GROWTH OF RANK 1 VALUATION SEMIGROUPS

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Let (R, m_R) be a local domain, with quotient field K. Suppose that ν is a valuation of K with valuation ring (V, m_V), and that ν dominates R; that is, $R \subset V$ and $m_V \cap R = m_R$. The possible value groups Γ of ν have been extensively studied and classified, including in the papers MacLane [7], MacLane and Schilling [8], Zariski and Samuel [10], and Kuhlmann [6]. Γ can be any ordered abelian group of finite rational rank (Theorem 1.1 [6]). The semigroup

$$S^R(\nu) = \{\nu(f) \mid f \in m_R \setminus \{0\}\}$$

is however not well understood, although it is known to encode important information about the topology and resolution of singularities of Spec(R) and the ideal theory of R.

In Zariski and Samuel's classic book on Commutative Algebra [10], two general facts about the semigroup $S^R(\nu)$ are proven (in Appendix 3 to Volume II).

1. $S^R(\nu)$ is a well ordered subset of the positive part of the value group Γ of ν of ordinal type at most ω^h, where ω is the ordinal type of the well ordered set $\mathbb{N}$, and h is the rank of ν.
2. The rational rank of ν plus the transcendence degree of V/m_V over R/m_R is less than or equal to the dimension of R.

The second condition is the Abhyankar inequality [1].

The only semigroups which are realized by a valuation on a one dimensional regular local ring are isomorphic to the natural numbers. The semigroups which are realized by a valuation on a regular local ring of dimension 2 with algebraically closed residue field are much more complicated, but are completely classified by Spivakovsky in [9]. A different proof is given by Favre and Jonsson in [5], and the theorem is formulated in the context of semigroups by Cutkosky and Teissier [3].

In [3], Teissier and the first author give some examples showing that some surprising semigroups of rank > 1 can occur as semigroups of valuations on noetherian domains, and raise the general questions of finding new constraints on value semigroups and classifying semigroups which occur as value semigroups.

In this paper, we consider semigroups of rank 1 valuations. We show in Theorem 2.1 that the Hilbert polynomial of R gives a bound on the growth of the valuation semigroup $S^R(\nu)$. This allows us to give (in Corollary 2.4) a very simple example of a well ordered subsemigroup of $\mathbb{Q}_+$ of ordinal type ω, which is not a value semigroup of a local domain. This shows that the above conditions 1 and 2 do not characterize value semigroups on local domains.

The simple bound of Theorem 2.1 of this paper is extended in the article [4] of Teissier and the first author to give a very general bound on the growth of a value semigroup of arbitrary rank, from which a rough description of the (extremely bizzare) shape of a higher rank valuation semigroup is derived.

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Prior to this paper and [4], no other general constraints were known on the value semigroups $S^R(\nu)$. In fact, it was even unknown if the above conditions 1 and 2 characterize value semigroups.

With our restriction that ν has rank 1, we can assume that $S^R(\nu)$ is embedded in $\mathbb{R}_+$. We can further assume that $s_0 = 1$ is the smallest element of $S^R(\nu)$. For $n \in \mathbb{N}$, let

$$\varphi(n) = |S^R(\nu) \cap (0, n)|.$$

Corollary 2.2 of Theorem 2.1 shows that for $n \gg 0$,

$$\varphi(n) < P_R(n),$$

where $P_R(n)$ is the Hilbert polynomial of R. This bound is the best possible for a one dimensional local domain, as we show after Corollary 2.4. However, this bound is far from being sharp for R of dimension larger than one. Let

$$\mathcal{P}_n = \{f \in R \mid \nu(f) \geq n\},$$

an ideal in R which contains m_R^n. Suppose that R contains a field k isomorphic to R/m_R, and $R/m_R \cong V/m_V$. Then for $n \gg 0$,

$$\varphi(n) = P_R(n) - \ell(\mathcal{P}_n/m_R^n),$$

where $\ell(\mathcal{P}_n/m_R^n)$ is the length of $\mathcal{P}_n/m^n$. We approximate $\ell(\mathcal{P}_n/m_R^n)$ to show in Corollary 3.4 that

$$\limsup \frac{\varphi(n)}{n^d} < \frac{e(R)}{d!} = \lim_{n\to\infty} \frac{P_R(n)}{n^d}$$

whenever R has dimension $d \geq 2$, where $e(R)$ is the multiplicity of R. When the dimension d of R is greater than 1, this is significantly smaller than the upper bound given by the Hilbert polynomial $P_R(n)$ of R.

In Section 4, we consider the rates of growth which are possible for the function $\varphi(n)$. We show that quite exotic behavior can occur, giving examples (Examples 4.4, 4.5 and 4.6) of valuations ν dominating a regular local ring R of dimension two which have growth rates n^α for any $\alpha \in \mathbb{Q}$ with $1 \leq \alpha \leq 2$. We also give an example of $n\log n$ growth in Example 4.7.

We show in Section 5 (Lemma 5.1 and Corollary 5.7) that the limit

$$\lim_{n\to\infty} \frac{\varphi(n)}{n^2}$$

actually exists, where R is a regular local ring of dimension two (with algebraically closed residue field). We also show (Lemma 5.1 and Proposition 5.8) that any real number β with $0 \leq \beta < \frac{1}{2}$ can be obtained as such a limit. By Corollary 2.3, these are the only limits which can be obtained.

In the final section, we consider the general question of characterizing rank 1 value semigroups, and ask if the necessary condition on a well ordered subsemigroup S of $\mathbb{R}_+$ that the growth of $|S \cap (0, n)|$ is bounded above by a polynomial in n is sufficient for S to be a valuation semigroup.

We give an example (Proposition 6.3) which shows that the characterization of semigroups of regular local rings of dimension two of Proposition 4.1 does not extend to higher dimensions.

1. Notation

The following conventions will hold throughout this paper.

If G is a totally ordered abelian group, then G_+ will denote the positive elements of G. $G_{\ge 0}$ will denote the non negative elements. If $a, b \in G$, we set

$$(a, b) = \{x \in G \mid a < x < b\}.$$

(R, m_R) will denote a (Noetherian) local ring with maximal ideal m_R, and $\ell(N)$ will denote the length of an R module N. Let

$$P_R(n) = \frac{e(R)}{d!} n^d + \text{ lower order terms}$$

be the Hilbert Samuel polynomial of R, where d is the dimension of R and $e(R)$ is the multiplicity of R. We have that $\ell(R/m_R^n) = P_R(n)$ for $n \gg 0$.

Suppose that R is a local domain with quotient field K, and ν is a valuation of K with valuation ring (V, m_V). We will say that ν dominates R if $R \subset V$ and $m_V \cap R = m_R$. We define the value semigroup of ν on R to be

$$S^R(\nu) = \nu(m_R - \{0\}).$$

Let Γ be the valuation group of ν. For $\lambda \in \Gamma$, we define ideals in R

$$\mathcal{P}_\lambda = \{f \in R \mid \nu(f) \ge \lambda\}.$$

2. bounds for growth of Semigroups of rank 1 valuations

The bounds in this section are valid for valuations of arbitrary rank, but since they give information about the smallest segment of the value group, they are essentially statements about rank 1 valuations. We use here, and in the remainder of this paper the notation introduced above in Section 1.

Theorem 2.1. *Suppose that R is a local domain which is dominated by a valuation ν, and suppose that s_0 is the smallest element of $S^R(\nu)$. Then*

$$|S^R(\nu) \cap (0, ns_0)| < \ell(R/m_R^n)$$

for all $n \in \mathbb{N}$.

Proof. Suppose that $n \in \mathbb{N}$. Since $S^R(\nu)$ is well ordered, $(0, ns_0) \cap S^R(\nu)$ is a finite set

$$\lambda_1 < \cdots < \lambda_r$$

for some $r \in \mathbb{N}$. Set $\lambda_{r+1} = ns_0$. We have a sequence of inclusions of ideals (as defined in Section 1)

$$m_R^n \subset \mathcal{P}_{ns_0} = \mathcal{P}_{\lambda_{r+1}} \subset \mathcal{P}_{\lambda_r} \subset \cdots \subset \mathcal{P}_{\lambda_1} = m_R. \tag{1}$$

Thus

$$\sum_{i=1}^{r} \ell(\mathcal{P}_{\lambda_i}/\mathcal{P}_{\lambda_{i+1}}) \le \ell(m_R/m_R^n). \tag{2}$$

Since $\ell(\mathcal{P}_{\lambda_i}/\mathcal{P}_{\lambda_{i+1}}) > 0$ for all i, we have the desired inequality. □

Recall that $P_R(n)$ is the Hilbert polynomial of a local ring R.

Corollary 2.2. *Suppose that R is a local domain of dimension d which is dominated by a valuation ν, and s_0 is the smallest element of $S^R(\nu)$. Then*

1. *For all positive integers $n \gg 0$, $|S^R(\nu) \cap (0, ns_0)| < P_R(n)$.*

2. *There exists $c > 0$ such that $|S^R(\nu) \cap (0, ns_0)| < cn^d$ for all $n \in \mathbb{N}$.*

Corollary 2.3. *Suppose that R is a regular local ring of dimension d which is dominated by a valuation ν, and s_0 is the smallest element of $S^R(\nu)$. Then*

$$|S^R(\nu) \cap (0, ns_0)| < \binom{d-1+n}{d}$$

for all $n \in \mathbb{N}$.

Corollary 2.4. *There exists a well ordered subsemigroup U of $\mathbb{Q}_+$ such that U has ordinal type ω and $U \neq S^R(\nu)$ for any valuation ν dominating a local domain R.*

Proof. Take any subset T of $\mathbb{Q}_+$ such that 1 is the smallest element of T and $n^n \leq |T \cap (0, n)| < \infty$ for all $n \in \mathbb{N}$. For all positive integers r, let

$$rT = \{a_1 + \cdots + a_r \mid a_1, \ldots, a_r \in T\}.$$

Let $U = \omega T = \cup_{r=1}^{\infty} rT$ be the semigroup generated by T. By our constraints, $|U \cap (0, r)| < \infty$ for all $r \in \mathbb{N}$. Thus U is well ordered and has ordinal type ω. By 2 of Corollary 2.2, U cannot be the semigroup of a valuation dominating a local domain. $\square$

We will now consider more closely the bound

$$|S^R(\nu) \cap (0, ns_0)| < P_R(n) \tag{3}$$

for $n \gg 0$ of 1 of Corollary 2.2.

In the case when R is a regular local ring of dimension 1, we have that

$$|S^R(\nu) \cap (0, ns_0)| = n - 1 = P_R(n)$$

for all $n \in \mathbb{N}$, so that the bound (3) is sharp.

When R is an arbitrary local domain of dimension 1, the bound (3) can be far from sharp, as is shown by the following example. Let R be the localization of $k[x,y]/y^2 - x^2 - x^3$ at the maximal ideal (x, y). Define a valuation ν which dominates R by embedding R into the power series ring $k[[t]]$ by the k-algebra homomorphism which maps x to t and y to $t\sqrt{1+t}$. Let ν be the restriction of the t-adic valuation to R. Then $S^R(\nu) = \mathbb{N}$ and $s_0 = 1$, and $|S^R(\nu) \cap (0, ns_0)| = n - 1$ for all positive n. But R has multiplicity 2, and $P_R(n) - 1 = 2(n-1)$ for all positive n.

However, (3) can be sharp for a one dimensional R which is not regular, as is illustrated by the following example. Let R be the localization of $k[x,y]/y^2 - x^3$ at the maximal ideal (x, y). Embed R into the valuation ring $V = k[t]_{(t)}$ by the k-algebra homomorphism which maps x to t^2 and y to t^3. Let ν be the restriction of $\frac{1}{2}$ times the t-adic valuation on V to R. Then $\nu(x) = 1$, $\nu(y) = \frac{3}{2}$ and $S^R(\nu) = \{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots\}$. Thus $s_0 = 1$ and

$$|S^R(\nu) \cap (0, ns_0)| = 2(n-1) = P_R(n) - 1$$

for all positive n.

Evidently, in the case of one dimensional domains, (3) is the best bound which is always valid.

In rings of dimension 2 and higher, (3) is always far from sharp, as we show in the next section.

3. A sharper bound

In this section, we assume that (R, m_R) is a local domain of dimension d and multiplicity $e = e(R)$. Suppose ν is a rank 1 valuation on the quotient field of R with valuation ring V such that ν dominates R, and R contains an infinite field k such that $k \cong R/m_R$ with $R/m_R \cong V/m_V$. Without loss of generality, we may assume that the smallest value of an element of m_R is $s_0 = 1$.
Let

$$\varphi(n) = |S^R(\nu) \cap (0, n)|$$

for $n \in \mathbb{N}$. We will measure the deviation of $\varphi(n)$ from the upper bound (3) given by the Hilbert-Samuel polynomial $P_R(n)$ of R.

We begin with another look at the proof of Theorem 2.1 with these assumptions on R. Since $k = V/m_V$, we have

$$\ell(\mathcal{P}_{\lambda_i}/\mathcal{P}_{\lambda_{i+1}}) = 1$$

for all i in the sequence (1). For $n \in \mathbb{N}$, let

$$\psi(n) = \ell(\mathcal{P}_n/m^n).$$

$\psi(n)$ measures the difference of $\psi(n)$ from the Hilbert-Samuel function as

$$\varphi(n) = \ell(R/m^n) - \psi(n)$$

for all $n \in \mathbb{N}$ (by (2)), and thus

$$\varphi(n) = P_R(n) - \psi(n)$$

for $n \gg 0$.

Let $A = \mathrm{gr}_m(R)$ be the associated graded ring of R and for nonzero $x \in m^i \setminus m^{i+1}$, let $\bar{x}$ denote the image of $x + m^{i+1}$ in A. We will call $\overline{x}$ the inital form of x and i the initial degree of x.

Lemma 3.1. *There exist $x_1, \dots, x_d \in m \setminus m^2$ such that $\bar{x}_i \in A$ form an algebraically independent set over k, and $\nu(x_i) \neq \nu(x_j)$ for $i \neq j$.*

Proof. Since k is infinite, A has a Noether Normalization in degree one. Let $y_1, \dots, y_d$ be elements of $m \setminus m^2$ such that $k[\bar{y}_1, \dots, \bar{y}_d]$ is a Noether Normalization of A.

Since $\nu(a) = \nu(b)$ implies the existence of $\lambda \in k$ such that $\nu(a + \lambda b) > \nu(a)$, we can find $\lambda_{ij} \in k$ such that $x_i = \sum \lambda_{ij} y_j$ satisfy the desired properties. ☐

Lemma 3.2. *Let x_i's be as in the previous lemma, and let K be the fraction field of $k[x_1, \dots, x_d]$. Then there are elements $m_1, \dots, m_e \in R$ (where the multiplicity of R is e) such that $\bar{m}_1, \dots, \bar{m}_e \in A$ are linearly independent over K.*

Proof. For $n \gg 0$, $\ell(m^n/m^{n+1})$ is a polynomial $Q(n)$ of degree $d - 1$. We will compute the leading coefficient of $Q(n)$ in two ways.

Let $B = k[\overline{x}_1, \dots, \overline{x}_n]$. First, observe that A is a finitely generated graded module over the standard graded ring B. Since A has dimension d, and B has multiplicty one, we can compute from tensoring a graded composition series of A as a B module with K (or from the graded version of the additivity formula given for instance in Corollary 4.7.8 [2]) that the multiplicity of A as a B module is $\dim_K(A \otimes_{k[\bar{x_1}, \dots, \bar{x_d}]} K)$.

For $n \gg 0$, $Q(n) = P_R(n+1) - P_R(n)$. Thus

$$Q(n) = \frac{e}{(d-1)!} n^{d-1} + \text{ lower order terms,}$$

and we conclude that

$$\dim_K(A \otimes_{k[\bar{x_1},\ldots,\bar{x_d}]} K) = e.$$

Choose a basis for the vector space $A \otimes_{k[\bar{x_1},\ldots,\bar{x_d}]} K$ of elements of the form $\bar{m}_i \otimes 1$. Such m_i's have the desired property. ☐

Proposition 3.3. *Suppose the x_i's and m_i's are as in the previous lemmas. Then the infimum limit*

$$\liminf \frac{\ell(\mathcal{P}_n/m^n)}{n^d} \geq \frac{e}{d!}\left(1 - \frac{1}{\nu(x_1)\ldots\nu(x_d)}\right).$$

Proof. Let α_i be the initial degree of m_i. Let

$$S = \{m_i x_1^{n_1} \cdots x_d^{nd} \mid \alpha_i + n_1 + \cdots + n_d < n\}.$$

We will first show that the classes of the elements of S in $\mathcal{P}_n/m^n$ are linearly independent over k. Suppose otherwise. Then there is a nontrivial sum

$$\sum \lambda_{i,n_1,\ldots,n_d} m_i x_1^{n_1} \cdots x_d^{n_d} \in m^n, \tag{4}$$

where $\alpha_i + n_1 + \cdots + n_d < n$ and $0 \neq \lambda_{i,n_1,\ldots,n_d} \in k$ for all terms in the sum. Let τ be the smallest value of $\alpha_i + n_1 + \cdots + n_d$ for a term appearing in (4).

Since $\tau < n$, we have that

$$\sum_{\alpha_i+n_1+\cdots+n_d=\tau} \lambda_{i,n_1,\ldots,n_d} m_i x_1^{n_1} \cdots x_d^{n_d} \in m^{\tau+1},$$

and thus

$$\sum_{\alpha_i+n_1+\cdots+n_d=\tau} \lambda_{i,n_1,\ldots,n_d} \overline{m}_i \overline{x}_1^{n_1} \cdots \overline{x}_d^{n_d} = 0$$

in $m^\tau/m^{\tau+1} \subset A$. But by Lemma 3.2, the elements $\overline{m}_i \overline{x}_1^{n_1} \cdots \overline{x}_d^{n_d}$ are linearly independent over k in A, which is a contradiction.

Our next goal is to determine which of the elements of S are in $\mathcal{P}_n \setminus m^n$. Note that since the classes of these elements in $\mathcal{P}_n/m^n$ are linearly independent over k, their number gives a lower bound on $\ell(\mathcal{P}_n/m^n)$.

For a fixed i, the condition that an element $m_i x_1^{n_1} \ldots x_d^{n_d}$ is in $\mathcal{P}_n \setminus m^n$ can be written as the following system of linear inequalities in terms of n_i's:

$$\begin{aligned} \nu(m_i) + \nu(x_1)n_1 + \cdots + \nu(x_d)n_d &\geq n \\ \alpha_i + n_1 + \cdots + n_d &< n. \end{aligned}$$

Now since $\alpha_i \leq \nu(m_i)$ every solution to the following two inequalities is also a solution to the above system.

$$\begin{array}{lll} \nu(x_1)n_1 + \cdots + \nu(x_d)n_d & \geq & n - \alpha_i \\ n_1 + \cdots + n_d & < & n - \alpha_i. \end{array} \tag{5}$$

We will now make an asymptotic approximation of the number of integral solutions to the system (5). To a polytope $P \subset \mathbb{R}^d$ and $n \in \mathbb{Z}_+$, we associate the Ehrhart function

$$E(p,n) = |\{z \in \mathbb{Z}^d \mid \frac{z}{n} \in P\}|.$$

By approximating P with d-cubes of small volume, we compute the volume of P as

$$\mathrm{vol}(P) = \lim_{n\to\infty} \frac{E(P,n)}{n^d} = \lim_{n\to\infty} \frac{|\{z \in \mathbb{Z}^d \mid z \in nP\}|}{n^d}.$$

The volume of the d-simplex Δ with vertices at the origin and at distance $c_1, \ldots, c_d$ along the coordinate axes is

$$\mathrm{vol}(\Delta) = \frac{1}{d!} c_1 \cdots c_d.$$

Let $\sigma(n)$ be the set of integral solutions to the system (5). We have that

$$\begin{aligned} \liminf \tfrac{\ell(\mathcal{P}_n/m^n)}{n^d} &\geq \lim_{n\to\infty} \tfrac{\sigma(n)}{n^d} \\ &= \lim_{n\to\infty} \tfrac{|\{(n_1,\ldots,n_d)\in\mathbb{N}^d | n_1+\cdots+n_d<n\}|}{n^d} \\ &\quad - \lim_{n\to\infty} \tfrac{\{(n_1,\ldots,n_d)\in\mathbb{N}^d | \nu(x_1)n_1+\cdots+\nu(x_d)n_d<n\}|}{n^d} \\ &= \tfrac{1}{d!}(1 - \tfrac{1}{\nu(x_1)\cdots\nu(x_d)}). \end{aligned}$$

□

Corollary 3.4. *Let assumptions be as introduced in the beginning of this section. If the first d elements in $S^R(\nu)$ are $1 = s_1, s_2, \ldots, s_d$, then the supremum limit*

$$\limsup \frac{\varphi(n)}{n^d} < \frac{e}{d!}\frac{1}{s_1 \ldots s_d},$$

and thus, if $d > 1$,

$$\limsup \frac{\varphi(n)}{n^d} < \frac{e}{d!} = \lim_{n\to\infty} \frac{P_R(n)}{n^d}.$$

Proof. From the proposition it follows that there are elements $x_1, \ldots x_d \in m$ such that $\nu(x_i) \neq \nu(x_j)$ and the number of elements in $S^R(\nu) \cap (0,n)$ is asymptotically smaller than

$$\frac{e}{d!}\frac{1}{\nu(x_1)\ldots\nu(x_d)} n^d.$$

Since x_i's have distinct values, we have $s_1 \ldots s_d \leq \nu(x_1) \ldots \nu(x_d)$. □

4. The rate of growth of value semigroups

In this section, we study the rate of growth of $\varphi(n) = |S^R(\nu) \cap (0,n)|$ when R is a regular local ring of dimension two. We show that a wide range of interesting growth occurs within the possible ranges of n and n^2.

We say that $\varphi(n)$ has the growth rate of the function $f(n)$ if there exist $0 < a \leq b$ such that $af(n) \leq \varphi(n) \leq bf(n)$ for all $n \gg 0$.

We can easily achieve growth of $\varphi(n) = |S^R(\nu) \cap (0,n)|$ of the rate n^d on a regular local ring R of dimension d. Choose d rationally independent real positive numbers $\gamma_1, \ldots, \gamma_d$, a regular system of parameters $x_1, \ldots, x_d$ of R and prescribe that $\nu(x_i) = \gamma_i$ for all i. It is also possible to achieve growth asymptotic to n^d from a rational rank 1 valuation, as we show in Example 4.6. We also give examples in this section showing that a wide range of interesting growth can occur.

We use the following characterization of value semigroups dominating a regular local ring of dimension two of [9], and as may also be found with a different treatment in [5]. We state the characterization in the notation of [3].

Proposition 4.1. *Let S be a well ordered subsemigroup of $\mathbb{Q}_+$ which is not isomorphic to $\mathbb{N}$ and whose minimal system of generators $1, a_1, \dots, a_i, \dots$ with*

$$1 < a_1 < a_2 < \cdots < a_i < \cdots$$

is of ordinal type $\le \omega$. Let S_i denote the semigroup generated by $1, a_1, \dots, a_i$, and G_i the subgroup of $\mathbb{Q}$ which it generates. Let $S_0 = \mathbb{N}_+$ and $G_0 = \mathbb{Z}$. Set $q_i = [G_i \colon G_{i-1}]$ for $i \ge 1$. Let s_i be the smallest positive integer s such that $sa_i \in S_{i-1}$. Then S is the semigroup $S^R(\nu)$ of a valuation ν dominating a regular local ring R of dimension 2 with algebraically closed residue field if and only if

$$\text{for each } i \ge 1 \text{ we have } s_i = q_i \text{ and } a_{i+1} > q_i a_i.$$

It follows that if $\{a_i\}$ is a minimal system of generators of a semigroup $S^R(\nu)$ of a valuation ν dominating a regular local ring R of dimension 2 with algebraically closed residue field, then $q_i \ge 2$ for all $i \ge 1$.

Much more complicated behavior occurs in rational semigroups of valuations in regular local rings of dimension 3. An example is given in Proposition 6.3.

We need the following two statements to estimate the number of terms of a rational rank 1 semigroup $S \subset \mathbb{R}$, which are contained in a fixed interval of length 1. Our notation is $\mathbb{N} = \mathbb{Z}_{\ge 0}$.

Lemma 4.2. *Suppose that $a_1, \dots, a_k$ are positive rational numbers. Let $G_0 = \mathbb{Z}$. For a fixed $1 \le i \le k$ let S_i be the subsemigroup of $\mathbb{Q}_{\ge 0}$ generated by $1, a_1, \dots, a_i$ and G_i be the group $S_i + (-S_i)$. Suppose that $q_i = [G_i \colon G_{i-1}]$ and $x_i = (q_1 - 1)a_1 + \cdots + (q_i - 1)a_i$.*

Then for all integers $1 \le i \le k$ we have $|S_i \cap (x_i - 1, x_i]| \ge q_1 \cdots q_i$. Moreover, for all integers $1 \le i \le k$ and real numbers $x > x_i$ we have $S_i \cap [x - 1, \infty) = G_i \cap [x - 1, \infty)$ and $|S_i \cap [x - 1, x)| = q_1 \cdots q_i$.

Proof. Notice that $G_i = \frac{1}{q_1 \cdots q_i}\mathbb{Z}$. Thus $S_i \cap [x - 1, \infty) = G_i \cap [x - 1, \infty)$ for all $x > x_i$ if and only if $|S_i \cap [x - 1, x)| = q_1 \cdots q_i$ for all $x > x_i$ if and only if $|S_i \cap [x - 1, x)| \ge q_1 \cdots q_i$ for all $x > x_i$. Also since $S_i + \mathbb{N} = S_i$, the fact that $|S_i \cap (x_i - 1, x_i]| \ge q_1 \cdots q_i$ implies $|S_i \cap (x - 1, x]| \ge q_1 \cdots q_i$ for all $x \ge x_i$. Moreover, if x is a fixed real number, since S_i is discrete there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ the following equality between sets holds $S_i \cap [x - 1, x) = S_i \cap (x - \varepsilon - 1, x - \varepsilon]$. Therefore, the fact that $|S_i \cap (x_i - 1, x_i]| \ge q_1 \cdots q_i$ also implies $|S_i \cap [x - 1, x)| \ge q_1 \cdots q_i$ for all $x > x_i$.

Assume that $k = 1$. Then S_1 can be presented as a disjoint union of $\mathbb{N}$-modules

$$S_1 = \mathbb{N} \cup (a_1 + \mathbb{N}) \cup \cdots \cup ((q_1 - 1)a_1 + \mathbb{N}).$$

If $0 \le j \le (q_1 - 1)$ then $(x_1 - ja_1) \ge 0$ and $|(ja_1 + \mathbb{N}) \cap (x_1 - 1, x_1]| = |\mathbb{N} \cap (x_1 - ja_1 - 1, x_1 - ja_1]| = 1$. Thus $|S_1 \cap (x_1 - 1, x_1]| = q_1$.

Assume that $k > 1$. By induction it suffices to assume that the statement is true for $i \le k - 1$. Notice that

$$S_k \supset S_{k-1} \cup (a_k + S_{k-1}) \cup \cdots \cup ((q_k - 1)a_k + S_{k-1}),$$

where the union on the right is a disjoint union of S_{k-1}-modules.

If $0 \le j \le (q_k - 1)$ then $(x_k - ja_k) \ge x_{k-1}$ and $|(ja_k + S_{k-1}) \cap (x_k - 1, x_k]| = |S_{k-1} \cap (x_k - ja_k - 1, x_k - ja_k]| \ge q_1 \cdots q_{k-1}$. Thus, $|S_k \cap (x_k - 1, x_k]| \ge q_1 \cdots q_k$. □

Corollary 4.3. *Under the assumptions of corollary 4.2 suppose also that $a_{i+1} > q_i a_i$ for all $1 \le i \le k - 1$. Then $|S_i \cap [x - 1, x)| = q_1 \cdots q_i$ for all real numbers $x \ge q_i a_i$.*

Proof. It suffices to notice that $(q_1 - 1)a_1 + \cdots + (q_i - 1)a_i < q_i a_i$ in this case. □

We will now give examples of value semigroups with unexpected rate of growth of the function $\varphi(n) = |S \cap (0, ns_0)|$.

Example 4.4. *($n\sqrt{n}$ rate of growth)*
Let $R = k[x,y]_{(x,y)}$ where k is an algebraically closed field. Let ν be a valuation of the quotient field of R defined by its generating sequence $\{P_i\}_{i\geq 0}$ as follows

$P_0 = x,$ $\quad\nu(P_0) = 1$
$P_1 = y,$ $\quad\nu(P_1) = 4 + \frac{1}{2}$
$P_2 = P_1^2 - x^9,$ $\quad\nu(P_2) = 16 + \frac{1}{2^2}$
$P_3 = P_2^2 - x^{28}P_1,$ $\quad\nu(P_3) = 64 + \frac{1}{2^3}$
$P_{k+1} = P_k^2 - x^{7\cdot 4^{k-1}}P_{k-1},$ $\quad\nu(P_{k+1}) = 4^{k+1} + \frac{1}{2^{k+1}}.$

Denote by S the semigroup $S_R(\nu) = \nu(m_R\backslash\{0\})$. Then $\varphi(n)$ grows like $n\sqrt{n}$.

Proof. We will show that $\frac{1}{6}n\sqrt{n} < \varphi(n) < \frac{4}{3}n\sqrt{n}$.

Set $a_i = \nu(P_i)$ for all $i \geq 1$. Then S is a subsemigroup of $\mathbb{Q}_+$ generated by $1, a_1, a_2, \ldots$. With notation of Proposition 4.1 we have $s_i = q_i = 2$ for all $i \geq 1$ and $q_i a_i \leq 2\cdot 4^i + 1 < 4^{i+1} < a_{i+1}$. This shows that ν is well defined. Also, by corollary 4.3 we find a lower bound on $|S \cap [n-1, n)|$ for $n \geq 4^i$:

$$|S \cap [n-1,n)| \geq |S_{i-1} \cap [n-1,n)| = 2^{i-1}.$$

If $n > 1$ set $i = \lfloor \log_4 n \rfloor$. Then $4^i \leq n < 4^{i+1}$ and $2^{i-1} \leq |S \cap [n-1,n)| \leq 2^i$. Thus for all $n \in \mathbb{N}_+$ we have

$$\int_{n-1}^{n} \frac{\sqrt{t}}{4}dt < \frac{\sqrt{n}}{4} < |S \cap [n-1,n)| \leq \sqrt{n} < \int_{n}^{n+1} \sqrt{t}dt.$$

Then

$$|S \cap (0,n)| = |S \cap [1,n)| < \int_2^{n+1} \sqrt{t}dt = \frac{2}{3}((n+1)\sqrt{n+1} - 2\sqrt{2}) < \frac{4}{3}n\sqrt{n}$$

and

$$|S \cap (0,n)| = 3 + |S \cap [4,n)| > 3 + \int_4^n \frac{\sqrt{t}}{4}dt > 3 + \frac{1}{6}(n\sqrt{n} - 8) > \frac{1}{6}n\sqrt{n}.$$

A more precise estimate can be obtained for $n = 4^k$. By induction on $k \in \mathbb{N}_+$ we see that $\frac{8^k}{3} < \varphi(4^k) < \frac{8^k}{2}$, since

$$\varphi(4) = 3, \qquad 8/3 < 3 < 4$$

and

$$\varphi(4^{k+1}) = \varphi(4^k) + |S \cap [4^k, 4^{k+1})| < \frac{8^k}{2} + 3\cdot 2^k \cdot 4^k < \frac{8^{k+1}}{2}$$

and

$$\varphi(4^{k+1}) = \varphi(4^k) + |S \cap [4^k, 2\cdot 4^k)| + |S \cap [2\cdot 4^k, 4^{k+1})| > \frac{8^k}{3} + 2^{k-1}\cdot 4^k + 2\cdot 2^k \cdot 4^k > \frac{8^{k+1}}{3}.$$

□

This example can be generalized to a construction of a value semigroup S such that $\varphi(n)$ grows like a power function n^α, where $\alpha \in \mathbb{Q}$, with natural restriction $1 < \alpha < 2$.

Example 4.5. *(n^α rate of growth)*

Suppose that $0 < p < q$ are coprime integers. Let $r = 2^q$ and $s = 2^p$. Let $R = k[x,y]_{(x,y)}$ where k is an algebraically closed field. Let ν be a valuation of the quotient field of R defined by its generating sequence $\{P_i\}_{i\geq 0}$ as follows

$P_0 = x,$ $\quad\nu(P_0) = 1$

$P_1 = y,$ $\quad\nu(P_1) = r + \frac{1}{s}$

$P_2 = P_1^s - x^{sr+1},$ $\quad\nu(P_2) = r^2 + \frac{1}{s^2}$

$P_3 = P_2^s - x^{(sr-1)r}P_1,$ $\quad\nu(P_3) = r^3 + \frac{1}{s^3}$

$P_{k+1} = P_k^s - x^{(sr-1)r^{k-1}}P_{k-1},$ $\quad\nu(P_{k+1}) = r^{k+1} + \frac{1}{s^{k+1}}.$

Denote by S the semigroup $S_R(\nu) = \nu(m_R\backslash\{0\})$. Then $\varphi(n)$ grows like $n^{1+p/q}$.

Proof. Set $a_i = \nu(P_i)$ for all $i \geq 1$. Then S is a subsemigroup of $\mathbb{Q}_+$ generated by $1, a_1, a_2, \ldots$. With notation of Proposition 4.1 we have $s_i = q_i = s$ for all $i \geq 1$ and $q_i a_i \leq s \cdot r^i + 1 < r^{i+1} < a_{i+1}$. This implies that ν is well defined. Also, by corollary 4.3 we find a lower bound on $|S \cap [n-1, n)|$ for $n \geq r^i$:

$$|S \cap [n-1, n)| \geq |S_{i-1} \cap [n-1, n)| = s^{i-1}.$$

If $n > 1$ set $i = \lfloor \log_r n \rfloor$. Then $r^i \leq n < r^{i+1}$ and $s^{i-1} \leq |S \cap [n-1, n)| \leq s^i$. Thus since $s^i = (r^i)^{p/q}$ and $s^{i-1} = \frac{(r^{i+1})^{p/q}}{s^2}$ for all $n \in \mathbb{N}_+$ we have

$$\int_{n-1}^{n} \frac{t^{p/q}}{s^2} dt < \frac{n^{p/q}}{s^2} < |S \cap [n-1, n)| \leq n^{p/q} < \int_{n}^{n+1} t^{p/q} dt.$$

Then

$$|S\cap(0,n)| = |S\cap[1,n)| < \int_2^{n+1} t^{p/q} dt = \frac{q}{p+q}((n+1)^{1+p/q} - 2^{1+p/q}) < \frac{3q}{p+q} n^{1+p/q} < 3n^{1+p/q}$$

and

$$|S\cap(0,n)| = r-1+|S\cap[r,n))| > r-1+\int_r^n \frac{t^{p/q}}{s^2} dt = r-1+\frac{q}{s^2(p+q)}(n^{1+p/q} - rs) > \frac{n^{1+p/q}}{2s^2}.$$

That is $\frac{1}{2s^2} n^{1+p/q} < \varphi(n) < 3n^{1+p/q}$. □

We remark that in the above construction it is necessary to have the strict inequality $p < q$. However, the maximal rate of growth of n^2 is also achievable on a rational rank 1 semigroup of a valuation centered in a 2-dimensional polynomial ring, as we show in the next example.

Example 4.6. *(n^2 rate of growth)*

Let $R = k[x,y]_{(x,y)}$ where k is an algebraically closed field. Let ν be a valuation of the quotient field of R defined by its generating sequence $\{P_i\}_{i\geq 0}$ as follows

$P_0 = x,$ $\quad\nu(P_0) = 1$

$P_1 = y,$ $\quad\nu(P_1) = 1 + \frac{1}{2}$

$P_2 = P_1^2 - x^{2+1},$ $\quad\nu(P_2) = 2 + 1 + \frac{1}{2^2}$

$P_3 = P_2^2 - x^{2^2+2-1}P_1,$ $\quad\nu(P_3) = 2^2 + 2 + \frac{1}{2} + \frac{1}{2^3}$

$P_{k+1} = P_k^2 - x^{2^k+2^{k-1}-2^{k-2}}P_{k-1},$ $\quad\nu(P_{k+1}) = 2^k + 2^{k-1} + 2^{k-3} + \cdots + 2^{-k-1}.$

Denote by S the semigroup $S_R(\nu) = \nu(m_R\backslash\{0\})$. Then $\varphi(n)$ grows like n^2.

Proof. Set $a_i = \nu(P_i)$ for all $i \geq 1$. Then S is the subsemigroup of $\mathbb{Q}_+$ generated by $1, a_1, a_2, \ldots$. We have $q_i = 2$ for all $i \geq 1$. Solving the recursion relation, we have

$$a_i = 2^{i-1} + \frac{1}{3}(2^i - \frac{1}{2^i})$$

for $i \geq 1$. We have $q_{i-1}a_{i-1} = 2a_{i-1} = \frac{5}{6}2^i - \frac{1}{3}\frac{1}{2^{i-2}} < 2^i$. Corollary 4.3 shows that

$$|S \cap [n-1, n)| \geq |S_{i-1} \cap [n-1, n)| = 2^{i-1}$$

for $n \geq 2^i$. For $n > 1$, set $i = \lfloor \log_2 n \rfloor$, so that $2^i \leq n < 2^{i+1}$. We have

$$\frac{1}{4}\int_{n-1}^{n} tdt < \frac{n}{4} < \frac{2^{i+1}}{4} = 2^{i-1} \leq |S \cap [n-1, n)|.$$

Thus for $n \geq 4$,

$$|S \cap (0, n)| = 3 + |S \cap [4, n)| > 3 + \frac{1}{4}\int_{4}^{n} tdt > \frac{n^2}{8}.$$

Since $|S \cap (0, n)| < \binom{1+n}{2}$ by Corollary 2.3, $\varphi(n)$ grows at the rate of n^2. □

Another interesting example is of logarithmic growth.

Example 4.7. *($n \log_{10} n$ rate of growth)*
Let $R = k[x, y]_{(x,y)}$ where k is an algebraically closed field. Let ν be a valuation of the quotient field of R defined by its generating sequence $\{P_i\}_{i \geq 0}$ as follows

$P_0 = x$, $\qquad \nu(P_0) = 1$
$P_1 = y$, $\qquad \nu(P_1) = 10 + \frac{1}{2}$
$P_2 = P_1^2 - x^{21}$, $\qquad \nu(P_2) = 10^2 + \frac{1}{2^2}$
$P_3 = P_2^2 - x^{190}P_1$, $\qquad \nu(P_3) = 10^4 + \frac{1}{2^3}$
$P_{k+1} = P_k^2 - x^{\alpha(k)}P_{k-1}$, $\qquad \nu(P_{k+1}) = 10^{2^k} + \frac{1}{2^{k+1}}$, $\qquad \alpha(k) = 2 \cdot 10^{2^{k-1}} - 10^{2^{k-2}}$.

Denote by S the semigroup $S_R(\nu) = \nu(m_R \backslash \{0\})$. Then $\varphi(n)$ grows like $n \log_{10} n$.

Proof. If $n \geq 10$ let $k = \lfloor \log_2 \log_{10} n \rfloor$. Then $10^{2^k} \leq n < 10^{2^{k+1}}$ and $2^k \leq |S \cap [n-1, n)| \leq 2^{k+1}$. Thus for all $n \geq 10$ we have

$$\int_{n-1}^{n} \frac{\log_{10} t}{2} dt < \frac{\log_{10} n}{2} \leq |S \cap [n-1, n)| \leq 2\log_{10} n < \int_{n}^{n+1} 2\log_{10} tdt$$

and

$$\frac{n \log_{10} n}{4} < 9 + \int_{10}^{n} \frac{\log_{10} t}{2} dt < |S \cap [0, n)| < 9 + \int_{11}^{n+1} 2\log_{10} tdt < 2n\log_{10} n.$$

That is $\frac{1}{4} n \log_{10} n < \varphi(n) < 2n \log_{10} n$. □

5. The existence of a limit of asymptotic growth on a 2 dimensional regular local ring

In this section (Lemma 5.1 and Corollary 5.7), we show that the limit

$$\lim_{n \to \infty} \frac{\varphi(n)}{n^2} \tag{6}$$

exists when R is a regular local ring of dimension two with algebraically closed residue field, and ν is a valuation dominating R with value semigroup S_ν and rational rank one. n^2 is the largest rate of growth of $\varphi(n) = |S^R(\nu) \cap (0, ns_0)|$ possible on a local ring of dimension 2. We further show (by Lemma 5.1 and Proposition 5.8) that any real number

β with $0 \leq \beta < \frac{1}{2}$ can be obtained as such a limit. By Corollary 2.3, these are the only limits which can be obtained.

If ν is discrete of rank 1, then the limit (6) exists and is zero. We will thus assume that ν is nondiscrete for the rest of this section. Let $S^R(\nu)$ be minimally generated by

$$\gamma_0 = 1 < \gamma_1 < \gamma_2 < \cdots .$$

Let S_i be the semigroup generated by $\gamma_0 = 1, \ldots, \gamma_i$ and $G_i =< 1/m_i >$ the group generated by the same elements. Also let $s_i = m_i/m_{i-1}$.

For any $x \geq 1$ define $\varphi(x) = |S^R(\nu) \cap [1, x)|$ and $\psi(x) = |S^R(\nu) \cap [x, x+1)|$.

The goal is to prove $\lim_{n\to\infty} \varphi(n)/n^2$ exists and to find its value. To that end we will prove that the above limit can be computed from $\lim_{n\to\infty} \psi(n)/n$ and that the latter limit has the same value as $\lim_{i\to\infty} m_{i-1}/\gamma_i$.

Note that $\varphi(n) = \sum_{i=1}^{n-1} \psi(i)$, therefore the following lemma essentially takes care of our first reduction:

Lemma 5.1. *Suppose a sequence a_i of positive numbers is given and that $\lim_{n\to\infty} a_n/n = \alpha$. If $b_n = \sum_{i=1}^{n-1} a_i$, then $\lim_{n\to\infty} b_n/n^2 = \alpha/2$.*

Proof. Suppose $\varepsilon > 0$ is given. There exists $N > 0$ so that for $i \geq N$,

$$\alpha - \varepsilon < \frac{a_i}{i} < \alpha + \varepsilon$$

Now for $n > N$ we have $b_n = \sum_{i=1}^{n-1} a_i$ therefore

$$\begin{aligned} \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n-1} a_i = \quad b_n \quad &= \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n-1} a_i \\ \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n-1} (\alpha - \varepsilon) i \leq \quad b_n \quad &\leq \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n-1} (\alpha + \varepsilon) i. \end{aligned}$$

Now let

$$C_1 = \sum_{i=1}^{N} a_i - \sum_{i=1}^{N} (\alpha - \varepsilon) i,$$

and

$$C_2 = \sum_{i=1}^{N} a_i - \sum_{i=1}^{N} (\alpha + \varepsilon) i,$$

to get

$$C_1 + (\alpha - \varepsilon)\frac{n^2 - n}{2} \leq b_n \leq C_2 + (\alpha + \varepsilon)\frac{n^2 - n}{2}.$$

Note that C_1 and C_2 are independent of n and only depend on ε. Now knowing the following four limits:

$$\lim_{n\to\infty} (\alpha - \varepsilon)\frac{n^2 - n}{2n^2} = \frac{\alpha - \varepsilon}{2},$$

$$\lim_{n\to\infty} (\alpha + \varepsilon)\frac{n^2 - n}{2n^2} = \frac{\alpha + \varepsilon}{2},$$

$$\lim_{n\to\infty} C_1/n^2 = 0,$$

and

$$\lim_{n\to\infty} C_2/n^2 = 0,$$

we can choose N' so that for all $n > N'$ the difference between the terms and their limit is smaller than $\varepsilon/4$ so for n larger than N and N' we have:

$$\begin{aligned} C_1/n^2 + (\alpha - \varepsilon)\frac{n^2 - n}{2n^2} \leq \quad & b_n/n^2 \quad \leq C_2/n^2 + (\alpha + \varepsilon)\frac{n^2 - n}{2n^2} \\ -\varepsilon/4 + \frac{\alpha - \varepsilon}{2} - \varepsilon/4 \leq \quad & b_n/n^2 \quad \leq \varepsilon/4 + \frac{\alpha + \varepsilon}{2} + \varepsilon/4. \\ \alpha/2 - \varepsilon \leq \quad & b_n/n^2 \quad \leq \alpha/2 + \varepsilon. \end{aligned}$$

and the assertion follows. □

Corollary 5.2. *If* $\lim_{n\to\infty} \psi(n)/n$ *exist, then so does* $\lim_{n\to\infty} \varphi(n)/n^2$*. Furthermore*

$$\lim_{n\to\infty} \varphi(n)/n^2 = \frac{\lim_{n\to\infty} \psi(n)/n}{2}$$

Proof. Immediate from the last lemma. □

Next we will show that the m_{i-1}/γ_i has a limit as i goes to infinity.

Lemma 5.3. *The sequence* m_{i-1}/γ_i *is decreasing and has a limit.*

Proof. By the classification of valuation semigroups dominating regular local rings of dimension 2, we know that $\gamma_{i+1} > s_i\gamma_i$ and $s_i = m_i/m_{i-1}$. Therefore

$$\gamma_{i+1} > \frac{m_i}{m_{i-1}}\gamma_i,$$

or

$$\frac{m_i}{\gamma_{i+1}} < \frac{m_{i-1}}{\gamma_i}.$$

Now we have a decreasing sequence of positive real numbers. Hence it has a limit. □

The last step in our proof is to show that $\lim_{n\to\infty} \psi(n)/n = \lim_{n\to\infty} \frac{m_n}{\gamma_{n+1}}$. Before doing so however we will find a recursive relation for computing $\psi(x)$.

Lemma 5.4. *If* $\gamma_i < x + 1 \leq \gamma_{i+1}$ *then* $\psi(x) = \min(\psi(x - \gamma_i) + m_{i-1}, m_i)$*.*

Proof. Since $x + 1 \leq \gamma_{i+1}$ we have $\psi(y) = |S_i \cap [y, y + 1)|$ for all $y \leq x$. In particular, this implies that $\psi(x) \leq m_i$. We will show that either $\psi(x) = \psi(x - \gamma_i) + m_{i-1}$ or the following two facts are true $\psi(x) < \psi(x - \gamma_i) + m_{i-1}$ and $\psi(x) = m_i$. This will imply the statement of the lemma.

Notice that $\psi(x) = |(S_i \setminus S_{i-1}) \cap [x, x+1)| + |S_{i-1} \cap [x, x+1)|$. Since $x+1 > \gamma_i > s_{i-1}\gamma_{i-1}$ by Corollary 4.3 we have $|S_{i-1} \cap [x, x + 1)| = m_{i-1}$. Now consider the following map

$$\begin{aligned} f : S_i \cap [x - \gamma_i, x - \gamma_i + 1) \quad & \rightarrow \quad S_i \cap [x, x + 1) \\ a \quad & \rightarrow \quad a + \gamma_i \end{aligned}$$

f is one to one and onto its image. The image of f contains $(S_i \setminus S_{i-1}) \cap [x, x + 1)$.

Case 1: The image of f is equal to $(S_i \setminus S_{i-1}) \cap [x, x+1)$. Then $\psi(x) = \psi(x - \gamma_i) + m_{i-1}$.

Case 2: The image of f intersects S_{i-1}. In this case $\psi(x) < \psi(x - \gamma_i) + m_{i-1}$. Then we can pick an element a in the intersection, being in the image of f it can be written as $c\gamma_i + c'$ with c a non-negative integer and $c' \in S_{i-1}$. Now since $s_i\gamma_i$ is the smallest multiple of γ_i in G_{i-1} we deduce that $a \geq s_i\gamma_i$. But $x + 1 > a$ and therefore $x + 1 > s_i\gamma_i$. Thus $\psi(x) = m_i$ by Corollary 4.3. □

Using this recursive formula we'll find two bounds for $\psi(x)/x$ which both tend to the same limit as x goes to infinity.

Proposition 5.5. *Let* $\alpha = \lim_{n\to\infty} \frac{m_{n-1}}{\gamma_n}$, *then for any* x *we have* $\alpha < \psi(x)/x$.

Proof. To prove the inequality we use induction on $[x]$. Note that if $0 < x \leq \gamma_1 - 1$ the inequality holds since $\alpha \leq m_1/\gamma_2 = s_1/\gamma_2 < 1/\gamma_1 < 1/x = \psi(x)/x$. Now suppose x is given and $\gamma_i < x + 1 \leq \gamma_{i+1}$. Using the recursive formula above we have

$$\psi(x) = \min(\psi(x - \gamma_i) + m_{i-1}, m_i).$$

If $\psi(x) = m_i$, then $\psi(x)/x = m_i/x > m_i/\gamma_{i+1} > \alpha$.

On the other hand if $\psi(x) = \psi(x - \gamma_i) + m_{i-1}$, then

$$\frac{\psi(x)}{x} = \frac{\psi(x - \gamma_i) + m_{i-1}}{(x - \gamma_i) + \gamma_i}$$

But $\psi(x - \gamma_i)/(x - \gamma_i) > \alpha$ by the induction hypothesis and $m_{i-1}/\gamma_i > \alpha$. Therefore

$$\frac{\psi(x - \gamma_i) + m_{i-1}}{(x - \gamma_i) + \gamma_i} > \alpha$$

We have shown that in either case $\psi(x)/x > \alpha$ and therefore by induction it holds for all x. □

Proposition 5.6. *For any given* ε *there exists* $C > 0$ *such that*

$$\psi(x) \leq (\alpha + \varepsilon)x + C$$

for all x.

Proof. Suppose ε is given, since the limit of m_{i-1}/γ_i is α there exists $N > 0$ such that for all $i > N$ we have $m_{i-1}/\gamma_i < \alpha + \varepsilon$.

Now $\psi(x)$ attains only finitely many different values for x between 0 and γ_N, therefore $\psi(x)$ is bounded above on the interval $[0, \gamma_N]$. Let C be an upper bound, that is $\psi(x) \leq C < (\alpha + \varepsilon)x + C$ for all $x \in [0, \gamma_N]$.

Now we will prove that the desired inequality holds for all values of x by induction on $[x]$. The assertion is already known if $x + 1 \leq \gamma_N$. So suppose $x + 1 > \gamma_N$ and suppose $\gamma_i < x + 1 \leq \gamma_{i+1}$ with $i \geq N$. Using the recursive formula for $\psi(x)$ and the induction hypothesis for $\psi(x - \gamma_i)$ we get

$$\begin{aligned}
\psi(x) &= \min(\psi(x - \gamma_i) + m_{i-1}, m_i) \\
&\leq \psi(x - \gamma_i) + m_{i-1} \\
&\leq (\alpha + \varepsilon)(x - \gamma_i) + C + (\alpha + \varepsilon)\gamma_i \\
&= (\alpha + \varepsilon)x + C.
\end{aligned}$$

□

Combining the last two propositions yields the desired result.

Corollary 5.7. *The sequence* $\psi(n)/n$ *approaches* α *as* n *goes to infinity.*

Proof. Let $\varepsilon > 0$ be given, and pick C such that $\psi(x) \leq (\alpha + \varepsilon/2)x + C$. Now for $n > 2C/\varepsilon$ we have

$$\alpha < \psi(n)/n \leq \frac{(\alpha + \varepsilon/2)n + C}{n} = \alpha + \varepsilon/2 + C/n < \alpha + \varepsilon.$$

Therefore for $n > 2C/\varepsilon$ we have $|\psi(n)/n - \alpha| < \varepsilon$. □

Note that the only restriction on the value of $\alpha = \lim_{n\to\infty} \frac{\psi(n)}{n}$ is that it should be a non-negative real number smaller than 1, as it is shown by the next proposition.

Proposition 5.8. *Let R be a regular local ring of dimension 2 with algebraically closed residue field, and let $0 \le \alpha < 1$ be a real number. There exists a valuation ν dominating R such that $\lim_{n\to\infty} \frac{\psi(n)}{n} = \alpha$.*

Proof. First we chose a sequence of rational numbers

$$1 = q_0 > q_1 > q_2 > \cdots > q_i > \cdots > \alpha,$$

such that $\lim_{i\to\infty} q_i = \alpha$. Next we chose sequences of natural numbers m_i, n_i such that

1) $q_i = \frac{m_i}{n_{i+1}}$
2) $m_0 = n_1 = 1$
3) $\frac{m_1 + 1}{n_2} < 1$
4) $m_i | m_{i+1}$
5) $n_i < n_{i+1}$ and $m_i < m_{i+1}$

Now we set $s_i = m_i/m_{i-1}$ and $\gamma_i = n_i + 1/m_i$ for $i \ge 1$ and $\gamma_0 = 1$. Now we claim that γ_i's satisfy the requirements of Proposition 4.1. First

$$s_i\gamma_i = \frac{m_i}{m_{i-1}}(n_i + \frac{1}{m_i}) = s_i n_i + \frac{1}{m_{i-1}} = \gamma_{i-1} + s_i n_i - n_{i-1}$$

But $s_i n_i > n_i > n_{i-1}$, therefore $s_i\gamma_i \in S_{i-1}$. Second s_i is the smallest integer multiplying γ_i into S_{i-1} since it is clearly the smallest integer multiplying γ_i into $G_{i-1} =< 1/m_{i-1} >$. And last $s_i\gamma_i < \gamma_{i+1}$ for $i \ge 1$ since $q_{i-1} > q_i$. (Note that the $i = 1$ case has to be checked separately by using the requirement (3) in the choice of n_i and m_i. The separate treatment is needed since this is the only case where $s_i\gamma_i$ is an integer.)

Therefore Proposition 4.1 implies that γ_i's generate the value semigroup of a valuation ν dominating R. Now we have

$$\lim_{n\to\infty} \frac{\psi(n)}{n} = \lim_{i\to\infty} \frac{m_i}{\gamma_{i+1}} = \lim_{i\to\infty} \frac{m_i}{n_{i+1} + \frac{1}{m_{i+1}}} = \lim_{i\to\infty} \frac{m_i}{n_{i+1}} = \lim_{i\to\infty} q_i = \alpha$$

□

6. When is a semigroup a value semigroup?

Corollary 2.2 gives a necessary condition for a rank 1 well ordered semigroup S consisting of positive elements of $\mathbb{R}$ to be the value semigroup $S^R(\nu)$ of a valuation dominating some local domain R. The condition is:

(7) There exists $c > 0$ and $d \in \mathbb{Z}_+$ such that $|S \cap (0, ns_0)| < cn^d$ for all n

where s_0 is the smallest element of S. An interesting question is if (7) is in fact sufficient. (7) is sufficient in the case when $d = 1$, as we now show. Suppose that $S \subset \mathbb{R}_+$ is a semigroup consisting of positive elements which contains a smallest element s_0. Suppose that there exists $c > 0$ such that

(8) $$|S \cap (0, ns_0)| < cn$$

for all $n \in \mathbb{N}$. By Lemma 6.2 below, we may assume that $S \subset \mathbb{Q}_+$ is finitely generated by some elements $\lambda_1, \ldots, \lambda_r$. There exists $\alpha \in \mathbb{Q}_+$ such that there exists $a_i \in \mathbb{N}_+$ with

$\lambda_i = \alpha a_i$ for $1 \le i \le r$, and $\gcd(a_1, \ldots, a_r) = 1$. Let $k[t]$ be a polynomial ring over a field k. Let $\nu(f(t)) = \alpha \mathrm{ord}(f(t))$ for $f(t) \in k[t]$. ν is a valuation of $k(t)$. Let R be the one dimensional local domain

$$R = k[t^{a_1}, \ldots, t^{a_r}]_{(t^{a_1}, \ldots, t^{a_r})}.$$

The quotient field of R is $k(t)$, and ν dominates R. We have that $S = \nu(m_R \setminus \{0\}) = S^R(\nu)$.

Lemma 6.1. *Suppose that $S \subset \mathbb{R}_+$ is a semigroup consisting of positive elements which contains a smallest element s_0. Suppose that there exists $c > 0$ and $d \in \mathbb{N}_+$ such that*

$$|S \cap (0, ns_0)| < cn^d \tag{9}$$

for all $n \in \mathbb{N}$. Then S is well ordered of ordinal type ω and has rational rank $\le d$.

Proof. The fact that S is well ordered of ordinal type ω is immediate from (9).

We will prove that the rational rank of S is $\le d$. After rescaling S by multiplying by $\frac{1}{s_0}$, we may assume that $s_0 = 1$. Suppose that $t \in \mathbb{N}$ and S has rational rank $\ge t$. Then there exist $\gamma_1, \ldots, \gamma_t \in S$ which are rationally independent. Let $b \in \mathbb{N}$ be such that $\max\{\gamma_1, \ldots, \gamma_t\} < b$. For $e \in \mathbb{R}_+$, we have

$$\begin{array}{lll} |S \cap (0, e)| & \ge & |\{a_1\gamma_1 + \cdots + a_t\gamma_t \mid a_1, \ldots, a_t \in \mathbb{N} \text{ and } a_1\gamma_1 + \cdots + a_t\gamma_t < e\}| - 1 \\ & \ge & |\{a_1\gamma_1 + \cdots + a_t\gamma_t \mid a_i \in \mathbb{N} \text{ and } 0 \le a_i < \frac{e}{tb} \text{ for } 1 \le i \le t\}| - 1 \\ & = & |\{(a_1, \ldots, a_t) \in \mathbb{N}^t \mid 0 \le a_i < \frac{e}{tb} \text{ for } 1 \le i < t\}| - 1 \end{array}$$

since $\gamma_1, \ldots, \gamma_t$ are rationally independent.

For $a \in \mathbb{N}$ let $n = abt$. Then we see that

$$|S \cap (0, (n+1))| \ge a^t - 1 = \left(\frac{1}{bt}\right)^t n^t - 1.$$

By (9), we see that $t \le d$. ☐

Lemma 6.2. *Suppose that $S \subset \mathbb{R}_+$ is a semigroup consisting of positive elements which contains a smallest element s_0. Suppose that there exists $c > 0$ such that*

$$|S \cap (0, ns_0)| < cn \tag{10}$$

for all $n \in \mathbb{N}$. Then S is finitely generated, and the group generated by S is isomorphic to $\mathbb{Z}$.

Proof. By Lemma 6.1, S has rational rank 1, so we may assume that S is contained in $\mathbb{Q}_+$. We may further assume that $s_0 = 1$. Suppose that S is not finitely generated. Then for $e \in \mathbb{N}$, we can find $\lambda_1, \ldots, \lambda_e \in S$ such that $\lambda_i = \frac{a_i}{b_i}$ with $a_i, b_i \in \mathbb{N}_+$, $b_i > 1$ for all i, $\gcd(a_i, b_i) = 1$ for all i and $b_1, \ldots, b_e$ all distinct.

Let $n_0 = \max\{\lfloor \lambda_i \rfloor \mid 1 \le i \le e\}$. For $n \ge n_0$, we have that $\lambda_i + n - \lfloor \lambda_i \rfloor \in S \cap (n, n+1)$ for all i. Thus $|S \cap (n, n+1)| \ge e$ for $n \ge n_0$, which implies that

$$|S \cap (0, n+1)| \ge en - en_0$$

for $n \ge n_0$. For $e > c$, we have a contradiction to (10). ☐

We conclude with an example which shows that the characterization of semigroups of regular local rings of dimension two of Proposition 4.1 does not extend so well to higher dimensions.

Suppose that R is a local domain and ν is a rational rank 1 valuation which dominates R. We can assume that the value group Γ_ν is contained in $\mathbb{Q}$. Let

$$\lambda_0 < \lambda_1 < \cdots < \lambda_r < \cdots$$

be the (discrete) set of minimal generators of the semigroup $S^R(\nu)$.

In the case when R is a RLR of dim 2 (with algebraically closed residue field) we always have that $\lambda_{i+1} \geq 2\lambda_i$ for $i \geq 1$ (as follows from Proposition 4.1). One can ask if a generalization of this bound is true for more general R. However, there is no bound of this type in dimension three, as is shown by the following example.

Proposition 6.3. *There exists a regular local ring R of dimension 3 dominated by a rational rank 1 valuation ν which has the property that given $\varepsilon > 0$, there exists an i such that $\lambda_{i+1} - \lambda_i < \varepsilon$.*

Proof. Let k be an algebraically closed field. Let n_i be an increasing sequence of natural numbers such that $n_0 = 0$ and $n_i > 3n_{i-1}$ for all i.

We first construct a valuation ν_1 of the two dimensional rational function field $k(x, y)$. We do this by constructing a generating sequence of the polynomial ring $k[x, y]$. Set $P_0 = x$, and $P_1 = y$. We prescribe that $\nu_1(P_0) = 1$ and $\nu_1(P_1) = n_1 + \frac{1}{3}$.

We construct inductively a minimal generating sequence $\{P_i\}$ for ν_1, such that $\deg_y(P_i) = 3^{i-1}$, P_i is monic in y, and $\nu_1(P_i) = n_i + \frac{1}{3^i}$ for $i \geq 1$.

We define inductively for all $i \geq 1$, $P_{i+1} = P_i^3 - P_0^{3n_i - n_{i-1}} P_{i-1}$. Since

$$\nu_1(P_0^{3n_i - n_{i-1}} P_{i-1}) = 3n_i + \frac{1}{3^{i-1}} = \nu_1(P_i^3),$$

we may thus prescribe that $\nu_1(P_{i+1}) = n_{i+1} + \frac{1}{3^{i+1}}$ since $n_{i+1} > 3n_i$.

The minimal generators of the semigroup of ν_1 on $k[x, y]_{(x,y)}$ are $\beta_i = \nu_1(P_i)$ for $i \geq 0$.

We now construct a valuation ν_2 of the two dimensional rational function field $k(x, z)$. We do this by constructing a generating sequence of the polynomial ring $k[x, z]$. Set $Q_0 = x$, and $Q_1 = z$. We prescribe that $\nu_2(Q_0) = 1$ and $\nu_2(Q_1) = n_1 + \frac{1}{2}$.

We construct inductively a minimal generating sequence $\{Q_i\}$ for ν_2, such that $\deg_z(Q_i) = 2^{i-1}$, Q_i is monic in z, and $\nu_2(Q_i) = n_i + \frac{1}{2^i}$ for $i \geq 1$.

We define inductively for all $i \geq 1$, $Q_{i+1} = Q_i^2 - Q_0^{2n_i - n_{i-1}} Q_{i-1}$. Since

$$\nu_2(Q_0^{2n_i - n_{i-1}} Q_{i-1}) = 2n_i + \frac{1}{2^{i-1}} = \nu_2(Q_i^2),$$

we may thus prescribe that $\nu_2(Q_{i+1}) = n_{i+1} + \frac{1}{2^{i+1}}$ since $n_{i+1} > 2n_i$.

The minimal generators of the semigroup of ν_2 on $k[x, z]_{(x,z)}$ are $\gamma_i = \nu(Q_i)$ for $i \geq 0$.

Suppose that $f \in k[x, y, z]$ is a nonzero polynomial. Since the P_i are monic in y and the Q_j are monic in z, we can apply the Euclidean algorithm to f to get a unique expansion

$$f = \sum \alpha_{m,i_1,\ldots,i_r,j_1,\ldots,j_s} x^m P_1^{i_1} \cdots P_r^{i_r} Q_1^{j_1} \cdots Q_s^{j_s} \tag{11}$$

with $\alpha_{m,i_1,\ldots,i_r,j_1,\ldots,j_s} \in k$, $m, s, r \in \mathbb{N}$, $0 \leq i_l < 3$ for all l, $0 \leq j_l < 2$ for all l, and either $r = 0$ or $i_r > 0$, either $s = 0$ or $j_s > 0$.

Define

$$\nu(x^m P_1^{i_1} \cdots P_r^{i_r} Q_1^{j_1} \cdots Q_s^{j_s}) = m + i_1(n_1 + \frac{1}{3}) + \cdots + i_r(n_r + \frac{1}{3^r}) + j_1(n_1 + \frac{1}{2}) + \cdots + j_s(n_s + \frac{1}{2^s}).$$

The ν value of all of the terms appearing in the expansion (11) of f are distinct. Thus (by the method of proof of Lemma 21 [4]), we may define a valuation on $k(x, y, z)$ which dominates $R = k[x, y, z]_{(x,y,z)}$ by defining

$$\nu(f) = \min\{\nu(x^m P_1^{i_1} \cdots P_r^{i_r} Q_1^{j_1} \cdots Q_s^{j_s}) \mid \alpha_{m,i_1,\ldots,i_r,j_1,\ldots,j_s} \neq 0 \text{ in the expansion (11)}\}.$$

By construction, the minimal generating sequence of $S^R(\nu)$ is the sequence

$$1, n_1 + \frac{1}{3}, n_1 + \frac{1}{2}, n_2 + \frac{1}{3^2}, \ldots, n_i + \frac{1}{3^i}, n_i + \frac{1}{2^i}, \ldots$$

□

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