HW 4

Due March 18

1. Prove that a group with no proper subgroups must be cyclic of prime order.

2. G is a group and H is a subgroup of G. Let $N_G(H) = \{g | gH = Hg\}$.
a, Prove that $N_G(H)$ is a subgroup of G that contains H,
b. Prove that H is normal in $N_G(H)$.
c. Prove that if H is normal in a subgroup K of G, then K is a subgroup of $N_G(H)$.
d. Prove that $N_G(N_G(H)) = N_G(H)$ for any subgroup H of G.

3. G is a group and e is the identity element of G. Prove that if G has a unique element a of order 2, then $\{e, a\}$ is a normal in G.

4. a. Prove that $m_x : G \to G$ given by $m_x(g) = xgx^{-1}$ is an isomorphism.
b. Let $Aut(G)$ be the set of all automorphisms on G. Prove that if G is not abelian then $Aut(G)$ is not trivial. Is the converse true? Why?

5. a. Prove that the order of ab is equal to order of ba.
b. Prove that $o(ab) = (o(a), o(b))$ if $ab = ba$.
c. Is $D_3 \times Z_2$ isomorphic to D_{12}? $D_4 \times Z_5$ isomorphic to D_{20}? Why?

6. a. If G is cyclic and H is a subgroup of G, does it follow that G/H is cyclic?
b. If G is a group with a normal subgroup H such that H and G/H are both abelian, does it follow that G is abelian?
c. How about if we replace abelian with cyclic in b.

7. Let $G = GL(n, \mathcal{R})$ be the group of all $n \times n$ invertible matrices over real numbers under multiplication and $S = SL(n, \mathcal{R})$ be the set of all matrices over real numbers with determinant 1.
a. Show that S is a subgroup of G and is normal in G.
b. What is G/S?
c. Answer questions a, and b with $\mathcal{R}$ replaced by integers $\mathcal{Z}$
d. For a prime number p, if S_p is the set of all matrices (over $\mathcal{R}$; over $\mathcal{Z}$) whose determinant is not divisible by p, is S_p a subgroup and is it normal in G? Why?