\left(\frac{\overline{y}_1^{\lambda_1^1}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}^1}}{\overline{y}_1^{\lambda_1^2}\cdots\overline{y}_{\overline{s}}^{\lambda_{\overline{s}}^2}}\right) \in \Gamma_\nu \otimes \mathbf{Q},$$

so that

$$(\lambda_1^1 - \lambda_1^2)\nu^*(\overline{y}_1) + \cdots + (\lambda_{\overline{s}}^1 - \lambda_{\overline{s}}^2)\nu^*(\overline{y}_{\overline{s}}) \in \Gamma_\nu \otimes \mathbf{Q}.$$

Thus, there exists $(a_1, \ldots, a_{\overline{r}}) \in \mathbf{Q}^{\overline{r}}$ such that

$$(\lambda_1^1 - \lambda_1^2)\nu^*(\overline{y}_1) + \cdots + (\lambda_{\overline{s}}^1 - \lambda_{\overline{s}}^2)\nu^*(\overline{y}_{\overline{s}}) = a_1\nu(\overline{x}_1) + \cdots + a_{\overline{r}}\nu(\overline{x}_{\overline{r}}).$$

Substituting from (11), we get

$$(a_1, \ldots, a_{\overline{r}})C = (\lambda_1^1 - \lambda_1^2, \ldots, \lambda_{\overline{s}}^1 - \lambda_{\overline{s}}^2)$$

and thus

$$\Lambda_1 - \Lambda_2 \in \Phi(\mathbf{Q}^{\overline{r}}) \cap \mathbf{Z}^{\overline{s}}.$$

□

Remark 7.4. *In the expansion (12), Let $\Lambda_0\in\mathbf{N}^{\overline{s}}$ be such that*

$$\nu^*(h_{[\Lambda_0]})=min\{\nu^*(h_{[\Lambda]})\mid\Lambda\in\mathbf{N}^{\overline{s}}\}.$$

This minimum exists since $\overline{U}$ is Noetherian. Then, by Lemma 7.3,

$$\nu^*(f)=\nu^*(h_{[\Lambda_0]}).$$

Lemma 7.5. *With the notation of (11), assume that $f\in\overline{T}''$ and*

$$f\in k(\overline{U})[[\overline{y}_1,\ldots,\overline{y}_{\overline{s}+l}]].$$

Then

$$f\in k(\overline{T})[[\overline{x}_1,\ldots,\overline{x}_{\overline{r}+l}]].$$

Proof. Let $h_{[\Lambda_0]}$ be the minimum value term of f in the expansion (12), so that

$$\nu^*(f)=\nu^*(h_{[\Lambda_0]})$$

(by Remark 7.4). Since $f\in\overline{T}''$, we have $\nu^*(f)<\infty$. Since

$$\nu^*(h_{[\Lambda_0]})=\nu(f)\in\Gamma_\nu,$$

we have $[\Lambda_0]=0$ by Lemma 7.3. Thus by Remark 7.1, there exists a CUTS of type I

$$\begin{array}{ccc}\overline{U}&\rightarrow&\overline{U}(1)\\ \uparrow&&\uparrow\\ \overline{T}&\rightarrow&\overline{T}(1)\end{array}$$

and d, a positive integer, such that

$$h_{[\Lambda_0]}=h_0\in A=k(\overline{U})[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}}][[\overline{x}_1(1)^{\frac{1}{d}},\ldots,\overline{x}_{\overline{r}}(1)^{\frac{1}{d}},\overline{x}_{\overline{r}+1}(1),\ldots,\overline{x}_{\overline{r}+l}(1)]].$$

Suppose that $f\neq h_0$. Then there exists $\Lambda_1\not\in(\mathbf{Q}^{\overline{r}}C)\cap\mathbf{Z}^{\overline{s}}$ in the expansion of f in (12), such that the minimal value term of $f-h_0$ is $h_{[\Lambda_1]}$. Write $\Lambda_1=(\lambda_1^1,\ldots,\lambda_{\overline{s}}^1)$.

Consider the k-derivation

$$\partial=\sum_{i=1}^{\overline{s}}e_i\overline{y}_i\frac{\partial y_i}{\partial\overline{y}_i}$$

on $\overline{U}''$ (and on $\overline{U}$) where $e_i\in\mathbf{Q}$ are chosen so that

$$\sum_{i=1}^{\overline{s}}e_ic_{ji}=0\text{ for }1\le j\le\overline{r}$$

and

$$\sum_{i=1}^{\overline{s}}e_i\lambda_i^1\neq 0.$$

We have

$$\partial(\overline{y}_1^{b_1}\cdots\overline{y}_n^{b_n})=(b_1e_1+\cdots+b_{\overline{s}}e_{\overline{s}})\overline{y}_1^{b_1}\cdots\overline{y}_n^{b_n}$$

for all monomials $\overline{y}_1^{b_1}\cdots\overline{y}_n^{b_n}$. From (13), we see that

$$\begin{array}{ll}\partial(h_{[\Lambda]})&=\sum_{\alpha\in\mathbf{N}^{\overline{s}}\mid[\alpha]=[\Lambda]}g_\alpha\partial(\overline{y}_1^{\alpha_1}\cdots\overline{y}_{\overline{s}}^{\alpha_{\overline{s}}})\\ &=\sum(\alpha_1e_1+\cdots+\alpha_{\overline{s}}e_{\overline{s}})g_\alpha\overline{y}_1^{\alpha_1}\cdots\overline{y}_{\overline{s}}^{\alpha_{\overline{s}}}\\ &=(\lambda_1e_1+\cdots+\lambda_{\overline{s}}e_{\overline{s}})h_{[\Lambda]}.\end{array}$$

In particular, $\partial(h_{[\Lambda_0]})=0$ and

$$\partial(h_{[\Lambda_1]})=(\lambda_1^1\alpha_1+\cdots+\lambda_{\overline{s}}^1\alpha_{\overline{s}})h_{[\Lambda_1]}\neq 0.$$

Thus

$$\partial(f-h_0)=\sum_{[\Lambda]\neq 0}(\lambda_1e_1+\cdots+\lambda_{\overline{s}}e_{\overline{s}})h_{[\Lambda]}$$

has value

$$\nu^*(\partial(f-h_0))=\nu^*(h_{[\Lambda_1]}).$$

But ∂ is a derivation of $\overline{U}''$, so that $\partial(f-h_0)=\partial(f)\in\overline{U}''$ has finite value, and

$$\nu^*(h_{[\Lambda_1]})=\nu^*(\partial(f-h_0))<\infty.$$

Thus $\nu^*(h_{[\Lambda_1]})\not\in\Gamma_\nu\otimes\mathbf{Q}$, by Lemma 7.3, but $\nu^*(h_{[\Lambda_1]})=\nu^*(f-h_0)\in\Gamma_\nu\otimes\mathbf{Q}$, by Lemma 7.2, since

$$f-h_0\in k(\overline{U})[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}}][[\overline{x}_1(1)^{\frac{1}{d}},\ldots,\overline{x}_{\overline{r}}(1)^{\frac{1}{d}},\overline{x}_{\overline{r}+1}(1),\ldots,\overline{x}_m(1)]]$$

which is a finite extension of $k(\overline{U})[[\overline{x}_1(1),\ldots,\overline{x}_m(1)]]$, a contradiction. Thus $f=h_0$. We have

$$f\in A\cap k(\overline{T})[[\overline{x}_1,\ldots,\overline{x}_m]]=k(\overline{T})[[\overline{x}_1,\ldots,\overline{x}_{\overline{r}+l}]].$$

□

8. CUTS of form $\overline{m}$

Let assumptions and notations be as in Section 4 throughout this section.

Theorem 8.1. *Suppose that $(R,\overline{T}'',\overline{T})$ and $(S,\overline{U}'',\overline{U})$ is a CUTS along ν^*, such that $\overline{T}''$ contains a subfield isomorphic to $k(c_0)$ for some $c_0\in\overline{T}''$ and $\overline{U}''$ contains a subfield isomorphic to $k(\overline{U})$. $\overline{T}''$ has regular parameters $(\overline{z}_1,\ldots,\overline{z}_m)$ and $\overline{U}''$ has regular parameters $(\overline{w}_1,\ldots,\overline{w}_n)$ with*

$$\begin{array}{ll}\overline{z}_1 & =\overline{w}_1^{c_{11}}\cdots\overline{w}_{\overline{s}}^{c_{1\overline{s}}}\phi_1\\ & \vdots\\ \overline{z}_{\overline{r}} & =\overline{w}_1^{c_{\overline{r}1}}\cdots\overline{w}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\phi_{\overline{r}}\\ \overline{z}_{\overline{r}+1} & =\overline{w}_{\overline{s}+1}\\ & \vdots\\ \overline{z}_{\overline{r}+l} & =\overline{w}_{\overline{s}+l}\end{array}$$

such that $\phi_1,\ldots,\phi_{\overline{r}}\in k(\overline{U})$, $\nu(\overline{z}_1),\ldots,\nu(\overline{z}_{\overline{r}})$ are rationally independent, $\nu^*(\overline{w}_1),\ldots,\nu^*(\overline{w}_{\overline{s}})$ are rationally independent and (c_{ij}) has rank $\overline{r}$.

Suppose that one of the following three conditions hold.

$$f\in k(\overline{U})[[\overline{w}_1,\ldots,\overline{w}_{\overline{s}+\overline{m}}]]\text{ for some }\overline{m}\text{ such that }0\le\overline{m}\le n-\overline{s}\text{ with }\nu^*(f)<\infty.\tag{18}$$

$$f\in k(\overline{U})[[\overline{w}_1,\ldots,\overline{w}_{\overline{s}+\overline{m}}]]\text{ for some }\overline{m}\text{ such that }0<\overline{m}\le n-\overline{s}\text{ with }\nu^*(f)=\infty\text{ and }A>0\text{ is given.}\tag{19}$$

$$f\in\left(k(\overline{U})[[\overline{w}_1,\ldots,\overline{w}_{\overline{s}+\overline{m}}]]-k(\overline{U})[[\overline{w}_1,\ldots,\overline{w}_{\overline{s}+l}]]\right)\cap\overline{U}''\text{ for some }\overline{m}\text{ such that }l<\overline{m}\le n-\overline{s}.\tag{20}$$

Then there exists a CUTS along ν^* $(R,\overline{T}''(t),\overline{T}(t))$ and $(S,\overline{U}''(t),\overline{U}(t))$

$$\begin{array}{ccccccccc}\overline{U}&=&\overline{U}(0)&\rightarrow&\overline{U}(1)&\rightarrow&\cdots&\rightarrow&\overline{U}(t)\\ &&\uparrow&&\uparrow&&&&\uparrow\\ \overline{T}&=&\overline{T}(0)&\rightarrow&\overline{T}(1)&\rightarrow&\cdots&\rightarrow&\overline{T}(t)\end{array}\tag{21}$$

such that $\overline{U}''(t)$ has regular parameters $(\overline{w}_1(t),\ldots,\overline{w}_n(t))$.

In case (18) we have

$$f=\overline{w}_1(t)^{d_1}\cdots\overline{w}_{\overline{s}}(t)^{d_{\overline{s}}}u$$

where $u \in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_{\overline{s}+\overline{m}}(t)]]$ *is a unit power series.*

In case (19) we have

$$f = \overline{w}_1(t)^{d_1}\cdots\overline{w}_{\overline{s}}(t)^{d_{\overline{s}}}\Sigma$$

where $\Sigma \in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_{\overline{s}+\overline{m}}(t)]]$, $\nu^*(\overline{w}_1(t)^{d_1}\cdots\overline{w}_{\overline{s}}(t)^{d_{\overline{s}}}) > A$.

In case (20) we have

$$f = P + \overline{w}_1(t)^{d_1}\cdots\overline{w}_{\overline{s}}(t)^{d_{\overline{s}}}H$$

for some powerseries $P \in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_{\overline{s}+l}(t)]]$,

$$H = u(\overline{w}_{\overline{s}+\overline{m}}(t) + \overline{w}_1(t)^{g_1}\cdots\overline{w}_{\overline{s}}(t)^{g_{\overline{s}}}\Sigma)$$

where $u \in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_{\overline{s}+\overline{m}}(t)]]$ *is a unit,* $\Sigma \in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_{\overline{s}+\overline{m}-1}(t)]]$ *and* $\nu^*(\overline{w}_{\overline{s}+\overline{m}}(t)) \le \nu^*(\overline{w}_1(t)^{g_1}\cdots\overline{w}_{\overline{s}}(t)^{g_{\overline{s}}})$.

(21) will be such that $\overline{T}''(\alpha)$ *has regular parameters*

$$(\overline{z}_1(\alpha),\ldots,\overline{z}_m(\alpha)) \text{ and } (\tilde{\overline{z}}'_1(\alpha),\ldots,\tilde{\overline{z}}'_m(\alpha)),$$

$\overline{U}''(\alpha)$ *has regular parameters*

$$(\overline{w}_1(\alpha),\ldots,\overline{w}_n(\alpha)) \text{ and } (\tilde{\overline{w}}'_1(\alpha),\ldots,\tilde{\overline{w}}'_n(\alpha))$$

where $\overline{z}_i(0) = \overline{z}_i$ *for* $1 \le i \le m$ *and* $\overline{w}_i(0) = \overline{w}_i$ *for* $1 \le i \le n$. *(21) will consist of three types of CUTS.*

- **(M1):** $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ *and* $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ *are of type I.*
- **(M2):** $\overline{T}(\alpha) \to \overline{T}(\alpha+1)$ *is of type* II_r, $1 \le r \le \min\{l,\overline{m}\}$, *and* $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ *is a transformation of type* II_r *followed by a transformation of type I.*
- **(M3):** $\overline{T}(\alpha) = \overline{T}(\alpha+1)$ *and* $\overline{U}(\alpha) \to \overline{U}(\alpha+1)$ *is of type* II_r *with* $l+1 \le r \le \overline{m}$.

$\overline{T}''(\alpha)$ *contains a subfield isomorphic to* $k(c_0,\ldots,c_\alpha)$ *and* $\overline{U}''(\alpha)$ *contains a subfield isomorphic to* $k(\overline{U}(\alpha))$.

We will find polynomials $P_{i,\alpha}$ *so that the variables will be related by:*

$$\tilde{\overline{z}}'_i(\alpha) = \begin{cases} \overline{z}_i(\alpha) - P_{i,\alpha} & \text{if } \overline{r}+1 \le i \le \overline{r}+\min\{l,\overline{m}\} \\ \overline{z}_i(\alpha) & \text{otherwise} \end{cases}$$

$$\tilde{\overline{w}}'_i(\alpha) = \begin{cases} \tilde{\overline{z}}_{i-\overline{s}+\overline{r}}(\alpha) & \text{if } \overline{s}+1 \le i \le \overline{s}+l \\ \overline{w}_i(\alpha) - P_{i,\alpha} & \text{if } \overline{s}+l+1 \le i \le \overline{s}+\overline{m} \\ \overline{w}_i(\alpha) & \text{otherwise} \end{cases}$$

We will have $P_{i,\alpha} \in k(c_0,\ldots,c_\alpha)[\overline{z}_1(\alpha),\ldots,\overline{z}_{i-1}(\alpha)]$ *if* $i \le \overline{r}+l$,

$$P_{i,\alpha} \in k(\overline{U}(\alpha))[\overline{w}_1(\alpha),\ldots,\overline{w}_{i-1}(\alpha)]$$

if $i > \overline{s}+l$. *For all* α *we will have*

$$\begin{array}{ll} \overline{z}_1(\alpha) & = \overline{w}_1(\alpha)^{c_{11}(\alpha)}\cdots\overline{w}_{\overline{s}}(\alpha)^{c_{1\overline{s}}(\alpha)}\phi_1(\alpha) \\ & \vdots \\ \overline{z}_{\overline{r}}(\alpha) & = \overline{w}_1(\alpha)^{c_{\overline{r}1}(\alpha)}\cdots\overline{w}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\phi_{\overline{r}}(\alpha) \\ \overline{z}_{\overline{r}+1}(\alpha) & = \overline{w}_{\overline{s}+1}(\alpha) \\ & \vdots \\ \overline{z}_{\overline{r}+l}(\alpha) & = \overline{w}_{\overline{s}+l}(\alpha) \end{array} \tag{22}$$

and

$$\begin{array}{ll} \tilde{\overline{z}}'_1(\alpha) & = \tilde{\overline{w}}'_1(\alpha)^{c_{11}(\alpha)}\cdots\tilde{\overline{w}}'_{\overline{s}}(\alpha)^{c_{1\overline{s}}(\alpha)}\phi_1(\alpha)\\ & \vdots\\ \tilde{\overline{z}}'_{\overline{r}}(\alpha) & = \tilde{\overline{w}}'_1(\alpha)^{c_{\overline{r}1}(\alpha)}\cdots\tilde{\overline{w}}'_{\overline{s}}(\alpha)^{c_{\overline{r}\overline{s}}(\alpha)}\phi_{\overline{r}}(\alpha)\\ \tilde{\overline{z}}'_{\overline{r}+1}(\alpha) & = \tilde{\overline{w}}'_{\overline{s}+1}(\alpha)\\ & \vdots\\ \tilde{\overline{z}}'_{\overline{r}+l}(\alpha) & = \tilde{\overline{w}}'_{\overline{s}+l}(\alpha) \end{array} \tag{23}$$

where $\phi_1(\alpha),\dots,\phi_{\overline{r}}(\alpha)\in k(\overline{U}(\alpha))$. $\nu(\overline{z}_1(\alpha)),\dots,\nu(\overline{z}_{\overline{r}}(\alpha))$ *are rationally independent,* $\nu^*(\overline{w}_1(\alpha)),\dots,\nu^*(\overline{w}_{\overline{s}}(\alpha))$ *are rationally independent and* $(c_{ij}(\alpha))$ *has rank* $\overline{r}$ *for* $1\le\alpha\le t$.

In a transformation $\overline{T}(\alpha)\to\overline{T}(\alpha+1)$ *of type I,* $\overline{T}''(\alpha+1)$ *will have regular parameters* $(\overline{z}_1(\alpha+1),\dots,\overline{z}_m(\alpha+1))$ *defined by*

$$\begin{array}{ll} \tilde{\overline{z}}'_1(\alpha) & = \overline{z}_1(\alpha+1)^{a_{11}(\alpha+1)}\cdots\overline{z}_{\overline{r}}(\alpha+1)^{a_{1\overline{r}}(\alpha+1)}\\ & \vdots\\ \tilde{\overline{z}}'_{\overline{r}}(\alpha) & = \overline{z}_1(\alpha+1)^{a_{\overline{r}1}(\alpha+1)}\cdots\overline{z}_{\overline{r}}(\alpha+1)^{a_{\overline{r}\overline{r}}(\alpha+1)}. \end{array} \tag{24}$$

and $c_{\alpha+1}$ *is defined to be 1. In a transformation* $\overline{T}(\alpha)\to\overline{T}(\alpha+1)$ *of type* II_r $(1\le r\le\ \min\{l,\overline{m}\})$ $\overline{T}''(\alpha+1)$ *will have regular parameters* $(\overline{z}_1(\alpha+1),\dots,\overline{z}_m(\alpha+1))$ *defined by*

$$\begin{array}{ll} \tilde{\overline{z}}'_1(\alpha) & = \overline{z}_1(\alpha+1)^{a_{11}(\alpha+1)}\cdots\overline{z}_{\overline{r}}(\alpha+1)^{a_{1\overline{r}}(\alpha+1)}c_{\alpha+1}^{a_{1\overline{r}+1}(\alpha+1)}\\ & \vdots\\ \tilde{\overline{z}}'_{\overline{r}}(\alpha) & = \overline{z}_1(\alpha+1)^{a_{\overline{r}1}(\alpha+1)}\cdots\overline{z}_{\overline{r}}(\alpha+1)^{a_{\overline{r}\overline{r}}(\alpha+1)}c_{\alpha+1}^{a_{\overline{r},\overline{r}+1}(\alpha+1)}\\ \tilde{\overline{z}}'_{\overline{r}+r}(\alpha) & = \overline{z}_1(\alpha+1)^{a_{\overline{r}+1,1}(\alpha+1)}\cdots\overline{z}_{\overline{r}}(\alpha+1)^{a_{\overline{r}+1,\overline{r}}(\alpha+1)}\\ & \quad\cdot(\overline{z}_{\overline{r}+r}(\alpha+1)+1)c_{\alpha+1}^{a_{\overline{r}+1,\overline{r}+1}(\alpha+1)} \end{array} \tag{25}$$

In a transformation $\overline{U}(\alpha)\to\overline{U}(\alpha+1)$ *of type I* $\overline{U}''(\alpha+1)$ *will have regular parameters* $(\overline{w}_1(\alpha+1),\dots,\overline{w}_n(\alpha+1))$ *defined by*

$$\begin{array}{ll} \tilde{\overline{w}}'_1(\alpha) & = \overline{w}_1(\alpha+1)^{b_{11}(\alpha+1)}\cdots\overline{w}_{\overline{s}}(\alpha+1)^{b_{1\overline{s}}(\alpha+1)}\\ & \vdots\\ \tilde{\overline{w}}'_{\overline{s}}(\alpha) & = \overline{w}_1(\alpha+1)^{b_{\overline{s}1}(\alpha+1)}\cdots\overline{w}_{\overline{s}}(\alpha+1)^{b_{\overline{s}\overline{s}}(\alpha+1)}. \end{array} \tag{26}$$

In a transformation $\overline{U}(\alpha)\to\overline{U}(\alpha+1)$ *of type* II_r $(1\le r\le\overline{m})$ $\overline{U}''(\alpha+1)$ *will have regular parameters* $(\overline{w}_1(\alpha+1),\dots,\overline{w}_n(\alpha+1))$ *defined by*

$$\begin{array}{ll} \tilde{\overline{w}}'_1(\alpha) & = \overline{w}_1(\alpha+1)^{b_{11}(\alpha+1)}\cdots\overline{w}_{\overline{s}}(\alpha+1)^{b_{1\overline{s}}(\alpha+1)}d_{\alpha+1}^{b_{1\overline{s}+1}(\alpha+1)}\\ & \vdots\\ \tilde{\overline{w}}'_{\overline{s}}(\alpha) & = \overline{w}_1(\alpha+1)^{b_{\overline{s}1}(\alpha+1)}\cdots\overline{w}_{\overline{s}}(\alpha+1)^{b_{\overline{s}\overline{s}}(\alpha+1)}d_{\alpha+1}^{b_{\overline{s},\overline{s}+1}(\alpha+1)}\\ \tilde{\overline{w}}'_{\overline{s}+r}(\alpha) & = \overline{w}_1(\alpha+1)^{b_{\overline{s}+1,1}(\alpha+1)}\cdots\overline{w}_{\overline{s}}(\alpha+1)^{b_{\overline{s}+1,\overline{s}}(\alpha+1)}\\ & \quad\cdot(\overline{w}_{\overline{s}+r}(\alpha+1)+1)d_{\alpha+1}^{b_{\overline{s}+1,\overline{s}+1}(\alpha+1)} \end{array} \tag{27}$$

We will call a CUTS as in (21) a CUTS of form $\overline{m}$. *Observe that the UTS* $\overline{T}\to\overline{T}(t)$ *is a UTS of form* $\min\{l,\overline{m}\}$.

Proof. We will first assume that f satisfies (18) or (19) with $0\le\overline{m}\le l$.

By (A1) of Theorem 6.2, after performing a CUTS of form $\overline{m}$, we may assume that

$$p_{\overline{m}} = (\overline{z}_{r(1)} - Q_{r(1)}(\overline{z}_1, \ldots, \overline{z}_{r(1)-1}), \ldots, \overline{z}_{r(\tilde{m})} - Q_{r(\tilde{m})}(\overline{z}_1, \ldots, \overline{z}_{r(\tilde{m})-1}) \tag{28}$$

where the coefficients of $Q_{r(i)}$ are in $k(c_0)$. We have that $f \in k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_{\overline{s}+\overline{m}}]]$.

Given a CUTS (21), define $\sigma(i)$ as in Lemma 6.3 for the UTS

$$\overline{T} \to \cdots \to \overline{T}(t).$$

If $\sigma(i)$ drops during the course of the proof, we can start the corresponding algorithm again with this smaller value of $\sigma(i)$. Eventually $\sigma(i)$ must stablize, so we may assume that $\sigma(i)$ is constant throughout the proof.

We have the expansion

$$f = \sum_{[\Lambda] \in \mathbf{Z}^{\overline{s}}/(\mathbf{Q}^{\overline{r}}C) \cap \mathbf{Z}^{\overline{s}}} h_{[\Lambda]}$$

of (12). Let $\Lambda_0 \in \mathbf{N}^{\overline{s}}$ be such that

$$\nu^*(h_{[\Lambda_0]}) = \min\{\nu^*(h_{[\Lambda]}) \mid h_{[\Lambda]} \neq 0\}.$$

By Remark 7.4, $\nu^*(f) = \nu^*(h_{[\Lambda_0]})$. We either have $\nu^*(f) < \infty$ and $\nu^*(h_{[\Lambda]}) > \nu^*(h_{[\Lambda_0]})$ if $[\Lambda] \neq [\Lambda_0]$ (by Lemma 7.3) or $\nu^*(f) = \infty$ and $\nu^*(h_{[\Lambda]}) = \infty$ for all $[\Lambda]$. Let I be the ideal

$$I = (h_{[\Lambda]} \mid [\Lambda] \neq [\Lambda_0]) \subset k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_{\overline{s}+\overline{m}}]].$$

Let $h_{[\Lambda_1]}, \ldots, h_{[\Lambda_\beta]}$ be generators of I. We will construct a CUTS along ν^* of form $\overline{m}$

$$\begin{array}{ccc} \overline{U} & \to & \overline{U}(\alpha) \\ \uparrow & & \uparrow \\ \overline{T} & \to & \overline{T}(\alpha) \end{array}$$

such that

$$h_{[\Lambda_i]} = \overline{w}_1(\alpha)^{b_1^i} \cdots \overline{w}_{\overline{s}}(\alpha)^{b_{\overline{s}}^i} \psi_i$$

for $0 \leq i \leq \beta$, with $\psi_i \in \overline{U}(\alpha)$. If $\nu^*(h_{[\Lambda_i]}) < \infty$, ψ_i will be a unit.

If $\nu^*(f) < \infty$, so that f satisfies the conditions of (18), then set

$$B = \nu^*(f) = \nu^*(h_{[\Lambda_0]}).$$

If $\nu^*(h_{[\Lambda_i]}) = \infty$, we will have $\nu^*(\overline{w}_1(\alpha)^{b_1^i} \cdots \overline{w}_{\overline{s}}(\alpha)^{b_{\overline{s}}^i}) > B$.

If $\nu^*(f) = \infty$, so that f satisfies the conditions of (19), we will have

$$\nu^*(\overline{w}_1(\alpha)^{b_1^i} \cdots \overline{w}_{\overline{s}}(\alpha)^{b_{\overline{s}}^i}) > A$$

for all i. In this case $\nu^*(h_{[\Lambda_i]}) = \infty$ for $0 \leq i \leq \beta$.

Assume that the above CUTS has been constructed. There exists (by Lemma 4.2 [14]) a CUTS of type (M1) along ν^*

$$\begin{array}{ccc} \overline{U}(\alpha) & \to & \overline{U}(\alpha+1) \\ \uparrow & & \uparrow \\ \overline{T}(\alpha) & = & \overline{T}(\alpha+1) \end{array}$$

such that if $\nu^*(f) < \infty$, then for $1 \leq i \leq \beta$, $h_{[\Lambda_0]}$ divides $h_{[\Lambda_i]}$ in $\overline{U}(\alpha+1)$. If $\nu^*(f) = \infty$, then there exists $\overline{w}_1(\alpha+1)^{a_1} \cdots \overline{w}_{\overline{s}}(\alpha+1)^{a_{\overline{s}}}$ such that

$$\nu^*(\overline{w}_1(\alpha+1)^{a_1} \cdots \overline{w}_{\overline{s}}(\alpha+1)^{a_{\overline{s}}}) > A$$

and $\overline{w}_1(\alpha+1)^{a_1} \cdots \overline{w}_{\overline{s}}(\alpha+1)^{a_{\overline{s}}}$ divides $h_{[\Lambda_i]}$ for $0 \leq i \leq \beta$ in $\overline{U}(\alpha+1)$. Thus the conclusions of the theorem hold for f satisfying the conditions (18) or (19) with $0 \leq \overline{m} \leq l$.

We are thus reduced to proving the theorem (with our assumption that f satisfies (18) or (19) and $0 \le \overline{m} \le l$) when $f = h_{[\Lambda]}$ for some $\Lambda \in \mathbf{N}^{\overline{s}}$. Assume that f has this form. There exists a CUTS along ν^* (using Lemma 5.1)

$$\begin{array}{ccc} \overline{U} & \rightarrow & \overline{U}(1)\\ \uparrow & & \uparrow\\ \overline{T} & \rightarrow & \overline{T}(1) \end{array}$$

of type (M1) such that there is an expression of the form of (16),

$$h_{[\Lambda]} = \overline{w}_1(1)^{\lambda_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)}\left(\sum_{i=1}^{\overline{b}}\overline{z}_1(1)^{\overline{u}_{i,1}}\cdots\overline{z}_{\overline{r}}(1)^{\overline{u}_{i,\overline{r}}}g_i\right),$$

$d \in \mathbf{N}$ with $\overline{u}_i = (\overline{u}_{i,1},\ldots,\overline{u}_{i,\overline{r}}) \in \frac{1}{d}\mathbf{Z}^{\overline{r}}$ for $1 \le i \le \overline{b}$, $g_i \in k(\overline{U})[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}}][[\overline{z}_1(1),\ldots,\overline{z}_{\overline{r}+\overline{m}}(1)]]$, and

$$\overline{w}_1^{\lambda_1}\cdots\overline{w}_{\overline{s}}^{\lambda_{\overline{s}}} = \overline{w}_1^{\lambda_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)}.$$

Set $L' = k(\overline{U})[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}}]$. There exists $a_1,\ldots,a_{\overline{r}} \in \mathbf{N}$ such that

$$a_j + \overline{u}_{i,j} \ge 0$$

for $1 \le j \le \overline{r}$ and $1 \le i \le \overline{b}$. Set

$$\delta = \frac{h_{[\Lambda]}\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}}}{\overline{w}_1(1)^{\lambda_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)}} \in L'[[\overline{z}_1(1)^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}(1)^{\frac{1}{d}},\overline{z}_{\overline{r}+1}(1),\ldots,\overline{z}_{\overline{r}+\overline{m}}(1)]]$$

and let

$$\sigma = \nu(\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}}).$$

Let ω be a primitive d-th root of unity (in an algebraic closure of L'). Set

$$\begin{aligned}\delta_{j_1,\ldots,j_{\overline{r}}} &= \delta(\omega^{j_1}\overline{z}_1(1)^{\frac{1}{d}},\ldots,\omega^{j_{\overline{r}}}\overline{z}_{\overline{r}}(1)^{\frac{1}{d}},\overline{z}_{\overline{r}+1}(1),\ldots,\overline{z}_{\overline{r}+\overline{m}}(1))\\ &\in B_1 = L'[\omega][[\overline{z}_1(1)^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}(1)^{\frac{1}{d}},\overline{z}_{\overline{r}+1}(1),\ldots,\overline{z}_{\overline{r}+\overline{m}}(1)]].\end{aligned}$$

Set

$$\epsilon = \prod_{j_1,\ldots,j_{\overline{r}}=1}^{d}\delta_{j_1,\ldots,j_{\overline{r}}} \in A_1 = L'[[\overline{z}_1(1),\ldots,\overline{z}_{\overline{r}+\overline{m}}(1)]].$$

Identify ν^* with an extension of (a restriction of ν^* to $Q(F)$) which dominates B_1.

We will now prove that

$$\nu^*(h_{[\Lambda]}) = \infty \Leftrightarrow \nu^*(\delta) = \infty \Leftrightarrow \nu^*(\epsilon) = \infty.$$

We certainly have that

$$\nu^*(h_{[\Lambda]}) = \infty \Leftrightarrow \nu^*(\delta) = \infty.$$

$\nu^*(\delta) = \infty \Rightarrow \nu^*(\epsilon) = \infty$ since $\delta \mid \epsilon$ in B_1.

Suppose that $\nu^*(\epsilon) = \infty$ and $\nu^*(\delta) < \infty$. We will derive a contradiction. $p_{\overline{m}}$ has the form of (28) and $\overline{T} \rightarrow \overline{T}(1)$ of type I implies $q = p_{\overline{m}}k(\overline{T}(1))[[\overline{z}_1(1),\ldots,\overline{z}_{\overline{r}+\overline{m}}(1)]]$ is a prime which is a complete intersection of height $\tilde{m} = m - \sigma(0)$. As $q \subset p_{\overline{m}}(1)$ and since, by assumption, $\sigma(1) = \sigma(0)$, so that $p_{\overline{m}}(1)$ is a prime ideal of the same height, we must have $q = p_{\overline{m}}(1)$.

By (9),

$$p_{A_1} = \{a \in A_1 \mid \nu^*(a) = \infty\} = p_{\overline{m}}(1)A_1.$$

Since $A_1 \rightarrow B_1$ is finite and $p_{\overline{m}}(1)B_1$ is a prime ideal,

$$p_{B_1} = \{a \in B_1 \mid \nu^*(a) = \infty\} = p_{\overline{m}}(1)B_1.$$

Since B_1 is Galois over A_1, the automorphisms of B_1 over A_1 fix p_{B_1} and $\delta \mid \epsilon$ in B_1, so that some conjugate of δ is in p_{B_1}, we have $\delta \in p_{B_1}$. Thus $\nu^*(\delta) = \infty$, a contradiction. We have completed the verification that

$$\nu^*(h_{[\Lambda]}) = \infty \Leftrightarrow \nu^*(\delta) \Leftrightarrow \nu^*(\epsilon) = \infty.$$

We now continue with our proof of the theorem for $h_{[\Lambda]}$ satisfying (18) or (19) when $\overline{m} \le l$.

By (A2) and (A3) of Theorem 6.2 and Lemma 4.2 of [14] there exists a UTS $\overline{T}(1) \rightarrow \overline{T}(2)$ of form $\overline{m}$ along ν such that

$$\epsilon = \overline{z}_1(2)^{g_1} \cdots \overline{z}_{\overline{r}}(2)^{g_{\overline{r}}} \Sigma$$

where

$$\Sigma \in C = L' * k(\overline{T}(2))[[\overline{z}_1(2), \ldots, \overline{z}_{\overline{r}+\overline{m}}(2)]]$$

with Σ a unit in C if $\nu^*(h_{[\Lambda]}) < \infty$ and

$$\nu(\overline{z}_1(2)^{g_1} \cdots \overline{z}_{\overline{r}}(2)^{g_{\overline{r}}}) > d^{\overline{r}}(A + \sigma)$$

if $\nu^*(h_{[\Lambda]}) = \infty$. The compositium of fields $*$ is defined in Section 2.

We can further assume that $\overline{z}_i(2)$ does not divide Σ if $1 \le i \le \overline{r}$. We have expressions

$$\begin{aligned} \overline{z}_1(1) &= \overline{z}_1(2)^{a_{11}} \cdots \overline{z}_{\overline{r}}(2)^{a_{1\overline{r}}} b_1 \\ &\vdots \\ \overline{z}_{\overline{r}}(1) &= \overline{z}_1(2)^{a_{\overline{r}1}} \cdots \overline{z}_{\overline{r}}(2)^{a_{\overline{r}\overline{r}}} b_{\overline{r}} \end{aligned}$$

with $b_1, \ldots, b_{\overline{r}} \in k(\overline{T}(2))$ and there exist polynomials

$$a_i \in k(\overline{T}(2))[\overline{z}_1(2), \ldots, \overline{z}_{\overline{r}+\overline{m}}(2)] \text{ for } \overline{r}+1 \le i \le \overline{r}+\overline{m}$$

such that $\overline{z}_i(1) = a_i$ for $\overline{r}+1 \le i \le \overline{r}+\overline{m}$. Thus there exists a series in indeterminates $x_1, \ldots, x_{2\overline{r}+\overline{m}}$

$$\delta' \in L' * k(\overline{T}(2))[[x_1, \ldots, x_{2\overline{r}+\overline{m}}]]$$

such that

$$\delta_{j_1 \cdots j_{\overline{r}}} = \delta'(\omega^{j_1} \overline{z}_1(2)^{\frac{a_{11}}{d}} \cdots \overline{z}_{\overline{r}}(2)^{\frac{a_{1\overline{r}}}{d}} b_1^{\frac{1}{d}}, \cdots, \omega^{j_{\overline{r}}} \overline{z}_1(2)^{\frac{a_{\overline{r}1}}{d}} \cdots \overline{z}_{\overline{r}}(2)^{\frac{a_{\overline{r}\overline{r}}}{d}} b_{\overline{r}}^{\frac{1}{d}}, \overline{z}_1(2), \ldots, \overline{z}_{\overline{r}+\overline{m}}(2))$$

for $1 \le j_1, \ldots, j_{\overline{r}} \le d$. Let

$$D = L' * k(\overline{T}(2))[\omega, b_1^{\frac{1}{d^{\overline{r}}}}, \ldots, b_{\overline{r}}^{\frac{1}{d^{\overline{r}}}}][[\overline{z}_1(2)^{\frac{1}{d^{\overline{r}}}}, \ldots, \overline{z}_{\overline{r}}(2)^{\frac{1}{d^{\overline{r}}}}, \overline{z}_{\overline{r}+1}(2), \ldots, \overline{z}_{\overline{r}+\overline{m}}(2)]].$$

We have $\delta_{j_1 \cdots j_{\overline{r}}} \in D$ for all $j_1, \ldots, j_{\overline{r}}$. Since for any natural numbers $a_1, \ldots, a_{\overline{r}}$ we have

$$\overline{z}_1(2)^{\frac{a_1}{d^{\overline{r}}}} \cdots \overline{z}_{\overline{r}}(2)^{\frac{a_{\overline{r}}}{d^{\overline{r}}}} \mid \delta_{j_1 \cdots j_{\overline{r}}} \Leftrightarrow \overline{z}_1(2)^{\frac{a_1}{d^{\overline{r}}}} \cdots \overline{z}_{\overline{r}}(2)^{\frac{a_{\overline{r}}}{d^{\overline{r}}}} \mid \delta$$

in D, we have

$$\overline{z}_1(2)^{\frac{g_1}{d^{\overline{r}}}} \cdots \overline{z}_{\overline{r}}(2)^{\frac{g_{\overline{r}}}{d^{\overline{r}}}} \mid \delta$$

in D, so that we have a factorization

$$\delta = \overline{z}_1(2)^{\frac{g_1}{d^{\overline{r}}}} \cdots \overline{z}_{\overline{r}}(2)^{\frac{g_{\overline{r}}}{d^{\overline{r}}}} \delta''$$

where $\delta'' \in D$ is such that $\overline{z}_i(2)^{\frac{1}{d^{\overline{r}}}}$ does not divide δ'' for $1 \le i \le \overline{r}$ and if $\nu^*(h_{[\Lambda]}) < \infty$, δ'' is a unit.

$\overline{T}(1) \rightarrow \overline{T}(2)$ extends to a CUTS of form $\overline{m}$

$$\begin{array}{ccc} \overline{U}(1) & \rightarrow & \overline{U}(2) \\ \uparrow & & \uparrow \\ \overline{T}(1) & \rightarrow & \overline{T}(2) \end{array}$$

by Lemma 5.1 and 5.2.

$$\begin{aligned}\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}}h_{[\Lambda]} &= \overline{w}_1(1)^{\lambda_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)}\delta\\ &= \overline{w}_1(1)^{\lambda_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)}\overline{z}_1(2)^{\frac{g_1}{d^{\overline{r}}}}\cdots\overline{z}_{\overline{r}}(2)^{\frac{g_{\overline{r}}}{d^{\overline{r}}}}\delta''\end{aligned}$$

in

$$E = L'*k(\overline{U}(2))[\omega, b_1^{\frac{1}{d^{\overline{r}}}},\ldots,b_{\overline{r}}^{\frac{1}{d^{\overline{r}}}},\phi_1(1)^{\frac{1}{d^{\overline{r}}}},\ldots,\phi_{\overline{r}}(1)^{\frac{1}{d^{\overline{r}}}}][[\overline{w}_1(2)^{\frac{1}{d^{\overline{r}}}},\ldots,\overline{w}_{\overline{r}}(2)^{\frac{1}{d^{\overline{r}}}},\overline{w}_{\overline{r}+1}(2),\ldots,\overline{w}_{\overline{r}+\overline{m}}(2)]].$$

Since we necessarily have that

$$\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}} \mid \overline{w}_1(1)^{\lambda_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)}\overline{z}_1(2)^{\frac{g_1}{d^{\overline{r}}}}\cdots\overline{z}_{\overline{r}}(2)^{\frac{g_{\overline{r}}}{d^{\overline{r}}}}$$

in E, there exist $\phi\in k(\overline{U}(2))[\phi_1(1)^{\frac{1}{d^{\overline{r}}}},\ldots,\phi_{\overline{r}}(1)^{\frac{1}{d^{\overline{r}}}}]$ and $e_1,\ldots,e_{\overline{r}}\in\mathbf{N}$ such that

$$\frac{\overline{w}_1(1)^{\lambda_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)}\overline{z}_1(2)^{\frac{g_1}{d^{\overline{r}}}}\cdots\overline{z}_{\overline{r}}(2)^{\frac{g_{\overline{r}}}{d^{\overline{r}}}}}{\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}}} = \overline{w}_1(2)^{\frac{e_1}{d^{\overline{r}}}}\cdots\overline{w}_{\overline{r}}(2)^{\frac{e_{\overline{r}}}{d^{\overline{r}}}}\phi.$$

We thus have that

$$f_1=\frac{e_1}{d^{\overline{r}}},\ldots,f_{\overline{r}}=\frac{e_{\overline{r}}}{d^{\overline{r}}}\in\mathbf{N}$$

and $\delta'''=\phi\delta''\in\overline{U}(2)$ since $h_{[\Lambda]}\in\overline{U}(2)$. Thus

$$h_{[\Lambda]}=\overline{w}_1(2)^{f_1}\cdots\overline{w}_{\overline{r}}(2)^{f_{\overline{r}}}\delta'''$$

in $\overline{U}(2)$. If $\nu^*(h_{[\Lambda]})<\infty$, we have that δ''' is a unit, and if $\nu^*(h_{[\Lambda]})=\infty$, we have that

$$\begin{aligned}\nu^*(\overline{w}_1(2)^{f_1}\cdots\overline{w}_{\overline{r}}(2)^{f_{\overline{r}}}) &= \nu^*(\overline{w}_1(1)^{\lambda_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{\lambda_{\overline{s}}(1)})-\nu^*(\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}})\\ &\quad+\nu^*(\overline{z}_1(2)^{\frac{g_1}{d^{\overline{r}}}}\cdots\overline{z}_{\overline{r}}(2)^{\frac{g_{\overline{r}}}{d^{\overline{r}}}})\\ &> \tfrac{1}{d^{\overline{r}}}[d^{\overline{r}}(A+\sigma)]-\sigma=A.\end{aligned}$$

This concludes the proof of the analysis of f satisfying (18) or (19) when $\overline{m}\le l$.

The proof of the theorem when f satisfies (18) or (19) with $\overline{m}>l$ is, with some obvious notational changes, the same as Case 2 of pages 59 -61 of [14]. The induction on line 9 of page 60 [14] is now on $\overline{m}$ in the conclusions of Theorem 8.1 of this paper.

The proof of the theorem when f satisfies (20) is the same, with obvious notational changes, and after replacing references to (42), (43) and (44) of [14] with (18) and (19) and (20) of this theorem, as "the proof when (44) holds" on pages 61-65 of [14]. ☐

Theorem 8.2. *Suppose that $(R,\overline{T}'',\overline{T})$ and $(S,\overline{U}'',\overline{U})$ is a CUTS along ν^*, such that $\overline{T}''$ contains the subfield $k(c_0)$ for some $c_0\in\overline{T}''$ and $\overline{U}''$ contains a subfield isomorphic to $k(\overline{U})$. $\overline{T}''$ has regular parameters $(\overline{z}_1,\ldots,\overline{z}_m)$ and $\overline{U}''$ has regular parameters $(\overline{w}_1,\ldots,\overline{w}_n)$ with*

$$\begin{aligned}\overline{z}_1 &= \overline{w}_1^{c_{11}}\cdots\overline{w}_{\overline{s}}^{c_{1\overline{s}}}\phi_1\\ &\ \vdots\\ \overline{z}_{\overline{r}} &= \overline{w}_1^{c_{\overline{r}1}}\cdots\overline{w}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\phi_{\overline{r}}\\ \overline{z}_{\overline{r}+1} &= \overline{w}_{\overline{s}+1}\\ &\ \vdots\\ \overline{z}_{\overline{r}+l} &= \overline{w}_{\overline{s}+l}\end{aligned}$$

such that $\phi_1,\ldots,\phi_{\overline{r}}\in k(\overline{U})$, $\nu(\overline{z}_1),\ldots,\nu(\overline{z}_{\overline{r}})$ are rationally independent, $\nu^*(\overline{w}_1),\ldots,\nu^*(\overline{w}_{\overline{s}})$ are rationally independent and (c_{ij}) has rank $\overline{r}$.

Suppose that $\overline{m}>l$ and $f\in\overline{T}''$ is such that

$$f = P + \overline{w}_1^{d_1} \cdots \overline{w}_{\overline{s}}^{d_{\overline{s}}} H \tag{29}$$

for some powerseries $P \in k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_{\overline{s}+l}]]$,

$$H = u(\overline{w}_{\overline{s}+\overline{m}} + \overline{w}_1^{g_1} \cdots \overline{w}_{\overline{s}}^{g_{\overline{s}}} \Sigma)$$

where $u \in k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_{\overline{s}+\overline{m}}]]$ *is a unit,* $\Sigma \in k(\overline{U})[[\overline{w}_1, \ldots, \overline{w}_{\overline{s}+\overline{m}-1}]]$ *and*

$$\nu^*(\overline{w}_1^{g_1} \cdots \overline{w}_{\overline{s}}^{g_{\overline{s}}}) \geq \nu^*(\overline{w}_{\overline{s}+\overline{m}}).$$

Then there exists a CUTS of form $\overline{m}$ *along* ν^* $(R, \overline{T}''(t), \overline{T}(t))$ *and* $(S, \overline{U}''(t), \overline{U}(t))$

$$\begin{array}{ccccccccc} \overline{U} & = & \overline{U}(0) & \to & \overline{U}(1) & \to & \cdots & \to & \overline{U}(t) \\ & & \uparrow & & \uparrow & & & & \uparrow \\ \overline{T} & = & \overline{T}(0) & \to & \overline{T}(1) & \to & \cdots & \to & \overline{T}(t) \end{array} \tag{30}$$

such that $\overline{T}''(i)$ *contains a subfield isomorphic to* $k(c_0, \ldots, c_i)$ *and* $\overline{U}''(i)$ *contains a subfield isomorphic to* $k(\overline{U}(i))$. $\overline{T}''(i)$ *has regular parameters* $(\overline{z}_1(i), \ldots, \overline{z}_m(i))$ *and* $\overline{U}''$ *has regular parameters* $(\overline{w}_1(i), \ldots, \overline{w}_n(i))$ *with*

$$\begin{array}{ll} \overline{z}_1(i) & = \overline{w}_1(i)^{c_{11}(i)} \cdots \overline{w}_{\overline{s}}(i)^{c_{1\overline{s}}(i)} \phi_1(i) \\ & \vdots \\ \overline{z}_{\overline{r}}(i) & = \overline{w}_1(i)^{c_{\overline{r}1}(i)} \cdots \overline{w}_{\overline{s}}(i)^{c_{\overline{r}\overline{s}}(i)} \phi_{\overline{r}}(i) \\ \overline{z}_{\overline{r}+1}(i) & = \overline{w}_{\overline{s}+1}(i) \\ & \vdots \\ \overline{z}_{\overline{r}+l}(i) & = \overline{w}_{\overline{s}+l}(i) \end{array}$$

such that $\phi_1(i), \ldots, \phi_{\overline{r}}(i) \in k(\overline{U}(i))$, $\nu(\overline{z}_1(i)), \ldots, \nu(\overline{z}_{\overline{r}}(i))$ *are rationally independent,* $\nu^*(\overline{w}_1(i)), \ldots, \nu^*(\overline{w}_{\overline{s}}(i))$ *are rationally independent and* $(c_{ij}(i))$ *has rank* $\overline{r}$ *for* $1 \leq i \leq t$.

We further have that

$$f = \overline{P} + \overline{w}_1(t)^{\overline{d}_1} \cdots \overline{w}_{\overline{s}}(t)^{\overline{d}_{\overline{s}}} \overline{H}$$

with $\overline{P} \in k(\overline{U}(t))[\overline{w}_1(t), \ldots, \overline{w}_{\overline{s}+l}(t)]$, *and there exists a finite extension* L *of the algebraic closure of* $k(\overline{T}(t))$ *in* $k(\overline{U}(t))$, *and a positive integer* d, *such that*

$$\overline{P} \in L[\overline{z}_1(t)^{\frac{1}{d}}, \ldots, \overline{z}_{\overline{r}}(1)^{\frac{1}{d}}, \overline{z}_{\overline{r}+1}(1), \ldots, \overline{z}_{\overline{r}+l}(1)],$$

$$\overline{H} = \overline{u}(\overline{w}_{\overline{s}+\overline{m}}(t) + \overline{w}_1(t)^{\overline{g}_1} \cdots \overline{w}_{\overline{s}}(t)^{\overline{g}_{\overline{s}}} \overline{\Sigma})$$

where $\overline{u} \in k(\overline{U}(t))[[\overline{w}_1(t), \ldots, \overline{w}_{\overline{s}+\overline{m}}(t)]]$ *is a unit,* $\overline{\Sigma} \in k(\overline{U}(t))[[\overline{w}_1(t), \ldots, \overline{w}_{\overline{s}+\overline{m}-1}(t)]]$ *and* $\nu^*(\overline{w}_1(t)^{\overline{g}_1} \cdots \overline{w}_{\overline{s}}(t)^{\overline{g}_{\overline{s}}}) \geq \nu^*(\overline{w}_{\overline{s}+\overline{m}}(t))$.

Proof. **Step 1.** In this step we perform CUTS to achieve that in (29), $P \in k(\overline{U})[\overline{w}_1, \ldots, \overline{w}_{\overline{s}+l}]$, and

$$P = \sum_{[\Lambda] \in \mathbf{Z}^{\overline{s}}/\mathbf{Q}^{\overline{r}} C \cap \mathbf{Z}^{\overline{s}}} h_{[\Lambda]}$$

with $\nu^*(h_{[\Lambda]}) \leq \rho$ for all Λ.

Set $\rho = \nu^*(\overline{w}_1^{d_1} \cdots \overline{w}_{\overline{s}}^{\overline{d}_{\overline{s}}})$. There is an expression

$$P = \sum_{\Lambda \in \mathbf{Z}^{\overline{s}}/(\mathbf{Q}^{\overline{r}} C) \cap \mathbf{Z}^{\overline{s}}} h_{[\Lambda]} \tag{31}$$

of the form (12). We can rewrite as

$$P=\sum_{\nu^*(h_{[\Lambda]})\le\rho}h_{[\Lambda]}+\Omega$$

with $\nu^*(\Omega)>\rho$. For each of the finitely many Λ with $\nu^*(h_{[\Lambda]})\le\rho$ (c.f. Lemma 2.3 [14]), we can further write

$$h_{[\Lambda]}=\overline{h}_{[\Lambda]}+\Omega_{[\Lambda]}$$

with $\nu^*(\Omega_{[\Lambda]})>\rho$ and $\overline{h}_{[\Lambda]}\in k(\overline{U})[\overline{w}_1,\ldots,\overline{w}_{\overline{s}+l}]$ of the form (13). Set

$$\overline{P}_1=\sum\overline{h}_{[\Lambda]},$$

$$\overline{P}_2=\sum\Omega_{[\Lambda]}+\Omega,$$

$$P=\overline{P}_1+\overline{P}_2.$$

Observe that $\nu^*(\overline{h}_{[\Lambda]})=\nu^*(h_{[\Lambda]})\le\rho$ for all Λ in the (finite) sum $\overline{P}_1$, and $\nu^*(\overline{P}_2)>\rho$.

By Theorem 8.1 applied to $\overline{P}_2$ in equations (19) or (20) and by Lemma 4.2 [14], there exists a CUTS $(R,\overline{T}''(1),\overline{T}(1))$ and $(S,\overline{U}''(1),\overline{U}(1))$ along ν^* of form l

$$\begin{array}{ccc}\overline{U}&\to&\overline{U}(1)\\ \uparrow&&\uparrow\\ \overline{T}&\to&\overline{T}(1)\end{array}$$

such that $\overline{P}_2=\overline{w}_1(1)^{e_1}\cdots\overline{w}_{\overline{s}}(1)^{e_{\overline{s}}}\Phi$ with $\Phi\in k(\overline{U}(1))[[\overline{w}_1(1),\ldots,\overline{w}_{\overline{s}+l}(1)]]$, and

$$\overline{w}_1^{d_1}\cdots\overline{w}_{\overline{s}}^{d_{\overline{s}}}=\overline{w}_1(1)^{d_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{d_{\overline{s}}(1)}$$

with $e_i>d_i(1)$ for all i. Thus

$$f=\overline{P}_1+\overline{w}_1(1)^{d_1(1)}\cdots\overline{w}_{\overline{s}}(1)^{d_{\overline{s}}(1)}\Omega'$$

where

$$\Omega'\in m(k(\overline{U}(1))[[\overline{w}_1(1),\ldots,\overline{w}_{\overline{s}+\overline{m}}(1)]]$$

and

$$\frac{\partial\Omega'}{\partial w_{\overline{s}+\overline{m}}}\not\in m\left(k(\overline{U}(1))[[\overline{w}_1(1),\ldots,\overline{w}_{\overline{s}+\overline{m}}(1)]]\right).$$

By the implicit function theorem (the case s=1 of the Weierstrass Preparation Theorem, Corollary 1 to Theorem 5, Section 1, Chapter VII [39]),

$$\Omega'=u'(\overline{w}_{\overline{s}+\overline{m}}(1)+\overline{\Sigma})$$

with $u'\in k(\overline{U}(1))[[\overline{w}_1(1),\ldots,\overline{w}_{\overline{s}+\overline{m}}(1)]]$ a unit, $\overline{\Sigma}\in k(\overline{U}(1))[[\overline{w}_1(1),\ldots,\overline{w}_{\overline{s}+\overline{m}-1}(1)]]$.

After replacing $\overline{w}_{\overline{s}+\overline{m}}(1)=\overline{w}_{\overline{s}+\overline{m}}$ with $\overline{w}_{\overline{s}+\overline{m}}+\Psi$ with

$$\Psi\in k(\overline{U}(1))[\overline{w}_1(1),\ldots,\overline{w}_{\overline{s}+\overline{m}-1}(1)]\subset\overline{U}''(1),$$

we may assume that

$$\overline{\Sigma}\in(\overline{w}_1(1),\ldots,\overline{w}_{\overline{s}+\overline{m}-1}(1))^B$$

where B is arbitrarily large. If $\nu^*(\Omega')<\infty$, we can choose B so large that

$$\nu^*(\Omega')=\nu^*(\overline{w}_{\overline{s}+\overline{m}}(1))<\nu^*(\overline{\Sigma}).$$

If $\nu^*(\Omega')=\infty$, we have $\nu^*(\overline{\Sigma})=\nu^*(\overline{w}_{\overline{s}+\overline{m}}(1))<\infty$. Then by cases (18) and (19) of Theorem 8.1, we can perform a CUTS of the form $\overline{m}-1$,

$$\begin{array}{ccc}\overline{U}(1)&\to&\overline{U}(2)\\ \uparrow&&\uparrow\\ \overline{T}(1)&\to&\overline{T}(2)\end{array}$$

to get

$$f=\overline{P}_1+\overline{w}_1(2)^{d_1(2)}\cdots\overline{w}_{\overline{s}}^{d_{\overline{s}}(2)}H'$$

where

$$H'=u'(\overline{w}_{\overline{s}+\overline{m}}(2)+\overline{w}_1(2)^{g_1(2)}\cdots\overline{w}_{\overline{s}}(1)^{g_{\overline{s}}(2)}\Sigma')$$

with u' a unit and $\nu^*(\overline{w}_1(1)^{g_1(2)}\cdots\overline{w}_{\overline{s}}(1)^{g_{\overline{s}}(2)})\ge\nu^*(\overline{w}_{\overline{s}+\overline{m}}(2))$.

Thus we may assume that in (29), $P\in k(\overline{U})[\overline{w}_1,\ldots,\overline{w}_{\overline{s}+l}]$, and

$$P=\sum_{[\Lambda]\in\mathbf{Z}^{\overline{s}}/\mathbf{Q}^{\overline{r}}C\cap\mathbf{Z}^{\overline{s}}}h_{[\Lambda]}$$

with $\nu^*(h_{[\Lambda]})\le\rho$ for all Λ.

Step 2. In this step we perform a CUTS to achieve that in (29), $P\in k(\overline{U})[\overline{w}_1,\ldots,\overline{w}_{\overline{s}+l}]$ is such that

$$P\in k(\overline{U})[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}}][\overline{z}_1^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}^{\frac{1}{d}},\overline{z}_{\overline{r}+1},\ldots,\overline{z}_{\overline{r}+l}]$$

and $\nu^*(P)\le\rho$.

Recall that $\nu^*(f)<\infty$ since $f\in\overline{T}''$. If $P=0$ we have proven the Theorem. Thus we may assume that $P\ne 0$. There exists Λ_0 such that $\nu^*(f)=\nu^*(P)=\nu^*(h_{[\Lambda_0]})$ where Λ_0 is such that

$$\nu^*(h_{[\Lambda_0]})=\ \min\ \{\nu^*(h_{[\Lambda]})\}$$

by Remark 7.4. Thus $[\Lambda_0]=0$ by Lemma 7.3. By Remark 7.1, there exists a CUTS of type (M1),

$$\begin{array}{ccc}\overline{U} & \to & \overline{U}(1)\\ \uparrow & & \uparrow\\ \overline{T} & \to & \overline{T}(1)\end{array}$$

and $d\in\mathbf{N}$, such that

$$h_0=h_{[\Lambda_0]}\in k(\overline{U})[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}}][\overline{z}_1(1)^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}(1)^{\frac{1}{d}},\overline{z}_{\overline{r}+1}(1),\ldots,\overline{z}_{\overline{r}+l}(1)].$$

Let

$$A_1=k(\overline{U})[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}}][[\overline{w}_1(1)^{\frac{1}{d}},\ldots,\overline{w}_{\overline{s}}(1)^{\frac{1}{d}},\overline{w}_{\overline{s}+1}(1),\ldots,\overline{w}_n(1)]].$$

Extend $\tilde{\nu}^*$ to the finite extension $Q(A_1)$ so that it dominates A_1. Let

$$B_1=\overline{T}(1)''\otimes_k k(t_1,\ldots,t_{\overline{\alpha}}).$$

Let $\nu'=\tilde{\nu}^*\mid Q(B_1)$. By Lemma 3.1, ν' is a rank one valuation with value group Γ_ν, since it extends the rank 1 valuation $\nu\mid Q(\overline{T}(1)'')$.

Let k' be the algebraic closure of k in $\overline{T}(1)''$, and let

$$C=\overline{T}(1)''\otimes_{k'}k(\overline{U}(1))[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}},\overline{z}_1^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}^{\frac{1}{d}}].$$

Let $\hat{\nu}$ be the restriction to $Q(C)$ of an extension of $\tilde{\nu}^*$ to $Q(A_1)$. $\hat{\nu}$ dominates C. Since C is finite over B_1, $\hat{\nu}$ has rank 1 and $\Gamma_{\hat{\nu}}\subset\Gamma_\nu\otimes\mathbf{Q}$.

If $P\ne h_0$, then there exists $h_{[\Lambda_1]}$ such that $\Lambda_1\notin\mathbf{Q}^{\overline{r}}C\cap\mathbf{Z}^{\overline{s}}$ and

$$\nu^*(f-h_0)=\nu^*(h_{[\Lambda_1]}).$$

$f-h_0\in C$ implies

$$\nu^*(f-h_0)\in\Gamma_\nu\otimes\mathbf{Q}.\tag{32}$$

But

$$\nu^*(h_{[\Lambda_1]})\notin\Gamma_\nu\otimes\mathbf{Q}$$

by Lemma 7.3, since $\Lambda_1\notin\mathbf{Q}^{\overline{r}}C\cap\mathbf{Z}^{\overline{s}}$, a contradiction.

Thus we may assume that in (29), $P\in k(\overline{U})[\overline{w}_1,\ldots,\overline{w}_{\overline{s}+l}]$ is such that

$$P\in k(\overline{U})[\phi_1^{\frac{1}{d}},\ldots,\phi_{\overline{r}}^{\frac{1}{d}}][\overline{z}_1^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}^{\frac{1}{d}},\overline{z}_{\overline{r}+1},\ldots,\overline{z}_{\overline{r}+l}] \tag{33}$$

and $\nu^*(P)\le\rho$.

Step 3. In this final step we perform a CUTS to achieve the conclusions of the theorem.

Recall that $t_1,\ldots,t_{\overline{\alpha}}$ is a transcendence basis of $k(\overline{U})$ over $k(\overline{T})$. Write

$$P=\sum a_I\overline{w}_1^{b_1(I)}\cdots\overline{w}_{\overline{s}+l}^{b_{\overline{s}+l}(I)} \tag{34}$$

with $a_I\in k(\overline{U})$. Let $a_{I_1},\ldots,a_{I_\gamma}$ be the (finitely many) nonzero terms in the sum. Let A be the integral closure of

$$k(\overline{T})[t_1,\ldots,t_{\overline{\alpha}},a_{I_1},\ldots,a_{I_\gamma},\phi_1,\frac{1}{\phi_1},\ldots,\phi_{\overline{r}},\frac{1}{\phi_{\overline{r}}}]$$

in $k(\overline{U})$. There exists an algebraic regular local ring B of $k(\overline{U})$ such that B dominates A, and the residue field of B is finite over $k(\overline{T})$ (c.f. Theorem 2.9 [14]).

Let $(v_1,\ldots,v_{\overline{\alpha}})$ be a regular system of parameters in B. We have an inclusion

$$B\rightarrow\hat{B}=L_1[[v_1,\ldots,v_{\overline{\alpha}}]]$$

where $L_1=k(B)$ is a finite extension of $k(\overline{T})$.

After reindexing $\overline{w}_1,\ldots,\overline{w}_{\overline{s}}$, we may assume that the matrix

$$\overline{C}=\begin{pmatrix} c_{11} & \cdots & c_{1\overline{r}}\\ \vdots & & \vdots\\ c_{\overline{r}1} & \cdots & c_{\overline{r}\overline{r}}\end{pmatrix}$$

has positive determinant $e=\det(\overline{C})$. Let

$$\overline{B}=(b_{ij})=\frac{1}{de}\mathrm{adj}\overline{C}=\frac{1}{d}\overline{C}^{-1}.$$

Let

$$\psi_i=\phi_1^{b_{i1}}\cdots\phi_{\overline{r}}^{b_{i\overline{r}}}$$

for $1\le i\le\overline{r}$, and let

$$\tilde{w}_i=\begin{cases}\psi_i^d\overline{w}_i & 1\le i\le\overline{r}\\ \overline{w}_i & \overline{r}<i\le\overline{s}.\end{cases}$$

We then have equations

$$\begin{aligned}\overline{z}_1&=\tilde{w}_1^{c_{11}}\cdots\tilde{w}_{\overline{s}}^{c_{1\overline{s}}}\\ &\vdots\\ \overline{z}_{\overline{r}}&=\tilde{w}_1^{c_{\overline{r}1}}\cdots\tilde{w}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\\ \overline{z}_{\overline{r}+1}&=\overline{w}_{\overline{s}+1}\\ &\vdots\\ \overline{z}_{\overline{r}+l}&=\overline{w}_{\overline{s}+l}.\end{aligned}$$

$\psi_i^{de}\in k(\overline{U})$ for $1\le i\le\overline{r}$. $\psi_i^{de}\in\hat{B}$ has residue $0\ne\lambda_i\in L_1$. Let $L''=L_1[\lambda_1^{\frac{1}{de}},\ldots,\lambda_{\overline{r}}^{\frac{1}{de}}]$, $E=L''[[v_1,\ldots,v_{\overline{\alpha}}]]$. Since

$$\frac{\psi_i^{de}}{\lambda_i}$$

has residue 1 in E, there exists a de-th root σ_i of

$$\frac{\psi_i^{de}}{\lambda_i}$$

in E, with residue 1 in L'' for $1 \le i \le \overline{r}$. Thus $\psi_i = \lambda_i^{\frac{1}{de}}\sigma_i \in E$ for $1 \le i \le \overline{r}$. Note that

$$\psi_1^{c_{i1}} \cdots \psi_{\overline{r}}^{c_{i\overline{r}}} = \phi_i^{\frac{1}{d}}$$

for $1 \le i \le \overline{r}$, so that we have natural inclusions

$$k(\overline{U}) \subset k(\overline{U})[\phi_1^{\frac{1}{d}}, \ldots, \phi_{\overline{r}}^{\frac{1}{d}}] \subset k(\overline{U})[\psi_1, \ldots, \psi_{\overline{r}}] \subset Q(E).$$

Let $\tau_1, \ldots, \tau_{\overline{\alpha}} \in \mathbf{R}_+$ be rationally independent. Let $\overline{\nu}$ be the L''-valuation on the quotient field of E defined by $\overline{\nu}(v_i) = \tau_i$ for $1 \le i \le \overline{\alpha}$. Let

$$A_2 = Q(E)[\overline{w}_1^{\frac{1}{d}}, \ldots, \overline{w}_{\overline{s}}^{\frac{1}{d}}, \overline{w}_{\overline{s}+1}, \cdots, \overline{w}_n]_{(\overline{w}_1^{\frac{1}{d}}, \ldots, \overline{w}_{\overline{s}}^{\frac{1}{d}}, \overline{w}_{\overline{s}+1}, \cdots, \overline{w}_n)}.$$

Let w be an extension of ν^* to $Q(A_2)$ which dominates A_2. Any element of A_2 is a quotient of elements of the form $g = \frac{\overline{x}}{\overline{a}}$ with $0 \ne \overline{a} \in L''[[v_1, \ldots, v_{\overline{\alpha}}]]$ and

$$\begin{aligned}\overline{x} &\in L''[[v_1, \ldots, v_{\overline{\alpha}}]][\overline{w}_1^{\frac{1}{d}}, \ldots, \overline{w}_{\overline{s}}^{\frac{1}{d}}, \overline{w}_{\overline{s}+1}, \ldots, \overline{w}_n] \\ &= L''[\overline{w}_1^{\frac{1}{d}}, \ldots, \overline{w}_{\overline{s}}^{\frac{1}{d}}, \overline{w}_{\overline{s}+1}, \ldots, \overline{w}_n][[v_1, \ldots, v_{\overline{\alpha}}]].\end{aligned}$$

We have that the value group of $w \mid Q(L''[\overline{w}_1^{\frac{1}{d}}, \ldots, \overline{w}_{\overline{s}}^{\frac{1}{d}}, \overline{w}_{\overline{s}+1}, \ldots, \overline{w}_n])$ is contained in $\Gamma_{\nu^*} \otimes \mathbf{Q}$ and thus $w(g) = w(\overline{x}) \in \Gamma_{\nu^*} \otimes \mathbf{Q}$ by Lemma 3.2. We further have that $k(V_w)$ is an algebraic extension of $Q(E)$. We thus have $\Gamma_w \subset \Gamma_{\nu^*} \otimes \mathbf{Q}$. Identify $\overline{\nu}$ with an extension of $\overline{\nu}$ to $k(V_w)$. Let $\tilde{\nu}'$ be the composite $w \circ \overline{\nu}$ of w and $\overline{\nu}$.

$$\tilde{\nu}'(v_1), \ldots, \tilde{\nu}'(v_{\overline{\alpha}}), \tilde{\nu}'(\tilde{w}_1), \ldots, \tilde{\nu}'(\tilde{w}_{\overline{s}})$$

is a rational basis of $\Gamma_{\tilde{\nu}'} \otimes \mathbf{Q}$. We have an equality

$$Q(E)[\overline{w}_1^{\frac{1}{d}}, \ldots, \overline{w}_{\overline{s}}^{\frac{1}{d}}, \overline{w}_{\overline{s}+1}, \ldots, \overline{w}_n] = Q(E)[\tilde{w}_1^{\frac{1}{d}}, \ldots, \tilde{w}_{\overline{s}}^{\frac{1}{d}}, \overline{w}_{\overline{s}+1}, \ldots, \overline{w}_n].$$

Let $F = k(\overline{T})[\overline{z}_1, \ldots, \overline{z}_m]$, $G = L''[\overline{z}_1^{\frac{1}{d}}, \ldots, \overline{z}_{\overline{r}}^{\frac{1}{d}}, \overline{z}_{\overline{r}+1}, \ldots, \overline{z}_m]$. The restriction of w to $Q(F)$ is the restriction of $\tilde{\nu}$ to $Q(F)$ so $w \mid Q(F)$ has residue field which is an algebraic extension of $k(\overline{T})$.

Now $Q(G)$ is finite over $Q(F)$, so that the restriction of w to $Q(G)$ has a residue field which is an algebraic extension of $k(\overline{T})$.

Thus if $h \in Q(G)$ is such that $w(h) = 0$, then if $[h]$ is the residue class of h in $k(V_w)$, we have that $[h]$ is contained in the algebraic closure M of $k(\overline{T})$ in $k(V_w)$, and thus by Lemma 1, Section 11, Chapter VI [39], $\overline{\nu}([h]) = 0$, since $\overline{\nu}$ is a $k(\overline{T})$ valuation.

For $I = (i_1, \ldots, i_{\overline{\alpha}}) \in \mathbf{N}^{\overline{\alpha}}$, let v^I denote $v_1^{i_1} \cdots v_{\overline{\alpha}}^{i_{\overline{\alpha}}}$. By (33) and (34), there is a series expansion

$$P = \sum_I g_I v^I$$

with each

$$g_I \in L''[\overline{z}_1^{\frac{1}{d}}, \ldots, \overline{z}_{\overline{r}}^{\frac{1}{d}}, \overline{z}_{\overline{r}+1}, \ldots, \overline{z}_{\overline{r}+l}].$$

Suppose that $\tilde{\nu}'(g_I v^I) = \tilde{\nu}'(g_J v^J)$. Then

$$\tilde{\nu}'(\frac{g_I}{g_J}) = \tilde{\nu}'(\frac{v^J}{v^I})$$

so that $w(\frac{g_I}{g_J}) = 0$. Thus

$$\tilde{\nu}'(\frac{g_I}{g_J}) = \overline{\nu}(\frac{g_I}{g_J}) = 0$$

since $\frac{g_I}{g_J}\in Q(G)$. We have $\overline{\nu}(\frac{v^J}{v^I})=0$, so that $I=J$. Thus

$$\tilde\nu'(g_Iv^I)=\tilde\nu'(g_Jv^J)\Leftrightarrow I=J. \tag{35}$$

Let $N=k(\overline{T})[\overline{z}_1,\ldots,\overline{z}_m]$. We will now establish that if $0\ne h\in Q(N)$, then

$$\tilde\nu'(h)\in \mathbf{Q}\tilde\nu'(\overline{z}_1)+\cdots+\mathbf{Q}\tilde\nu'(\overline{z}_{\overline{r}}). \tag{36}$$

To establish (36), we first observe that since $\nu(h)<\infty$, there is a UTS $\overline{T}\rightarrow\overline{T}_1$ along ν, such that $\overline{T}_1$ has regular parameters $(\overline{z}_1(1),\ldots,\overline{z}_m(1))$ and

$$h=\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}}u$$

where $a_1,\ldots,a_{\overline{r}}\in\mathbf{Z}$, $u\in\overline{T}_1$ is a unit, and

$$\begin{aligned}\overline{z}_1&=\overline{z}_1(1)^{b_{11}}\cdots\overline{z}_{\overline{r}}(1)^{b_{1\overline{r}}}c_1\\ &\vdots\\ \overline{z}_{\overline{r}}&=\overline{z}_1(1)^{b_{\overline{r}1}}\cdots\overline{z}_{\overline{r}}(1)^{b_{\overline{r}}}c_{\overline{r}}\end{aligned}$$

with $b_{ij}\in\mathbf{N}$ for all i,j, Det $(b_{ij})\ne 0$ and $c_1,\ldots,c_{\overline{r}}\in k(\overline{T}_1)$.

We identify $\tilde\nu'$ with an extension of $\tilde\nu'$ to $Q(\hat A_2)$ which dominates $\hat A_2$. Now $\overline{T}\rightarrow\overline{T}_1$ is also a UTS along $\tilde\nu'$ since the center of $\tilde\nu'$ on a UTS of $\overline{T}$ must be the center of ν.

We thus have that

$$\begin{aligned}\tilde\nu'(h)&=\tilde\nu'(\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}})+\overline{\nu}(u)\\ &=\tilde\nu'(\overline{z}_1(1)^{a_1}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}}})\in\mathbf{Q}\tilde\nu'(\overline{z}_1)+\cdots+\mathbf{Q}\tilde\nu'(\overline{z}_{\overline{r}})\end{aligned}$$

Here $\overline{\nu}(u)=0$ since $\nu(u)=0$ and $k(V_\nu)$ is algebraic over k. We have thus established (36).

Since $Q(G)$ is a finite extension of $Q(N)$, we have that

$$\tilde\nu'(h)\in\mathbf{Q}\tilde\nu'(\overline{z}_1)+\cdots+\mathbf{Q}\tilde\nu'(\overline{z}_{\overline{r}})\text{ if }0\ne h\in Q(G).$$

Since $L''[\overline{z}_1^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}^{\frac{1}{d}},\overline{z}_{\overline{r}+1},\ldots,\overline{z}_{\overline{r}+l}][[v_1,\ldots,v_{\overline{\alpha}}]]$ is a Noetherian ring, there exists I_0 such that

$$\tilde\nu'(g_{I_0}v^{I_0})=\ \min\ \{\tilde\nu'(g_Iv^I)\mid g_I\ne 0\}.$$

We necessarily have that $\tilde\nu'(P)=\tilde\nu'(g_{I_0}v^{I_0})$ by (35). Thus

$$\tilde\nu'(P)\in\mathbf{Q}\tilde\nu'(\overline{z}_1)+\cdots+\mathbf{Q}\tilde\nu'(\overline{z}_{\overline{r}})$$

if and only if $I_0=0$.

Write $P=P_1+P_2$ where

$$P_1=\sum_{w(g_I)\le\rho}g_Iv^I,$$

$$P_2=\sum_{w(f_{g_I})>\rho}g_Iv^I$$

(each sum is possibly infinite). If $P_1\ne 0$, let I_0 be such that $\tilde\nu'(v^{I_0}g_{I_0})=\tilde\nu'(P_1)$. Then $\tilde\nu'(v^{I_0}g_{I_0})=\nu(f)\in\Gamma_\nu\otimes\mathbf{Q}$ implies

$$\tilde\nu'(v^{I_0})\in\mathbf{Q}\tilde\nu'(\overline{z}_1)+\cdots+\mathbf{Q}\tilde\nu'(\overline{z}_{\overline{r}})\subset\tilde\nu'(\tilde w_1)\mathbf{Q}+\cdots+\tilde\nu'(\tilde w_s)\mathbf{Q}$$

which implies $I_0=0$. Thus $v^{I_0}g_{I_0}=g_{I_0}\in L''[\overline{z}_1^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}^{\frac{1}{d}},\overline{z}_{\overline{r}+1},\ldots,\overline{z}_{\overline{r}+l}]$. Now $\nu(f-g_{I_0})\in\Gamma_\nu\otimes\mathbf{Q}$ implies $P_1=g_{I_0}$.

In the sum P_2, let $g_1,\ldots,g_\beta$ be generators of the ideal

$$(g_I\mid P_2=\sum g_Iv^I)\subset L''[\overline{z}_1^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}^{\frac{1}{d}},\overline{z}_{\overline{r}+1},\ldots,\overline{z}_{\overline{r}+l}].$$

Let ω be a primitive dth root of unity. Let

$$d_j=\prod_{i_1,\ldots,i_{\overline{r}}=1}^{d} g_j(\omega^{i_1}\overline{z}_1^{\frac{1}{d}},\ldots,\omega^{i_{\overline{r}}}\overline{z}_{\overline{r}}^{\frac{1}{d}},\overline{z}_{\overline{r}+1},\ldots,\overline{z}_{\overline{r}+l})\in L''[\overline{z}_1,\ldots,\overline{z}_{\overline{r}+l}]$$

for $1\le j\le\beta$. The d_j are of the form of (A2) of Theorem 6.2 with $\overline{m}=l$.

Now apply Theorem 6.2 to d_j for $1\le j\le\beta$ (and Lemmas 5.1 and 5.2) to construct a CUTS of form l along ν^*,

$$\begin{array}{ccc}\overline{T}(1) & \rightarrow & \overline{U}(1)\\ \uparrow & & \uparrow\\ \overline{T} & \rightarrow & \overline{U}\end{array}\tag{37}$$

so that for $1\le j\le\beta$,

$$d_j=\overline{z}_1(1)^{e_1(j)}\cdots\overline{z}_{\overline{r}}(1)^{e_{\overline{r}}(j)}u_j$$

with u_j a unit in $k(T(1))*L''[\overline{z}_1(1),\ldots,\overline{z}_{\overline{r}+l}(1)]$. The compositium of fields $*$ is defined in Section 2. We have

$$\begin{array}{ll}\overline{z}_1 & =\overline{z}_1(1)^{a_{11}}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}1}}c_1\\ \vdots & \\ \overline{z}_r & =\overline{z}_1(1)^{a_{11}}\cdots\overline{z}_{\overline{r}}(1)^{a_{\overline{r}\overline{r}}}c_{\overline{r}}\end{array}$$

for some $c_1,\ldots,c_{\overline{r}}\in k(\overline{T}(1))$.

We have

$$P_1\in D=L''*k(\overline{T}(1))[c_1^{\frac{1}{d}},\ldots,c_{\overline{r}}^{\frac{1}{d}}][\overline{z}_1(1)^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}(1)^{\frac{1}{d}},\overline{z}_{\overline{r}+1}(1),\ldots,\overline{z}_{\overline{r}+l}(1)].$$

g_j divides d_j in D implies $g_j=\overline{w}_1(1)^{\overline{e}_1(j)}\cdots\overline{w}_{\overline{s}}(1)^{\overline{e}_{\overline{s}}(j)}f_j$ in

$$M=L''*k(\overline{U}(1))[c_1^{\frac{1}{d}},\ldots,c_{\overline{r}}^{\frac{1}{d}},\phi_1(1)^{\frac{1}{d}},\ldots,\phi_{\overline{r}}(1)^{\frac{1}{d}}][\overline{w}_1(1)^{\frac{1}{d}},\ldots,\overline{w}_{\overline{s}}(1)^{\frac{1}{d}},\overline{w}_{\overline{s}+1}(1),\ldots,\overline{w}_{\overline{s}+l}(1)]$$

for $1\le j\le d$ where f_j is a unit, $\overline{e}_i(j)\in\frac{1}{d}\mathbf{N}$, and $\nu^*(\overline{w}_1(1)^{\overline{e}_1(j)}\cdots\overline{w}_{\overline{s}}(1)^{\overline{e}_{\overline{s}}(j)})>\rho$ for all $1\le j\le\beta$, where ν^* is identified with an extension of ν^* to $Q(M)$ which dominates M.

There exists a CUTS of type (M1) along ν^*

$$\begin{array}{ccc}\overline{U}(1) & \rightarrow & \overline{U}(2)\\ \uparrow & & \uparrow\\ \overline{T}(1) & = & \overline{T}(2)\end{array}$$

so that if

$$\overline{w}_1^{d_1}\cdots\overline{w}_{\overline{s}}^{d_{\overline{s}}}=\overline{w}_1(2)^{d_1(2)}\cdots\overline{w}_{\overline{s}}(2)^{d_{\overline{s}}(2)},$$

$\overline{w}_1(2)^{d_1(2)+1}\cdots\overline{w}_{\overline{s}}(2)^{d_{\overline{s}}(2)+1}$ divides P_2 in

$$F=L''*k(\overline{U}(2))[c_1^{\frac{1}{d}},\ldots,c_{\overline{r}}^{\frac{1}{d}},\phi_1(1)^{\frac{1}{d}},\ldots,\phi_{\overline{r}}(1)^{\frac{1}{d}}][\overline{w}_1(2)^{\frac{1}{d}},\ldots,\overline{w}_{\overline{s}}(2)^{\frac{1}{d}},\overline{w}_{\overline{s}+1}(2),\ldots,\overline{w}_n(2)].$$

$$P=P_1+\overline{w}_1(2)^{d_1(2)+1}\cdots\overline{w}_{\overline{s}}(2)^{d_{\overline{s}}(2)+1}\Phi$$

with

$$P_1\in F'=L''*k(\overline{T}(2))[c_1^{\frac{1}{d}},\ldots,c_{\overline{r}}^{\frac{1}{d}}][\overline{z}_1(2)^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}(2)^{\frac{1}{d}},\overline{z}_{\overline{r}+1}(2),\ldots,\overline{z}_{\overline{r}+l}(2)].$$

We can thus rewrite this equation to get

$$P=\tilde{P}_1+\overline{w}_1(2)^{d_1(2)+1}\cdots\overline{w}_{\overline{s}}(2)^{d_{\overline{s}}(2)+1}\tilde{\Phi}$$

with $\tilde{P}_1\in F',\tilde{\Phi}\in F$ and where $\overline{w}_1(2)^{d_1(2)+1}\cdots\overline{w}_{\overline{s}}(2)^{d_{\overline{s}}(2)+1}$ does not divide any monomial in the expansion of $\tilde{P}_1$ in F.

Comparing with the extension of the expansion of P in $k(\overline{U}(2))[[\overline{w}_1(2),\ldots,\overline{w}_{\overline{s}+l}(2)]]$ to the expansion in $\hat{F}$, we see that

$$\tilde{\Phi}\in k(\overline{U}(2))[[\overline{w}_1(2),\ldots,\overline{w}_{\overline{s}+l}(2)]]\cap F=k(\overline{U}(2))[\overline{w}_1(2),\ldots,\overline{w}_{\overline{s}+l}(2)]$$

and

$$\tilde{P}_1\in k(\overline{U}(2))[[\overline{w}_1(2),\ldots,\overline{w}_{\overline{s}+l}(2)]]\cap F'\subset L[\overline{z}_1(2)^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}(2)^{\frac{1}{d}},\overline{z}_{\overline{r}+1}(2),\ldots,\overline{z}_{\overline{r}+l}(2)]$$

where L is a finite extension of the algebraic closure of $k(\overline{T}(2))$ in $k(\overline{U}(2))$. As in the first part of this proof, we can make a change of variables in $\overline{w}_{\overline{s}+\overline{m}}(2)$ to get the conclusions of the theorem. □

9. Conclusion of the proof for rank 1 valuations

In this section, assumptions and notations will be as in Section 4.

Theorem 9.1. *Suppose that $T''(0)\subset\hat{R}$ is a regular local ring, essentially of finite type over R such that the quotient field of $T''(0)$ is finite over K, $U''(0)\subset\hat{S}$ is a regular local ring, essentially of finite type over S such that the quotient field of $U''(0)$ is finite over K^*, $U''(0)$ dominates $T''(0)$, $T''(0)$ contains a subfield isomorphic to $k(c_0)$, for some $c_0\in k(T''(0))$, $U''(0)$ contains a subfield isomorphic to $k(U''(0))$. Suppose that R has regular parameters $(x_1,\ldots,x_m)$, S has regular parameters $(y_1,\ldots,y_n)$, $T''(0)$ has regular parameters $(\tilde{\overline{x}}_1,\ldots,\tilde{\overline{x}}_m)$ and $U''(0)$ has regular parameters $(\tilde{\overline{y}}_1,\ldots,\tilde{\overline{y}}_n)$ such that*

$$\begin{aligned}
\tilde{\overline{x}}_1 &= \tilde{\overline{y}}_1^{c_{11}}\cdots\tilde{\overline{y}}_{\overline{s}}^{c_{1\overline{s}}}\phi_1\\
&\ \vdots\\
\tilde{\overline{x}}_{\overline{r}} &= \tilde{\overline{y}}_1^{c_{\overline{r}1}}\cdots\tilde{\overline{y}}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\phi_{\overline{r}}\\
\tilde{\overline{x}}_{\overline{r}+1} &= \tilde{\overline{y}}_{\overline{s}+1}\\
&\ \vdots\\
\tilde{\overline{x}}_{\overline{r}+l} &= \tilde{\overline{y}}_{\overline{s}+l}
\end{aligned}$$

where $\phi_1,\ldots,\phi_{\overline{r}}\in k(U''(0))$, $\nu(\tilde{\overline{x}}_1),\ldots,\nu(\tilde{\overline{x}}_{\overline{r}})$ are rationally independent, $\nu^*(\tilde{\overline{y}}_1),\ldots,\nu^*(\tilde{\overline{y}}_{\overline{s}})$ are rationally independent and (c_{ij}) has rank $\overline{r}$.

Suppose that there exists an algebraic regular local ring $\tilde{R}\subset R$ such that $(x_1,\ldots,x_{\overline{r}+l})$ are regular parameters in $\tilde{R}$, $k(\tilde{R})\cong k(c_0)$ and

$$x_i=\begin{cases}\gamma_i\tilde{\overline{x}}_i & 1\le i\le\overline{r}+l\\ \overline{x}_i & \overline{r}+l<i\le m\end{cases}$$

with $\gamma_i\in k(c_0)[[x_1,\ldots,x_{\overline{r}+l}]\cap T''(0)$ for $1\le i\le\overline{r}+l$ and $\gamma_i\equiv 1 \bmod (x_1,\ldots,x_{\overline{r}+l})$, there exist $\gamma_i^y\in U''(0)$ such that $y_i=\gamma_i^y\tilde{\overline{y}}_i$, $\gamma_i^y\equiv 1 \bmod m(U''(0))$ for $1\le i\le n$.

Suppose that one of the following three conditions holds

$$f\in k(U''(0))[[\tilde{\overline{y}}_1,\ldots,\tilde{\overline{y}}_{\overline{s}+\overline{m}}]] \text{ for some } \overline{m} \text{ with } l\le\overline{m}\le n-\overline{s} \text{ and } \nu^*(f)<\infty. \tag{38}$$

$$f\in k(U''(0))[[\tilde{\overline{y}}_1,\ldots,\tilde{\overline{y}}_{\overline{s}+\overline{m}}]] \text{ for some } \overline{m} \text{ with } l<\overline{m}\le n-\overline{s},\nu^*(f)=\infty, \text{ and } A\in\mathbf{N} \text{ is given.} \tag{39}$$

$$f\in U''(0)-k(U''(0))[[\tilde{\overline{y}}_1,\ldots,\tilde{\overline{y}}_{\overline{s}+\overline{l}}]]. \tag{40}$$

Then there exists a positive integer N_0 such that for $N \geq N_0$, we can construct a CRUTS along ν^* $(R, T''(t), T(t))$ and $(S, U''(t), U(t))$ with associated MTSs

$$\begin{array}{ccc} S & \rightarrow & S(t) \\ \uparrow & & \uparrow \\ R & \rightarrow & R(t) \end{array}$$

such that the following holds. $T''(t)$ contains a subfield isomorphic to $k(c_0, \ldots, c_t)$, $U''(t)$ contains a subfield isomorphic to $k(U(t))$, $R(t)$ has regular parameters $(x_1(t), \ldots, x_m(t))$, $T''(t)$ has regular parameters $(\tilde{\overline{x}}_1(t), \ldots, \tilde{\overline{x}}_m(t))$, $S(t)$ has regular parameters $(y_1(1), \ldots, y_n(t))$, $U''(t)$ has regular parameters $(\tilde{\overline{y}}_1(t), \ldots \tilde{\overline{y}}_n(t))$ such that

$$x_i(t) = \begin{cases} \gamma_i(t)\tilde{\overline{x}}_i(t) & 1 \leq i \leq \overline{r} + l \\ \tilde{\overline{x}}_i(t) & \overline{r} + l < i \leq m \end{cases}$$

where $\gamma_i(t) \in k(c_0, \ldots, c_t)[[x_1(t), \ldots, x_{\overline{r}+l}(t)]]$ are units such that

$$\gamma_i(t) \equiv 1 \ mod \ (x_1(t), \ldots, x_{\overline{r}+l}(t)).$$

In particular,

$$k(c_0, \ldots, c_t)[[x_1(t), \ldots, x_{\overline{r}+l}(t)]] = k(c_0, \ldots, c_t)[[\tilde{\overline{x}}_1(t), \ldots, \tilde{\overline{x}}_{\overline{r}+l}(t)]].$$

For $1 \leq i \leq n$ there exists $\gamma_i^y(t) \in U''(t)$ such that $y_i(t {=} \gamma_i^y(t)\tilde{\overline{y}}_i(t)$,

$$\gamma_i^y(t) \equiv 1 \ mod \ m(U''(t)).$$

$$\begin{array}{ll} \tilde{\overline{x}}_1(t) & = \tilde{\overline{y}}_1(t)^{c_{11}(t)} \cdots \tilde{\overline{y}}_{\overline{s}}(t)^{c_{1\overline{s}}(t)} \phi_1(t) \\ & \vdots \\ \tilde{\overline{x}}_{\overline{r}}(t) & = \tilde{\overline{y}}_1(t)^{c_{\overline{r}1}} \cdots \tilde{\overline{y}}_{\overline{s}}^{c_{\overline{r}\overline{s}}} \phi_{\overline{r}}(t) \\ \tilde{\overline{x}}_{\overline{r}+1}(t) & = \tilde{\overline{y}}_{\overline{s}+1}(t) \\ & \vdots \\ \tilde{\overline{x}}_{\overline{r}+l} & = \tilde{\overline{y}}_{\overline{s}+l}(t) \end{array} \tag{41}$$

$\phi_1(t), \ldots \phi_{\overline{r}}(t) \in k(U(t))$, $\nu(\tilde{\overline{x}}_1(t)), \ldots, \nu(\tilde{\overline{x}}_{\overline{r}}(t))$ are rationally independent, $\nu^*(\tilde{\overline{y}}_1(t)), \ldots, \nu^*(\tilde{\overline{y}}_{\overline{s}}(t))$ are rationally independent and $(c_{ij}(t))$ has rank $\overline{r}$. There exists an algebraic regular local ring $\tilde{R}(t) \subset R(t)$ such that $(x_1(t), \ldots, x_{\overline{r}+l})$ are regular parameters in $\tilde{R}(t)$ and $k(\tilde{R}(t)) \cong k(c_0, \ldots, c_t)$. Furthermore, $x_i(t) = x_i$ for $\overline{r} + l + 1 \leq i \leq m$, $y_i(t) = y_i$ for $\overline{s} + \overline{m} + 1 \leq i \leq n$, so that the CRUTS is of the form $\overline{m}$ where $\overline{s} + \overline{m} = n$ in case (40). Set $n_{t,l} = m(k(U(t))[[\tilde{\overline{y}}_1(t), \ldots, \tilde{\overline{y}}_{\overline{s}+l}(t)]])$.

In case (38) we have

$$f \equiv \tilde{\overline{y}}_1(t)^{d_1} \cdots \tilde{\overline{y}}_{\overline{s}}(t)^{d_{\overline{s}}} u \ mod \ m(U(t))^N \tag{42}$$

where $u \in k(U(t))[[\tilde{\overline{y}}_1(t), \ldots, \tilde{\overline{y}}_{\overline{s}+\overline{m}}(t)]]$ is a unit power series. Further, if $f \in k(U)[[\tilde{\overline{y}}_1, \ldots, \tilde{\overline{y}}_{\overline{s}+l}]]$, then

$$f \equiv \tilde{\overline{y}}_1(t)^{d_1} \cdots \tilde{\overline{y}}_{\overline{s}}(t)^{d_{\overline{s}}} u \ mod \ n_{t,l}^N$$

where $u \in k(U(t))[[\tilde{\overline{y}}_1(t), \ldots, \tilde{\overline{y}}_{\overline{s}+\overline{l}}(t)]]$ is a unit power series.

In case (39) we have

$$f \equiv \tilde{\overline{y}}_1(t)^{d_1} \cdots \tilde{\overline{y}}_{\overline{s}}(t)^{d_{\overline{s}}} \Sigma \ mod \ m(U(t))^N \tag{43}$$

where $\Sigma \in k(U(t))[[\tilde{\overline{y}}_1(t), \ldots, \tilde{\overline{y}}_{\overline{s}+\overline{m}}(t)]]$ and $\nu^*(\tilde{\overline{y}}_1(t)^{d_1} \cdots \tilde{\overline{y}}_{\overline{s}}^{d_{\overline{s}}}(t)) > A$. Further, if $f \in k(U)[[\tilde{\overline{y}}_1, \ldots, \tilde{\overline{y}}_{\overline{s}+l}]]$, then

$$f \equiv \tilde{\overline{y}}_1(t)^{d_1} \cdots \tilde{\overline{y}}_{\overline{s}}(t)^{d_{\overline{s}}} \Sigma \ mod \ n_{t,l}^N$$

where $\Sigma\in k(U(t))[[\tilde{\overline{y}}_1(t),\ldots,\tilde{\overline{y}}_{\overline{s}+\overline{l}}(t)]]$ *and* $\nu^*(\tilde{\overline{y}}_1(t)^{d_1}\cdots\tilde{\overline{y}}_{\overline{s}}^{d_{\overline{s}}}(t))>A$.

In case (40) we have

$$f\equiv P+\tilde{\overline{y}}_1(t)^{d_1}\cdots\tilde{\overline{y}}_{\overline{s}}(t)^{d_{\overline{s}}}H \ mod\ m(U(t))^N \tag{44}$$

where $P\in k(U(t))[\tilde{\overline{y}}_1(t),\ldots\tilde{\overline{y}}_{\overline{s}+l}(t)]$ *and there exists a finite extension* L *of the algebraic closure of* $k(T(t))$ *in* $k(U(t))$ *and a positive integer* d *such that*

$$P\in L[\tilde{\overline{x}}_1(t)^{\frac{1}{d}},\ldots,\tilde{\overline{x}}_{\overline{r}}(t)^{\frac{1}{d}},\tilde{\overline{x}}_{\overline{r}+1}(t),\ldots,\tilde{\overline{x}}_{\overline{r}+l}(t)],$$

$$H=u(\tilde{\overline{y}}_{\overline{s}+l+1}(t)+\tilde{\overline{y}}_1(t)^{g_1}\cdots\tilde{\overline{y}}_{\overline{s}}(t)^{g_{\overline{s}}}\Sigma)$$

where $u\in U(t)$ *is a unit series,* $\Sigma\in k(U(t))[[\tilde{\overline{y}}_1(t),\ldots\tilde{\overline{y}}_{\overline{s}+l}(t),\tilde{\overline{y}}_{\overline{s}+l+2}(t),\ldots,\tilde{\overline{y}}_n(t)]]$ *and* $\nu^*(\tilde{\overline{y}}_{\overline{s}+l+1}(t))\le\nu^*(\tilde{\overline{y}}_1(t)^{g_1}\cdots\tilde{\overline{y}}_{\overline{s}}(t)^{g_{\overline{s}}})$.

We further have that $\nu^*(m(U(t))$ *is a constant which is independent of* N *for* $N\ge N_0$.

Proof. The proof is essentially the same as the proof of Theorem 4.8 [14], with some modification of notation. On page 67, line 2 of the proof, "By Theorem 4.7 there is a CUTS" should be replaced with "By Theorems 8.1 and 8.2 there is a CUTS". On page 67, line 7 of the proof in [14], "notation of Theorem 4.7" should be "notation of Theorems 8.1 and 8.2". On page 67, line 18 of the proof, replace "$P\in k(\overline{U}(t))[[\overline{w}_1(t),\ldots,\overline{w}_l(t)]]$" with "$P\in k(\overline{U}(t))[\overline{w}_1(t),\ldots,\overline{w}_{\overline{s}+l}(t)]$ and there exists a finite extension L of the algebraic closure of $k(\overline{T}(t))$ in $k(\overline{U}(t))$, and a positive integer d such that

$$P\in L[\overline{z}_1(t)^{\frac{1}{d}},\ldots,\overline{z}_{\overline{r}}(t)^{\frac{1}{d}},\overline{z}_{\overline{r}+1}(1),\ldots,\overline{z}_{\overline{r}+l}(1)].\text{”}$$

All later references in this proof to Theorem 4.7 and to equations (46), (48), (49), (50) and (51) should be replaced with references to Theorem 8.1, (22), (24), (25), (26) and (27) of this paper. References to Lemma 4.4 should be replaced with references to Lemma 5.2 of this paper.

The independence of $\nu^*(m(U(t))=\min\{\nu^*(f)\mid f\in m(U(t))\}$ of N follows from (A3) of page 83 of the proof of Theorem 4.8 in [14]. □

Theorem 9.2. *Suppose that* $T''(0)\subset\hat{R}$ *is a regular local ring, essentially of finite type over* R *such that the quotient field of* $T''(0)$ *is finite over* K, $U''(0)\subset\hat{S}$ *is a regular local ring, essentially of finite type over* S *such that the quotient field of* $U''(0)$ *is finite over* K^*, $U''(0)$ *dominates* $T''(0)$, $T''(0)$ *contains a subfield isomorphic to* $k(c_0)$ *for some* $c_0\in k(T''(0))$, $U''(0)$ *contains a subfield isomorphic to* $k(U''(0))$. *Suppose that* R *has regular parameters* $(x_1,\ldots,x_m)$, S *has regular parameters* $(y_1,\ldots,y_n)$, $T''(0)$ *has regular parameters* $(\overline{x}_1,\ldots,\overline{x}_m)$ *and* $U''(0)$ *has regular parameters* $(\overline{y}_1,\ldots,\overline{y}_n)$ *such that*

$$\begin{array}{ll}
\overline{x}_1 &= \overline{y}_1^{c_{11}}\cdots\overline{y}_{\overline{s}}^{c_{1\overline{s}}}\phi_1\\
&\vdots\\
\overline{x}_{\overline{r}} &= \overline{y}_1^{c_{\overline{r}1}}\cdots\overline{y}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\phi_{\overline{r}}\\
\overline{x}_{\overline{r}+1} &= \overline{y}_{\overline{s}+1}\\
&\vdots\\
\overline{x}_{\overline{r}+l} &= \overline{y}_{\overline{s}+l}
\end{array}$$

where $\phi_1,\ldots,\phi_{\overline{r}}\in k(U''(0))$, $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_{\overline{r}})$ *are rationally independent,* $\nu^*(\overline{y}_1),\ldots,\nu^*(\overline{y}_{\overline{s}})$ *are rationally independent and* (c_{ij}) *has rank* $\overline{r}$.

Suppose that there exists an algebraic regular local ring $\tilde{R} \subset R$ such that $(x_1, \ldots, x_{\overline{r}+l})$ are regular parameters in $\tilde{R}$, $k(\tilde{R}) \cong k(c_0)$ and

$$x_i = \begin{cases} \gamma_i \overline{x}_i & 1 \le i \le \overline{r}+l \\ \overline{x}_i & \overline{r}+l < i \le m \end{cases}$$

with $\gamma_i \in k(c_0)[[x_1, \ldots, x_{\overline{r}+l}] \cap T''(0)$ for $1 \le i \le \overline{r}+l$ and $\gamma_i \equiv 1 \mod (x_1, \ldots, x_{\overline{r}+l})$, there exist $\gamma_i^y \in U''(0)$ such that $y_i = \gamma_i^y \overline{y}_i$, $\gamma_i^y \equiv 1 \mod m(U''(0))$ for $1 \le i \le n$.

Further, suppose that

$$x_{\overline{r}+l+1} = \overline{P} + \overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}} \overline{H} + \Omega$$

where $\overline{P} \in k(U''(0))[\overline{y}_1, \ldots, \overline{y}_{\overline{s}+l}]$ and $\overline{P} \in L[\overline{x}_1^{\frac{1}{d}}, \ldots, \overline{x}_{\overline{r}}^{\frac{1}{d}}, \overline{x}_{\overline{r}+1}, \ldots, \overline{x}_{\overline{s}+l}]$, where d is a positive integer, L is a finite extension of the algebraic closure of $k(T''(0))$ in $k(U''(0))$,

$$\overline{H} = \overline{u}(\overline{y}_{\overline{s}+l+1} + \overline{y}_1^{\overline{g}_1} \cdots \overline{y}_{\overline{s}}^{\overline{g}_{\overline{s}}} \overline{\Sigma})$$

where $\overline{u} \in U''(0)^\frown$ is a unit,

$$\overline{\Sigma} \in k(U''(0))[[\overline{y}_1, \ldots, \overline{y}_{\overline{s}+l}, \overline{y}_{\overline{s}+l+2}, \ldots, \overline{y}_n]],$$

$$\nu^*(\overline{y}_{\overline{s}+l+1}) \le \nu(\overline{y}_1^{\overline{g}_1} \cdots \overline{y}_{\overline{s}}^{\overline{g}_{\overline{s}}}) \text{ and}$$

$$\Omega \in m(U''(0))^N \text{ with } N\nu^*(m(U''(0))) > \nu^*(\overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}} \overline{y}_{\overline{s}+l+1}).$$

Then there exists a CRUTS along ν $(R, T''(t), T(t))$ and $(S, U''(t), U(t))$ with associated MTSs

$$\begin{array}{ccc} S & \rightarrow & S(t') \\ \uparrow & & \uparrow \\ R & \rightarrow & R(t') \end{array}$$

such that the following holds. $T''(t')$ contains a subfield isomorphic to $k(c_0, \ldots, c_{t'})$, $U''(t')$ contains a subfield isomorphic to $k(U(t'))$, $R(t')$ has regular parameters $(x_1(t'), \ldots, x_m(t'))$, $T''(t')$ has regular parameters $(\tilde{\overline{x}}_1(t'), \ldots, \tilde{\overline{x}}_m(t'))$, $S(t')$ has regular parameters $(y_1(t'), \ldots, y_n(t'))$, $U''(t')$ has regular parameters $(\tilde{\overline{y}}_1(t'), \ldots \tilde{\overline{y}}_n(t'))$ where

$$\begin{array}{ll} \tilde{\overline{x}}_1(t') & = \tilde{\overline{y}}_1(t')^{c_{11}(t')} \cdots \tilde{\overline{y}}_{\overline{s}}(t')^{c_{1\overline{s}}(t')} \phi_1(t') \\ & \vdots \\ \tilde{\overline{x}}_{\overline{r}}(t') & = \tilde{\overline{y}}_1(t')^{c_{\overline{r}1}(t')} \cdots \tilde{\overline{y}}_{\overline{s}}(t')^{c_{\overline{r}\overline{s}}} \phi_{\overline{r}}(t') \\ \tilde{\overline{x}}_{\overline{r}+1}(t') & = \tilde{\overline{y}}_{\overline{s}+1}(t') \\ & \vdots \\ \tilde{\overline{x}}_{\overline{r}+l}(t') & = \tilde{\overline{y}}_{\overline{s}+l}(t') \\ x_{\overline{r}+l+1}(t') = \tilde{\overline{x}}_{\overline{r}+l+1}(t') & = P + \tilde{\overline{y}}_1(t')^{d_1(t')} \cdots \tilde{\overline{y}}_{\overline{s}}(t')^{d_{\overline{s}}(t')} H \end{array}$$

where $P \in k(U(t'))[\tilde{\overline{y}}_1(t'), \ldots, \tilde{\overline{y}}_{\overline{s}+l}(t')]$ and

$$\overline{P} \in L'[\tilde{\overline{x}}_1(t')^{\frac{1}{d}}, \ldots, \tilde{\overline{x}}_{\overline{r}}(t')^{\frac{1}{d}}, \tilde{\overline{x}}_{\overline{r}+1}(t'), \ldots, \tilde{\overline{x}}_{\overline{r}+l}(t')],$$

where L' is a finite extension of the algebraic closure of $k(T(t'))$ in $k(U(t'))$, $H \in k(U(t'))[[\tilde{\overline{y}}_1(t'), \ldots, \tilde{\overline{y}}_n(t')]]$ is such that

$$\text{mult } H(0, \ldots, 0, \tilde{\overline{y}}_{\overline{s}+l+1}(t'), 0, \ldots, 0) = 1,$$

$\phi_1(t'), \ldots \phi_{\overline{r}}(t') \in k(U(t'))$, $\nu(\tilde{\overline{x}}_1(t')), \ldots, \nu(\tilde{\overline{x}}_{\overline{r}}(t'))$ are rationally independent, $\nu^*(\tilde{\overline{y}}_1(t')), \ldots, \nu^*(\tilde{\overline{y}}_{\overline{s}}(t'))$ are rationally independent and $(c_{ij}(t'))$ has rank $\overline{r}$. There

exists an algebraic regular local ring $\tilde{R}(t') \subset R(t')$ such that $(x_1(t'), \ldots, x_{\overline{r}+l}(t'))$ are regular parameters in $\tilde{R}(t')$ and $k(\tilde{R}(t')) \cong k(c_0, \ldots, c_{t'})$.

$$x_i(t') = \begin{cases} \gamma_i(t')\tilde{\overline{x}}_i(t') & 1 \le i \le \overline{r}+l \\ \tilde{\overline{x}}_i(t') & \overline{r}+l < i \le m \end{cases}$$

with $\gamma_i(t') \in k(c_0, \ldots, c_{t'})[[x_1(t'), \ldots, x_{\overline{r}+l}(t')]] \cap T''(t')$ units for $1 \le i \le \overline{r}+l$, such that

$$\gamma_i(t') \equiv 1 \text{ mod } (x_1(t'), \ldots, x_{\overline{r}+l}(t'))$$

and for $1 \le i \le n$ there exists $\gamma_i^y(t') \in U''(t')$ such that $y_i(t') = \gamma_i^y(t')\tilde{\overline{y}}_i(t')$,

$$\gamma_i^y(t') \equiv 1 \text{ mod } m(U''(t')).$$

The proof of Theorem 9.2 is similar to the proof of Theorem 4.9 of [14].

Theorem 9.3. *Let $n_{0,l} = m(k(U''(0))[[\overline{y}_1, \ldots, \overline{y}_{\overline{s}+l}]]$ in the assumptions of Theorem 9.2.*

(1) *If $\Omega \in n_{0,l}^N$ in the assumptions of Theorem 9.2, then a sequence of MTSs of type (M2) and a MTS of type (M1) (so that the CRUTS along ν^* is of form l) are sufficient to transform $x_{\overline{r}+l+1}$ into the form of the conclusions of Theorem 9.2.*

(2) *Suppose that*

$$g = \overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}} u + \Omega$$

where $u \in k(U''(0))[[\overline{y}_1, \ldots, \overline{y}_{\overline{s}+l}]]$ is a unit power series and $\Omega \in n_{0,l}^N$ with $N\nu^*(n_{0,l}) > \nu(\overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}})$. Then a sequence of MTSs of type (M2) and a MTS of type (M1) (so that the CRUTS along ν^* is of form l) are sufficient to transform g into the form

$$g = \overline{y}_1(t')^{d_1(t')} \cdots \overline{y}_{\overline{s}}(t')^{d_{\overline{s}}(t')} \overline{u}$$

where $\overline{u} \in k(U''(t'))[[\overline{y}_1(t'), \ldots \overline{y}_{\overline{s}+l}(t')]]$ is a unit power series.

(3) *Suppose that*

$$g = \overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}} \Sigma + \Omega$$

where $\Sigma \in k(U''(0))[[\overline{y}_1, \ldots, \overline{y}_{\overline{s}+l}]]$, $\nu^*(\overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}}) > A$ and $\Omega \in n_{0,l}^N$ with $N\nu^*(n_{0,l}) > \nu(\overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}})$. Then a sequence of MTSs of type (M2) and a MTS of type (M1) (so that the CRUTS along ν^* is of form l) are sufficient to transform g into the form

$$g = \overline{y}_1(t')^{d_1(t')} \cdots \overline{y}_{\overline{s}}(t')^{d_{\overline{s}}(t')} \overline{\Sigma}$$

where $\overline{\Sigma} \in k(U''(t'))[[\overline{y}_1(t'), \ldots \overline{y}_{\overline{s}+l}]]$ and $\nu^*(\overline{y}_1(t')^{d_1(t')} \cdots \overline{y}_{\overline{s}}(t')^{d_{\overline{s}}(t')}) > A$

(4) *Suppose that*

$$g = \overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}} u + \Omega$$

where $u \in k(U''(0))[[\overline{y}_1, \ldots, \overline{y}_{\overline{s}+l}]]$ is a unit power series and $\Omega \in m(U(0))^N$ with $N\nu^*(m(U''(0))) > \nu^*(\overline{y}_1^{d_1} \cdots \overline{y}_{\overline{s}}^{d_{\overline{s}}})$. Then there exists a CRUTS along ν^* as in the conclusions of Theorem 9.2 such that

$$g = \overline{y}_1(t')^{d_1(t')} \cdots \overline{y}_{\overline{s}}(t')^{d_{\overline{s}}(t')} \overline{u}$$

where $\overline{u} \in k(U(t'))[[\overline{y}_1(t'), \ldots \overline{y}_{\overline{s}+l}]]$ is a unit power series.

(5) *Suppose that*

$$g = \overline{y}_1^{d_1}\cdots\overline{y}_{\overline{s}}^{d_{\overline{s}}}\Sigma+\Omega$$

where $\Sigma\in k(U''(0))[[\overline{y}_1,\ldots,\overline{y}_{\overline{s}+l}]]$, $\nu^*(\overline{y}_1^{d_1}\cdots\overline{y}_{\overline{s}}^{d_{\overline{s}}})>A$ *and* $\Omega\in m(U(0))^N$ *with* $N\nu^*(m(U''(0)))>\nu^*(\overline{y}_1^{d_1}\cdots\overline{y}_{\overline{s}}^{d_{\overline{s}}})$. *Then there exists a CRUTS along* ν^* *as in the conclusions of Theorem 9.2 such that*

$$g=\overline{y}_1(t')^{d_1(t')}\cdots\overline{y}_{\overline{s}}(t')^{d_{\overline{s}}(t')}\overline{\Sigma}$$

where $\overline{\Sigma}\in k(U(t'))[[\overline{y}_1(t'),\ldots\overline{y}_{\overline{s}+l}]]$ *and* $\nu^*(\overline{y}_1(t')^{d_1(t')}\cdots\overline{y}_{\overline{s}}(t')^{d_{\overline{s}}(t')})>A$.

Theorem 9.3 and its proof is a modification of the statement and proof of Theorem 4.10 of [14].

Theorem 9.4. *Suppose that* $T''(0)\subset\hat{R}$ *is a regular local ring, essentially of finite type over* R *such that the quotient field of* $T''(0)$ *is finite over* K, $U''(0)\subset\hat{S}$ *is a regular local ring, essentially of finite type over* S *such that the quotient field of* $U''(0)$ *is finite over* K^*, $U''(0)$ *dominates* $T''(0)$, $T''(0)$ *contains a subfield isomorphic to* $k(c_0)$ *for some* $c_0\in k(T''(0))$, $U''(0)$ *contains a subfield isomorphic to* $k(U''(0))$. *Suppose that* R *has regular parameters* $(x_1,\ldots,x_m)$, S *has regular parameters* $(y_1,\ldots,y_n)$, $T''(0)$ *has regular parameters* $(\overline{x}_1,\ldots,\overline{x}_m)$ *and* $U''(0)$ *has regular parameters* $(\overline{y}_1,\ldots,\overline{y}_n)$ *such that*

$$\begin{array}{ll}
\overline{x}_1 & =\tilde{\overline{y}}_1^{c_{11}}\cdots\overline{y}_{\overline{s}}^{c_{1\overline{s}}}\phi_1\\
 & \vdots\\
\overline{x}_{\overline{r}} & =\overline{y}_1^{c_{\overline{r}1}}\cdots\overline{y}_{\overline{s}}^{c_{\overline{r}\overline{s}}}\phi_{\overline{r}}\\
\overline{x}_{\overline{r}+1} & =\overline{y}_{\overline{s}+1}\\
 & \vdots\\
\overline{x}_{\overline{r}+l} & =\overline{y}_{\overline{s}+l}
\end{array}$$

where $\phi_1,\ldots,\phi_{\overline{r}}\in k(U''(0))$, $\nu(\overline{x}_1),\ldots,\nu(\overline{x}_{\overline{r}})$ *are rationally independent,* $\nu^*(\overline{y}_1),\ldots,\nu^*(\overline{y}_{\overline{s}})$ *are rationally independent and* (c_{ij}) *has rank* $\overline{r}$. *Further suppose that* $l<m-\overline{r}$.

Suppose that there exists an algebraic regular local ring $\tilde{R}\subset R$ *such that* $(x_1,\ldots,x_{\overline{r}+l})$ *are regular parameters in* $\tilde{R}$, $k(\tilde{R})\cong k(c_0)$ *and*

$$x_i=\left\{\begin{array}{ll}\gamma_i\overline{x}_i & 1\le i\le\overline{r}+l\\ \overline{x}_i & \overline{r}+l<i\le m\end{array}\right.$$

with $\gamma_i\in k(c_0)[[x_1,\ldots,x_{\overline{r}+l}]]\cap T''(0)$ *for* $1\le i\le\overline{r}+l$ *and* $\gamma_i\equiv 1 \mod (x_1,\ldots,x_{\overline{r}+l})$, *there exist* $\gamma_i^y\in U''(0)$ *such that* $y_i=\gamma_i^y\overline{y}_i$, $\gamma_i^y\equiv 1 \mod m(U''(0))$ *for* $1\le i\le n$.

Then there exists a CRUTS along ν^* $(R,T''(t),T(t))$ *and* $(S,U''(t),U(t))$ *with associated MTSs*

$$\begin{array}{ccc} S & \rightarrow & S(t)\\ \uparrow & & \uparrow\\ R & \rightarrow & R(t)\end{array}$$

such that the following holds. $T''(t)$ *contains a subfield isomorphic to* $k(c_0,\ldots,c_t)$, $U''(t)$ *contains a subfield isomorphic to* $k(U(t))$, $R(t)$ *has regular parameters* $(x_1(t),\ldots,x_m(t))$, $S(t)$ *has regular parameters* $(y_1(t),\ldots,y_n(t))$, $T''(t)$ *has regular parameters* $(\tilde{\overline{x}}_1(t),\ldots,\tilde{\overline{x}}_m(t))$, $U''(t)$ *has regular parameters* $(\tilde{\overline{y}}_1(t),\ldots\tilde{\overline{y}}_n(t))$ *where*

$$\begin{aligned}
\tilde{\overline{x}}_1(t) &= \tilde{\overline{y}}_1(t)^{c_{11}(t)}\cdots\tilde{\overline{y}}_{\overline{s}}(t)^{c_{1\overline{s}}(t)}\phi_1(t)\\
&\vdots\\
\tilde{\overline{x}}_{\overline{r}}(t) &= \tilde{\overline{y}}_1(t)^{c_{\overline{r}1}(t)}\cdots\tilde{\overline{y}}_{\overline{s}}(t)^{c_{\overline{r}\overline{s}}(t)}\phi_{\overline{r}}(t)\\
\tilde{\overline{x}}_{\overline{r}+1}(t) &= \tilde{\overline{y}}_{\overline{s}+1}(t)\\
&\vdots\\
\tilde{\overline{x}}_{\overline{r}+l+1}(t) &= \tilde{\overline{y}}_{\overline{s}+l+1}(t)
\end{aligned}$$

such that $\phi_1(t),\ldots\phi_{\overline{r}}(t)\in k(U(t))$, $\nu(\tilde{\overline{x}}_1(t)),\ldots,\nu(\tilde{\overline{x}}_{\overline{r}}(t))$ *are rationally independent,* $\nu^*(\tilde{\overline{y}}_1(t)),\ldots,\nu^*(\tilde{\overline{y}}_{\overline{s}}(t))$ *are rationally independent and* $(c_{ij}(t))$ *has rank* $\overline{r}$.

$$x_i(t)=\tilde{\overline{x}}_i(t) \text{ for } 1\le i\le m.$$

For $1\le i\le n$ *there exists* $\gamma_i^y(t)\in U''(t)$ *such that* $y_i(t)=\gamma_i^y(t)\tilde{\overline{y}}_i(t)$,

$$\gamma_i^y(t)\equiv 1 \text{ mod } m(U''(t)).$$

Proof. The proof of this theorem is similar to that of Theorem 4.11 [14]. References to Theorems 4.8, 4.9 and 4.10 must be replaced with references to Theorems 9.1, 9.2, 9.3 of this paper. The argument of Lines 11-13 of page 95 [14], "By Theorem 2.12 ...... mult $\Sigma(0,\ldots,0,\overline{y}_{l+1},0\ldots,0)=1$", must be replaced with:

"By Lemma 7.5 and Theorem 9.1 (with $f=x_{\overline{r}+l+1}$ in (40)) and Theorem 9.2, we may assume that

$$x_{\overline{r}+l+1}=\overline{x}_{\overline{r}+l+1}=P+\overline{y}_1^{d_1}\cdots\overline{y}_{\overline{s}}^{d_{\overline{s}}}\Sigma_0$$

where $P\in k(U''(0))[\overline{y}_1,\ldots,\overline{y}_{\overline{r}+l}]$ and $P\in L[\overline{x}_1^{\frac{1}{d}},\ldots,\overline{x}_{\overline{r}}^{\frac{1}{d}},\overline{x}_{\overline{r}+1},\ldots,\overline{x}_{\overline{r}+l}]$, with L a finite extension of the algebraic closure of $k(T''(0))$ in $k(U''(0))$, d a natural number. Σ_0 is a series with mult $\Sigma(0,\ldots,0,\overline{y}_{\overline{r}+l+1},0\ldots,0)=1$".

On lines 21-31 of page 95 [14], "Suppose that $\nu(P)<\infty$ ...... $g\in k(c_0)[[\overline{x}_1(1),\ldots,\overline{x}_l(1)]][x_{l+1}]$" must be replaced with:

"Suppose that $\nu^*(P)<\infty$. Let ω be a primitive dth root of unity in an algebraic closure of L. Set

$$g'=\prod_{i_1,\ldots,i_{\overline{r}}=1}^{d}(x_{\overline{r}+l+1}-P(\omega^{i_1}\overline{x}_1^{\frac{1}{d}},\ldots,\omega^{i_{\overline{r}}}\overline{x}_{\overline{r}}^{\frac{1}{d}},\overline{x}_{\overline{r}+1},\ldots,\overline{x}_{\overline{r}+l})).$$

$$g'\in L[\overline{x}_1,\ldots,\overline{x}_{\overline{r}+l},x_{\overline{r}+l+1}].$$

Let G be the Galois group of a Galois closure of L over $k(c_0)$. We can define

$$g=\prod_{\tau\in G}\tau(g')$$

where G acts on the coefficients of g'.

$$g\in k(c_0)[\overline{x}_1,\ldots,\overline{x}_{\overline{r}+l},x_{\overline{r}+l+1}]."$$

On page 97, lines 8-20 of [14], "Set $(e_{ij})=$ ...... $x_l(\alpha+1)=\tilde{\overline{y}}_l(\alpha+1)$" should be replaced with: "After possibly interchanging $\tilde{\overline{y}}_1(\alpha+1),\ldots,\tilde{\overline{y}}_{\overline{s}}(\alpha+1)$, we may assume that $\mathrm{Det}(\tilde{C})\ne 0$ where

$$\tilde{C}=\begin{pmatrix} c_{11}(\alpha+1) & \cdots & c_{1\overline{r}}(\alpha+1)\\ \vdots & & \vdots\\ c_{\overline{r}1} & \cdots & c_{\overline{r}\overline{r}}(\alpha+1)\end{pmatrix}$$

Set $(e_{ij})=\tilde{C}^{-1}$, $d=\mathrm{Det}(\tilde{C})$. We can replace $\tilde{\overline{y}}_i(\alpha+1)$ with

$$\tilde{\overline{y}}_i(\alpha+1)\gamma_1(\alpha+1)^{e_{i1}}\cdots\gamma_{\overline{r}}(\alpha+1)^{e_{i\overline{r}}}$$

for $1\le i\le \overline{r}$, $\tilde{\overline{y}}_i(\alpha+1)$ with $\tilde{\overline{y}}_i(\alpha+1)\gamma_i(\alpha+1)$ for $\overline{s}+1\le i\le \overline{s}+l$ and replace $U''(\alpha+1)$ with $U''(\alpha+1)[\gamma_1(\alpha+1)^{\frac{1}{d}},\ldots,\gamma_{\overline{r}}(\alpha+1)^{\frac{1}{d}}]_q$ where

$$q=m(U(\alpha+1))\cap\left(U''(\alpha+1)[\gamma_1(\alpha+1)^{\frac{1}{d}},\ldots,\gamma_{\overline{r}}(\alpha+1)^{\frac{1}{d}}]\right)$$

to get

$$\begin{array}{ll}
x_1(\alpha+1) & =\tilde{\overline{y}}_1(\alpha+1)^{c_{11}(\alpha+1)}\cdots\tilde{\overline{y}}_{\overline{s}}(\alpha+1)^{c_{1\overline{s}}(\alpha+1)}\phi_1(\alpha+1)\\
 & \vdots\\
x_{\overline{r}}(\alpha+1) & =\tilde{\overline{y}}_1(\alpha+1)^{c_{\overline{r}1}(\alpha+1)}\cdots\tilde{\overline{y}}_{\overline{s}}(\alpha+1)^{c_{\overline{r}\overline{s}}(\alpha+1)}\phi_{\overline{r}}(\alpha+1)\\
 & \vdots\\
x_{\overline{r}+1}(\alpha+1) & =\tilde{\overline{y}}_{\overline{s}+1}(\alpha+1)\\
x_{\overline{r}+l}(\alpha+1) & =\tilde{\overline{y}}_{\overline{s}+l}(\alpha+1)\text{''}
\end{array}$$

On page 98, line 18 to page 99 line 8, "By construction, ... $x_s^*(\alpha+2)=\hat{y}_1(\alpha+2)^{c_{s1}(\alpha+2)}\cdots\hat{y}_s(\alpha+2)^{c_{ss}(\alpha+2)}\psi_s$" should be replaced with "By construction there are positive integers f_{ij} such that

$$\begin{array}{ll}
x_1^*(\alpha+2) & =\overline{y}_1(\alpha+2)^{f_{11}}\cdots\overline{y}_{\overline{s}}(\alpha+2)^{f_{1\overline{s}}}\gamma^{e_{1,\overline{r}+1}}\tau^{e_{1,\overline{r}+1}}\\
 & \quad\cdot\phi_1(\alpha+1)^{e_{11}}\cdots\phi_{\overline{r}}(\alpha+1)^{e_{1\overline{r}}}\\
 & \vdots\\
x_{\overline{r}}^*(\alpha+2) & =\overline{y}_1(\alpha+2)^{f_{\overline{r}1}}\cdots\overline{y}_{\overline{s}}(\alpha+2)^{f_{\overline{r}\overline{s}}}\gamma^{e_{\overline{r},\overline{r}+1}}\tau^{e_{\overline{r},\overline{r}+1}}\\
 & \quad\cdot\phi_1(\alpha+1)^{e_{\overline{r}1}}\cdots\phi_{\overline{r}}(\alpha+1)^{e_{\overline{r}\overline{r}}}\\
x_{\overline{r}+l+1}^*(\alpha+2)+c_{\alpha+2} & =\overline{y}_1(\alpha+2)^{f_{\overline{r}+1,1}}\cdots\overline{y}_{\overline{s}}(\alpha+2)^{f_{\overline{r}+1,\overline{s}}}\gamma^{e_{\overline{r}+1,\overline{r}+1}}\tau^{e_{\overline{r}+1,\overline{r}+1}}\\
 & \quad\cdot\phi_1(\alpha+1)^{e_{\overline{r}+1,1}}\cdots\phi_{\overline{r}}(\alpha+1)^{e_{\overline{r}+1,\overline{r}}}
\end{array}$$

in $S(\alpha+2)\hat{}$. $\nu(x^*_{\overline{r}+1}(\alpha+2)+c_{\alpha+2})=0$ implies

$$f_{\overline{r}+1,1}=\cdots=f_{\overline{r}+1,\overline{s}}=0.$$

Set

$$\tilde{\omega}=\phi_1(\alpha+1)^{e_{\overline{r}+1,1}}\cdots\phi_{\overline{r}}(\alpha+1)^{e_{\overline{r}+1,\overline{r}}}\tau^{e_{\overline{r}+1,\overline{r}+1}}\in k(U(\alpha+1)).$$

Substituting

$$\gamma=\overline{P}_{\alpha+1}+\tilde{\overline{y}}_1(\alpha+1)^{\epsilon_1(\alpha+1)}\cdots\tilde{\overline{y}}_{\overline{s}}(\alpha+1)^{\epsilon_{\overline{s}}(\alpha+1)}\Sigma_{\alpha+1}$$

we have

$$\begin{array}{ll}
x^*_{\overline{r}+l+1}(\alpha+2)+c_{\alpha+2} & =\tilde{\omega}(\overline{P}_{\alpha+1}+\tilde{\overline{y}}_1(\alpha+1)^{\epsilon_1(\alpha+1)}\cdots\tilde{\overline{y}}_{\overline{s}}(\alpha+1)^{\epsilon_{\overline{s}}(\alpha+1)}\Sigma_{\alpha+1})^{e_{\overline{r}+1,\overline{r}+1}}\\
 & =Q_0(\overline{y}_1(\alpha+2),\ldots,\overline{y}_{\overline{s}+l}(\alpha+2))\\
 & \quad+\overline{y}_1(\alpha+2)^{\epsilon_1(\alpha+2)}\cdots\overline{y}_{\overline{s}}(\alpha+2)^{\epsilon_{\overline{s}}(\alpha+2)}\Lambda_0(\overline{y}_1(\alpha+2),\ldots,\overline{y}_n(\alpha+2))
\end{array}$$

where Q_0 is a unit series and

$$\operatorname{mult}\Lambda_0(0,\ldots,0,\overline{y}_{\overline{s}+l+1}(\alpha+2),0,\ldots,0)=1.$$

The $\overline{r}\times\overline{s}$ matrix (f_{ij}) with $1\le i\le\overline{r}$, $1\le j\le\overline{s}$, has rank $\overline{r}$, so after possibly reindexing $\overline{y}_1(\alpha+2),\ldots,\overline{y}_{\overline{s}}(\alpha+2)$, we may assume that

$$\begin{pmatrix} f_{11} & \cdots & f_{1\overline{r}}\\ \vdots & & \vdots\\ f_{\overline{r}1} & \cdots & f_{\overline{r}\overline{r}}\end{pmatrix}$$

has rank $\overline{r}$. Define $\alpha_i\in\mathbf{Q}$ by

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_{\overline{r}}\end{pmatrix}=\begin{pmatrix} f_{11} & \cdots & f_{1\overline{r}}\\ \vdots & & \vdots\\ f_{\overline{r}1} & \cdots & f_{\overline{r}\overline{r}}\end{pmatrix}^{-1}\begin{pmatrix}-e_{1,\overline{r}+1}\\ \vdots\\ -e_{\overline{r},\overline{r}+1}\end{pmatrix}$$

and set

$$\hat{y}_i(\alpha+2) = \begin{cases} \gamma^{-\alpha_i}\overline{y}_i(\alpha+2) & \text{for } 1 \le i \le \overline{r} \\ \overline{y}_i(\alpha+2) & \text{for } \overline{r} < i \end{cases}$$

to get

$$\begin{aligned} x_1^*(\alpha+2) &= \hat{y}_1(\alpha+2)^{c_{11}(\alpha+2)} \cdots \hat{y}_{\overline{s}}(\alpha+2)^{c_{1\overline{s}}(\alpha+2)}\psi_1 \\ &\vdots \\ x_{\overline{r}}^*(\alpha+2) &= \hat{y}_1(\alpha+2)^{c_{\overline{r}1}(\alpha+2)} \cdots \hat{y}_{\overline{s}}(\alpha+2)^{c_{\overline{r}\overline{s}}(\alpha+2)}\psi_{\overline{r}} \end{aligned}$$

where $c_{ij}(\alpha+2) = f_{ij}$, $(\hat{y}_1(\alpha+2), \ldots, \hat{y}_i(\alpha+2))$ are regular parameters in $S(\alpha+2)\hat{}$, $\psi_1, \ldots, \psi_{\overline{r}} \in k(S(\alpha+2))$."

On page 107, lines 3-12 substitute for "$x_1(t) = \ldots\ldots i = l+1$" the following:

$$\begin{aligned} \text{“}x_1(t') &= y_1(t')^{c_{11}(t')} \ldots y_{\overline{s}}(t')^{c_{1\overline{s}}(t')}\tau_1(t') \\ &\vdots \\ x_{\overline{r}}(t') &= y_1(t')^{c_{\overline{r}1}(t')} \ldots y_{\overline{s}}(t')^{c_{\overline{r}\overline{s}}(t')}\tau_{\overline{r}}(t') \\ x_{\overline{r}+1}(t') &= y_{\overline{s}+1}(t') \\ &\vdots \\ x_{\overline{r}+l}(t') &= y_{\overline{s}+l}(t') \\ x_{\overline{r}+l+1}(t') &= y_1(t')^{d_1(t')} \cdots y_{\overline{s}}(t')^{d_{\overline{s}}(t')}\overline{y}_{\overline{s}+l+1}(t') \end{aligned}$$

Let $\phi_i(t')$ be the residue of $\tau_i(t')$ in $k(S(t'))$,

$$\overline{\tau}_i = \frac{\tau_i(t')}{\phi_i(t')}.$$

After possibly reindexing $y_1(t'), \ldots, y_{\overline{s}}(t')$, we may assume that

$$\tilde{C} = \begin{pmatrix} c_{11}(t') & \cdots & c_{1\overline{r}}(t') \\ \vdots & & \vdots \\ c_{\overline{r}1}(t') & \cdots & c_{\overline{r}\overline{r}}(t') \end{pmatrix}$$

has rank $\overline{r}$. Let $(e_{ij}) = \tilde{C}^{-1}$. Define

$$\overline{y}_i(t') = \begin{cases} \overline{\tau}_1^{e_{i1}} \cdots \overline{\tau}_{\overline{r}}^{e_{i\overline{r}}} y_i(t') & \text{if } 1 \le i \le \overline{r} \\ y_i(t') & \text{if } \overline{r} < i, i \ne \overline{s}+l+1 \\ \overline{\tau}_1^{-e_{11}d_1(t') - \cdots - e_{\overline{r}1}d_{\overline{r}}(t')} \cdots \overline{\tau}_{\overline{r}}^{-e_{1\overline{r}}d_1(t') - \cdots - e_{\overline{r}\overline{r}}d_{\overline{r}}(t')} y_{\overline{s}+l+1}(t') & \text{if } i = \overline{r}+l+1'' \end{cases}$$

□

10. Monomialization

Theorem 10.1. *Suppose that k is a field of characteristic zero, $K \to K^*$ is a (possibly transcendental) extension of algebraic function fields over k. Suppose that ν^* is a rank 1 valuation of K^* which is trivial on k. Suppose that R is an algebraic local ring of K, S is an algebraic local ring of K^* such that S dominates R and ν^* dominates S. Let $\nu = \nu^* \mid K$,*

$$\overline{s} = \textit{ratrank } \nu^* \ge \overline{r} = \textit{ratrank } \nu.$$

Then there exist sequences of monoidal transforms $R \to R'$ and $S \to S'$ along ν^* such that R' and S' are regular local rings, S' dominates R', there exist regular parameters $(y_1', \ldots, y_n')$ in S', $(x_1', \ldots, x_m')$ in R', where

$$n = \textit{trdeg}_k K^* - \textit{trdeg}_k k(V^*),$$

$$m = \textit{trdeg}_k K - \textit{trdeg}_k k(V),$$

$\nu(x_1'),\ldots,\nu(x_{\overline{r}}')$ is a rational basis of $\Gamma_\nu\otimes\mathbf{Q}$, $\nu^*(y_1'),\ldots,\nu^*(y_{\overline{s}}')$ is a rational basis of $\Gamma_{\nu^*}\otimes\mathbf{Q}$, there are units $\delta_1,\ldots,\delta_{\overline{r}}\in S'$ and an $\overline{r}\times\overline{s}$ matrix (c_{ij}) of nonnegative integers such that (c_{ij}) has rank $\overline{r}$, and

$$\begin{aligned} x_1' &= (y_1')^{c_{11}}\cdots(y_{\overline{s}}')^{c_{1\overline{s}}}\delta_1\\ &\vdots\\ x_{\overline{r}}' &= (y_1')^{c_{\overline{r}1}}\cdots(y_{\overline{s}}')^{c_{\overline{r}\overline{s}}}\delta_{\overline{r}}\\ x_{\overline{r}+1}' &= y_{\overline{s}+1}'\\ &\vdots\\ x_m' &= y_{\overline{s}+m-\overline{r}}'. \end{aligned}$$

Proof. $k(V)$ and $k(V^*)$ have finite transcendence degree over k by Theorem 1 [2] or Appendix 2 [39]. We have rank $\nu\le$ rank $\nu^*=1$. By Hironaka's theorems on resolution, resolution of singularities Theorem $I_2^{m,n}$ [26] (c.f. Theorem 2.9 [14]) and resolution of indeterminancy (c.f. Theorem 2.6 [C1], the statement and proof of Theorem 2.6 are valid if R is not regular) we can assume that R and S are regular local rings.

By resolution of indeterminancy (c.f. Theorem 2.6 [14]) and Theorem 2.7 [14], applied to a lift to V of a transcendence basis of $k(V)$ over k, and Theorem 2.7 [14] applied to a lift to V^* of a transcendence basis of $k(V^*)$ over k, there exist MTSs along ν^* $R\to R(1)$ and $S\to S(1)$ such that $R(1)$ and $S(1)$ are regular local rings, V^* dominates $S(1)$, $S(1)$ dominates $R(1)$ and

$$\mathrm{trdeg}_{k(R(1))}k(V)=0,$$

$$\mathrm{trdeg}_{k(S(1))}k(V^*)=0.$$

First assume that rank $\nu=1$. Let $\{t_1,\ldots,t_\beta\}$ be a lift of a transcendence basis of $k(R(1))$ over k to $R(1)$. Let $L=k(t_1,\ldots,t_\beta)\subset R(1)$. By replacing k with L, we may assume that

$$\mathrm{trdeg}_k k(R(1))=\mathrm{trdeg}_k k(V)=0.$$

There exist $f_1,\ldots,f_{\overline{r}}\in K$ such that $\nu(f_1),\ldots,\nu(f_{\overline{r}})$ are positive and rationally independent. By Theorem 2.7 [14], there exists a MTS $R(1)\to R(2)$ along ν such that $f_1,\ldots,f_{\overline{r}}\in R(2)$. By Theorem 2.5 [14], there exists a MTS $R(2)\to R(3)$ along ν such that $f_1\cdots f_{\overline{r}}$ is a SNC divisor in $R(3)$. Thus $R(3)$ has regular parameters $(x_1(3),\ldots,x_m(3))$ such that $\nu(x_1(3)),\ldots,\nu(x_{\overline{r}}(3))$ are a rational basis of $\Gamma_\nu\otimes\mathbf{Q}$.

By Theorem 2.6, there exists a MTS $S(1)\to S(2)$ along ν^* such that $S(2)$ dominates $R(3)$.

As in the construct of $R(1)\to R(3)$ there exists a MTS $S(2)\to S(3)$ along ν^* such that $S(3)$ has regular parameters $(y_1(3),\ldots,y_n(3))$ such that $\nu^*(y_1(3)),\ldots,\nu^*(y_{\overline{s}}(3))$ are a rational basis of $\Gamma_{\nu^*}\otimes\mathbf{Q}$.

By (38) of Theorem 9.1, with the R, S, f, $\overline{m}$, l of the hypothesis of that theorem set to $R=S(3)$ and $S=S(3)$, $f=x_1(3)\cdots x_{\overline{r}}(3)$, $\overline{m}=n-\overline{s}$, $l=0$, and by (4) of Theorem 9.3, there exists a MTS $S(3)\to S(4)$ along ν^* such that

$$x_i(3)=y_1(4)^{c_{i1}}\cdots y_{\overline{s}}(4)^{c_{i\overline{s}}}\psi_i$$

where $\psi_i\in S(4)$ are units for $1\le i\le\overline{r}$, $\nu^*(y_1(4)),\ldots,\nu^*(y_{\overline{s}}(4))$ are a rational basis of $\Gamma_{\nu^*}\otimes\mathbf{Q}$, and rank $(c_{ij})=\overline{r}$.

After possibly permuting the first $\overline{s}$ variables $y_i(4)$, we may assume that the matrix

$$\tilde{C}=\begin{pmatrix} c_{11} & \cdots & c_{1\overline{r}}\\ \vdots & & \vdots\\ c_{\overline{r}1} & \cdots & c_{\overline{r}\overline{r}}\end{pmatrix}$$

has nonzero determinant.

Let ϕ_i be the residue of ψ_i in $k(S(4))$. For $1 \le i \le \overline{r}$, set

$$\epsilon_i = \prod_{j=1}^{\overline{r}} \left(\frac{\psi_j}{\phi_j}\right)^{e_{ij}}$$

where $(e_{ij}) = \tilde{C}^{-1}$, a matrix with rational coefficients. We have $\epsilon_j \in S(4)\hat{}$ for $1 \le j \le \overline{r}$.

Set

$$\overline{y}_j(4) = \begin{cases} \epsilon_j y_j(4) & 1 \le j \le \overline{r} \\ y_j(4) & \overline{r} < j \end{cases}$$

we have

$$x_i(3) = \prod_{j=1}^{\overline{s}} \overline{y}_j(4)^{c_{ij}} \phi_i$$

for $1 \le i \le \overline{r}$.

In the notation of Theorem 9.4, set $R = R(3)$, $T''(0) = R(3)$, $x_i = x_i(3)$ for $1 \le i \le m$, $\overline{x}_i = x_i(3)$ for $1 \le i \le m$, $c_0 = 1$, $\tilde{R} = k[x_1(3), \dots, x_{\overline{r}}(3)]_q$ where $q = m(R(3)) \cap k[x_1(3), \dots, x_{\overline{r}}(3)]$, $\gamma_i = 1$ for $1 \le i \le \overline{r}$. Set $S = S(4)$, $U''(0) = S(4)[d_0, \epsilon_1, \dots, \epsilon_{\overline{r}}]_p$ where $k(d_0) \cong k(S(4))$, $p = m(S(4)\hat{}) \cap S(4)[d_0, \epsilon_1, \dots, \epsilon_{\overline{r}}]$, $y_i = y_i(4)$ for $1 \le i \le n$, $\overline{y}_i = \overline{y}_i(4)$ for $1 \le i \le n$.

Then the assumptions of Theorem 9.4 are satisfied with $l = 0$. By induction on l in Theorem 9.4, we construct the desired MTSs, and finish the proof of the Theorem when rank $\nu = 1$.

If rank $\nu = 0$, then ν is trivial, so that $V = K$, $R = K$ and $\overline{r} = 0$. We can then construct a MTS $S \to S'$ along ν^* as in the first part of the proof, so that S' has a regular system of parameters $(y_1', \dots, y_n')$ such that $\nu^*(y_1'), \dots, \nu^*(y_{\overline{s}}')$ is a rational basis of $\Gamma_{\nu^*} \otimes \mathbf{Q}$ and reach the conclusions of the theorem (see Remark 10.2 below). ☐

Remark 10.2. *The degenerate case of Theorem 10.1 is when ν is trivial on K. This case can only occur if K^* is transcendental over K. When ν is trivial, $V = K$, so we must have that $R = K$. The conclusions of Theorem 10.1 are in this case that there exists a sequence of monoidal transforms $S \to S'$ along ν^* such that S' is a regular local ring (which contains K), and there exist regular parameters $(y_1', \dots, y_n')$ in S', where*

$$n = \mathit{trdeg}_k K^* - \mathit{trdeg}_k k(V^*),$$

such that $\nu^*(y_1'), \dots, \nu^*(y_n')$ is a rational basis of $\Gamma_{\nu^*} \otimes \mathbf{Q}$.

We now introduce notation that will be used in the proof of Theorem 10.5. Suppose that k is a field of characteristic zero, $K \to K^*$ is a (possibly transcendental) extension of algebraic function fields over k. Suppose that ν^* is a valuation of K^* of arbitrary (but necessarily finite) rank which is trivial on k. Suppose that R is an algebraic local ring of K, S is an algebraic local ring of K^* such that S dominates R and ν^* dominates S. Let $\nu = \nu^* \mid K$. Let V^* be the valuation rings of ν^* and V be the valuation ring of ν.

Lemma 10.3. *Suppose that p is a prime ideal of V. Then there exists a prime ideal q of V^* such that $q \cap V = p$.*

Proof. By Theorem 15 of Section 10, Chapter VI [39], there exists an isolated subgroup Δ of Γ_ν such that

$$p = \{a \in K \mid \nu(a) = \beta \text{ for some } \beta \in \Gamma_\nu - \Delta \text{ with } \beta \ge 0\} \cup \{0\}.$$

Set

$$\Delta^* = \{\beta \in \Gamma_{\nu^*} \mid\mid \beta \mid\leq\mid \alpha \mid \text{ for some } \alpha \in \Delta\}.$$

Δ^* is an isolated subgroup of Γ_{ν^*}. Theorem 15 of Section 10, Chapter VI [39] implies that

$$q \in \{a \in K \mid \nu(a) = \beta \text{ for some } \beta \in \Gamma_{\nu^*} - \Delta^* \text{ with } \beta \geq 0\} \cup \{0\}$$

is a prime ideal of V^*.

$$(\Gamma_{\nu^*} - \Delta^*) \cap \Gamma_\nu = \Gamma_\nu - \Delta$$

implies $q \cap V = p$. □

Let $\beta = \text{rank } V$. The primes of V are a finite chain

$$0 = p_0 \subset \cdots \subset p_\beta \subset V.$$

Note that if $\beta = 0$ then $V = K$. The primes of V^* are a finite chain

$$0 = q_{0,1} \subset \cdots \subset q_{0,\sigma(0)} \subset q_{1,1} \subset \cdots \subset q_{\beta,\sigma(\beta)}$$

where $p_i = q_{i,j} \cap V$ for $1 \leq j \leq \sigma(i)$.

The isolated subgroups of Γ_ν are a chain

$$0 = \Gamma_\beta \subset \cdots \subset \Gamma_0 = \Gamma_\nu.$$

There is a corresponding chain of isolated subgroups of Γ_{ν^*}

$$0 = \Gamma_{\beta,\sigma(\beta)} \subset \cdots \subset \Gamma_{0,1} = \Gamma_{\nu^*}.$$

For $i \leq j$, ν induces a valuation on the field $(V/p_i)_{p_i}$ with valuation ring $(V/p_i)_{p_j}$ and value group Γ_i/Γ_j. If $j = i+1$ then Γ_i/Γ_j has rank 1. For $i < j$, $1 \leq a \leq \sigma(i)$ and $1 \leq b \leq \sigma(j)$, ν^* induces a valuation on the field $(V^*/q_{i,a})_{q_{i,a}}$ with valuation ring $(V^*/q_{i,a})_{q_{j,b}}$ and value group $\Gamma_{i,a}/\Gamma_{j,b}$. If $i = j$ and $1 \leq a \leq b \leq \sigma(i)$, ν^* induces a valuation on the field $(V^*/q_{i,a})_{q_{i,a}}$ with valuation ring $(V^*/q_{i,a})_{q_{i,b}}$ and value group $\Gamma_{i,a}/\Gamma_{i,b}$.

We have dominant inclusions of valuation rings

$$(V/p_i)_{p_j} \to (V^*/q_{ia})_{q_{jb}}$$

if $i < j$, $1 \leq a \leq \sigma(i)$, $1 \leq b \leq \sigma(j)$ which induce inclusions of valuation groups

$$\Gamma_i/\Gamma_j \to \Gamma_{i,a}/\Gamma_{j,b}.$$

We also have dominant inclusions of valuation rings

$$(V/p_i)_{p_i} \to (V^*/q_{ia})_{q_{ib}}$$

if $i = j$, $1 \leq a \leq b \leq \sigma(i)$. Note that the value group of the field $(V/p_i)_{p_i}$ is $\Gamma_i/\Gamma_i = 0$.

Lemma 10.4. *There exist MTSs along* ν^*

$$\begin{array}{ccc} R' & \rightarrow & S' \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

such that R' *and* S' *are regular,*

$$\text{trdeg}_{k(R'_{p_i \cap R'})} k(V_{p_i}) = 0$$

for all i *and*

$$\text{trdeg}_{k(S'_{q_{ij} \cap S'})} k(V^*_{q_{ij}}) = 0$$

for all i, j.

Proof. By Hironaka's theorem on resolution of singularities (Theorem $I_2^{m,n}$[26] or Theorem 2.9 [14]) and resolution of indeterminancy (c.f. Theorem 2.6 [14], the statement and proof are valid if R is not regular) we can assume that R and S are regular local rings.

For all i, V_{p_i} is a valuation ring of K dominating $R_{p_i\cap R}$. Thus

$$\mathrm{trdeg}_{(R/p_i\cap R)_{p_i\cap R}}(V/p_i)_{p_i} < \infty$$

by Theorem 1 [2] or Appendix 2 [39]. We can lift transcendence bases of $(V/p_i)_{p_i}$ over $(R/p_i\cap R)_{p_i\cap R}$ for $1\le i\le\beta$ to $t_1,\ldots,t_a\in V$. After possibly replacing the t_i with $\frac{1}{t_i}$, we have $\nu(t_i)\ge 0$ for all t_i.

By Theorem 2.7 [14], there exists a MTS $R\rightarrow R'$ along ν such that $t_i\in R'$ for all i. Let $p_i'=R'\cap p_i$. Then

$$\mathrm{trdeg}_{(R'/p_i')_{p_i'}}(V/p_i)_{p_i}=0 \text{ for } 1\le i\le\beta.$$

By Theorem 2.6 [14], there exists a MTS $S\rightarrow S''$ along ν^* such that S'' dominates R'. As argued above for R, there exists a MTS $S''\rightarrow S'$ along ν^* such that if $q_{ij}'=S'\cap q_{ij}$, then

$$\mathrm{trdeg}_{(S'/q_{ij}')_{q_{ij}'}}(V^*/q_{ij})_{q_{ij}}=0 \text{ for all } i,j.$$

□

Theorem 10.5. *Let notation be as above. Suppose that R and S are regular,*

$$trdeg_{k(R_{p_i\cap R})}k(V_{p_i})=0$$

for all i and

$$trdeg_{k(S_{q_{ij}\cap S})}k(V^*_{q_{ij}})=0$$

for all i,j. Suppose that the rank 1 valuation groups Γ_{i-1}/Γ_i has rational rank $\overline{r}_i$ for $1\le i\le\beta$, $\Gamma_{i-1,\sigma(i-1)}/\Gamma_{i,1}$ have rational rank $\overline{s}_i=\overline{s}_{i1}$ for $1\le i\le\beta$ and $\Gamma_{i,a-1}/\Gamma_{i,a}$ have rational rank $\overline{s}_{ia}$ for $1\le i\le\beta$ and $2\le a\le\sigma(i)$.

Set $t_i=dim(R/p_{i-1}\cap R)_{p_i\cap R}$ for $1\le i\le\beta$, so that

$$m=dim\ R=t_1+\cdots+t_\beta.$$

Set

$$\overline{t}_{ij}=\begin{cases}0 & \text{if } i=0, j=1\\ dim\ (S/q_{i-1,\sigma(i-1)}\cap S)_{q_{i,1}\cap S} & \text{if } 1\le i\le\beta, j=1\\ dim\ (S/q_{i,j-1}\cap S)_{q_{i,j}\cap S} & \text{if } 0\le i\le\beta, 2\le j\le\sigma(i).\end{cases}$$

For $1\le i\le\beta$ set

$$\overline{t}_i=\overline{t}_{i,1}+\cdots+\overline{t}_{i,\sigma(i)}=dim\ (S/q_{i-1,\sigma(i-1)}\cap S)_{q_{i\sigma(i)}\cap S},$$

and set

$$\overline{t}_0=\overline{t}_{0,1}+\cdots+\overline{t}_{0,\sigma(0)}=\ dim\ S_{q_{0,\sigma(0)}\cap S}$$

so that $n=\ dim\ S=\overline{t}_0+\cdots+\overline{t}_\beta$.

Then there exist sequences of monoidal transforms $R\rightarrow R'$ and $S\rightarrow S'$ such that V^* dominates S', S' dominates R', R' has regular parameters $(z_1,\ldots,z_m)$, S' has regular parameters $(w_1,\ldots,w_n)$ and there are units $\delta_j\in S'$ such that

$$p_i\cap R'=(z_1,\ldots,z_{t_1+\cdots+t_i})$$

for $1\le i\le\beta$ and

$$q_{ij}\cap S'=(w_1,\ldots,w_{\overline{t}_0+\cdots+\overline{t}_{i-1}+\overline{t}_{i,1}+\cdots+\overline{t}_{i,j}})$$

for $0\le i\le\beta$ and $1\le j\le\sigma(i)$.

$$\nu(z_{t_1+\cdots+t_{i-1}+1}),\cdots,\nu(z_{t_1+\cdots+t_{i-1}+\overline{r}_i})$$

is a rational basis of $\Gamma_{i-1}/\Gamma_i \otimes \mathbf{Q}$ *for* $1 \le i \le \beta$,

$$\nu^*(w_{\overline{t}_0+\cdots+\overline{t}_{i-1}+1}), \ldots, \nu^*(w_{\overline{t}_0+\cdots+\overline{t}_{i-1}+\overline{s}_i})$$

is a rational basis of $\Gamma_{i-1,\sigma(i-1)}/\Gamma_{i,1} \otimes \mathbf{Q}$ *for* $1 \le i \le \beta$ *and*

$$\nu^*(w_{\overline{t}_0+\cdots+\overline{t}_{i-1}+\overline{t}_{i,1}+\cdots+\overline{t}_{i,j-1}+1}), \ldots, \nu^*(w_{\overline{t}_0+\cdots+\overline{t}_{i-1}+\overline{t}_{i,1}+\cdots+\overline{t}_{i,j}+\overline{s}_{i,j}})$$

is a rational basis of $\Gamma_{i,j-1}/\Gamma_{i,j} \otimes \mathbf{Q}$ *for* $0 \le i \le \beta$ *and* $2 \le j \le \sigma(i)$. *Furthermore*

$$\begin{aligned}
z_1 &= w_{\overline{t}_0+1}^{g_{11}(1)} \cdots w_{\overline{t}_0+\overline{s}_1}^{g_{1\overline{s}_1}(1)} w_{\overline{t}_0+\overline{t}_{1,1}+1}^{h_{1,\overline{t}_0+\overline{t}_{1,1}+1}(1)} \cdots w_n^{h_{1,n}(1)} \delta_1 \\
&\vdots \\
z_{\overline{r}_1} &= w_{\overline{t}_0+1}^{g_{\overline{r}_1,1}(1)} \cdots w_{\overline{t}_0+\overline{s}_1}^{g_{\overline{r}_1,\overline{s}_1}(1)} w_{\overline{t}_0+\overline{t}_{1,1}+1}^{h_{t_1,\overline{t}_0+\overline{t}_{1,1}+1}(1)} \cdots w_n^{h_{\overline{r}_1,n}(1)} \delta_{\overline{r}_1} \\
z_{\overline{r}_1+1} &= w_{\overline{t}_0+\overline{s}_1+1} w_{\overline{t}_0+\overline{t}_{1,1}+1}^{h_{\overline{r}_1+1,\overline{t}_0+\overline{t}_{1,1}+1}(1)} \cdots w_n^{h_{\overline{r}_1+1,n}(1)} \delta_{\overline{r}_1+1} \\
&\vdots \\
z_{t_1} &= w_{\overline{t}_0+\overline{s}_1+t_1-\overline{r}_1} w_{\overline{t}_0+\overline{t}_{1,1}+1}^{h_{t_1,\overline{t}_0+\overline{t}_{1,1}+1}(1)} \cdots w_n^{h_{t_1,n}(1)} \delta_{t_1} \\
z_{t_1+1} &= w_{\overline{t}_0+\overline{t}_1+1}^{g_{1,1}(2)} \cdots w_{\overline{t}_0+\overline{t}_1+\overline{s}_2}^{g_{1,\overline{s}_2}(2)} w_{\overline{t}_0+\overline{t}_1+\overline{t}_{2,1}+1}^{h_{1,\overline{t}_0+\overline{t}_1+\overline{t}_{2,1}+1}(2)} \cdots w_n^{h_{1,n}(2)} \delta_{t_1+1} \\
&\vdots \\
z_{t_1+\overline{r}_2} &= w_{\overline{t}_0+\overline{t}_1+1}^{g_{\overline{r}_2,1}(2)} \cdots w_{\overline{t}_0+\overline{t}_1+\overline{s}_2}^{g_{\overline{r}_2,\overline{s}_2}(2)} w_{\overline{t}_0+\overline{t}_1+\overline{t}_{2,1}+1}^{h_{\overline{r}_2,\overline{t}_0+\overline{t}_1+\overline{t}_{2,1}+1}(2)} \cdots w_n^{h_{\overline{r}_2,n}(2)} \delta_{t_1+\overline{r}_2} \\
z_{t_1+\overline{r}_2+1} &= w_{\overline{t}_0+\overline{t}_1+\overline{s}_2+1} w_{\overline{t}_0+\overline{t}_1+\overline{t}_{2,1}+1}^{h_{\overline{r}_2+1,\overline{t}_0+\overline{t}_1+\overline{t}_{2,1}+1}(2)} \cdots w_n^{h_{\overline{r}_2+1,n}(2)} \delta_{t_1+\overline{r}_2+1} \\
&\vdots \\
z_{t_1+t_2} &= w_{\overline{t}_0+\overline{t}_0+t_2+\overline{s}_2-\overline{r}_2} w_{\overline{t}_0+\overline{t}_1+\overline{t}_{2,1}+1}^{h_{t_2,\overline{t}_0+\overline{t}_1+\overline{t}_{2,1}+1}(2)} \cdots w_n^{h_{t_2,n}(2)} \delta_{t_1+t_2} \\
&\vdots \\
z_{t_0+\cdots+t_{\beta-1}+1} &= w_{\overline{t}_0+\cdots+\overline{t}_{\beta-1}+1}^{g_{11}(\beta)} \cdots w_{\overline{t}_0+\cdots+\overline{t}_{\beta-1}+\overline{s}_\beta}^{g_{1,\overline{s}_\beta}(\beta)} w_{\overline{t}_0+\cdots+\overline{t}_{\beta,1}+1}^{h_{1,\overline{t}_0+\cdots+\overline{t}_{\beta,1}+1}(\beta)} \cdots w_n^{h_{1,n}(\beta)} \delta_{t_1+\cdots+t_{\beta-1}+1} \\
&\vdots \\
z_{t_1+\cdots+t_{\beta-1}+\overline{r}_\beta} &= w_{\overline{t}_0+\cdots+\overline{t}_{\beta-1}+1}^{g_{\overline{r}_\beta,1}(\beta)} \cdots w_{\overline{t}_0+\cdots+\overline{t}_{\beta-1}+\overline{s}_\beta}^{g_{\overline{r}_\beta,\overline{s}_\beta}(\beta)} w_{\overline{t}_0+\cdots+\overline{t}_{\beta,1}+1}^{h_{\overline{r}_\beta,\overline{t}_0+\cdots+t_{\beta,1}+1}(\beta)} \cdots w_n^{h_{\overline{r}_\beta,n}(\beta)} \delta_{t_1+\cdots+t_{\beta-1}+\overline{r}_\beta} \\
z_{t_1+\cdots+t_{\beta-1}+\overline{r}_\beta+1} &= w_{\overline{t}_0+\cdots+\overline{t}_{\beta-1}+\overline{s}_\beta+1} w_{\overline{t}_0+\cdots+\overline{t}_{\beta,1}+1}^{h_{\overline{r}_\beta+1,\overline{t}_0+\cdots+\overline{t}_{\beta,1}+1}(\beta)} \cdots w_n^{h_{\overline{r}_\beta+1,n}(\beta)} \delta_{t_1+\cdots+t_{\beta-1}+\overline{r}_\beta+1} \\
z_{t_1+\cdots+t_\beta} &= w_{\overline{t}_0+\cdots+\overline{t}_{\beta-1}+t_\beta+\overline{s}_\beta-\overline{r}_\beta} w_{\overline{t}_0+\cdots+\overline{t}_{\beta,1}+1}^{h_{t_\beta,\overline{t}_0+\cdots+\overline{t}_{\beta,1}+1}(\beta)} \cdots w_n^{h_{t_\beta,n}(\beta)} \delta_{t_1+\cdots+t_\beta}
\end{aligned}$$

Where

$$\text{rank} \begin{pmatrix} g_{11}(i) & \cdots & g_{1\overline{s}_i}(i) \\ \vdots & & \vdots \\ g_{\overline{r}_i1}(i) & \vdots & g_{\overline{r}_i\overline{s}_i}(i) \end{pmatrix} = \overline{r}_i$$

for $1 \le i \le \beta$.

Proof. We prove the theorem by induction on rank V^*. If rank $V^* = 1$, then the theorem is immediate from Theorem 10.1.

By induction on $\gamma =$ rank V^*, we may assume that the theorem is true whenever rank $V^* = \gamma - 1$. We are reduced to proving the theorem in the following two cases.

Case 1: $\sigma(\beta) = 1$

Case 2: $\sigma(\beta) > 1$.

Suppose that we are in Case 1. Then $V^*/q_{\beta-1,\sigma(\beta-1)}$ is a rank 1 valuation ring which dominates the rank 1 valuation ring $V/p_{\beta-1}$. $V^*/q_{\beta-1,\sigma(\beta-1)}$ has rational rank $\overline{s}_\beta$ and $V/p_{\beta-1}$ has rational rank $\overline{r}_\beta$.

The proof of Theorem 10.5 in case 1 follows from the proof of Theorem 5.3 [14] with some changes in notation and references to supporting lemmas and theorems. We must replace r with β, $p_{r-1}(i)$ with $p_{\beta-1}\cap R(i)$ and $q_{r-1}(i)$ with $q_{\beta-1,\sigma(\beta-1)}\cap S(i)$ throughout the proof. Then $R(i)_{p_{\beta-1}(i)}$ has a system of $\lambda = t_1+\cdots+t_{\beta-1}$ regular parameters, while $S(i)_{q_{\beta-1}}(i)$ has a system of $\overline{\lambda}=\overline{t}_0+\cdots+\overline{t}_{\beta-1}\geq\lambda$ regular parameters.

References to Theorems 4.8, 4.10 and 5.1 of [14] must be replaced with references to Theorems 9.1, 9.3 and 10.1 of this paper.

Now suppose that we are Case 2, $\sigma(\beta)>1$. Then $V^*/q_{\beta,\sigma(\beta)-1}$ is a rank 1, rational rank $\overline{s}_{\beta,\sigma(\beta)}$ valuation ring which dominates the rank 0 valuation ring V/p_β. That is, V/p_β is a field. The proof in Case 2 is thus a substantial simplification of the proof in Case 1. □

The proof of Theorem 1.4 is now an immediate corollary.

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