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Inner Product Spaces and Orthonormal bases.

Let V be a vector space with inner product $*$. Let $B = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\}$ be a basis for V.

Two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are said to be orthogonal to each other if $\mathbf{v_1} * \mathbf{v_2} = 0$.

A basis such as B is said to be an orthogonal basis for V if the vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}$ are pairwise mutually orthogonal.

Length of a vector $\mathbf{v_1}$ is $\sqrt{\mathbf{v} * \mathbf{v}} = ||\mathbf{v}||$.

A basis $B = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\}$ is said to be an orthonormal basis for V if the vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}$ are pairwise mutually orthogonal and are all of length 1. In other words, if $*$ is the inner product on V, B is an orthonormal basis if $\mathbf{v_i} * \mathbf{v_j} = 0, i \neq j$ and $\mathbf{v_i} * \mathbf{v_i} = 1, 1 \leq i \leq n$.

Theorem: Any mutually orthogonal set of non-zero vectors in V is linearly independent.

Proof: Let $B = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\}$ be a mutually orthogonal set of non-zero vectors in V. So, $\mathbf{v_i} * \mathbf{v_j} = 0, i \neq j$ and none of the vectors $\mathbf{v_i}$ are zero. Consider $c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n} = \mathbf{0}$.

Let i be arbitrary. Taking inner product with $\mathbf{v_i}$, we get

$(c_1\mathbf{v_1} + \cdots + c_n\mathbf{v_n}) * \mathbf{v_i} = \mathbf{0} * \mathbf{v_i}$.

$c_1\mathbf{v_1} * \mathbf{v_i} + \cdots + c_n\mathbf{v_n} * \mathbf{v_i} = \mathbf{0}$.

Since, $\mathbf{v_i} * \mathbf{v_j} = 0, i \neq j$, we get,

$c_i\mathbf{v_i} * \mathbf{v_i} = 0$

But $\mathbf{v_i} * \mathbf{v_i} \neq 0$, because $\mathbf{v_i} \neq \mathbf{0}$

So, $c_i = 0$. This is true for all i. Hence, B is linearly independent.

This theorem tells us that an orthonormal basis for an inner product space V of dimension n is a set of vectors $B = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\}$ such that $\mathbf{v_i} * \mathbf{v_j} = 0, i \neq j$ and $\mathbf{v_i} * \mathbf{v_i} = 1$, for all i and j.

Examples

1. R^n with the dot product is an inner product spcae and the standard basis $E = \{\mathbf{e_1}, \mathbf{e_2}, \cdots \mathbf{e_n}\}$ is an orthonormal basis for R^n.

2. The ortno normal basis depends on the inner product. R^n with an inner product different from dot product will not have the usual basis E as an orthonormal basis. Let A be a non singular symmetric $n \times n$ matrix.

Define $\mathbf{v} * \mathbf{w} = \mathbf{v}^T A\mathbf{w}$. Then * defines an inner product on R^n.

If $A = (a_{ij})_{n \times n}$, then $\mathbf{e_i} * \mathbf{e_j} = a_{ij}$.

Thus unless $A = I$, E is not an orthonormal basis for R^n with this inner product.

3. P_3 with the inner product $(ax^2 + bx + c).(a'x^2 + b'x + c') = aa' + bb' + cc'$ is an inner product space and the standard basis $\{1, x, x^2\}$ is an orthonormal basis for P_3 with this inner product.

4. P_3 with the inner product $f * g = f(0)g(0) + f(1)g(1) + f(2)g(2)$ is a an inner product space. This inner product is different from the one in 2. For, $1 * x = 3$ and $1.x = 0$.

Remark : An orthonormal basis for V converts the inner product to a dot product. Let V be an inner product space with an orthonormal basis B. Let Let $[\mathbf{v}]_B$ denote the coordinate vector in R^n of $\mathbf{v}$ in the basis B. Then,

$$\mathbf{v} * \mathbf{w} = [\mathbf{v}]_B . [\mathbf{w}]_B$$

Gram-Schmidt Orthonormalization process

Gram Schmidt algorithm is one which converts any ordered basis of an inner product space in to an orthonormal basis.

Let V be an inner product space with inner product $*$. Let $B = \{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_n}\}$ be an ordered basis for V.

The Gram Schmidt process produces an orthogonal basis of vectors $\mathbf{w_i}, i = 1, \cdots n$ and an orthonormal basis of vectors $\mathbf{u_i}, i = 1, \cdots n$. These vectors are computed as follows:

$$\begin{array}{rcl}
\mathbf{w_1} = \mathbf{v_1} & \text{and} & \mathbf{u_1} = \frac{\mathbf{w_1}}{\sqrt{\mathbf{w_1} * \mathbf{w_1}}} \\
\mathbf{w_2} = \mathbf{v_2} - \frac{\mathbf{v_2} * \mathbf{w_1}}{\mathbf{w_1} * \mathbf{w_1}} \mathbf{w_1} & \text{and} & \mathbf{u_2} = \frac{\mathbf{w_2}}{\sqrt{\mathbf{w_2} * \mathbf{w_2}}} \\
\mathbf{w_3} = \mathbf{v_3} - \frac{\mathbf{v_3} * \mathbf{w_1}}{\mathbf{w_1} * \mathbf{w_1}} \mathbf{w_1} - \frac{\mathbf{v_3} * \mathbf{w_2}}{\mathbf{w_2} * \mathbf{w_2}} \mathbf{w_2} & \text{and} & \mathbf{u_3} = \frac{\mathbf{w_3}}{\sqrt{\mathbf{w_3} * \mathbf{w_3}}} \\
.. & .. & .. \\
.. & .. & .. \\
\mathbf{w_k} = \mathbf{v_k} - \frac{\mathbf{v_k} * \mathbf{w_1}}{\mathbf{w_1} * \mathbf{w_1}} \mathbf{w_1} - \cdots - \frac{\mathbf{v_k} * \mathbf{w_{k-1}}}{\mathbf{w_{k-1}} * \mathbf{w_{k-1}}} \mathbf{w_{k-1}} & \text{and} & \mathbf{u_k} = \frac{\mathbf{w_k}}{\sqrt{\mathbf{w_k} * \mathbf{w_k}}} \\
.. & .. & .. \\
.. & .. & .. \\
\mathbf{w_n} = \mathbf{v_n} - \frac{\mathbf{v_n} * \mathbf{w_1}}{\mathbf{w_1} * \mathbf{w_1}} \mathbf{w_1} - \cdots - \frac{\mathbf{v_n} * \mathbf{w_{n-1}}}{\mathbf{w_{n-1}} * \mathbf{w_{n-1}}} \mathbf{w_{n-1}} & \text{and} & \mathbf{u_n} = \frac{\mathbf{w_n}}{\sqrt{\mathbf{w_n} * \mathbf{w_n}}}
\end{array}$$

This process ensures that for every k,

1. $\mathbf{u_1}, \mathbf{u_2}, \cdots \mathbf{u_k}$ are mutually orthogonal and of length one and

2. $Span\{\mathbf{v_1}, \mathbf{v_2}, \cdots \mathbf{v_k}\} = Span\{\mathbf{u_1}, \mathbf{u_2}, \cdots \mathbf{u_k}\}$.

Thus, $U = \{\mathbf{u_1}, \mathbf{u_2}, \cdots \mathbf{u_n}\}$ is an orthonormal basis for V.

Further, if one started from a given ordered basis B of an inner product space V, Gram Schmidt process will produce a unique orthonormal basis. In particular, if one started from an orthonormal basis and applied Gram Schmidt process it will return the same basis. Any mutually orthogonal set of nonzero vectors in an inner product space can be enlarged to form an orthogonal basis.

If $U = \{\mathbf{u_1}, \mathbf{u_2}, \cdots \mathbf{u_n}\}$ is an orthonormal basis for V with inner product $*$ and $\mathbf{v}$ is any vector in V, then $\mathbf{v} = \sum_{i=1}^{n} \mathbf{v} * \mathbf{u_i}$.

Thus,

$$[\mathbf{v}]_U = \begin{bmatrix} \mathbf{v} * \mathbf{u_1} \\ \cdots \\ \mathbf{v} * \mathbf{u_i} \\ \cdots \\ \mathbf{v} * \mathbf{u_n} \end{bmatrix}$$

So, $\mathbf{v} * \mathbf{w} = \sum_{i=1}^{n} (\mathbf{v} * \mathbf{u_i})(\mathbf{w} * \mathbf{u_i})$.

Example

: P_3 with inner product $*$ as in example 3 above. That is, $f * g = f(0)g(0) + f(1)g(1) + f(2)g(2)$. Then $\{1, x, x^2\}$ is not an orthonormal basis. Applying Gram Schmidt process to this basis, we will get $\mathbf{w_1} = 1, \mathbf{w_2} = x - 1, \mathbf{w_3} = x^2 - 2x + \frac{1}{3}$

and hence the orthonormal basis $\{\frac{1}{\sqrt{3}}, \frac{x-1}{\sqrt{2}}, \frac{x^2-2x+\frac{1}{3}}{\sqrt{\frac{2}{3}}}\}$.

Orthogonal Subspaces

Let V be a vector space with an inner product $*$. Let W be a subspace of V. Let $\mathbf{v}$ be a vector in V.

Theorem The set of all vectors in V, which are orthogonal to a given vector $\mathbf{v}$ is a subspace of V.

That is, $\{\mathbf{w} \in V | \mathbf{w} * \mathbf{v} = 0\}$ is a subspace of V.

This subspace is called the subspace orthogonal to $\mathbf{v}$. In general, two subspaces W_1 and W_2 of V are said to be orthogonal to each other, if every vector in W_1 is orthogonal to every vector in W_2 The set of all vectors in V, which are orthogonal to every vector in W is a subspace of V. This is the largest subspace of V that is orthogonal to W . This is called the orthogonal complement of W in V and is denoted by W^{perp}. Also, every vector in V can be uniquely written as the sum of a vector in W and a vector in W^{perp}.

Example

Consider R^3 with dot product. Let W be the x-axis. Then the orthogonal complement of W is the y-z plane. If $S = \{[a, 2a, 3a]^t | a \in R\}$ is the line through the origin and (1, 2, 3), then the orthogonal complement of S is the plane perpendicular $S^{\perp} = \{[x, y, z]^t | x + 2y + 3z = 0\}$.

In general, if S is the subspace spanned by a set of vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}$ in R^n, then the orthogonal complement of S is the null space of the matrix with $\mathbf{v_i}$ written as rows.

Thus, the orthogonal complement of Span $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k}\}$ is

$$N\left(\begin{bmatrix} \mathbf{v_1}^t \\ \vdots \\ \mathbf{v_k}^t \end{bmatrix}\right).$$

Thus the $dim S^{\perp} = n - dim S$.

Methods to find a basis for the orthogonal complement of S in V.

Let Dim S = t and dim V = n.

I.a. Find a basis $B = \{\mathbf{v_1}, \cdots \mathbf{v_t}\}$ for S.

b. Write the equations $\mathbf{v} * \mathbf{v_i} = 0, 1 \leq i \leq t$, where $\mathbf{v}$ is an arbitrary vector in V. Solve. The space of solutions is the orthogonal complement.

Alternately, extend the basis B to a basis $\{\mathbf{v_1}, \cdots \mathbf{v_t}, \mathbf{v_{t+1}} \cdots \mathbf{v_n}\}$ of V and apply Gram Schmdt to this basis of V. The last n-t vectors in the ortho-normal basis obtained is a basis for the orthogonal complement of S.

II. Another method: i. Choose a basis for V and convert the problem to R^n. Convert the basis vectors of S to vectors in R^n.

ii. Then write these t vectors as rows of a $t \times n$ matrix.

iii. Find a basis for the null space of this matrix using Gaussian Elimination or Row operations.

iv. Convert these vectors in the null space of the matrix to vectors in V.

This is a basis for the orthogonal complement of S in V.

Note : The steps i and iv will be omitted if the vector space V is R^n.

Some computational questions.

1. Find a basis for the orthogonal complement of the Span $\{(1,0,1,2)^t, (2,1,1,1)^t\}$.

2. Convert the given basis to an orthonormal basis.

3. Trace of a matrix is the sum of the diagonal entries. $< A, B >= Trace(AB^t)$ is an inner product on $R^{3\times 3}$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

a. Is A orthogonal to B?

b. What is the length of C ?