

# An approach to prove Hot-Spots type results using level curves besides the nodal line

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Consider the heat equation on a bounded, simply connected region  $D$  with Neumann boundary conditions. Denote by  $\lambda_2$  the second eigenvalue of the Neumann Laplacian, which we will assume is simple. Denote by  $u_2(x)$  the corresponding second eigenfunction (Of course the eigenfunction is only defined up to multiplication by scalar constants, but here we \*fix\*  $u_2(x)$  to be that with unit  $L^2$ -norm).

Suppose that there exists a curve (denoted by the dotted curve in Figure 1) which divides  $D$  into two subregions  $D_1$  and  $D_2$ , and along which  $u_2(x) \equiv a$  is constant. (We might call this the “ $\{u_2 = a\}$  level curve”, but that would include all such points whereas here we might only consider a piece of the level curve). We now restrict our focus to one of the subregions, say  $D_1$ . If

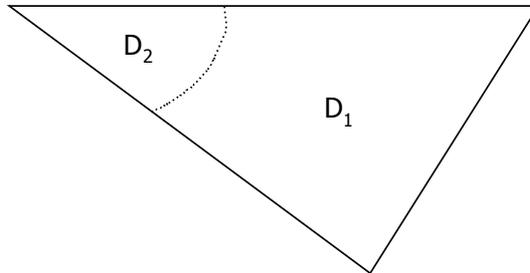


Figure 1: The domain  $D$  divided into two subdomains along a curve where  $u_2(x) \equiv a$ .

we consider the function  $u(x, t) = e^{-\lambda_2 t} u_2(x)$ , we see that  $u$  solves the PDE

$$\begin{cases} u_t = \Delta u, & \text{in } D_1 \\ \frac{\partial u}{\partial \nu} = 0, & \text{on the filled part of } \partial D_1 \\ u(x, t) = a e^{-\lambda_2 t}, & \text{on the dotted part of } \partial D_1 \\ u(x, 0) = u_2(x) \end{cases} \quad (1)$$

Now, suppose we consider this PDE but replace the initial data by the zero function. That is, let  $v$

solve the PDE

$$\begin{cases} v_t = \Delta v, & \text{in } D_1 \\ \frac{\partial v}{\partial \nu} = 0, & \text{on the filled part of } \partial D_1 \\ v(x, t) = g(t) := ae^{-\lambda_2 t}, & \text{on the dotted part of } \partial D_1 \\ v(x, 0) \equiv 0 \end{cases} \quad (2)$$

How does  $v$  differ from  $u$ ? Letting  $w = u - v$ , we see that  $w$  solves the PDE

$$\begin{cases} w_t = \Delta w, & \text{in } D_1 \\ \frac{\partial w}{\partial \nu} = 0, & \text{on the filled part of } \partial D_1 \\ w(x, t) = 0, & \text{on the dotted part of } \partial D_1 \\ w(x, 0) \equiv u_2(x) \end{cases} \quad (3)$$

Now, let's construct explicit expressions for  $v$  and  $w$ ! Denote by  $\mu_k$  and  $v_k(x)$  the  $k$ -th eigenvalue/vector of the mixed-boundary (Dirichlet on the dotted boundary, Neumann on the filled boundary) Laplacian for  $D_1$ . We then have that the solution to (2) is given by

$$\begin{aligned} v(x, t) &= \int_0^t g'(s) \left( 1 - \sum_{k=1}^{\infty} \langle v_k, 1 \rangle e^{-\mu_k(t-s)} v_k(x) \right) ds + g(0) \left( 1 - \sum_{k=1}^{\infty} \langle v_k, 1 \rangle g(0) e^{-\mu_k t} v_k(x) \right) \\ &= g(t) - \sum_{k=1}^{\infty} \left[ \int_0^t -g(s) \langle v_k, 1 \rangle e^{-\mu_k(t-s)} ds - g(0) \langle v_k, 1 \rangle e^{-\mu_k t} \right] v_k(x) \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the  $L^2$  inner product on  $D_1$ . At this point we plug in  $g(t) = ae^{-\lambda_2 t}$  to get that

$$v(x, t) = g(t) + \sum_{k=1}^{\infty} \left[ a \langle v_k, 1 \rangle \left( \frac{\lambda_2}{\mu_k - \lambda_2} \left( e^{(\mu_k - \lambda_2)t} - 1 \right) + 1 \right) e^{-\mu_k t} \right] v_k(x)$$

On the other hand, the solution to (3) is given by

$$w(x, t) = \langle u_2, v_1 \rangle e^{-\mu_1 t} v_1(x) + \sum_{k=2}^{\infty} \langle u_2, v_k \rangle e^{-\mu_k t} v_k(x).$$

Since  $u = v + w$ , the behavior of  $u(x, t)$  as  $t \rightarrow \infty$  is governed by that of  $v$  and  $w$ . Looking at  $v$ , we see that the coefficient of  $v_k(x)$  decays at rate  $\mathcal{O}(e^{-(\lambda_2 \wedge \mu_k)t})$ . Looking at  $w$ , we see that the coefficient of  $v_k(x)$  decays at rate  $\mathcal{O}(e^{-\mu_k t})$ .

Therefore, **provided**  $\mu_1 < \lambda_2$ , in the long run the extrema of  $u(x, t)$  will match the extrema of  $v_1(x)$ . That is, we can find the extrema of the second Neumann eigenfunction  $u_2(x)$  of  $D$  by finding the extrema of the first mixed eigenfunction  $v_1(x)$  of  $D_1$ !

## References