RESOLUTION OF MORPHISMS

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1. Introduction

Suppose that X is a nonsingular variety, over an algebraically closed field k of characteristic zero.

If $V \subset X$ is a nonsingular subvariety, the monoidal transform of X with center V is the blow up

$$\pi : Y = \mathrm{Proj}(\oplus_{n\geq 0}\mathcal{I}_V^n) \rightarrow X.$$

If p is a closed point of Y such that $\pi(p) = q$, there exist regular parameters $(x_1,\ldots,x_n)$ at q and regular parameters $(y_1,\ldots,y_n)$ at p such that

$$x_1 = x_2 = \cdots = x_r = 0$$

(with $r \leq n$) are local equations of V at q and

$$x_1 = y_1, x_2 = y_1y_2, \cdots, x_r = y_1y_r, x_{r+1} = y_{r=1}, \ldots, x_n = y_n.$$

If $V = q$, so that $r = n$, π is called a quadratic transform.

Suppose that

$$\Phi : X \rightarrow Y \tag{1}$$

is a morphism of k-varieties, where k is a field of characteristic 0. The structure of Φ is extremely complicated. However, we can hope to construct a commutative diagram

$$\begin{array}{ccc} X_1 & \overset{\Psi}{\rightarrow} & Y_1\\ \downarrow & & \downarrow\\ X & \overset{\Phi}{\rightarrow} & Y \end{array} \tag{2}$$

where the vertical maps are products of monoidal transforms, to obtain a morphism $\Psi : X_1 \rightarrow Y_1$ which has a relatively simple structure.

Definition 1.1. *Suppose that $\Phi : X \rightarrow Y$ is a dominant morphism of nonsingular integral finite type k schemes. Φ is* **monomial** *if for every $p \in X$ there exist regular parameters $(y_1,\ldots,y_m)$ in $\mathcal{O}_{Y,\Phi(p)}$, and an étale cover U of an affine neighborhood of p, uniformizing parameters $(x_1,\ldots,x_n)$ on U and a matrix a_{ij} such that*

$$\begin{array}{rl} y_1 = & x_1^{a_{11}}\cdots x_n^{a_{1n}}\\ & \vdots\\ y_m = & x_1^{a_{m1}}\cdots x_n^{a_{mn}} \end{array}$$

Since Φ is dominant, the matrix (a_{ij}) must have maximal rank m.

Definition 1.2. *Suppose that $\Phi : X \rightarrow Y$ is a dominant morphism of k-varieties. A morphism $\Psi : X_1 \rightarrow Y_1$ is a* **monomialization** *of Φ if there are sequences of*

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monoidal transforms $\alpha : X_1 \to X$ and $\beta : Y_1 \to Y$, and a morphism $\Psi : X_1 \to Y_1$ such that the diagram

$$\begin{array}{ccc} X_1 & \overset{\Psi}{\to} & Y_1 \\ \downarrow & & \downarrow \\ X & \overset{\Phi}{\to} & Y \end{array}$$

commutes, and Ψ is a monomial morphism.

Question *Suppose that $\Phi : X \to Y$ is a dominant morphism of varieties over a field k of characteristic zero. Does there exist a monomialization of Φ?*

By resolution of singularities and elimination of indeterminacy, we easily reduce to the case where X and Y are nonsingular.

The characteristic of k must be zero in the question. If k has characteristic $p > 0$, global monomializations may not exist even for curves.

$$t = x^p + x^{p+1}$$

gives a simple example of a mapping of curves which cannot be monomialized, since $\sqrt[p]{1+x}$ is inseparable over $k[x]$.

The obstruction to monomialization in positive characteristic is thus wild ramification.

In section 3, we discuss our positive answer ([16]) to a local analogue of the Question for generically finite morphisms.

In section 4, we outline proofs of the positive answer to the question in the previously known cases, a morphism to a curve and a morphism of surfaces.

In sections 5 and 6 we outline some aspects of our recent proof of monomialization of proper morphisms from 3 folds to surfaces ([18]).

Theorem 1.3. *(Theorem 18.21,* [18]*) Suppose that $\Phi : X \to S$ is a dominant morphism from a 3 fold X to a surface S (over an algebraically closed field k of characteristic zero). Then there exist sequences of blow ups of nonsingular subvarieties $X_1 \to X$ and $S_1 \to S$ such that the induced map $\Phi_1 : X_1 \to S_1$ is a monomial morphism.*

From this we deduce that it is possible to toroidalize ([26], [8], Definition 19.1 [18]) a dominant morphism from a 3 fold to a surface. A toroidal morphism $X \to Y$ is a morphism which is monomial with respect to fixed SNC divisors on X and Y.

Theorem 1.4. *(Theorem 19.11* [18]*) Suppose that $\Phi : X \to S$ is a dominant morphism from a 3 fold X to a surface S (over an algebraically closed field k of characteristic zero) and D_S is a reduced 1 cycle on S such that $E_X = \Phi^{-1}(D_S)_{red}$ contains sing(X) and sing(Φ). Then there exist sequences of blow ups of nonsingular subvarieties $\pi_1 : X_1 \to X$ and $\pi_2 : S_1 \to S$ such that the induced morphism $X_1 \to S_1$ is a toroidal morphism with respect to $\pi_2^{-1}(D_S)_{red}$ and $\pi_1^{-1}(E_X)_{red}$.*

Suppose that $\Phi : X \to Y$ is a dominant morphism of nonsingular k-varieties, and $\dim(Y) > 1$.

To begin with, we point out that monomialization is not a direct consequence of embedded resolution of singularities and principalization of ideals.

Suppose that $p \in X$ is a point where Φ is not smooth, and $q = \Phi(p)$. Let $(y_1, \ldots, y_m)$ be regular parameters in $\mathcal{O}_{Y,q}$. By standard theorems on resolution, we have a sequence of blow ups of nonsingular subvarieties $\pi : X_1 \to X$ such that if $p_1 \in \pi^{-1}(p)$, then there exist regular parameters $(x_1, \ldots, x_n)$ in $\mathcal{O}_{X_1,p_1}$, a matrix

(a_{ij}) with nonnegative coefficients and units $\delta_1, \dots, \delta_m \in \mathcal{O}_{X_1,p_1}$ such that

$$\begin{aligned} y_1 &= x_1^{a_{11}} \cdots x_n^{a_{1n}} \delta_1 \\ &\vdots \\ y_m &= x_1^{a_{m1}} \cdots x_n^{a_{mn}} \delta_n \end{aligned} \tag{3}$$

In general, p_1 will lie on a single exceptional component of π, and p_1 will be disjoint from the strict transforms of codimension 1 subschemes of X determined by $y_i = 0$, $1 \le i \le m$, on a neighborhood of $\Phi^{-1}(q)$. In this case we will have $a_{ij} = 0$ if $j > 1$, since the $x_i = 0$ are either local equations of exceptional components of $X_1 \to X$ or are local equations of the strict transforms of irreducible components of $y_i = 0$. Thus (a_{ij}) will have rank 1.

There thus cannot exist regular parameters $(\overline{x}_1, \dots, \overline{x}_n)$ in $\hat{\mathcal{O}}_{X_1,p_1}$ such that

$$\begin{aligned} y_1 &= \overline{x}_1^{a_{11}} \cdots \overline{x}_n^{a_{1n}} \\ &\vdots \\ y_m &= \overline{x}_1^{a_{m1}} \cdots \overline{x}_n^{a_{mn}} \end{aligned}$$

since this would imply that $\mathrm{rank}(a_{ij}) = m > 1$. $\mathrm{rank}(a_{ij}) < m$ would imply, by Zariski's subspace theorem [4], that Φ is not dominant.

In fact, in general it is necessary to blow up in both X and Y to construct a monomialization. For instance, if we blow up a point p on a nonsingular surface S, blow up a point on the exceptional curve E_1, blow up the intersection point of the new exceptional curve E_2 with the strict transform of E_1, then blow up a general point on the new exceptional curve E_3 with exceptional curve E_4, we get a birational map $\pi : S_1 \to S$ such that if $p_1 \in E_4$ is a general point we have regular parameters (u, v) in $\mathcal{O}_{S,p}$ and regular parameters (x, y) in $\hat{\mathcal{O}}_{S_1,p_1}$ such that

$$u = x^2, v = \alpha x^3 + x^4 y.$$

π is not monomial at p_1 and further blow ups over S_1 will produce a morphism which is further from being monomial.

Suppose that Y is a nonsingular surface. If $\pi_2 : Y_1 \to Y$ is a sequence of blow ups of points over $q \in Y$, and $q_1 \in \pi_2^{-1}(q)$ is a point which only lies on a single exceptional component E of π_2, then there exist regular parameters (u, v) in $\mathcal{O}_{Y,q}$ and $(\overline{x}, \overline{y})$ in $\hat{\mathcal{O}}_{Y_1,q_1}$ such that

$$\begin{aligned} u &= \overline{x}^a \\ v &= P(\overline{x}) + \overline{x}^b \overline{y} \end{aligned} \tag{4}$$

where $a, b \in \mathbf{N}$ and $P(\overline{x})$ is a polynomial of degree $\le b$. This follows since $Y_1 \to Y$ can be factored near p_1 by sequences of blow ups of the form

$$x_i = x_{i+1}^{a_{i+1}}, y_i = x_{i+1}^{b_{i+1}}(x_{i+1} + \alpha_{i+1}).$$

If we perform a sequence of blow ups of nonsingular subvarieties $\pi_1 : X_1 \to X$, and if $p_1 \in (\Phi \circ \pi_1)^{-1}(q)$ is such that $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters $(\overline{x}_1, \overline{x}_2, \overline{x}_3, \cdots, \overline{x}_n)$ such that

$$\begin{aligned} u &= \overline{x}_1^a \\ v &= P(\overline{x}_1) + \overline{x}_1^b \overline{x}_2 \end{aligned} \tag{5}$$

of the form of (4), we will have a factorization $X_1 \to S_1$ which is a morphism in a neighborhood of p_1, and $X_1 \to S_1$ will be monomial at p_1.

A strategy for monomializing a dominant morphism from a nonsingular variety X to a nonsingular surface S is thus to first perform a sequence of blow ups of nonsingular subvarieties $\pi_1 : X_1 \to X$ so that for all points p of X_1, appropriate regular parameters

(u,v) in $\mathcal{O}_{S_1,q}$ where $q = \Phi \circ \pi_1(p)$ will have simple forms which we will call prepared, which include the form of (5). This is accomplished if $\dim(X) = 3$ in Theorem 17.3 of [18]. Almost the entirety of [18] is devoted to proving this theorem.

An interesting case when the existence of a global monomialization is still open is for birational morphisms of nonsingular, characteristic 0 varieties of dimension ≥ 3. Such birational maps are known to have a simple structure, since they can be factored by alternating sequences of blow ups and blowdowns [7]. A local form of factorization along a valuation was proven earlier by us in Theorem 1.6 [16]. Our local proof of "strong factorization" in dimension 3 appears in [15].

2. Local Uniformization and Resolution of Singularities

A basic problem is to prove Local Uniformization (resolution of singularities) along an arbitrary valuation. This was first proved in characteristic zero by Zariski in 1940.

Theorem 2.1. *(Local Uniformization, Zariski* [37]*) Suppose that K is a field of algebraic functions over a field k of characteristic zero, and V is a valuation ring of K (containing k). Then there exists a regular local ring which is essentially of finite type over k, with quotient field K, such that V dominates R ($R \subset V$ and $m_V \cap R = m_R$).*

The proof is by a case by case analysis of the different types of valuations which can occur.

A quasi-complete variety over a field k is an integral finite type k-scheme which satisfies the existence part of the valuative criterion for properness (c.f. Chapter 0 [21]). Quasi-complete and separated is equivalent to proper. In arbitrary dimension, Zariski proves resolution by a quasi-complete variety.

Theorem 2.2. *(Zariski) If X is a proper variety of characteristic zero, then there exists a complete nonsingular variety Y, and a birational morphism $Y \rightarrow X$.*

The theorem follows from Local Uniformization, Theorem 2.1, and the compactness of the Zariski Riemann manifold [39].

Resolution of singularities in all dimensions and characteristic zero has been proven by Hironaka [21]. Resolution in dimension ≤ 3 and positive characteristic has been proven by Abhyankar [4].

Some recent papers on local uniformization and resolutions of singularities are Heinzer, Rotthaus, Wiegand [20], Kuhlmann [27], Teissier [34]. Some other proofs of resolution are Hironaka [22], Lipman [29], Hironaka [23], Spivakovsky [33], Villamayor [35], Moh [30], Bierstone and Milman [11], De Jong [24], Abromovich and De Jong [6], Bogomolov and Pantev [12]. Lipman's article [28] is an excellent introduction to resolution of singularities.

Let K/k be an algebraic function field,

$$Z(K) = \{ \text{ The set of valuation rings } V \text{ of } K\}.$$

$Z(K)$ is a ringed space, with $\mathcal{O}_{Z(K),V} = V$. We can define a map $\pi_X : Z(K) \rightarrow X$ for any model X of K by $\pi_X(V) = p$ if the center of V on X is p. We can define a topology on $Z(K)$ by taking as a basis for the topology the open sets $\pi_X^{-1}(U)$ where X is a model of K, and U is an open subset of X. The maps π_X are morphisms of ringed spaces. If $f : X \rightarrow Y$ is a birational map of models of K, then we have $f \circ \pi_X = \pi_Y$. For any valuation ring V of $Z(K)$, we then have inclusions of local rings

$$\mathcal{O}_{X,p_1} \subset V \supset \mathcal{O}_{Y,p_2}$$

where p_1 is the center of V on X, and p_2 is the center of V on Y. This gives a local formulation of a rational map in commutative algebra.

If K has dimension 1, then $Z(K) = X$ where X is the unique nonsingular model of K. If $\dim(K) > 1$, then we know that K has nonnoetherian valuations ring, so $Z(K)$ cannot be a variety. However, $Z(K)$ is quasi-compact, ([39]; Theorem 40, Section 17, Chapter VI [41]).

The Zariski-Riemann manifold can be used to convert a birational problem of projective varieties into

(1) A local analogue of the problem along an arbitrary valuation, and
(2) A patching problem

The method is to first solve the problem locally along a valuation. Then for each valuation ring V of K, there exists a model X_V of K such that the problem has been solved at the center of V on X_V. Thus the problem has been solved in some open neighborhood U_V of the center of V in X_V. $\{\pi_{X_V}^{-1}(U_V)\}$ is an open cover of $Z(K)$. Since $Z(K)$ is quasi-compact, there exists a finite subcover, say $\{\pi_{X_{V_1}}^{-1}(U_{V_1}), \cdots, \pi_{X_{V_m}}^{-1}(U_{V_m})\}$.

If we can now patch $V_1, \ldots, V_m$ to obtain a separated algebraic variety V, then we will know (by the valuative criterion for properness) that V is proper. By construction, we will have a global solution to our problem.

There is an easy partial solution to the patching problem. We can always patch the V_i along the open sets where they are isomorphic. This is a finite type k scheme, with ring of rational functions K. The only problem is that it will probably not be separated! The difficulty is that the valuative criterion for separatedness could fail. However, the existence part of the valuative criterion for properness will hold, so it is quasi-complete.

3. Local Monomialization

Suppose that $R \subset S$ is a local homomorphism of local rings essentially of finite type over a field k and that V is a valuation ring of the quotient field K of S, such that V dominates S. Then we can ask if there are sequences of monoidal transforms $R \to R'$ and $S \to S'$ such that V dominates S', S' dominates R', and $R \to R'$ is a monomial mapping.

$$\begin{array}{ccc} R' & \to & S' \subset V \\ \uparrow & & \uparrow \\ R & \to & S \end{array} \tag{6}$$

Theorem 3.1. *(Monomialization)(Theorem 1.1 [16]) Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field k of characteristic zero, such that the quotient field K of S is a finite extension of the quotient field J of R.*

Let V be a valuation ring of K which dominates S. Then there exist sequences of monoidal transforms $R \to R'$ and $S \to S'$ such that V dominates S', S' dominates R' and there are regular parameters $(x_1, ..., x_n)$ in R', $(y_1, ..., y_n)$ in S', units $\delta_1, \ldots, \delta_n \in S'$ and a matrix (a_{ij}) of nonnegative integers such that $Det(a_{ij}) \neq 0$ and

$$\begin{array}{rcl} x_1 & = & y_1^{a_{11}} \cdots y_n^{a_{1n}} \delta_1 \\ & \vdots & \\ x_n & = & y_1^{a_{n1}} \cdots y_n^{a_{nn}} \delta_n. \end{array} \tag{7}$$

Thus (since $\mathrm{char}(k) = 0$) there exists an étale extension $S' \to S''$ where S'' has regular parameters $\overline{y}_1, \ldots, \overline{y}_n$ such that $x_1, \ldots, x_n$ are pure monomials in $\overline{y}_1, \ldots, \overline{y}_n$.

The standard theorems on resolution of singularities allow one to easily find R' and S' such that (7) holds, but, in general, we will not have the essential condition $\mathrm{Det}(a_{ij}) \neq 0$. The difficulty of the problem is to achieve this condition.

It is an interesting open problem to prove Theorem 3.1 in positive characteristic, even in dimension 2. Theorem 3.1 implies simultaneous resolution from above [17], which is a key step in a program of Abhyankar's for proving resolution in positive characteristic. This method is completely worked out by Abhyankar in dimension 2 [1].

The construction of a monomialization by quasi-complete varieties follows from Theorem 3.1.

Theorem 3.2. *(Theorem 1.2* [16]*) Let k be a field of characteristic zero, $\Phi : X \to Y$ a generically finite morphism of nonsingular proper k-varieties. Then there are birational morphisms of nonsingular quasi-complete k-varieties $\alpha : X_1 \to X$ and $\beta : Y_1 \to Y$, and a locally monomial morphism $\Psi : X_1 \to Y_1$ such that the diagram*

$$\begin{array}{ccc} X_1 & \stackrel{\Psi}{\to} & Y_1 \\ \downarrow & & \downarrow \\ X & \stackrel{\Phi}{\to} & Y \end{array}$$

commutes and α and β are locally products of blow ups of nonsingular subvarieties. That is, for every $z \in X_1$, there exist affine neighborhoods V_1 of z, V of $x = \alpha(z)$, such that $\alpha : V_1 \to V$ is a finite product of monoidal transforms, and there exist affine neighborhoods W_1 of $\Psi(z)$, W of $y = \beta(\Psi(z))$, such that $\beta : W_1 \to W$ is a finite product of monoidal transforms.

A monoidal transform of a nonsingular k-scheme S is the map $T \to S$ induced by an open subset T of $\mathrm{Proj}(\oplus \mathcal{I}^n)$, where $\mathcal{I}$ is the ideal sheaf of a nonsingular subvariety of S.

The proof of Theorem 3.2 follows from Theorem 3.1, by the patching argument explained in Section 2 and the following lemma.

Lemma 3.3. *Suppose that $\Phi : X \to Y$ is a morphism, $p \in X$ is a closed point such that there exist regular parameters $(y_1, \dots, y_n)$ in $\mathcal{O}_{Y,\Phi(p)}$, regular parameters $(x_1, \dots, x_n)$ in $\mathcal{O}_{X,p}$ and units $\delta_1, \dots, \delta_n \in \mathcal{O}_{X,p}$ such that*

$$y_i = x_1^{a_{i1}} \cdots x_n^{a_{in}} \delta_i$$

$1 \le i \le n$, and $d = det(a_{ij}) \ne 0$. Then Φ is locally monomial at p.

Proof. Let (b_{ij}) be the adjoint matrix of $A = (a_{ij})$. Set

$$\overline{x}_i = \prod_j \delta_j^{\frac{b_{ij}}{d}} x_i$$

$1 \le i \le n$. Then

$$y_i = \overline{x}_1^{a_{i1}} \cdots \overline{x}_n^{a_{in}} \delta_i$$

$1 \le i \le n$.

Let $U_1 = \mathrm{spec}(R)$ be an affine neighborhood of p on which $(x_1, \dots, x_n)$ are uniformizing parameters, and $\delta_1, \dots, \delta_n$ are units. Let

$$S = R[\delta_1^{\frac{1}{d}}, \dots, \delta_n^{\frac{1}{d}}],$$

$V_1 = \mathrm{spec}(S)$. $f : V_1 \to U_1$ is an étale cover. $k[\overline{x}_1, \dots, \overline{x}_n] \subset S$ defines a morphism $g : V_1 \to \mathbf{A^n}$. Let a be the origin of $\mathbf{A^n}$. $q \in g^{-1}(a)$ if and only if $\overline{x}_1, \dots, \overline{x}_n \in m_q$, which holds if and only if $x_1, \dots, x_n \in m_q$. Thus $g^{-1}(a) = f^{-1}(p)$.

$$\hat{\mathcal{O}}_{V_1,q} = k[[x_1, \dots, x_n]] = k[[\overline{x}_1, \dots, \overline{x}_n]]$$

for all $q \in g^{-1}(a)$. Thus g is étale at all points of $g^{-1}(a)$. Since this is an open condition, there exists a closed set Z_1 of V_1 which is disjoint from $f^{-1}(p)$ such that $g \mid V_1 - Z_1$ is étale. Let U be an affine neighborhood of p in U_1 which is disjoint from $f(Z_1)$. Let $V = f^{-1}(U)$. Then V is an étale cover of U on which $\overline{x}_1, \ldots, \overline{x}_n$ are uniformizing parameters. □

4. Monomialization of Morphisms in Low Dimensions

We will outline proofs of monomialization in the previously known cases. Suppose that k is an algebraically closed field of characteristic zero and $\Phi : X \to Y$ is a dominant morphism of nonsingular k varieties.

Let $\mathrm{sing}(\Phi)$ be the closed subset of X where Φ is not smooth.

If Φ is a dominant morphism from a variety to a curve, the existence of a global monomialization follows immediately from resolution of singularities. In fact, it is really a restatement of embedded resolution of hypersurface singularities.

Theorem 4.1. *Suppose that $\Phi : X \to C$ is a dominant morphism from a k-variety to a curve. Then Φ has a monomialization.*

Proof. Suppose that $\Phi : X \to C$ where C is a nonsingular curve, X is a nonsingular n fold. $\Phi(\mathrm{sing}(\Phi))$ is a finite number of points of C, so we may fix a regular parameter t at a point $q \in \Phi(\mathrm{sing}(\Phi))$, and monomialize the mapping above q.

By embedded resolution of hypersurfaces, there exists a sequence of blow ups of nonsingular subvarieties which dominate subvarieties of $\Phi^{-1}(q)$, $\pi : X_1 \to X$ such that for all $p \in (\Phi \circ \pi)^{-1}(q)$, there exist regular parameters $(x_1, \ldots, x_n)$ at p such that

$$t = u x_1^{a_1} \cdots x_n^{a_n}$$

where $a_1 > 0$, $u \in \mathcal{O}_{X_1,p}$ is a unit. If $x_1 = \overline{x}_1 u^{-\frac{1}{a_1}}$, we have

$$t = \overline{x}_1^{a_1} \cdots x_n^{a_n}.$$

□

If $\Phi : T \to S$ is a dominant morphism of surfaces, monomialization is not a direct corollary of resolution of singularities. One proof of monomialization in this case (over $\mathbf{C}$) is given by Akbulut and King in [9].

In our paper [19] with Olivier Piltant, we show that if L is a perfect field and $\Phi : T \to S$ is a dominant morphism of L-surfaces, then Φ can be monomialized if Φ is unramified. That is, no wild ramification occurs with respect to any divisorial valuation of $L(T)$ over $L(S)$. This condition occurs, for instance, if $p \not| [K : L(S)]$ where K is a Galois closure of $L(T)$ over $L(S)$.

We will now outline a simple proof of monomialization for morphisms of surfaces (when k is algebraically closed of characteristic zero).

Theorem 4.2. *Suppose that $\Phi : T \to S$ is a dominant morphism of surfaces over k. Then Φ has a monomialization.*

If Φ is a monomial mapping, then Φ comes from an expression

$$\begin{aligned} u &= x^a y^b \\ v &= x^c y^d \end{aligned} \tag{8}$$

where $ad - bc \neq 0$.

$\mathrm{sing}(\Phi)$ must be contained in $xy = 0$. At a point p on $x = 0$ we have regular parameters $(\hat{x}, \hat{y})$ in $\hat{\mathcal{O}}_{T,p}$ such that

$$\hat{x} = x, \hat{y} = y - \alpha$$

for some $\alpha \in k$. If $a > 0$ and $c > 0$ we have

$$\begin{aligned} u &= \hat{x}^a(\hat{y} + \alpha)^b = \overline{x}^a \\ v &= \hat{x}^c(\hat{y} + \alpha)^d = \beta \overline{x}^c + \overline{x}^c \overline{y} \end{aligned} \tag{9}$$

where
$$\hat{x} = \overline{x}(\hat{y}+\alpha)^{-\frac{b}{a}}, \overline{y} = (\hat{y}+\alpha)^{d-\frac{cb}{a}} - \beta$$
with $\beta = \alpha^{d-\frac{cb}{a}}$.

If $a = 0$ or $c = 0$ we also obtain a form (9) with respect to regular parameters (u_1, v_1) in $\mathcal{O}_{S,\Phi(p)}$.

Thus Φ is monomial at a point p if and only if there exist regular parameters in $\hat{\mathcal{O}}_{T,p}$ such that one of the forms (8) or (9) hold.

We will say that Φ is prepared at $p \in T$ if there exist regular parameters (u, v) in $\mathcal{O}_{S,\Phi(p)}$, regular parameters (x, y) in $\hat{\mathcal{O}}_{T,p}$, and a power series P such that one of the following forms holds at p.
$$\begin{aligned} u &= x^a \\ v &= P(x) + x^c y \end{aligned} \tag{10}$$
or
$$\begin{aligned} u &= (x^a y^b)^m \\ v &= P(x^a y^b) + x^c y^d \end{aligned} \tag{11}$$
where $(a, b) = 1$ and $ad - bc \neq 0$.

We first observe that by resolution of singularities and elimination of indeterminacy, there exists a commutative diagram
$$\begin{array}{ccc} T_1 & \stackrel{\Phi_1}{\rightarrow} & S_1 \\ \downarrow & & \downarrow \\ T & \stackrel{\Phi}{\rightarrow} & S \end{array}$$
where the vertical maps are products of blow ups of points, $\mathrm{sing}(\Phi_1)$ is a simple normal crossings (SNC) divisor, and for all $p \in \mathrm{sing}(\Phi_1)$, there exist regular parameters (u, v) at $\Phi_1(p)$ such that $u = 0$ is a local equation of $\mathrm{sing}(\Phi_1)$ at p.

The essential observation is that Φ_1 is now prepared. We give a simple proof that appears in [9].

Lemma 4.3. Φ_1 *is prepared.*

Proof. Suppose that $p \in T_1$. With our assumptions, one of the following must hold at p.
$$\begin{aligned} u &= x^a \\ u_x v_y - u_y v_x &= \delta x^e \end{aligned} \tag{12}$$
where δ is a unit or
$$\begin{aligned} u &= (x^a y^b)^m \\ u_x v_y - u_y v_x &= \delta x^e y^f \end{aligned} \tag{13}$$
where $a, b, e, f > 0$, $(a, b) = 1$ and δ is a unit.

Write $v = \sum a_{ij} x^i y^j$ with $a_{ij} \in k$. First suppose that (12) holds. Then $a x^{a-1} v_y = \delta x^e$ implies we have the form (10) (after making a change of parameters in $\hat{\mathcal{O}}_{T_1,p}$). Now suppose that (13) holds.
$$u_x v_y - u_y v_x = \sum m(aj - bi) a_{ij} x^{am+i-1} y^{bm+j-1} = \delta x^e y^f.$$
Thus
$$v = \sum_{aj-bi=0} a_{ij} x^i y^j + \epsilon x^{e+1-am} y^{f+1-bm}$$
where ϵ is a unit. After making a change of parameters, multiplying x by a unit, and multiplying y by a unit, we get the form (11). $\square$

It is now not difficult to construct a monomialization. We must blow up points q on S_1 over which the map is not monomial at some point over q, and blow up points on T_1 to make $m_q\mathcal{O}_{T_1}$ principal. If we iterate this procedure, it can be shown that we construct a commutative diagram

$$\begin{array}{ccc} T_2 & \stackrel{\Phi_2}{\to} & S_2 \\ \downarrow & & \downarrow \\ T_1 & \stackrel{\Phi_1}{\to} & S_1 \end{array}$$

such that Φ_2 is monomial.

5. An overview of the proof of Monomialization of morphisms from 3 folds to surfaces

Suppose that k is an algebraically closed field of characteristic zero, and $\Phi : X \to Y$ is a dominant morphism of nonsingular k-varieties.

A natural first step in monomializing a morphism $\Phi : X \to Y$ is to use resolution of singularities and elimination of indeterminacy to construct a commutative diagram

$$\begin{array}{ccc} X_1 & \stackrel{\Phi_1}{\to} & Y_1 \\ \downarrow & & \downarrow \\ X & \stackrel{\Phi}{\to} & Y \end{array}$$

where the vertical maps are products of blow ups of nonsingular subvarieties, $\mathrm{sing}(\Phi_1)$ is a simple normal crossings (SNC) divisor, and for all $p \in \mathrm{sing}(\Phi_1)$, there exist regular parameters $(u_1, \dots, u_n)$ at $\Phi_1(p)$ such that $u_1 = 0$ is a local equation of $\mathrm{sing}(\Phi_1)$ at p.

We observed that if X and Y are surfaces, then Φ_1 is prepared. Unfortunately, even for morphisms from a 3 fold to a surface, Φ_1 may be quite complicated. We will give some examples later on in this section.

A key step in the local proof of monomialization, Theorem 3.1, is to define a new invariant, which measures how far the situation is from a specific form which is close to being monomial. In the local valuation theoretic proof we make use of special products of monoidal transforms defined by Zariski called Perron transforms [37]. Under appropriate application of Perron transforms our invariant does not increase, and we can in fact make the invariant decrease, by an appropriate algorithm.

An essential difficulty globally is that our invariant can increase after a permissible monoidal transform. This is a significant difference from resolution of singularities, where a foundational result is that the multiplicity of an ideal does not go up under permissible blow ups.

We will give a brief overview of the proof of Theorem 1.3 on monomialization of morphisms from 3 folds to surfaces (Theorem 18.21 [18]).

Step 1. First construct a diagram

$$\begin{array}{ccc} X' & \stackrel{\Phi'}{\to} & S' \\ \downarrow & & \downarrow \\ X & \stackrel{\Phi}{\to} & S \end{array}$$

where the vertical maps are products of blow ups of nonsingular subvarieties such that X', S' are nonsingular, there exist reduced SNCS divisors $D_{S'}$ on S', $E_{X'} = (\Phi')^{-1}(D_{S'})_{red}$ on X' such that $\mathrm{sing}(\Phi') \subset E_{X'}$ and components of $E_{X'}$ on X' dominating distinct components of $D_{S'}$ are disjoint. Such a morphism Φ' will be called weakly prepared.

For all $p \in X'$ there exist regular parameters (u,v) in $\mathcal{O}_{S',q}$ $(q = \Phi'(p))$ and regular parameters (x,y,z) in $\hat{\mathcal{O}}_{X',p}$ such that $u = 0$ is a local equation of $E_{X'}$, $u = 0$ (or $uv = 0$) is a local equation of $D_{S'}$ and exactly one of the following cases hold:

(1)
$$u = x^a, v = P(x) + x^b F$$
where $x \not| F$, F has no terms which are monomials in x.

(2)
$$u = (x^a y^b)^m, v = P(x^a y^b) + x^c y^d F$$
where $(a,b) = 1$, $x \not| F$, $y \not| F$, $x^c y^d F$ has no terms which are monomials in $x^a y^b$.

(3)
$$u = (x^a y^b z^c)^m, v = P(x^a y^b z^c) + x^d y^e z^f F$$
where $(a,b,c) = 1$, $x \not| F$, $y \not| F$, $z \not| F$, $x^d y^e z^f F$ has no terms which are monomials in $x^a y^b z^c$.

The structure of the singularities of F can be very complicated as the following examples show. This is in sharp contrast to the case of a morphism of surfaces (Lemma 4.3).

Consider the germ of maps
$$u = x^a, v = x^c F$$
with $a \geq 2$, $c \geq 0$ where
$$F = x^r z + h(x,y)$$
where h is arbitrary. The singular locus of this map germ is the variety defined by the ideal where the jacobian has rank < 2. That is, the variety with ideal J= $\sqrt{(x^{a+c-1}F_y, x^{a+c-1}F_z)}$. Since $F_z = x^r$, we have that $\sqrt{J} = (x)$.

For another example, consider the (formal) germ of maps
$$u = x^a, v = x^c F$$
where
$$F = \sum_{i>0, j\geq 0} \frac{i^j}{j!} a_i(x) y^i z^j + x^r z$$
where $a_i(x)$ are arbitrary series, $a \geq 2$, $c \geq 0$. The singular locus of this map germ is defined by $J = x^{a+c-1}(F_y, F_z)$. Since $F_z - yF_y = x^r$, $\sqrt{J} = (x)$.

Our main invariant is
$$\nu(p) = \text{mult}(F).$$
This invariant is independent of parameters in the forms above.
$$S_r(X') = \{p \in X' \mid \nu(p) \geq r\}$$
is a constructible (but not Zariski closed) subset of X'.

$S_r(X)$ is in general not Zariski closed. Consider the 2 point p with local equations
$$\begin{aligned} u &= xy \\ v &= x^2 y. \end{aligned}$$
$\nu(p) = 0$. At 1 points q on the surface $x = 0$ there are regular parameters (x, y_1, z) with $y = y_1 + \alpha$ for some $0 \neq \alpha \in k$. Set $\overline{x} = x(y_1 + \alpha)$. There are permissible parameters $(\overline{x}, \overline{y}, z)$ at q such that
$$\begin{aligned} u &= \overline{x} \\ v &= \alpha^{-1}\overline{x}^2 + \overline{x}^2 \overline{y}. \end{aligned}$$
Thus $\nu(q) = 1$.

Step 2. This is the difficult step. We construct a commutative diagram

$$\begin{array}{ccc} X'' & & \\ \downarrow\lambda & \overset{\Phi''}{\searrow} & \\ X' & \overset{\Phi'}{\rightarrow} & S' \end{array}$$

so that everywhere we have one of the forms:

(1) $u = x^a, v = P(x) + x^b y$,
(2) $u = (x^a y^b)^m, v = P(x^a y^b) + x^c y^d$,
(3) $u = (x^a y^b)^m, v = P(x^a y^b) + x^c y^d z$,
(4) $u = (x^a y^b z^c)^m, v = P(x^a y^b z^c) + x^d y^e z^f$ with

$$\text{rank}\left\{\begin{array}{ccc} a & b & c \\ d & e & f \end{array}\right\} = 2.$$

We impose the further condition that 1. - 4. are compatible with the reduced SNC divisors $D_{S'}$ and $E_{X''} = (\Phi'')^{-1}(D_{S'})_{red}$. $u = 0$ is a local equation of $E_{X''}$, $u = 0$ (or $uv = 0$) are local equations of $D_{S'}$ in the above forms. We will say that Φ'' is prepared.

We use induction on

$$r = \max\{t \mid \nu(p) = t \text{ for some } p \in X\}$$

to achieve the conclusions of the theorem. A major difficulty is that, unlike in the case of resolution of singularities, $\nu(p)$ can go up after blowing up a point or a nonsingular curve.

We can construct an example as follows.

$$u = xy, v = x^2 y$$

has $F = 1$. blow up p and consider the point p_1 above p with regular parameters (x_1, y_1, z_1) defined by $x = x_1, y = x_1(y_1 + \alpha)$, $\alpha \neq 0$, $z = x_1 z_1$. Set $\overline{x}_1 = x_1(y_1 + \alpha)^{\frac{1}{2}}$, $\overline{y}_1 = (y_1 + \alpha)^{-\frac{1}{2}} - \alpha^{-\frac{1}{2}}$. Then

$$u = \overline{x}_1^2, v = \alpha^{-\frac{1}{2}}\overline{x}_1^3 + \overline{x}_1^3\overline{y}_1,$$

so that $F_1 = \overline{y}_1$.

However, $\nu(p)$ can go up by at most 1, and some other invariants get better, or at least no worse. For a local resolution, we reduce to two difficult cases (Sections 11 and 12 of [18]) which we settle by blowing up generic curves on $E_{X'}$ through a particular point, and use a generalization of Abhyankar's Good Point Algorithm ([5], [28]) to achieve an improvement. This depends on arithmetic information which is captured in this algorithm.

Step 3. We construct a commutative diagram

$$\begin{array}{ccc} X''' & \overset{\Phi'''}{\rightarrow} & S'' \\ \downarrow & & \downarrow \\ X'' & \overset{\Phi''}{\rightarrow} & S' \end{array}$$

such that $X''' \rightarrow X''$ is a product of blow ups of nonsingular curves, $S'' \rightarrow S'$ is a product of blow ups of points and Φ''' is monomial.

$\pi : S'' \rightarrow S'$ is a sequence of blow ups of points. If $q \in S'$ and $q_1 \in \pi^{-1}(q)$ then there exist regular parameters (u, v) in $\mathcal{O}_{S',q}$ and (u_1, v_1) in $\hat{\mathcal{O}}_{S'',q_1}$ such that

$$\begin{array}{ll} u & = u_1^a \\ v & = P(u_1) + u_1^b v_1 \end{array}$$

or

$$\begin{aligned} u &= (u_1^a v_1^b)^m \\ v &= P(u_1^a v_1^b) + u_1^c v_1^d \end{aligned}$$

with $ad - bc \neq 0$ and $(a, b) = 1$.

If $p \in X''$ is a point of the form 1. of Step 2, then there exists $\overline{\pi} : S_1 \rightarrow S'$ and $q_1 \in \overline{\pi}^{-1}(q)$ with regular parameters (u_1, v_1) in $\mathcal{O}_{S_1,q_1}$, $(\overline{x}, \overline{y}, \overline{z})$ in $\hat{\mathcal{O}}_{X'',p}$ such that

$$\begin{aligned} u_1 &= \overline{x}^{\overline{a}} \\ v_1 &= \overline{x}^b(\alpha + \overline{y}). \end{aligned}$$

A similar argument holds in the other cases.

We have an essentially canonical procedure for achieving Step 3. We blow up on S' the (finitely many) images of all non monomial points of X'', then blow up nonsingular curves on X'' to eliminate the indeterminacy of the resulting rational map. An invariant improves. By induction we eventually construct Φ'''.

6. Resolution of Generically Finite Morphisms of Surfaces

In this section, a surface is a nonsingular 2-dimensional integral scheme, of finite type over an algebraically closed field k of characteristic 0.

Suppose that Z and Y are surfaces and $f : Z \rightarrow Y$ is a generically finite morphism.

Definition 6.1. *We will say that f is resolved if for all $q \in Y$ there exist regular parameters (u, v) at q such that for all $p \in f^{-1}(q)$ there exist regular parameters (x, y) in $\hat{\mathcal{O}}_{Z,p}$ and a series P such that one of the following forms hold.*

$$\begin{aligned} u &= x^a \\ v &= P(x) + x^b y^c \end{aligned} \tag{14}$$

with $c > 0$ or

$$\begin{aligned} u &= (x^a y^b)^m \\ v &= P(x^a y^b) + x^c y^d \end{aligned} \tag{15}$$

where $(a, b) = 1$ and $ad - bc \neq 0$.

We will prove the following Theorem.

Theorem 6.2. *Suppose that Z and Y are surfaces and $f : Z \rightarrow Y$ is a generically finite morphism. Then there exists a sequence of quadratic transforms $\pi : X \rightarrow Z$ such that $f \circ \pi$ is resolved.*

Theorem 6.2 is achieved without any blowing up in the base S. The "resolved form" of Theorem 6.2 is slightly weaker that that of the conclusions of Lemma 4.3. However, the conclusions of Lemma 4.3 cannot be achieved without blowing up in S. The proof of Theorem 6.2 is substantially more difficult. However, this generalized theorem is a necessary step in the proof of Theorem 1.3.

Theorem 6.2 is an ingredient in the proof of Theorems 1.3 and 1.4. The proof of Theorem 6.2 is based on a classical method for resolution of plane curve singularities (c.f. [10]). However, there are significant differences.

Let S be the (finitely many) points of Y over which f is not finite. Let B be the branch curve of $Z - f^{-1}(S) \rightarrow Y - S$, T be the set of (finitely many) singular points of $f(B)$. By Abhyankar's Lemma, for each $q \in f(B) - T$ there are regular parameters (u, v) in $\mathcal{O}_{Y,q}$ such that for all $p \in f^{-1}(p)$, there exists regular parameters (x, y) in $\hat{\mathcal{O}}_{Z,p}$ such that

$$\begin{aligned} u &= x^n \\ v &= y. \end{aligned}$$

At each of the finitely many points q of $S\cup T$, fix regular parameters $(u_q,v_q)=(u,v)$ in $\mathcal{O}_{Y,q}$. We will preserve this notation and this choice through out this section.

After performing a sequence of quadratic transforms $Z_1\rightarrow Z$, and replacing f with $Z_1\rightarrow Y$, we can assume that $u=u_q=0$ is a SNC divisor everywhere above q, for $q\in S\cup T$. Suppose that $f(p)=q$. Then there are regular parameters (x,y) in $\hat{\mathcal{O}}_{X,p}$ such that $u=x^{\overline{a}}y^{\overline{b}}$, and $\overline{a}>0$.

Suppose that $\overline{b}>0$. Let $m=(\overline{a},\overline{b})$, let $a=\frac{\overline{a}}{m}$, $b=\frac{\overline{b}}{m}$. There are power series $P(t)$ and $F(x,y)$ such that x does not divide F, y does not divide F, x^cy^dF has no nonzero terms which are powers of x^ay^b and in $\hat{\mathcal{O}}_{X,p}$

$$\begin{array}{l} u=(x^ay^b)^m\\ v=P(x^ay^b)+x^cy^dF(x,y)\end{array} \tag{16}$$

We will say that p is a 2 point.

If $\overline{b}=0$, there are power series $P(t)$ and $F(x,y)$ such that x does not divide F, F has no nonzero terms which are powers of x and in $\hat{\mathcal{O}}_{X,p}$

$$\begin{array}{l} u=x^a\\ v=P(x)+x^cF(x,y)\end{array} \tag{17}$$

We will say that p is a 1 point.

For the rest of this section, we will assume that $\Lambda:X\rightarrow Z$ is a sequence of quadratic transforms, centered at points over $S\cup T$. let $f_X=f\circ\Lambda$.

Lemma 6.3. *Suppose that (R,m) is the local ring of a point on a surface, (u,v) are regular parameters in R and $g:W\rightarrow Spec(R)$ is a sequence of quadratic transforms. Suppose that $p\in g^{-1}(m)$ is a closed point. Then there exist a series P and regular parameters (u_1,v_1) in $\hat{\mathcal{O}}_{W,p}$ such that*

$$\begin{array}{l} u=u_1^a\\ v=P(u_1)+u_1^cv_1\end{array} \tag{18}$$

with $c>0$ or

$$\begin{array}{l} u=(u_1^av_1^b)^m\\ v=P(u_1^av_1^b)+u_1^cv_1^d\end{array} \tag{19}$$

with $b>0$, $(a,b)=1$, $ad-bc=\pm1$.

Proof. $R\rightarrow\mathcal{O}_{W,p}$ can be factored as a sequence of quadratic transforms, and can thus be factored as a sequence of transformations of intermediate local rings

$$R=R_0\rightarrow R_1\rightarrow\cdots R_n=\mathcal{O}_{W,p}$$

where R_i has regular parameters (x_i,y_i) such that

$$x_0=u,y_0=v$$

$$x_{i-1}=x_i^{\beta_i}(y_i+c_i)^{\beta_i'},y_{i=1}=x_i^{\alpha_i}(y_i+c_i)^{\alpha_i'}$$

with $c_i\in k$ and $\alpha_i'\beta_i-\alpha_i\beta_i'=\pm1$ for $i\le n$, and $0\ne c_i$ if $i<n$. We can then find regular parameters $(\overline{x}_i,\overline{y}_i)$ in $\hat{R}_i$ such that

$$\overline{x}_0=u,\overline{y}_0=v$$

$$\overline{x}_{i-1}=\overline{x}_i^{\beta_i},\overline{y}_{i-1}=\overline{x}_i^{\alpha_i}(\overline{y}_i+\overline{c}_i).$$

with $\overline{c}_i\ne0$ for $i\le n-1$ and $\overline{x}_{n-1},\overline{y}_{n-1}$ have this form, or

$$\overline{x}_{n-1}=\overline{x}_n^{\beta_n}\overline{y}_n^{\beta_n'},\overline{y}_{n-1}=\overline{x}_n^{\alpha_n}\overline{y}_n^{\alpha_n'}.$$

Setting $u_1=\overline{x}_n$, $v_1=\overline{y}_n$ we get the conclusions of the Lemma. ∎

Lemma 6.4. *$f_X : X \to Y$ is resolved at all but finitely many points of X.*

Proof. We need only show that if E is an exceptional curve of f_X, then f_X is resolved at all but finitely many points of E. Let ν be the (dimension 1) valuation of $k(X)$ corresponding to E. ν restricts to a (dimension 1) valuation ν' of $k(Y)$. Let $W \to Y$ be the minimal sequence of quadratic transforms such that the center of ν' on W is a curve E'. The rational map $\Phi : X \to W$ is defined and is finite in a neighborhood of a generic point q of E. By Lemma 6.3

There exist regular parameters (u_1, v_1) at $\Phi(q)$ such that $u_1 = 0$ is a local equation of E_1 in W at $\Lambda(q)$ and the form (18) holds at $\Lambda(q)$.

By Abhyankar's Lemma, there exist regular parameters (x_1, y_1) at q such that

$$u_1 = x_1^e, v_1 = y_1$$

for some $e \in \mathbf{N}$. Thus the form (14) holds at q. □

Suppose that $p \in X$, and (x, y) are regular parameters in $\hat{\mathcal{O}}_{X,p}$ such that (u, v) have one of the forms (16) or (17). Set

$$\nu(p) = \begin{cases} \text{mult}(F) - 1 & \text{if } p \text{ is a 1 point} \\ \text{mult}(F) & \text{if } p \text{ is a 2 point} \end{cases}$$

Lemma 6.5. *$\nu(p)$ is independent of the choice of regular parameters (x, y) in (16) or (17).*

Proof. First suppose that p is a 2 point. To express u and v in the form (16) we can only make a permissible change of variables in x and y, where a permissible change of variables is one of the following two forms:

$$x = \omega_x \overline{x}, y = \omega_y \overline{y} \text{ where } \omega_x^{ma} \omega_y^{mb} = 1 \tag{20}$$

or

$$y = \omega_y \overline{x}, x = \omega_x \overline{y} \text{ where } \omega_x^{ma} \omega_y^{mb} = 1. \tag{21}$$

ν does not change after a change of variables of one of these forms.

Now suppose p is a 1 point. To preserve the form (17), we can only make a permissible change of variables, where a permissible change of variables is of the form:

$$x = \omega_x \overline{x}, y = \phi(\overline{x}, \overline{y}) \text{ where } \text{mult}(\phi(0, \overline{y})) = 1. \tag{22}$$

and $\omega_x \in k$ is an a-th root of unity. Then

$$\phi(\overline{x}, \overline{y}) = \overline{\phi}(\overline{x}, \overline{y})(\overline{y} + \psi(\overline{x}))$$

where $\overline{\phi}$ is a unit. Write

$$\psi(\overline{x}) = \sum b_i \overline{x}^i.$$

$$\begin{aligned} u &= \overline{x}^a \\ v &= \overline{P}(\overline{x}) + \overline{x}^c \overline{F}(\overline{x}, \overline{y}) \end{aligned} \tag{23}$$

where

$$\overline{F} = \omega_x^c (F(\omega_x \overline{x}, \phi(\overline{x}, \overline{y})) - F(\omega_x \overline{x}, \phi(\overline{x}, 0)))$$
$$\overline{P}(\overline{x}) = P(\omega_x \overline{x}) + \overline{x}^c \omega_x^c F(\omega_x \overline{x}, \phi(\overline{x}, 0))$$

Suppose that the leading form of F is

$$L = \sum_{i+j=r} a_{ij} x^i y^j.$$

The leading form $\overline{L}$ of $\overline{F}$ is then

$$\omega_x^c(\sum_{i+j=r} a_{ij}\omega_x\overline{x}^i(e(\overline{y}-b_1\overline{x}))^j - \sum_{i+j=r} a_{ij}\omega_x\overline{x}^i(-eb_1\overline{x})^j)$$

where $e=\overline{\phi}(0,0)$, which is nonzero since $a_{ij}\neq 0$ for some $j>0$. □

A 2 point p is resolved if and only if F is a unit in (16). A 1 point p is resolved if and only if in (17) $F(x,y)=g(x,y)^d+h(x)$ for some series $g(x,y)$ with $\mathrm{mult}(g(0,y))=1$, and positive integer d.

Theorem 6.6. *Suppose that $g: X_1\rightarrow X$ is a quadratic transform, centered at a closed point p of X, and $p_1\in X_1$ is a closed point such that $g(p_1)=p$. Then*

$$\nu(p_1)\le\nu(p).$$

If p is resolved, then p_1 is resolved.

Proof. First suppose that p is a 2 point. Write

$$F=\sum_{i+j\ge r} a_{ij}x^iy^j.$$

in $\hat{\mathcal{O}}_{X,p}$, where $r=\mathrm{mult}(F)=\nu(p)$. Suppose that $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters (x_1,y_1) such that $x=x_1, y=x_1(y_1+\alpha)$ with $\alpha\neq 0$. Define $\overline{x}_1$ by

$$x_1=\overline{x}_1(y_1+\alpha)^{\frac{-b}{a+b}}.$$

Then $(\overline{x}_1,y_1)$ are regular parameters in $\hat{\mathcal{O}}_{X_1,p_1}$.

$$u=x_1^{m(a+b)}(y_1+\alpha)^{mb}=\overline{x}_1^{m(a+b)}.$$

$$v=P(\overline{x}_1^{a+b})+\overline{x}_1^{c+d+r}(y_1+\alpha)^{\lambda}(\frac{F}{x_1^r})$$

where $\lambda=d-\frac{b(c+d+r)}{a+b}$.

$$F=\sum_{i+j=r} a_{ij}x_1^r(y_1+\alpha)^j+x_1^{r+1}\Omega.$$

$$\frac{F}{x_1^r}=\sum_{j=0}^{r} a_j(y_1+\alpha)^j+x_1\Omega.$$

where $a_j=a_{r-j,j}$. We have

$$\begin{array}{l} u=\overline{x}_1^{m(a+b)}\\ v=\overline{P}(\overline{x}_1)+\overline{x}_1^{c+d+r}\overline{F}(\overline{x}_1,y_1)\end{array}\qquad(24)$$

where

$$\overline{P}=P(\overline{x}_1^{a+b})+\overline{x}_1^{c+d+r}(\alpha)^{\lambda}(\frac{F(\alpha^{\frac{-b}{a+b}}\overline{x}_1,\alpha^{\frac{a}{a+b}}\overline{x}_1)}{\alpha^{\frac{-rb}{a+b}}\overline{x}_1^r})$$

$$\overline{F}=(y_1+\alpha)^{\lambda}(\frac{F((y_1+\alpha)^{\frac{-b}{a+b}}\overline{x}_1,(y_1+\alpha)^{\frac{a}{a+b}}\overline{x}_1)}{(y_1+\alpha)^{\frac{-rb}{a+b}}\overline{x}_1^r})-(\alpha)^{\lambda}(\frac{F(\alpha^{\frac{-b}{a+b}}\overline{x}_1,\alpha^{\frac{a}{a+b}}\overline{x}_1)}{\alpha^{\frac{-rb}{a+b}}\overline{x}_1^r})$$

Set

$$\beta=(\sum_{j=0}^{r} a_j\alpha^j)\alpha^{\lambda}.$$

Suppose that $\nu(p_1)>\nu(p)=r$, so that

$$\mathrm{mult}(\overline{F})\ge\mathrm{mult}(F)+2=r+2.$$

Then
$$(y_1+\alpha)^\lambda(\sum_{j=0}^r a_j(y_1+\alpha)^j)-\beta\equiv 0 \bmod (y_1)^{r+2}.$$
$\beta\ne 0$ since $\sum_{j=0}^r a_j(y_1+\alpha)^j\ne 0$. We have
$$\sum_{j=0}^r a_j(y_1+\alpha)^j\equiv\beta(y_1+\alpha)^{-\lambda} \bmod (y_1)^{r+2} \tag{25}$$

First suppose that $-\lambda$ is a natural number. Then
$$\sum_{i=0}^r a_i(y_1+\alpha)^j=\beta(y_1+\alpha)^{-\lambda}$$
where $t=-\lambda\le r$. Thus the leading form of F is
$$\begin{array}{lll} L &=& \sum_{i+j=r}a_{ij}x_1^r(y_1+\alpha)^j\\ &=& \beta x_1^r(y_1+\alpha)^{-\lambda}\\ &=& \beta x^{r+\lambda}y^{-\lambda}\\ &=& \beta x^{r-t}y^t\end{array}$$
So the leading form of F is $\beta x^{r-t}y^t$. Thus $\beta x^{c+r-t}y^{d+t}$ is a nonzero term of x^cy^dF. Since
$$t=\frac{b(c+d+r)}{a+b}-d,$$
we have
$$b(c+r-t)-a(d+t)=0$$
so that $x^{c+r-t}y^{d+t}$ is a power of x^ay^b, a contradiction.

We must then have $-\lambda\not\in\mathbf{N}$. But then the y_1^{r+1} coefficient of $\beta(y_1+\alpha)^{-\lambda}$ is non zero, a contradiction to (25).

Now suppose that p is a 2 point and $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters (x_1,y_1) such that $x=x_1, y=x_1y_1$. Write
$$F=\sum_{i+j\ge r}a_{ij}x^iy^j.$$
Then
$$\begin{array}{l} u=x_1^{m(a+b)}y_1^{mb}\\ v=\overline{P}(x_1^{a+b}y_1^b)+x_1^{c+d+r}y_1^d\overline{F}(x_1,y_1)\end{array} \tag{26}$$
where $\overline{P}=P$, $\overline{F}=\frac{F}{x_1^r}$. We need only check that $\frac{F}{x_1^r}$ has no nonzero $x_1^\alpha y_1^\beta$ terms with $b(c+d+r+\alpha)=(a+b)(d+\beta)$. We have that $a_{ij}=0$ if $b(c+i)-a(d+j)=0$.
$$\frac{F}{x_1^r}=\sum a_{ij}x_1^{i+j-r}y_1^j.$$
Suppose that $b(c+d+r+\alpha)=(a+b)(d+\beta)$. Set $i=\alpha-\beta+r$, $j=\beta$. Then $b(c+i)-a(d+j)=0$, and $a_{ij}=0$. But this is the coefficient of $x_1^\alpha y_1^\beta$ in $\frac{F}{x_1^r}$. We have $\mathrm{mult}(\overline{F})\le\mathrm{mult}(F)$.

The above argument also works, by interchanging the variables x and y, in the case where p is a 2 point and $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters (x_1,y_1) such that $x=x_1y_1, y=y_1$.

Now suppose that p is a 1 point and $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters (x_1,y_1) such that $x=x_1y_1, y=y_1$. Write
$$F=\sum_{i+j\ge r}a_{ij}x^iy^j.$$

Then

$$\begin{aligned} u &= x_1^a y_1^a \\ v &= \overline{P}(x_1 y_1) + x_1^c y_1^{c+r} \overline{F}(x_1, y_1) \end{aligned} \tag{27}$$

where $\overline{P} = P$, $\overline{F} = \frac{F}{y_1^r}$. We must show that $\overline{F}$ has no nonzero terms $x_1^\alpha y_1^\beta$ terms with $\alpha = r + \beta$. But this is impossible since F has no nonzero x^i terms, with $i \geq 0$.

The leading form of $\overline{F}$ is

$$\overline{F} = \sum_{i=0}^{r-1} a_{i,r-i} x_1^i + y_1 \Omega$$

since $a_{r0} = 0$, where some $a_{ij} \neq 0$ with $i + j = r$, $j > 0$. Thus $\mathrm{mult}(\overline{F}) \leq \mathrm{mult}(F) - 1$.

Now suppose that p is a 1 point and $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters (x_1, y_1) such that $x = x_1, y = x_1(y_1 + \alpha)$. By making if necessary a permissible change of variables at p, replacing y with $y - \alpha x$, we may assume that $x = x_1, y = x_1 y_1$. Write

$$F = \sum_{i+j \geq r} a_{ij} x^i y^j.$$

where $a_{i0} = 0$ for all i.

$$\begin{aligned} u &= x_1^a \\ v &= \overline{P}(x_1) + x_1^{c+r} \overline{F}(x_1, y_1) \end{aligned}$$

where $\overline{P} = P$, $\overline{F} = \frac{F}{x_1^r}$. $\overline{F}$ has no nonzero terms which are powers of x_1. Thus $\mathrm{mult}(\overline{F}) \leq \mathrm{mult}(F)$. ☐

Suppose that $p \in X$. Set

$$\sigma(p) = \begin{cases} 0 & \text{if } p \text{ is a 1 point and } \mathrm{mult}(F) = \mathrm{mult}(F(0, y) \\ \frac{1}{2} & \text{if } p \text{ is a 2 point} \\ 1 & \text{ if } p \text{ is a 1 point and } \mathrm{mult}(F) < \mathrm{mult}(F(0, y) \end{cases}$$

Lemma 6.7. *σ is independent of the choice of permissible parameters (x, y).*

Proof. The proof of Lemma 6.5 shows that $\mathrm{mult}(F(0, y))$ is independent of the choice of permissible parameters at a 1 point. ☐

Lemma 6.8. *Suppose that $g : X_1 \to X$ is a quadratic transform, centered at a point p of X, and p_1 is a closed point such that $g(p_1) = p$. Further suppose that p is a 2 point, p_1 is a 1 point and $\nu(p_1) = \nu(p)$. Then $\sigma(p_1) = 0$.*

Proof. $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters (x_1, y_1) such that $x = x_1, y = x_1(y_1 + \alpha)$ with $\alpha \neq 0$. Let $r = \mathrm{mult}(F)$. $\mathrm{mult}(F_1) = \mathrm{mult}(F) + 1 = r + 1$. Let

$$F = \sum_{i+j \geq r} a_{ij} x^i y^j$$

As in the analysis leading to (25),

$$F_1 \equiv (y_1 + \alpha)^\lambda \left(\sum_{i=0}^{r} a_{i,r-i} (y_1 + \alpha)^j \right) - \beta \bmod (x_1, y_1^{r+1}) \tag{28}$$

for some $\beta \in k$. If $F_1 \equiv 0 \bmod (x_1, y_1^{r+1})$, then $\beta \neq 0$ and $-\lambda \notin \{0, 1 \ldots, r\}$, as in the proof of Theorem 6.6. Then

$$\begin{aligned} F_1 &\equiv (y_1 + \alpha)^\lambda \left(-\beta \tfrac{-\lambda(-\lambda-1)\cdots(-\lambda-r)}{(r+1)!} \alpha^{-\lambda-r-1} \right) y_1^{r+1} \bmod (x_1, y_1^{r+2}) \\ &\equiv -\beta \tfrac{-\lambda(-\lambda-1)\cdots(-\lambda-r)}{(r+1)!} \alpha^{-r-1} y_1^{r+1} \bmod (x_1, y_1^{r+2}) \end{aligned} \tag{29}$$

Thus $\mathrm{mult}(F_1(0, y_1)) = r + 1$. ☐

Lemma 6.9. *Suppose that $g : X_1 \to X$ is a quadratic transform, centered at a 1 point p of X and p_1 is a closed point above p such that $g(p_1) = p$.*

If p_1 is a 1 point and $\nu(p_1) = \nu(p)$, then $\sigma(p_1) = 0$. If $\sigma(p) = 0$ and p_1 is a 2 point then $\nu(p_1) = 0$.

Proof. First suppose that p_1 is a 2 point. Then $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters (x_1, y_1) such that $x = x_1y_1, y = y_1$.

$$F = \sum_{i+j\ge r} a_{ij}x^iy^j$$

with $a_{0r} \ne 0$.

$$F_1 = \sum_{i=0}^{r-1} a_{i,r-i}x_1^i + y_1\Omega.$$

is then a unit.

Now suppose that p_1 is a 1 point and $\nu(p_1) = \nu(p)$. After appropriate choice of permissible variables (x, y) at p, $\hat{\mathcal{O}}_{X_1,p_1}$ has regular parameters (x_1, y_1) such that $x = x_1, y = x_1y_1$. Set $r = \mathrm{mult}(F) = \mathrm{mult}(F_1)$. Then $F_1 = \frac{F(x_1,x_1y_1)}{x_1^r}$ and $\mathrm{mult}(F_1(0, y_1)) = r$. ☐

Theorem 6.10. *Suppose that $g : X_1 \to X$ is a quadratic transform, centered at a point p of X, and p_1 is a closed point such that $g(p_1) = p$. If $\nu(p_1) = \nu(p)$, then $\sigma(p_1) \le \sigma(p)$.*

Proof. This is immediate from Lemmas 6.8 and 6.9. ☐

Suppose that

$$F = \sum_{i+j\ge r} a_{ij}x^iy^j$$

has multiplicity r. Define

$$\delta(F; x, y) = \min(\frac{i}{r-j} \mid j < r, a_{ij} \ne 0).$$

$\delta(F; x, y) = \infty$ if and only if $F = y^r\omega$, where ω is a unit. If $\delta(F; x, y) < \infty$, then $\delta(F; x, y) \in \frac{1}{r!}\mathbf{N}$.

Suppose that $p \in X$. If (x, y) are permissible parameters at p with one of the forms (16) or (17), set

$$\delta(p; x, y) = \delta(F; x, y).$$

Then set

$$\delta(p) = \sup(\delta(p; x, y))$$

where the sup is over all permissible parameters at p. Note that if p is a 2 point, then

$$\delta(p) = \max(\delta(p; x, y), \delta(p; y, x))$$

if (x, y) are a particular choice of permissible parameters at p.

If p is a 2 point and $\nu(p) > 0$, then $\delta(p) < \infty$. If p is a 1 point and $\sigma(p) = 1$, then $\delta(p) = 1$, since $\delta(p; x, y) = 1$ for all permissible parameters (x, y).

Lemma 6.11. *Suppose that p is a 1 point, $\sigma(p) = 0$ and (x, y) are fixed permissible parameters at p. Then there exists a power series $t(x)$ such that*

$$\delta(p) = \delta(p; x, y - t(x)).$$

If $\delta(p) < \infty$, then $t(x)$ is a polynomial.

$\delta(p) > \delta = \delta(p;x,y)$ if and only if $\delta \in \mathbf{N}$ and

$$\sum_{i+\delta j=r\delta} a_{ij}x^iy^j = \tau(y-cx^\delta)^r + \lambda x^{r\delta}$$

for some $\tau, c, \lambda \in k$ with $c \neq 0$ (so that $\lambda = -\tau(-c)^r$).

Proof. Suppose that $(\overline{x}, \overline{y})$ are also permisible parameters at p. Then $\overline{x} = \lambda x$, with $\lambda^a = 1$ and $\overline{y} = \overline{\phi}(y - t(x))$ for some unit series $\overline{\phi}$ and series $t(x)$.

$$\delta(p; \overline{x}, \overline{y}) = \delta(p; x, y - t(x))$$

Thus

$$\delta(p) = \sup(\delta(p; x, y - t(x) \mid t(x) \text{ is a polynomial of positive order}). \tag{30}$$

Let $\delta = \delta(p; x, y)$,

$$\overline{L} = \sum_{i+\delta j=r\delta} a_{ij}x^iy^j.$$

so that

$$F = \overline{L} + \sum_{i+\delta j>r\delta} a_{ij}x^iy^j.$$

Suppose that

$$\overline{L} = \tau(y-cx^\delta)^r + \lambda x^{r\delta}$$

for some $\tau, c, \lambda \in k$ with $0 \neq c$. Set $y_1 = y - cx^\delta$. Then $\delta \in \mathbf{N}$ and $\delta(p; x, y_1) > \delta(p; x, y)$ since

$$F_1 = \tau y_1^r + \sum_{i+\delta j>r\delta} \overline{a}_{ij}x^iy_1^j.$$

where

$$v = P_1(x) + x_1^c F_1$$

is the normalized form of v with respect to (x, y_1). We can repeat this process, with y replaced by $y - cx^\delta$. The process will either produce a polynomial $t(x)$ such that if $y_1 = y - t(x)$, and $\delta_1 = \delta(p; x, y_1)$, then $\delta_1 \notin \mathbf{N}$, or $\delta_1 \in \mathbf{N}$ and

$$\sum_{i+\delta_1 j=r\delta_1} \overline{a}_{ij}x^iy_1^j \neq \tau(y_1 - cx^{\delta_1})^r + \lambda x^{r\delta_1} \tag{31}$$

for any $\tau, c, \lambda \in k$ with $0 \neq c$, or we will produce a series $t(x)$ such that if $y_1 = y - t(x)$, then $\delta(p; x, y_1) = \delta(p) = \infty$, so that $F_1 = y_1^r\phi$, where ϕ is a unit series.

Suppose that we have produced y_1 such that $\delta(p; x, y_1) \notin \mathbf{N}$ or $\delta(p; x, y_1) \in \mathbf{N}$ and (31) holds. We will show that $\delta(p) = \delta(p; x, y_1)$. Suppose that $\delta_1 = \delta(p; x, y_1) < \delta(p)$. By (30), there is a polynomial

$$t(x) = \sum e_ix^i$$

such that if $y_2 = y_1 - t(x)$, then $\delta(p; x, y_2) > \delta(p; x, y_1)$. Substitute $y_1 = y_2 + t(x)$ into

$$F_1 = \sum_{i+\delta_1 j=r\delta_1} \overline{a}_{ij}x^iy_1^j + \sum_{i+\delta_1 j>r\delta_1} \overline{a}_{ij}x^iy_1^j,$$

and normalize with respect to the permissible parameters to get

$$v = P_2(x) + x^cF_2(x, y_2).$$

Let $\overline{d} = \text{ord } (t(x))$. $x^iy_1^j = x^i(y_2 + t(x))^j$ has nonzero $x^{i+m\overline{d}}y_2^{j-m}$ terms with $0 \leq m \leq j$, and may have other nonzero $x^{i+m\overline{d}+\gamma}y_2^{j-m}$ terms with $0 \leq m \leq j$, $\gamma \geq 0$.

Suppose that $\overline{d} < \delta_1 = \delta(p; x, y_1)$. The expansion of y_1^r has a nontrivial $x^{\overline{d}}y_1^{r-1}$ term. Suppose that $x^iy_1^j$ is such that its expansion has a nontrivial $x^{\overline{d}}y_1^{r-1}$ term.

Then $\overline{d} = i + m\overline{d} + \gamma$, $r - 1 = j - m$ with $0 \le m \le j$, $i, \gamma \ge 0$. $\overline{d}(1-m) = i + \gamma \ge 0$ implies $m = 0$ or 1. $m = 0$ implies $j = r-1$, $i \le \overline{d}$. $\overline{a}_{ij} = 0$ in this case since

$$i + \delta_1 j \le \overline{d} + \delta_1(r-1) < \delta_1 r.$$

$m = 1$ implies $i = 0$, $j = r$. Thus there exists a nontrivial $x^{\overline{d}}y_1^{r-1}$ term in $F_2(x, y_1)$ which implies that $\delta_2 < \delta_1$, a contradiction. Thus $\overline{d} \ge \delta_1$.

We then see that if $i + \delta_1 j > r\delta_1$, then all terms $x^\alpha y^\beta$ in the expansion of $x^i y_1^j = x^i(y_2 + t(x))^j$ satisfy $\alpha + \delta_1\beta > r\delta_1$. Since $\delta_2 < \delta_1$, we see that

$$\sum_{i+\delta_1 j = r\delta_1} \overline{a}_{ij} x^i (y_2 + t(x))^j = \begin{cases} cy_2^r + \text{terms with } i + \delta_1 j > r\delta_1 \text{ if } \delta_1 \notin \mathbf{N}, \\ cy_2^r + dx^{r\delta_1} + \text{terms with } i + \delta_1 j > r\delta_1 \text{ if } \delta_1 \in \mathbf{N}. \end{cases}$$

Thus $\mathrm{mult}(t) = \delta_1$ and

$$\sum_{i+\delta_1 j = r\delta_1} \overline{a}_{ij} x^i y_1^j = c(y_1 - e_{\delta_1} x^{\delta_1})^r + dx^{r\delta_1}$$

a contradiction. □

Lemma 6.12. *Suppose that $g : X_1 \to X$ is a quadratic transform, centered at a point p of X, and $p_1 \in X_1$ is a closed point above p such that $g(p_1) = p$ and $\overline{\nu}(p_1) = \overline{\nu}(p)$.*

Suppose that p and p_1 are both 2 points. Then $\delta(p_1) = \delta(p) - 1$.

Suppose that p and p_1 are both 1 points, $\sigma(p) = 0$ and $\delta(p) < \infty$. Then $\delta(p_1) = \delta(p) - 1$.

Proof. Suppose that $r = \mathrm{mult}(F)$.

First suppose that p and p_1 are both 2 points. Then p has permissible parameters (x, y) and $\hat{\mathcal{O}}_{X_1,p_1}$ has permissible parameters (x_1, y_1) such that $x = x_1, y = x_1 y_1$. Since $F_1 = \frac{F}{x_1^r}$, $\delta(p_1; x_1, y_1) = \delta(p; x, y) - 1$. Since $\overline{\nu}(p_1) = \overline{\nu}(p)$, we have $F = \sum a_{ij} x^i y^j$ with $a_{ij} = 0$ if $i + j \le r$ and $j < r$. Thus $a_{0r} \ne 0$, so that $\delta(p; y, x) = 1$ and $\delta(p; x, y) > 1$. Thus $\delta(p) = \delta(p; x, y)$. Since $\mathrm{mult}(F_1) = r$ and $\mathrm{mult}(F_1(0, y_1)) = r$, $\delta(p_1; y_1, x_1) = 1$ and $\delta(p_1; x_1, y_1) \ge 1$. Then $\delta(p_1) = \delta(p_1; x_1, y_1) = \delta(p) - 1$.

Now suppose that p and p_1 are both 1 points, $\sigma(p) = 0$ and $\delta(p) < \infty$. We can suppose that we have permissible coordinates (x, y) at p such that $\delta = \delta(p) = \delta(F; x, y)$ and $\mathrm{mult}(F(0, y)) = \mathrm{mult}(F)$. p_1 has permissible parameters (x_1, y_1) such that $x = x_1, y = x_1(y_1 + \gamma)$ for some $\gamma \in k$.

First suppose that $\gamma \ne 0$.

$$F_1 = \sum_{i+j=r} a_{ij}(y_1 + \gamma)^j - \overline{a} + x_1\Omega$$

where

$$\overline{a} = \sum_{i+j=r} a_{ij}\gamma^j.$$

$\mathrm{mult}(F_1) = \mathrm{mult}(F)$ implies

$$\sum_{i+j=r} a_{ij}(y_1 + \gamma)^j - \overline{a} = a_{0r} y_1^r.$$

Thus

$$\sum_{i+j=r} a_{ij} x^i y^j = a_{0r}(y - \gamma x)^r + \overline{a} x^r.$$

This is a contradiction to the assumption that $\delta(p; x, y) = \delta(p)$ by Lemma 6.11.

Now suppose that $\gamma = 0$. Then $F_1 = \frac{F}{x_1^r}$ and $\delta(p_1; x_1, y_1) = \delta(p; x, y) - 1$. If $\delta(p_1; x_1, y_1) < \delta(p_1)$, then we must also have $\delta(p; x, y) < \delta(p)$ By Lemma 6.11. Thus $\delta(p_1) = \delta(p) - 1$. □

If $p \in X$ is a 2 point, then p is resolved precisely when $\nu(p) = 0$. If $p \in X$ is a 1 point then p is resolved precisely when $\delta(p) = \infty$. Thus $p \in X$ is not resolved at p if and only if $\nu(p) > 0$ and $\delta(p) < \infty$.

We can define an invariant

$$\mathrm{Inv}(p) = (\nu(p), \sigma(p), \delta(p))$$

for $p \in X$.

Theorem 6.13. *Suppose that $g : X_1 \rightarrow X$ is a quadratic transform, centered at a point p of X, and p_1 is a closed point such that $g(p_1) = p$. Suppose that $\nu(p) > 0$ and $\delta(p) < \infty$. Then*

$$Inv(p_1) < Inv(p)$$

in the lexicographic ordering.

Proof. The Theorem follows from Theorem 6.6, and Lemmas 6.8, 6.9, 6.12. □

The proof of Theorem 6.2 is immediate from Theorem 6.13.

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