Math 331 - Summary of Chapter VI -

1. Eigen Values and Eigen Vectors

V is a vector space over **R**, the real numbers (or **C** , the complext numers). Let $T : V \to V$ be a linear transformation.

A scalar λ is called an **eigenValue** of T if there is a non zero vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \lambda\mathbf{v}$. This nonzero vector $\mathbf{v}$ is called an **eigenvector** of T with the eigen value λ.

If A is a square matrix of size n over **R**, the real numbers (or **C** , the complext numers), then the eigen values and eigen vectors of A are the eigen values and the eigen vectors of the linear transformation on $\mathbf{R}^n$ (or $\mathbf{C}^n$ defined by multiplication by A. Let $V = \mathbf{R}^n$ (or $\mathbf{C}^n$

A scalar λ is called an **eigenValue** of A if there is a non zero vector $\mathbf{v}$ in V such that $A\mathbf{v} = \lambda\mathbf{v}$. This nonzero vector $\mathbf{v}$ is called an **eigenvector** of A with the eigen value λ.

The **characteristic polynomial** $P_A(\lambda)$ of A is the polynomial
determinant $(A - \lambda I) = (-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_o$

For a matrix A, and a scalar λ,

$E_\lambda(A) = \{\mathbf{v}|A(\mathbf{v} = \lambda\mathbf{v}\}$ is the subspace of all eigen vectors with eigen value λ. This is called the eigen space of λ.

Thus, dimension of $E_\lambda(A)$ is not zero if and only if λ is an eigen value of A.

Theorem 1. *: x is an eigen value of A if and only if x is the root of the characteristic polynomial of A, that is if and only if $P_A(x) = 0$.*

Theorem 2. *Similar matrices have the same characteristic polynomial and hence the same eigen values*

Theorem 3. *A and A^t, the transpose of A, have the same characteristic polynomial and the same eigen values.*

As a result of Theorem 2, the eigen values and eigen vectors of a linear transformation T as defined above can be computed by finding those of the matrix of T with respect to some basis B of the vector space V. Theorem 2 says that the eigen values and eigen vectors do not depend on the choice of the basis.

To Compute the eigen values and the eigen vectors of a matrix A.

1. Compute A-xI
2. Compute determinant (A-xI) = $P_A(x)$
3. Solve $P_A(x) = 0$ to find the roots $x = \lambda_1, \cdots \lambda_t$

$\lambda_1, \cdots \lambda_t$ are the eigen values of A.

4. Proceed to find the eigen vectors . $E_{\lambda_i}(A) =$ Null space of $(A - \lambda_i I)$.

Use Gaussian elimination to find a basis for the null space of $A - \lambda_i I$ for each i.

Theorem 4. *If $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v_n}$ are eigen vectors of A with distinct eigen values $\lambda_1, \cdots \lambda_n$, then $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v_n}$ are linearly independent.*

An eigen value λ of a matrix A is said to be of multiplicity k if $(\lambda - x)^k$, divides the characteristic polynomial $P_A(x)$ and $(\lambda - x)^{k+1}$ does not divide $P_A(x)$. That is, λ is an eigen value with multiplicity k if λ is a root of $P_A(x) = 0$ with multiplicity k.

Thoeorem 5: If λ is an eigen value of A with multiplicity k then dim $E_\lambda(A) \leq k$

To compute the multiplicity of the eigen value λ and dim $E_\lambda(A)$.

Factor the characterisitic polynomial $P_A(x)$. Then the multiplicity of λ equals the power of $(x - \lambda)$ in the factorizartion of $P_A(x)$.

Compute a row echelon form of matrix $A - \lambda I$. Dimension of the eigne space of λ is the number of columns without a leading 1.

2. Diagonalization

A matrix A is **diagonalizable** if there is an invertible matrix X such that $X^{-1}AX$ is a diagonal matrix.

THEOREM 6. *An n x n matrix A is diagonalizable over* $\mathbf{R}$ *(or* $\mathbf{C}$ *) if and only if there is a basis for* $\mathbf{R}^n$ *(respectively* $\mathbf{C}^n$*) consisting of eigen vectors of A.*

To determine if a given matrix $A = (a_{ij})_{n \times n}$ is diagonalizable:

1. Compute the eigen values and their multiplicties as in the previous section.

2. Let $\lambda_1, \cdots \lambda_t$ be the distinct eigen values of A with multiplicities $k_1, k_2, \cdots k_n$ respectively.

Compute the basis for the eigen spaces $E_{\lambda_i}(A)$ as before. If dim $(E_{\lambda_i}(A) < k_i$ for any i, then A is not diagonalizable. If dim $E_{\lambda_i}(A) = k_i$ for all i, then A is diagonalizable. Collect all the vectors in the bases of $E_{\lambda_i}(A)$ for each i, to get a $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v_n}$ consisting of eigen vectors. Let X = $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v_n}$. Then $X^{-1}AX$ is diagonal.

3. Orthogonal diagonalization

A matrix A is said to be orthogonally diagonalizable if there is an orthogonal matrix X such that XAX^{-1} is diagonal. A is orthogonally diagonalizable if there is an orothonormal basis consisting of eigen vectors.

THEOREM 7. *Suppose that A is a symmetric matrix. If* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are eigen vectors with distinct eigen values* t_1 *and* t_2 *respectively, then* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are orthogonal to each other.*

THEOREM 8. *If A is orthogonally diagonalizable then A must be symmetric.*

Suppose that A is symmetric. To orthogonally diagonalize A: 1. Proceed as in section 2 to find the eigen values and bases for eignen spaces.

2. Use Gram-Schmidt process to find orthonormal basis for each eigen space. Putting these together gives us an orthonormal basis of eigen vectors.