Some Algorithms for problems from Chapter III and IV
Vector Spaces

1. To find a basis for a subspace S and compute its dimension:

Write the subspace S in terms of some variables $a_1, a_2, \cdots a_n$ Then in turn set each variable equals 1 and the rest of the variables equal zero. You get n vectors this way. Delete any repetitions. Check to see they are linearly independent. Then they form a basis. The number of vectors in the basis is its dimension.

Example: Find a basis for the subspace S of R^4 given by $S = \{[a, a+b-c, 2a-3b, a+c]^T | a, b, c \in R\}$.

Set a=1, b=c=0. We get $\mathbf{v_1} = [1, 1, 2, 1]^T$. Set a=c=0, b=1. We get $\mathbf{v_2} = [0, 1, 3, 0]^T$ Set a=b=0, c=1. We get $\mathbf{v_3} = [1, -1, 0, 1]^T$.

These three vectors are linearly independent for if we write them as rows and perform $R_3 = R_3 - R_1$ followed by $R_3 = R_3 + 2R_2$, we see that there will be three leading ones in a row echelon form. So a basis for S is

$$\{\mathbf{v_1} = [1, 1, 2, 1]^T, \mathbf{v_2} = [0, 1, 3, 0]^T, \mathbf{v_3} = [1, -1, 0, 1]^T\}$$

and dimension of S is 3.

2. Given a basis $B = \{\mathbf{v_1}, \cdots \mathbf{v_n}\}$ of a V and a vector $\mathbf{v}$, to compute the Co-ordinate vector $X_B(\mathbf{v})$:

i. Specify an easy basis E of V. This is a basis with respect to which you can easily find the Co-ordinates. For instance, for R^n, it is the usual basis, for P_n, it can be $\{1, x, x^2, \cdots x^{n-1}\}$ and for $R^{m \times n}$ it is $E = \{\mathbf{E_{ij}} | 1 \leq i \leq m, 1 \leq j \leq n\}$.

b. Write B and $\mathbf{v}$ in this basis as column vectors .

c. Start with the augumented matrix $[B|\mathbf{v}]$ and do row operations to arrive at $[I|X_B(\mathbf{v})]$

d. Write down the answer $X_B(\mathbf{v})$ from the previous step.

ii. Given B and $X_B(\mathbf{v})$ find $\mathbf{v}$.

If $X_B(\mathbf{v}) = \begin{bmatrix} c_1 \\ \vdots c_n \end{bmatrix}$, then $\mathbf{v} = c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}$

3. Given two bases B and C , to find the change of basis matrix P

a. Specify an easy basis for the vector space V as in 1.

b. Write both B and C in that basis.

c. Start with the augumented matrix $[C|B]$ and do row operations to arrive at $[I|T]$. d. the change off basis matrix from B to C is P = T the answer.

Remember to write your vectors as columns and also that the matrix you start with is $[To|from]$.

4. To find a basis for the row space of A.

Start with A and find a row echelon form of A. The non zero rows in the row echelon form form a basis for the row space of A. List them as columns.

Rank A = number of leading ones in a row echelon form of A.

5. To find a basis for the column space of A.

Method 1. Take the transpose of A. Do row operations to arrive at a row echelon form of A^T, The non zero rows, re written as columns form a basis for the column space of A.

Method 2. Let U be a row echelon form of A. Then the columns of A corresponding to the columns in U with leading 1's form a basis for the column space of A.

6. To find the matrix of a linear transformation $L : V \to W$ with respect to a basis $B = \{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ of V and $C = \{\mathbf{w_1}, \cdots \mathbf{w_m}\}$ of W.

a. Compute $L(\mathbf{v}_i)$, $1 \le i \le n$. b. Pick an easy basis for W. Then write the vectors in C and $L(B)$ in terms of this basis. c. Start with the augumented matrix $[C|L(B)]$ and do row operations to get to $[I|M]$. d. Then M is the $m \times n$ matrix that represents L in the basis B and C.

7. If M is the matrix of $L : V \to W$ in the basis B of V and C of W, then for any $\mathbf{v} \in V$, $X_C(L(\mathbf{v})) = M.X_B(\mathbf{v})$.

8. To find a basis for the Kernel and Image of $L : V \to W$.

Choose a basis for V and W and write the matrix of L. Call it M.

Then the column space of M is the image of L and the Null space of M is the Kernel of L.

So, find the row echelon form of M. Call it U. The columns of M corresponding to the columns in U with a leading 1, form a basis for the column space of M. Convert these columns to vectors in W. This is a basis for the Image of L.

Solve $MX = 0$. Find a basis for the Null space of M from $[U|0]$. Convert these to vectors in V. This gives a basis for the Kernel of L.

The rank of L is the rank of M and is the number of leading 1's in U. The dimension of the Kernel of L is the number of columns in U without a leading 1.

9. Isomorphism: $L : V \to W$ is an isomorphism of vector spaces if L is a one-one and onto linear transformation.

Theorem: Two finite dimensional vector spaces V and W are isomorphic to each other if and only if dim V = dim W.

Theorem: $L : V \to W$ is an isomorphism if and only if the matrix representing L in any choice of bases is invertible (nonsingular).