Sample Final Exam

how all your work. If you use a theorem in the text, you should quote it and not say by a theorem in the text or by theorem 2.15 in the text
All answers must be justified properly to get credit.

1.(40) Prove or Disprove:

a. All finite integral domains are fields.

b. $f : R \to S$ is a ring homomorphism. If I is an ideal of R, then $f(I)$ is an ideal of S.

c. $x^{211} + 33x^{111} - 27x^{92} + 54x^{21} - 123$ is irreducible over the rational numbers.

d. There are no fields with exactly 49 elements.

e. The symmetric group S_5 has subgroups isomorphic to the dihedral groups of order 8, 10, 12 but not 24.

2.(25) Recall that isomorphisms from a group G to a group G are called automorphisms. Let $Aut(G)$ be the set of all automorphisms from a group G to a group G. For a group G, and an element $g \in G$, let $\phi_g : G \to G$ denotes the homomorphism $\phi_g(x) = gxg^{-1}$ and let $Con(G) = \{\phi_g | g \in G\}$. (i). Show that Aut(G) is a group under composition.

(ii)Show that $Con(G)$ is a normal subgroup of Aut (G). (Show that it is a subgroup and that it is normal).

(iii) For the group $Z_3 = \{0, 1, 2\}$ under addition modulo 3, compute, $Aut(Z_3)$ and $Con(Z_3)$

(iv) Is $Aut(Z_3)$ isomorphic to Z_3 or S_3 or Z_4? Justify your answers.

(v) Compute $Con(Z_3)$ and its order.

3. (15) How many polynomials of degree 2 are there in $Z_3[x]$? How many polynomials of degree 2 in $Z_3[x]$ are irreducible. Justify your answers completely.

4.(15) Prove using induction that the sum of the cubes of the first n positive integers is the square of the sum of the first n positive integers.

5. (15) $f(x)$ is a monic polynomial of degree 5. We know that $f'(0) = f'''(0) = 0$ and $f''(0) = 24; f^{iv}(0) = 600; f^{v}(0) = 120$ and that f(x) is not irreducible over the integers. We also know that $f(0)$ is one of the following: 14, 15, 16 and 18. Can you reveal the polynomial? Why?

6.(15) Consider the ring $R = Z_5[x]/(x^2 + 3x + 1)$

What is the cardinality of R?

What is R, Is R a commutative ring, an integral domain, a field ?

Is $\psi : R \to Z_5[x]$ given by $\psi(f(x)) = f(1)$ a ring homomorphism?