Basics of Bessel Functions

June 10, 2012

Abstract

These are just some basic notes on Bessel functions and their application to finding the eigenfunctions of the Laplacian.

1 Where Bessel Functions Arise

Consider the problem of finding the eigenvalues/vectors for the Laplacian in 2 dimensions: We wish to find a pair (u, λ) which solves

$$\Delta u = -\lambda u,\tag{1}$$

(since the eigenvalues of Δ will turn out to be non-positive, we express them as $-\lambda$ where λ will be non-negative).

Let's express $u = u(r, \theta)$ as a function in polar coordinates and recall that in polar coordinates the Laplacian can be expressed as

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

Now let's guess¹ a solution to (1) of the form $u(r, \theta) = R(r)T(\theta)$. Plugging this into (1) we see that in order for our guess to work, we need that

$$\frac{1}{r}R'T + R''T + \frac{1}{r^2}RT'' = -\lambda RT.$$

Rearranging terms, this becomes

$$\frac{rR' + r^2R''}{R} + \lambda r^2 = \frac{-T''}{T}$$

Aha, the term on the left is a function purely of r while the term on the right is a function purely of θ . Therefore, the only way we could get equality is if the left and right side are constant. We denote this constant by α^2 (i.e. we will take the constant to be positive²) and so have

$$\frac{rR'+r^2R''}{R}+\lambda r^2=\frac{-T''}{T}=\alpha^2$$

¹This technique is known as *separation of variables*

 $^{^{2}}$ I am not entirely sure why... presumably the case of a negative constant fails or is unpleasant somewhere down the line.

Now we solve each of these ODE, one for R and one for T. For T we have

$$T'' + \alpha^2 T = 0$$

which has the general solution

$$T(\theta) = d_1 \cos(\alpha t) + d_2 \sin(\alpha t).$$

For R, we have the less trivial ODE

$$r^{2}R'' + rR' + (\lambda r^{2} - \alpha^{2})R = 0.$$
 (2)

This ODE depends on λ (the eigenvalue we are looking for eigenfunctions for) and α (the constant that we have free choice over) so we would expect to have to find the (two) solutions to this ODE separately for each choice of λ and α . But wait, we can make a clever change of variables to "get rid of λ "... we let $x = \sqrt{\lambda}$ and $\tilde{R}(x) = R\left(\frac{x}{\sqrt{\lambda}}\right)$. Then \tilde{R} solves the ODE

$$x^{2}\tilde{R}'' + x\tilde{R}' + (x^{2} - \alpha^{2})\tilde{R} = 0.$$
(3)

The ODE (3) is called **Bessel's differential equation**. It has been well studied and has two solutions (for each choice of α) which are denoted as $J_{\alpha}(x)$ and $Y_{\alpha}(x)$. They are respectively called the **Bessel function of the first kind** and the **Bessel function of the second kind**. Now, recalling how R is related to \tilde{R} , we have that the two solutions to (2) are $J_{\alpha}(\sqrt{\lambda}r)$ and $Y_{\alpha}(\sqrt{\lambda}r)$. Hence, the general solution to (2) is

$$c_1 J_{\alpha}(\sqrt{\lambda}r) + c_2 Y_{\alpha}(\sqrt{\lambda}r).$$

So putting this all together we see that any solution of the form

$$u(r,\theta) = \left(c_1 J_\alpha(\sqrt{\lambda}r) + c_2 Y_\alpha(\sqrt{\lambda}r)\right) \left(d_1 \cos(\alpha t) + d_2 \sin(\alpha t)\right) \tag{4}$$

solves (1). Of course, by linearity, any linear combination (with respect to α) of these also solves (1).

2 Specific Domains

Remark 2.1. I am not an expert on this material by any means so I am mostly just playing around trying to fit (4) to various domains. Better sources should be consulted for more complete details.

Of course when we consider a particular domain (along with its boundary conditions), we cannot just take any choice of the parameters in (4).

2.1 Domain is \mathbb{R}^2

For example, if we take all of \mathbb{R}^2 as our domain, then we want that $\theta = 0$ and $\theta = 2\pi$ lead to the same output value (i.e. we want $u(r, \theta)$ to be well defined). For this, we need that α be an integer, i.e. $\alpha = n \in \mathbb{Z}$. We therefore have solutions of the form

$$u(r,\theta) = \left(c_1 J_n(\sqrt{\lambda}r) + c_2 Y_n(\sqrt{\lambda}r)\right) \left(d_1 \cos(nt) + d_2 \sin(nt)\right), \quad n \in \mathbb{Z}, \lambda \in \mathbb{R}^+$$

2.2 Domain is the ball B(0, R) with Dirichlet boundary

In this case we also need that $u(R, \theta) = 0$. This boils down to considering the zeros of the Bessel functions J_n and Y_n . At this point I will dispense with the Y_n for the rest of the paper and consider only the J_n^3 . We see that we need $J_n(\sqrt{\lambda}R) = 0$. This imposes a condition on λ (which is reassuring as we don't expect just any eigenvalue to be in the spectrum of the Laplacian!), but for those pair of n and λ ,

$$u(r,\theta) = c_1 J_n(\sqrt{\lambda r}) \cos(nt) + c_2 J_n(\sqrt{\lambda r}) \sin(nt)$$

is a solution.

Now what is the first non-trivial eigenvalue for this domain? This comes down to the question, what is the smallest we can take λ such that $J_n(\sqrt{\lambda}R) = 0$ for some choice of n? Looking at the graphs of the Bessel functions $J_n(x)$ (e.g. on the Wikipedia page) we see that $J_0(x)$ has the smallest zero, so taking n = 0 and letting λ be (the smallest choice) such that $J_0(\sqrt{\lambda}R) = 0$, we have that this λ is the first non-trivial eigenvalue, and its corresponding eigenfunction is

$$u(r,\theta) = J_0(\sqrt{\lambda}r)$$

which is in fact radial! (This makes intuitive sense... long time heat flow in a ball with Dirichlet boundary should look like a decaying mound of heat in the middle)

2.3 Domain is the ball B(0, R) with Neumann boundary

This is mostly the same as the previous subsection, only now we need to choose λ as small as possible such that there is an n with $J'_n(\sqrt{\lambda}R) = 0$. Looking at the graphs of the Bessel functions $J_n(x)$ we see that $J'_0(x)$ is actually zero for x = 0... so we can take $\lambda = 0$ and n = 0 to get the corresponding eigenfunction

$$u(r,\theta)\equiv 1$$

But this is just the trivial eigenvalue/eigenfunction. So lets find the next smallest λ . Looking at the graphs of the Bessel functions $J_n(x)$ we see that, indeed, $J_1(x)$ has the earliest peak among them and so the first non-trivial eigenvalue corresponds to the $\lambda > 0$ such that $J'_1(\sqrt{\lambda}R) = 0$ and its corresponding eigenfunction is

$$u(r,\theta) = c_1 J_1(\sqrt{\lambda r}) \cos(t) + c_2 J_1(\sqrt{\lambda r}) \sin(t)$$

which in particular is two dimensional.

2.4 Domain is the sector $S(\gamma, R)$ with Neumann boundary

Let $S(\gamma, R) = \{(r, \theta) | 0 \le r \le R, 0 \le \theta \le \gamma\}$ denote a sector of angle $\gamma < 2\pi$ and radius R. Let's try and fit (4) to $S(\gamma, R)$ with Neumann boundary conditions.

First of all, since our angle γ is strictly less than 2π , we need no longer worry about $\theta = 0$ and $\theta = 2\pi$ syncing up to give the same output... so we no longer need that α be an integer! Instead we now need to pick α carefully so that the Neumann boundary condition is satisfied.

³The Y_n are a bit nasty in that they shoot to $-\infty$ as the input goes to 0. Therefore we might ignore them if we only wish to consider solutions $u(r, \theta)$ which are defined at the origin.

Considering the Neumann boundary condition along $\{\theta = 0\}$ we see that the sine term is no good so we consider only solutions of the form

$$u(r,\theta) = c_1 J_\alpha(\sqrt{\lambda r}) \cos(\alpha t)$$

Next, considering the Neumann boundary condition along $\{\theta = \gamma\}$, we see that we need $\alpha = \frac{n\pi}{\gamma}, \quad n \in \mathbb{Z}.$

We have, as before, the trivial solution $\lambda = 0$, $u \equiv 1$. To find the first non-trivial eigenvalue, we need to find $\lambda > 0$ as small as possible such that there exists an n such that $J'_{\frac{n\pi}{\gamma}}(\sqrt{\lambda}R) = 0$. Since the peaks of $J_{\alpha}(x)$ move to the right as α increases, the earliest peak we can find will be for n = 1. Then the corresponding λ will be the first non-trivial eigenvalue and its corresponding eigenfunction will be

$$u(r,\theta) = J_{\frac{\pi}{\gamma}}(\sqrt{\lambda}r)\cos(\frac{\pi}{\gamma}t).$$

But ACTUALLY, if γ is sufficiently small it could be that the first valley of $J_0(x)$ comes before the first peak of $J_{\frac{\pi}{\gamma}}(x)$. In this case, the first non-trivial eigenvalue would be the $\lambda > 0$ such that $J'_0(\sqrt{\lambda}R) = 0$ (i.e. the λ corresponding to the first valley of $J_0(x)$) and its corresponding eigenfunction would be

$$u(r,\theta) = J_0(\sqrt{\lambda r}).$$

So for a sector of a very narrow angle, the first non-trivial eigenfunction is radial (and so has its extrema in the corner and along the curved edge).

References