

# Probabilistic Proof of Mini-Lemma about Mixed Dirichlet-Neumann Boundary Triangles

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## 1 The Lemma

Let  $D \subset \mathbb{R}^2$  be a triangular domain and consider the Laplacian  $\Delta$  with mixed boundary conditions – on two sides of the triangle take Neumann boundary conditions and on the other side take Dirichlet boundary conditions (See Figure 1).

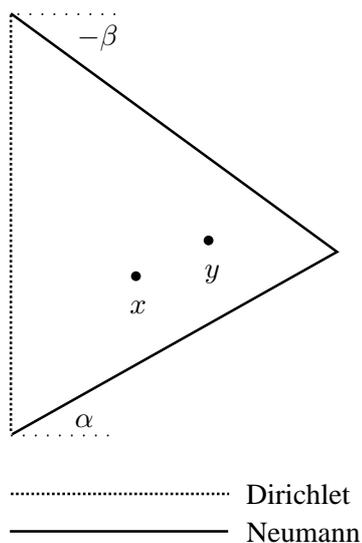


Figure 1: The triangle.

**Lemma 1.1.** *The first eigenfunction of  $\Delta$  has its extremum at the corner opposite the side with Dirichlet boundary.*

We give a probabilistic argument: Orient the domain  $D$  as in Figure 1 so that the Dirichlet side is a vertical line on the left and its opposite corner is therefore the right-most point. Let  $\alpha$  and  $-\beta$

be the angles denoted in the figure. Now let  $x, y \in D$  and let  $(X_t, Y_t)$  be a synchronous coupling (c.f. [1]) of reflected Brownian motion started from the points  $x$  and  $y$ . Intuitively,  $X_t$  and  $Y_t$  will have the same motion (only translated) except of course that they will reflect off the boundary at different times and this will make their trajectories slightly dissimilar.

Let  $\Theta_t$  be the angle that the vector  $Y_t - X_t$  makes with the horizontal (in particular  $\Theta_0$  is the angles that  $y - x$  makes with the horizontal). If we let  $\tau_x$  and  $\tau_y$  be the first time that  $X_t$  and  $Y_t$  hit the Dirichlet boundary respectively, and let  $\tau = \min(\tau_x, \tau_y)$  then we can make the following claim.

**Claim 1.2.** *Suppose  $-\beta \leq \Theta_0 \leq \alpha$ . Then  $-\beta \leq \Theta_t \leq \alpha$  for all  $t \leq \tau$ .*

**Proof 1.3.** I think a full proof of this should be possible but I won't make any claims for now. I do think that the case when the angle opposite the Dirichlet side (which is  $\alpha + \beta$  in our notation) is acute, the claim holds.

If  $\alpha + \beta < \frac{\pi}{2}$ , we can reuse the argument from [1] as follows: Rotate the triangle clockwise so that the Dirichlet side is flat and embed the triangle in a larger obtuse triangle (whose which we take to have Neumann boundary condition (See Figure 2). Now we ignore the dotted line and just think of reflected Brownian motion in this larger obtuse triangle. Let  $\tilde{\Theta}_t$  denote the angle between  $X_t$  and  $Y_t$  in this rotated coordinate system. By our assumption on  $\Theta_0$ , we have that  $-\alpha - \beta \leq \tilde{\Theta}_0 \leq 0$

Now, the results of [1] imply that  $-\alpha - \beta \leq \tilde{\Theta}_t \leq \gamma$ . In our original coordinates, this says that  $-\beta \leq \Theta_t \leq \alpha + \gamma$ . Now the same must hold for the mixed Dirichlet-Neumann triangle for  $t < \tau$  (because until we cross the dotted line nothing distinguishes the triangles)... and in fact, as  $\gamma$  could be taken arbitrarily small, we have that  $-\beta \leq \Theta_t \leq \alpha$  for  $t < \tau$  (if not then we could have taken a sufficiently small  $\gamma$  and the results of [1] would be violated).

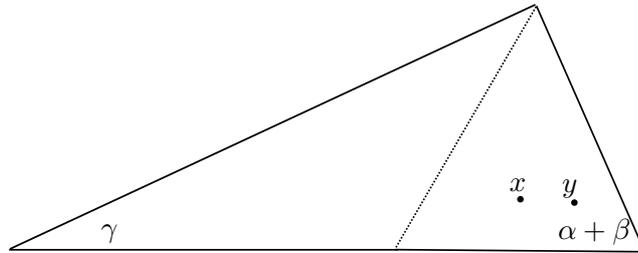


Figure 2: The triangle embedded in a larger obtuse triangle.

With this claim in hand we can now prove the original theorem: Consider the heat equation with initial condition  $u_0 \equiv 1$

$$\begin{cases} \partial_t u = \Delta u \\ u(x, 0) = u_0(x) \equiv 1 \end{cases}$$

Recall that the solution is given in probabilistic terms as

$$u(x, t) = \mathbb{P}(\tau_x > t) = \mathbb{E} [\chi_{\{\tau_x > t\}}]$$

Hence,

$$\begin{aligned}u(y, t) - u(x, t) &= \mathbb{E} [\chi_{\{\tau_y > t\}}] - \mathbb{E} [\chi_{\{\tau_x > t\}}] \\ &= \mathbb{E} [\chi_{\{\tau_y > t\}} - \chi_{\{\tau_x > t\}}] \\ &\geq 0,\end{aligned}$$

where the last inequality follows from the claim (The claim implies that  $\tau_x < \tau_y$ ).

It follows that the maximum of  $u(x, t)$  is achieved at the right-most point. As  $u_0 \equiv 0$  is not orthogonal to the first eigenfunction, and the first eigenfunction is simple<sup>1</sup>, it follows that the first eigenfunction also achieves its maximum at the right-most point.

## References

- [1] Rodrigo Bañuelos and Krzysztof Burdzy. On the “hot spots” conjecture of J. Rauch. *J. Funct. Anal.*, 164(1):1–33, 1999.

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<sup>1</sup>I am pretty sure these are both consequences of the Nodal Line Theorem: The first eigenfunction is strictly positive or strictly negative. If there were more than one then we could find a non-trivial linear combination which was zero in the interior of  $D$  violating the Nodal line theorem.