BETTI NUMBERS AND DEGREE BOUNDS FOR SOME LINKED ZERO-SCHEMES

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Abstract. In [8], Herzog and Srinivasan study the relationship between the graded Betti numbers of a homogeneous ideal I in a polynomial ring R and the degree of I. For certain classes of ideals, they prove a bound on the degree in terms of the largest and smallest Betti numbers, generalizing results of Huneke and Miller in [9]. The bound is conjectured to hold in general; we study this using linkage. If R/I is Cohen-Macaulay, we may reduce to the case where I defines a zero-dimensional subscheme Y. If Y is residual to a zero-scheme Z of a certain type (low degree or points in special position), then we show that the conjecture is true for I_Y.

1. Introduction

Let R be a polynomial ring over a field $\mathbb{K}$, and let I be a homogeneous ideal. Then the module R/I admits a finite minimal graded free resolution over R:

$$\mathbb{F}: \ \cdots \rightarrow \bigoplus_{j\in J_2} R(-d_{2,j}) \rightarrow \bigoplus_{j\in J_1} R(-d_{1,j}) \rightarrow R \rightarrow R/I \rightarrow 0.$$

Many important numerical invariants of I and the associated scheme can be read off from the free resolution. For example, the *Hilbert polynomial* is the polynomial $f(t) \in \mathbb{Q}[t]$ such that for all $m \gg 0$, $\dim_{\mathbb{K}}(R/I)_m = f(m)$; if $f(t)$ has degree n and lead coefficient d, then the *degree* of I is $n!d$. When one has an explicit free resolution in hand, then it is possible to write down the Hilbert polynomial, and hence the degree, in terms of the shifts $d_{i,j}$ which appear in the free resolution.

If R/I is Cohen-Macaulay and has a *pure resolution*

$$0 \rightarrow R^{e_p}(-d_p) \cdots \rightarrow R^{e_2}(-d_2) \rightarrow R^{e_1}(-d_1) \rightarrow R \rightarrow R/I \rightarrow 0,$$

then Huneke and Miller show in [9] that $deg(I) = (\prod_{i=1}^{p} d_i)/p!$. Their result points to a more general possibility:

Conjecture 1.1 (Huneke & Srinivasan)**.** *Let R/I be a Cohen-Macaulay algebra with minimal free resolution of the form*

$$0 \rightarrow \bigoplus_{j\in J_p} R(-d_{p,j}) \rightarrow \cdots \rightarrow \bigoplus_{j\in J_2} R(-d_{2,j}) \rightarrow \bigoplus_{j\in J_1} R(-d_{1,j}) \rightarrow R \rightarrow R/I \rightarrow 0.$$

Let $m_i = \min\,\{d_{i,j} \mid j \in J_i\}$ be the minimum degree shift at the ith step and let $M_i = \max\,\{d_{i,j} \mid j \in J_i\}$ be the maximum degree shift at the ith step. Then

$$\frac{\prod_{i=1}^{p} m_i}{p!} \ \leq \ \deg(I) \ \leq \ \frac{\prod_{i=1}^{p} M_i}{p!}.$$

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When R/I is not Cohen-Macaulay, it is easy to see that the lower bound fails; for example if $I = (x^2, xy) \subset k[x, y]$, then $\deg(I) = 1$, $m_1 = 2$ and $m_2 = 3$, but $\frac{(2)(3)}{2!} \geq 1$. However, in [8], Herzog and Srinivasan conjecture that even if R/I is not Cohen-Macaulay, the upper bound is still valid if one takes $p = \mathrm{codim}(I)$. Conjecture 1.1 is verified in [8] in a number of situations: when I is codimension two; for codimension three Gorenstein ideals with five generators (in fact, the upper bound holds for codimension three Gorenstein with no restriction on the number of generators); when I is a complete intersection, and also for certain classes of monomial ideals. Additional cases where Conjecture 1.1 has been verified appear in [5], [6], [7]. In the non-Cohen-Macaulay case, [8] proves the bound for stable monomial ideals [4], squarefree strongly stable monomial ideals [1], and ideals with a pure resolution; [15] proves it for codimension two. In fact, in the codimension two Cohen-Macaulay and codimension three Gorenstein cases, a stronger version of the conjecture holds, see [12].

Most of the situations where the conjecture is known to be true are when the entire minimal free resolution is known; the work in proving the conjecture generally involves a complicated analysis translating the numbers $d_{i,j}$ to the actual degree. In this paper we take a different approach. Our goal is to obtain *only* the information germane to the conjecture; in particular we need the smallest and biggest shift at each step. When I is Cohen-Macaulay we can always slice with hyperplanes without changing the degree or free resolution, hence the study of the conjecture, in the Cohen-Macaulay case, always reduces to the study of zero-schemes.

Suppose Y is a zero-scheme, and Z is a zero-scheme residual to Y inside a complete intersection X. The resolution for I_X is known, so if one has some control over Z, (for example, when Z consists of a small number of points, or points in special position), then linkage allows us to say something about the resolution for I_Y. Central to this are the results of Peskine-Szpiro [14] connecting resolutions and linkage.

1.1. **Resolutions and linkage.**

Two codimension r subschemes Y and Z of $\mathbb{P}^n$ are *linked* in a complete intersection X if $I_Y = I_X : I_Z$ and $I_Z = I_X : I_Y$. The most familiar form of linkage is the Cayley-Bacharach theorem [2], which was our original motivation.

Theorem 1.2 (see [14] or [13]). *Let $X \subset \mathbb{P}^n$ be an arithmetically Gorenstein scheme of codimension n, with minimal free resolution*

$$0 \to R(-\alpha) \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to R \to R/I_X \to 0.$$

Suppose that Z and Y are linked in X, and that the minimal free resolution of R/I_Z is given by:

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to R \to R/I_Z \to 0.$$

Then there is a free resolution for R/I_Y given by

$$0 \to G_1^\vee(-\alpha) \to \begin{matrix} G_2^\vee(-\alpha) \\ \oplus \\ F_1^\vee(-\alpha) \end{matrix} \to \begin{matrix} G_3^\vee(-\alpha) \\ \oplus \\ F_2^\vee(-\alpha) \end{matrix} \to \cdots \to \begin{matrix} G_n^\vee(-\alpha) \\ \oplus \\ F_{n-1}^\vee(-\alpha) \end{matrix} \to R \to R/I_Y \to 0.$$

It turns out that in certain situations the shifts in the mapping cone resolution for Y given by the theorem above are such that no cancellation of the relevant shifts can occur.

2. Ideals linked to a collinear subscheme

We assume for the remainder of the paper that $n \geq 3$ and that X is a non-degenerate (all the $d_i > 1$) complete intersection zero-scheme of type $(d_1, d_2, \dots, d_n)$; let d_X denote the degree of X, and $\alpha_X = \sum_{i=1}^n d_i$. Suppose Z is a complete intersection subscheme of X, of type $(e_1, \dots, e_n)$; with d_Z and α_Z as above. A minimal free resolution for R/I_X is given by $F_i = \wedge^i(\oplus_{j=1}^n R(-d_j))$, and a minimal free resolution for R/I_Z is given by $G_i = \wedge^i(\oplus_{j=1}^n R(-e_j))$. In this case it is easy to see that Theorem 1.2 implies that there exists f of degree $a = \alpha_X - \alpha_Z$ such that $I_Y = I_X : I_Z = (I_X + f)$ and $I_Z = I_X : f$; in particular, I_Y is an almost complete intersection. Since $I_X \subseteq I_Z$, $R/I_X \to R/I_Z$; the mapping cone of Theorem 1.2 comes from a map of complexes which begins:

$$\begin{tikzcd}
{} \arrow[r] & \wedge^2(\oplus_{i=1}^n R(-d_i)) \arrow[r] & \oplus_{i=1}^n R(-d_i) \arrow[r] \arrow[d, "\phi"] & R \arrow[r] \arrow[d] & R/I_X \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & \wedge^2(\oplus_{i=1}^n R(-e_i)) \arrow[r] & \oplus_{i=1}^n R(-e_i) \arrow[r] & R \arrow[r] & R/I_Z \arrow[r] & 0
\end{tikzcd}$$

The comparison map ϕ which makes the diagram commute is simply an expression of the generators of I_X in terms of the generators of I_Z (e.g.[3], Exercise 21.23). If $I_X \subseteq \mathfrak{m} I_Z$ then ϕ has entries in $\mathfrak{m}$; in the construction of Theorem 1.2 the map $G_{n-1}^\vee \to F_{n-1}^\vee$ is the transpose of ϕ. Since the comparison maps further back in the resolution are simply exterior powers of ϕ, we have:

Lemma 2.1. *If $I_X \subseteq \mathfrak{m} I_Z$, then the mapping cone resolution is in fact a minimal free resolution for I_Y.*

So if $I_X \subseteq \mathfrak{m} I_Z$, then the minimal free resolution $H_\bullet$ for R/I_Y has $H_n = \oplus_{i=1}^n R(e_i - \alpha_X))$, and for $i \in \{1, \dots, n-1\}$,

$$H_i = \wedge^{n-i}(\oplus_{i=1}^n R(d_i)) \bigoplus \wedge^{n-i+1}(\oplus_{i=1}^n R(e_i))(-\alpha_X).$$

If $I_X \not\subseteq \mathfrak{m} I_Z$, then I_X and I_Z share some minimal generators; in this case, there can be cancellation in the mapping cone resolution:

Example 2.2. Let $I_X = \langle x^2, y^2, z^6 \rangle \subseteq k[x, y, z, w]$, and let $I_Z = \langle x, y, z^6 \rangle$. Then we find that $I_Y = I_X + \langle xy \rangle$. In betti diagram notation the mapping cone resolution of R/I_Y is:

degree	1	4	6	3
0	1	–	–	–
1	–	3	2	1
2	–	–	1	–
3	–	–	–	–
4	–	–	–	–
5	–	1	–	–
6	–	–	3	2

This is not a minimal resolution; the $R(-4)$ summand can be pruned off. The degree of I_Y is 18. Checking, we obtain $\prod_{i=1}^3 m_i = 54$, $\prod_{i=1}^3 M_i = 432$, and indeed $9 \leq 18 \leq 72$. Notice that the upper bound was not affected when we pruned the resolution, and the value of $\prod_{i=1}^3 m_i$ increased after pruning.

Example 2.3. Let Z be a single point. For Y, Lemma 2.1 implies that $M_n = m_n = \alpha_X - 1$, and for $i < n$, $M_i = \alpha_X - n + i - 1$ and $m_i = \sum_{j=1}^{i} d_j$ (where $d_i \leq d_j$ if $i \leq j$). We want to show that

$$(\prod_{j=1}^{n-1} \sum_{i=1}^{j} d_i)(\sum_{i=1}^{n} d_i - 1) \ \leq \ n!(d_X - 1) \ \leq \ \prod_{i=1}^{n}(\alpha_X - i).$$

For the upper bound there are two cases. If $d_1 < d_n$, then we have the following inequalities:

$$\begin{array}{rcl} nd_1 & \leq & d_1 + d_2 + \cdots + d_{n-1} + d_n - 1 = \alpha_X - 1 \\ (n-1)d_2 & \leq & (d_2 + \cdots + d_n) + (d_1 - 2) = \alpha_X - 2 \\ \vdots & & \vdots \\ 2d_{n-1} & \leq & (d_{n-1} + d_n) + (d_1 + d_2 + \cdots + d_{n-2} - (n-1)) = \alpha_X - (n-1) \\ d_n & \leq & (d_n) + (d_1 + d_2 + \cdots + d_{n-1} - n) = \alpha_X - n \end{array}$$

So it follows that $n!(d_X - 1) \ \leq \ n! d_1 d_2 \cdots d_n \ \leq \ \prod_{i=1}^{n}(\alpha_X - i)$. If $d_1 = d_n = \delta$, then

$$\begin{array}{rcl} n\delta & \leq & n\delta = \alpha_X \\ (n-1)\delta & \leq & (n-1)\delta + (\delta - 2) = \alpha_X - 2(1) \ \leq \ \alpha_X - 2 \\ (n-2)\delta & \leq & (n-2)\delta + (2)(\delta - 2) = \alpha_X - 2(2) \ \leq \ \alpha_X - 3 \\ \vdots & & \vdots \\ 2\delta & \leq & 2\delta + (n-2)(\delta - 2) = \alpha_X - 2(n-2) \ \leq \ \alpha_X - (n-1) \\ \delta & \leq & \delta + (n-1)(\delta - 2) = \alpha_X - 2(n-1) \end{array}$$

So $n!(\delta^n - 1) \ \leq \ n!\delta^n \ \leq \ (\alpha_X)\left(\prod_{i=2}^{n-1}(\alpha_X - i)\right)(\alpha_X - 2n + 2)$. To finish the upper bound, we must verify that $\alpha_X(\alpha_X - 2n + 2) \ \leq \ (\alpha_X - 1)(\alpha_X - n)$; this follows since $n \geq 3$.

The lower bound is easier: it holds for a complete intersection, and by assumption $d_j \geq 2$ for all j, so we have

$$\prod_{j=1}^{n} \sum_{i=1}^{j} d_i \leq n! d_X \quad \text{and} \quad j + 1 \ \leq \ 2j \ \leq \ \sum_{i=1}^{j} d_i.$$

Thus

$$n! = \prod_{j=1}^{n-1}(j+1) \ \leq \ \prod_{j=1}^{n-1} 2j \ \leq \ \prod_{j=1}^{n-1} \sum_{i=1}^{j} d_i.$$

Combining these two inequalities yields the lower bound.

Lemma 2.4. *If X is a non-degenerate zero-dimensional complete intersection in $\mathbb{P}^n$, with $n \geq 3$, then $d_X \leq \binom{\alpha_X - 1}{n}$, i.e. $d_X n! \leq (\alpha_X - 1)(\alpha_X - 2) \cdots (\alpha_X - n)$.*

Proof. The bounds in Conjecture 1.1 hold for a $(d_1, d_2, \cdots, d_n)$ complete intersection, so $d_X\, n! \leq \alpha_X (\sum_{i=2}^{n} d_i)(\sum_{i=3}^{n} d_i) \cdots d_n$. If $d_1 < d_n$, then as in the first case of Example 2.3, $d_X\, n! \leq (\alpha_X - 1)(\sum_{i=2}^{n} d_i)(\sum_{i=3}^{n} d_i) \cdots d_n$. Hence it suffices to show

$$\alpha_X(\sum_{i=2}^{n} d_i)(\sum_{i=3}^{n} d_i) \cdots (\sum_{i=n}^{n} d_i) \leq \prod_{j=1}^{n}(\alpha_X - j)$$

Case 1: $d_1 > 2$. Then $(\sum_{i=2}^n d_i) \leq (\alpha_X - 3)$ and $(\sum_{i=j}^n d_i) \leq (\alpha_X - j)$ for all $j \geq 3$. So since $\alpha_X(\alpha_X - 3) \leq (\alpha_X - 1)(\alpha_X - 2)$, we obtain:

$$\begin{array}{rcl} \alpha_X(\sum_{i=2}^n d_i)(\sum_{i=3}^n d_i)\cdots(\sum_{i=n}^n d_i) & \leq & \alpha_X(\alpha_X - 3)(\alpha_X - 3)(\alpha_X - 4)\cdots(\alpha_X - n) \\ & \leq & \prod_{j=1}^n (\alpha_X - j) \end{array}$$

Case 2: $d_1 = 2$. Then $(\sum_{i=3}^n d_i) \leq (\alpha_X - 4)$ and $(\sum_{i=j}^n d_i) \leq (\alpha_X - j)$ for all $j \geq 2$, so

$$\alpha_X(\sum_{i=2}^n d_i)(\sum_{i=3}^n d_i)\cdots(\sum_{i=n}^n d_i) \leq \alpha_X(\alpha_X - 2)(\alpha_X - 4)(\alpha_X - 4)\cdots(\alpha_X - n).$$

Since $\alpha_X(\alpha_X - 4) \leq (\alpha_X - 1)(\alpha_X - 3)$, we obtain $\alpha_X(\alpha_X - 2)(\alpha_X - 4)(\alpha_X - 4)\cdots(\alpha_X - n) \leq \prod_{j=1}^n (\alpha_X - j)$. □

The proof of the next lemma is similar so we omit it.

Lemma 2.5. *With the same hypothesis as Lemma 2.4,* $d_X n! \leq \alpha_X(\alpha_X - 2)(\alpha_X - 4)(\alpha_X - 6)\cdots(\alpha_X - 2(n-1))$.

Definition 2.6. *A subscheme* $Z \subseteq \mathbb{P}^n$ *is* collinear *if* $I_Z = \langle l_1, \ldots, l_{n-1}, f\rangle$, *where the* l_i *are linearly independent linear forms and* $\deg f = t$.

We now use linkage to study the case where Y is linked in X to a collinear subscheme Z. While we expect our methods to work more generally, this case is already complicated enough to be interesting. Since the line $V(l_1, \ldots, l_{n-1})$ cannot be contained in each of the hypersurfaces defining X (or X would contain the whole line), the line on which Z is supported must intersect one of the hypersurfaces defining X in a zero-scheme. Thus, Z is of degree at most d_n. Henceforth we write α for α_X.

Theorem 2.7. *Let* X *be a zero-dimensional complete intersection of type* $d_1, d_2, \ldots, d_n$ *in* $\mathbb{P}^n$. *Let* $Z \subset X$ *be a collinear subscheme of degree* t, *and let* Y *be residual to* Z. *Then Conjecture 1.1 holds for* R/I_Y.

Proof. **Upper bound.** Because $d_j \geq 2$ for all j, even if cancellation occurs we have $M_i = \alpha - n + i - 1$ for $i \in \{2, \ldots, n\}$, as in Example 2.3. For $i = 1$, $M_1 \geq d_n$ or $M_1 = d_n - 1$, depending on the amount of cancellation. If $t \leq \sum_{i=1}^{n-1}(d_i - 1)$, then $\alpha - n - t + 1 \geq d_n$ and so $M_1 \geq d_n$. If $\sum_{i=1}^{n-1}(d_i - 1) < t$, then cancellation can occur.

Case 1: $M_1 \geq d_n$. In this case, since

$$n!(d - t) \leq n!d \leq \alpha(\alpha - d_1)(\alpha - d_1 - d_2)\cdots(d_n),$$

it suffices to show that

$$\alpha(\alpha - d_1)(\alpha - d_1 - d_2)\cdots(d_{n-1} + d_n)(d_n) \leq (\alpha - 1)(\alpha - 2)(\alpha - 3)\cdots(\alpha - (n-1))M_1$$

Since $d_j \geq 2$ for all j, $\alpha(\alpha - d_1 - d_2) \leq (\alpha - 1)(\alpha - 3)$, and

$$\begin{array}{rcl} (\alpha - d_1) & \leq & (\alpha - 2) \\ (\alpha - d_1 - d_2 - d_3) & \leq & (\alpha - 4) \\ (\alpha - d_1 - d_2 - d_3 - d_4) & \leq & (\alpha - 5) \\ & \vdots, & \end{array}$$

the result follows if $n \geq 5$. If $n = 4$, then we must replace the $\alpha - 4$ above with M_1. The result holds since $M_1 \geq d_4 = \alpha - d_1 - d_2 - d_3$.

For $n = 3$, there are four cases to analyze. If $d_1 \geq 3$, then $\alpha(\alpha - d_1) \leq (\alpha - 1)(\alpha - 2)$. If $d_1 = 2$, then if $d_2 \geq 3$ we find that $6d \leq (\alpha-1)(\alpha-2)d_3$ because $11d_2 \leq d_2^2 + 2d_2d_3 + d_3^2 + d_3$. If $d_1 = 2$ and $d_2 = 2$, but $d_3 \geq 3$, then we find that $24d_3 \leq d_3^3 + 5d_3^2 + 6d_3$. Since $d_3 \geq 3$, $18 \leq d_3^2 + 5d_3$ so the inequality is true.

Finally, if $d_1 = d_2 = d_3 = 2$, then as long as $t > 1$ we have $6(8-t) \leq (5)(4)(2)$, so the bound holds when $t > 1$. The case $t = 1$ is covered by Example 2.3, which concludes Case 1.

Case 2: $d_n > M_1$. Then $\alpha - t - n + 1 = d_n - 1$. If $d_1 = d_n$, then since at most $n-1$ of the d_i's can cancel, this forces $M_1 = d_1 = d_n$ and the inequalities from the previous case apply. So henceforth we assume $d_1 < d_n$, which as noted in Lemma 2.4 implies $d\,n! \leq (\alpha-1)(\sum_{i=2}^{n} d_i)(\sum_{i=3}^{n} d_i)\cdots d_n$. We wish to show

$$n!(d-t) \leq (\alpha - t - n + 1)\prod_{i=2}^{n}(\alpha - n + i - 1) = (\alpha - t - n + 1)\prod_{i=1}^{n-1}(\alpha - i)$$

Suppose $n \geq 5$. We claim that $d_n(d_n + d_{n-1}) \leq (d_n - 1)(\alpha - n + 2) = (d_n - 1)(d_n + t)$. This follows from the inequalities

$$\begin{array}{rcl} (d_n - 1)(d_n + t) - d_n(d_n + d_{n-1}) & = & -d_n + t(d_n - 1) - d_{n-1}d_n \\ & \geq & -d_n + (d_n - 1)(d_{n-1} + n - 2) - d_{n-1}d_n \end{array}$$

because $t = \alpha - d_n - n + 2 = d_{n-1} + \sum_{i=1}^{n-2}(d_i - 1) \geq d_{n-1} + n - 2$. Then

$$\begin{array}{rcl} -d_n + (d_n - 1)(d_{n-1} + n - 2) - d_{n-1}d_n & = & -d_n + (n-2)d_n - d_{n-1} - (n-2) \\ & = & (n-4)d_n + (d_n - d_{n-1}) - (n-2) \\ & \geq & (n-4)d_n - (n-2) \\ & = & (n-4)(d_n - 1) - 2. \end{array}$$

Finally $(n-4)(d_n - 1) \geq 2$ because $n \geq 5$ and $d_n > d_1 \geq 2$, so we obtain

$$\begin{array}{rcl} n!d & \leq & d_n(d_n + d_{n-1})(d_n + d_{n-1} + d_{n-2})\cdots(\alpha - d_1)(\alpha - 1) \\ & \leq & (d_n - 1)(d_n + t)(d_n + d_{n-1} + d_{n-2})\cdots(\alpha - d_1)(\alpha - 1) \\ & = & (d_n - 1)(\alpha - n + 2)(d_n + d_{n-1} + d_{n-2})\cdots(\alpha - d_1)(\alpha - 1) \\ & \leq & (d_n - 1)(\alpha - n + 2)(\alpha - n + 1)(d_n + d_{n-1} + d_{n-2} + d_{n-3})\cdots(\alpha - d_1)(\alpha - 1) \\ & \leq & (d_n - 1)(\alpha - n + 2)(\alpha - n + 1)(\alpha - (n-3))(\alpha - (n-4))\cdots(\alpha - 2)(\alpha - 1). \end{array}$$

Hence, the upper bound holds if $n \geq 5$.

If $n = 4$ and $d_2 < d_4$, then $3d_2 \leq d_2 + d_3 + d_4 - 1 + d_1 - 2 = \alpha - 3$. If $d_4 = d_3$, then since $d_1 < d_4$, we also have $4d_1 \leq \alpha - 2$. So, $12d_1d_2 \leq (\alpha-2)(\alpha-3)$. On the other hand, if $d_2 = d_4$, then $3d_2 \leq \alpha - 2$ and $4d_1 \leq \alpha - 3$ so we also find that $12d_1d_2 \leq (\alpha-2)(\alpha-3)$. It just remains to show that $2d_3d_4 \leq (\alpha-1)(d_4-1)$. But $(d_4 - 1)(\alpha - 1) - 2d_3d_4 \geq (d_4 - 1)(2d_4 + 3) - 2d_4^2 = d_4 - 3 \geq 0$. Thus the upper bound holds when $d_4 = d_3$. If $d_3 < d_4$, we may only have $4d_1 \leq (\alpha - 1)$. Nevertheless,

$$\begin{array}{rcl} (\alpha - 2)(d_4 - 1) - 2d_3d_4 & = & (d_1 + d_2 + d_4 - d_3 - 2)(d_4 - 1) - 2d_3 \\ & \geq & (d_1 + d_2 + d_4 - d_3 - 2)(d_4 - 1) - 2(d_4 - 1) \\ & = & (d_1 + d_2 + d_4 - d_3 - 4)(d_4 - 1) \\ & = & (d_1 + d_2 - 4 + d_4 - d_3)(d_4 - 1) \geq 0. \end{array}$$

Thus, the upper bound holds when $n = 4$.

If $n = 3$, then since $M_1 = d_3 - 1$, $d_2 \neq d_3$. If $3d_1 \leq (\alpha - 2)$ then as before, $(\alpha - 1)(d_3 - 1) - 2d_2d_3 \geq (d_1 - d_2 + d_3 - 3)(d_3 - 1) \geq 0$. If $3d_1 = \alpha - 1$, we must have $d_1 = d_2 = d_3 - 1$. In this case, using the fact that $t = 2d_1 - 1$, we calculate the inequality directly: $6(d_1^2(d_1+1)-(2d_1-1)) \leq (d_1)(3d_1+1-2)(3d_1+1-1)$ simplifies to the true statement $0 \leq 3(d_1-1)(d_1-2d_1+2)$.

Lower bound. If there is no cancellation, then $m_n = \alpha - t$ and for $i < n$ we have $m_i = \min\{\alpha-n-t+i, \sum_{j=1}^{i} d_j\}$. In particular, $m_i \leq \sum_{j=1}^{i} d_j$, for $i \in \{1,\dots,n-1\}$, and so

$$\prod_{i=1}^{n} m_i = (\prod_{i=1}^{n-1} m_i)m_n \leq (\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j)m_n.$$

Hence it is sufficient to prove that

$$(\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j)(\alpha-t) \leq n!(d-t).$$

Exactly as in Example 2.3, we have

$$\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j\alpha \leq n!d \quad \text{and} \quad i+1 \;\leq\; 2i \;\leq\; \sum_{j=1}^{i} d_j.$$

So $n!t = t\prod_{i=1}^{n-1}(i+1) \;\leq\; t\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j$. Subtracting this inequality from the left hand inequality above yields the desired inequality, so the lower bound holds for R/I_Y if there is no cancellation.

Now let us look at where cancellation can occur. We only care about cancellation when a term of some degree that shows up in the set of minimums disappears. We can break it up into two cases:

Case 1: $t < d_n$. Then $\alpha-t > \alpha-d_n$, and so $\alpha-t-1 \geq \alpha-d_n$, hence $m_{n-1} \leq \alpha-d_n$. Also $\alpha-t-1 \geq \alpha-d_n$ implies $\alpha-t-1 > \alpha-d_n-d_{n-1}$, so that $m_{n-2} \leq \alpha-d_n-d_{n-1}$, and in general $m_{n-i} \leq \alpha-d_n-\cdots d_{n-i+1}$. So if $m_n = \alpha-t$, then the argument from the previous case holds.

However, if $t = d_l$ for some $l < n$, then it is possible that $m_n = \alpha-1$. So in this case, we need to show that

$$(\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j)(\alpha-1) \leq n!(d-d_l).$$

We have the inequalities

$$\begin{array}{ccc} d_1 & \leq & d_1 \\ d_1+d_2 & \leq & 2d_2 \\ \vdots & & \\ d_1+d_2+\cdots+d_{n-2}+d_{n-1} & \leq & (n-1)d_{n-1} \\ \alpha+1 & \leq & nd_n, \end{array}$$

where the last row follows since $d_l < d_n$. Subtracting $2\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j$ from the product of the left hand column and $n!d_l$ from the product of the right hand column would yield the desired inequality, so it suffices to show that $n!d_l \leq 2\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j$. Let $\beta = \prod_{i=1}^{n-2}\sum_{j=1}^{i} d_j$, so

$$2\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j = 2(d_{n-1}+\sum_{j=1}^{n-2} d_j)\beta.$$

Since $d_l \leq d_{n-1}$, it is enough to show that $n! \leq 2\beta$. Since the d_i are at least two,

$$2^{n-1}(n-2)! \leq 2\beta,$$

and the inequality holds if $n \geq 6$. For $n \in \{3,4,5\}$, a case analysis shows we have to verify the bound directly for

$$\begin{array}{ll} n=3 & d_1 = 2 \\ n=4 & (d_1,d_2) = (2,2) \text{ or } (2,3) \\ n=5 & (d_1,d_2,d_3) = (2,2,2) \text{ or } (2,2,3). \end{array}$$

For example, if $n = 3$ and $d_1 = 2$, we must verify that

$$2(2+d_2)(2+d_2+d_3-1) \leq 6(2d_2d_3 - d_2).$$

This follows by summing the inequalities:

$$\begin{array}{rcl} (2+d_2)d_3 & \leq & (2d_2)d_3 \\ (2+d_2)(d_2+1) & \leq & (2d_2)d_3, \end{array}$$

and observing that $2d_2d_3 - 3d_2 = d_2(2d_3 - 3) \geq 0$. The other cases are similar so we omit them.

Case 2: $t = d_n$. The $\alpha - d_n$ term cancels with $\alpha - t$, and so $m_n = \alpha - 1$. Also $m_{n-1} = \min\{\alpha - d_{n-1}, \alpha - t - 1\} \leq \alpha - t - 1 = \alpha - d_n - 1$. Since all the $d_i \geq 2$, we cannot have $\alpha - d_n - \cdots - d_{k+1} = \alpha - n - t + k + 1$ for any $k \leq n-2$, and hence we always have $m_i \leq \sum_{j=1}^{i} d_j$ for $i \leq n-2$. In order to prove the lower bound, we need to show

$$(\alpha - 1)(\alpha - d_n - 1)\prod_{i=1}^{n-2}\sum_{j=1}^{i} d_j \leq n!(d - d_n)$$

We can write

$$n!(d - d_n) = d_n n(n-1)!(d' - 1)$$

where $d' = \prod_{i=1}^{n-1} d_i$. By the bound on the complete intersection of type $d_1, d_2, \ldots, d_{n-1}$, we know that

$$(n-1)!d' \geq \prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j = (\alpha - d_n)\prod_{i=1}^{n-2}\sum_{j=1}^{i} d_j.$$

It is also true that $n - 1 \leq 2^{n-2}$ for all $n \geq 2$, so

$$(n-1)! \leq 2^{n-2}(n-2)! \leq \prod_{i=1}^{n-2}\sum_{j=1}^{i} d_j, \text{ since } d_i \geq 2.$$

Therefore

$$(n-1)!(d'-1) = (n-1)!d' - (n-1)! \geq (\alpha - d_n)\prod_{i=1}^{n-2}\sum_{j=1}^{i} d_j - \prod_{i=1}^{n-2}\sum_{j=1}^{i} d_j = (\alpha - d_n - 1)\prod_{i=1}^{n-2}\sum_{j=1}^{i} d_j.$$

But since $nd_n \geq \alpha \geq \alpha - 1$, this gives

$$n!(d - d_n) = d_n n(n-1)!(d' - 1) \geq (\alpha - 1)(\alpha - d_n - 1)\prod_{i=1}^{n-2}\sum_{j=1}^{i} d_j.$$

□

3. Y IS LINKED TO 3 GENERAL POINTS

In this section, we study the simplest Z which is not a collinear scheme: three general points. While we are able to carry out the degree analysis in this case, it also serves to illustrate that this type of argument will become increasingly complex.

Theorem 3.1. *Let X be a zero-dimensional complete intersection of type $d_1, d_2, \ldots, d_n$ in $\mathbb{P}^n$, $n > 2$. Let $Z \subset X$ be a set of 3 non-collinear points, and suppose Y is linked to Z in X. Then Conjecture 1.1 holds for R/I_Y.*

By Theorem 1.2, the mapping cone resolution of $I_Y = I_X : I_Z$ is

$$0 \to \begin{array}{c} R^{n-2}(-(\alpha-1)) \\ \oplus \\ R^3(-(\alpha-2)) \end{array} \to \cdots \to \begin{array}{c} R^{\binom{n-2}{n-i+1}}(-(\alpha-n-1+i)) \\ \oplus \\ R^{3\binom{n-2}{n-i}+2\binom{n-2}{n-i-1}}(-(\alpha-n-2+i)) \\ \oplus \\ \oplus R(-(\sum\limits_{\substack{j\in\Lambda \\ |\Lambda|=i}} d_j)) \end{array} \to \cdots$$

$$\cdots \to \begin{array}{c} R^{\binom{n-2}{n-2}}(-(\alpha-n+2)) \\ \oplus \\ R^{3\binom{n-2}{n-3}+2\binom{n-2}{n-4}}(-(\alpha-n+1)) \\ \oplus \\ \oplus R(-(\sum\limits_{\substack{j\in\Lambda \\ |\Lambda|=3}} d_j)) \end{array} \to \begin{array}{c} R^{3\binom{n-2}{n-2}+2\binom{n-2}{n-3}}(-(\alpha-n)) \\ \oplus \\ \oplus R(-(d_j+d_k)) \end{array} \to \begin{array}{c} R^{2\binom{n-2}{n-2}}(-(\alpha-n-1)) \\ \oplus \\ \oplus R(-d_j) \end{array} \to I_Y$$

Proof. **Upper bound.** We begin with the upper bound. If $n \geq 4$, then there is no cancellation of terms which affect the upper bound, and for $i \in \{3, \ldots, n-1\}$, $M_i = \max\{\sum_{j=n-i+1}^{n} d_j, \alpha - n + i - 1\} = \alpha - n + i - 1$, while

$$\begin{aligned} M_1 &= \max\{d_n, \alpha-n-1\} = \alpha-n-1 \\ M_2 &= \max\{d_{n-1}+d_n, \alpha-n\} = \alpha-n \\ M_n &= \alpha-1. \end{aligned}$$

So we want to show that

$$n!(d-3) \leq (\alpha-n)(\alpha-(n+1)) \prod_{i=1}^{n-2} (\alpha-i).$$

Since we know that

$$n!(d-3) \leq n!d \leq \alpha \prod_{i=2}^{n} \sum_{j=i}^{n} d_j,$$

it is enough to show that

$$\alpha \prod_{i=2}^{n} \sum_{j=i}^{n} d_j \leq (\alpha-n)(\alpha-(n+1)) \prod_{i=1}^{n-2} (\alpha-i).$$

By Lemma 2.5, we know that

$$\alpha \prod_{i=2}^{n} \sum_{j=i}^{n} d_j \leq \alpha(\alpha-2)(\alpha-4)(\alpha-6)\cdots(\alpha-2(n-1)),$$

so it is enough to show that

$$\alpha(\alpha-2)(\alpha-4)(\alpha-6)\cdots(\alpha-2(n-1)) \leq (\alpha-n)(\alpha-(n+1)) \prod_{i=1}^{n-2} (\alpha-i).$$

If $n > 4$, then

$$\begin{array}{rcl} \alpha - 2 & \leq & \alpha - 2 \\ \alpha - 2(3) & \leq & \alpha - 4 \\ \alpha - 2(4) & \leq & \alpha - 5 \\ & \vdots & \\ \alpha - 2(n-3) & \leq & \alpha - (n-2) \\ \alpha - 2(n-2) & \leq & \alpha - n \\ \alpha - 2(n-1) & \leq & \alpha - (n+1) \end{array}$$

and

$$\alpha(\alpha - 4) \quad \leq \quad (\alpha - 1)(\alpha - 3).$$

Taking the product, we see that the bound holds if $n > 4$. If $n = 4$, then we must show that

$$\alpha(\alpha - 2)(\alpha - 4)(\alpha - 6) \leq (\alpha - 1)(\alpha - 2)(\alpha - 4)(\alpha - 5);$$

which is true since $\alpha(\alpha - 6) \leq (\alpha - 1)(\alpha - 5)$ for all α.

Finally, if $n = 3$, then we have to be a bit more careful. It is always true that $M_1 = \alpha - 4$ and $M_3 = \alpha - 1$. The value of M_2 is either $\alpha - 2$ or $\alpha - 3$ depending on cancellation.

Case 1: $d_1 = d_2 = d_3 = 2$. We check directly that

$$30 = 3!(8 - 3) = (2)(6 - 3)(5) = (\alpha - 4)(\alpha - 3)(\alpha - 1) \leq M_1 M_2 M_3.$$

Case 2: $d_1 = d_2 = 2, d_3 > 2$. In this case $\alpha = d_3 + 4$, and so $M_2 \geq d_3 + 1$. Again we plug in values, and check to see that the resulting inequality is true. Is $6(d - 3) = 6(4d_3 - 3) \leq (d_3)(d_3 + 1)(d_3 + 3)$? This is equivalent to $0 \leq d_3^3 + 4d_3^2 - 21d_3 + 18 = (d_3 - 2)(d_3^2 + 6d_3 - 9)$, which is true for $d_3 \geq 3$.

Case 3: $d_1 = 2, d_2 > 2$. Here $\alpha = d_2 + d_3 + 2$ and $M_2 \geq d_2 + d_3 - 1$, so we need to check that $6(2d_2d_3 - 3) \leq (d_2 + d_3 - 2)(d_2 + d_3 - 1)(d_2 + d_3 + 1)$. This inequality reduces to checking that $d_2^3 + 3d_2^2d_3 + 3d_2d_3^2 + d_3^3 - 2d_2^2 - 16d_2d_3 - 2d_3^2 - d_2 - d_3 + 20 \geq 0$, which is true since for $3 \leq d_2 \leq d_3$,

$$\begin{array}{rcl} d_2^3 + 3d_2^2d_3 + 3d_2d_3^2 + d_3^3 & \geq & 3d_2^2 + 9d_2d_3 + 9d_3^2 + 3d_3^2 \\ & = & 2d_2^2 + 2d_3^2 + d_2^2 + 9d_2d_3 + 8d_3^2 \\ & \geq & 2d_2^2 + 2d_3^2 + d_2^2 + 9d_2d_3 + 7d_2d_3 + d_3^2 \\ & = & 2d_2^2 + 2d_3^2 + 16d_2d_3 + d_2^2 + d_3^2 \\ & \geq & 2d_2^2 + 2d_3^2 + 16d_2d_3 + d_2 + d_3. \end{array}$$

Case 4: $d_1 > 2$. In this case, we check directly that

$$\alpha(\alpha - d_1)(\alpha - d_1 - d_2) \leq (\alpha - 1)(\alpha - d_1)(\alpha - 4) \leq M_1 M_2 M_3.$$

The left expression is the familiar product from I_X, so it is bigger than $3!d$, and hence also $3!(d - 3)$. So the upper bound holds.

Lower bound Now we will prove the lower bound. Notice that the only cancellation that is numerically feasible is at the last step because $d_j \geq 2$ for all j. So cancellation can only happen if d_1, d_2, and possibly d_3 are all 2. Such a cancellation will affect m_n only if all three terms of degree $\alpha - 2$ cancel, that is, if $d_1 = d_2 = d_3 = 2$ and all possible cancellation occurs, and $d_4 \geq 3$ when $n \geq 4$. Therefore for $i < n$ we have $m_i = \min\{\sum_{j=1}^{i} d_j, \alpha - n + i - 2\}$, and m_n is either $\alpha - 1$ or $\alpha - 2$. If we assume $m_n = \alpha - 2$, then there are four cases to consider.

Case $n \geq 4$: We know that

$$(\alpha-2)\prod_{i=1}^{n-1} m_i \leq (\alpha-2)\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j,$$

so we need to show that the rightmost expression is less than or equal to $n!(d-3)$. Since $d_j \geq 2$, $2i \leq \sum_{j=1}^{i} d_j$, so $n!3 \leq 2^n(n-1)! \leq 2\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j$. Thus

$$(\alpha-2)\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j \leq n!d - 2\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j \leq n!d - n!3 = n!(d-3)$$

Case $n=3$, $d_1=d_2=d_3=2$: In this case $m_1=2$, $m_2=3$, and $m_3=4$, so we check directly that

$$24 = (2)(3)(4) \leq 3!(2^3-3) = 30.$$

Case $n=3$, $d_1=d_2=2$, $d_3>2$: In this case we check directly that $(2)(4)(\alpha-2) \leq 3!(d-3)$. Since $\alpha = d_3+4$, this inequality holds as long as $d_3 \geq \frac{17}{8}$, which it is.

Case $n=3$, $d_2>2$: In this case $m_1 \leq d_1$, $m_2 \leq d_1+d_2$, and $m_3 = \alpha-2$. Using the bound for the complete intersection of type d_1, d_2, d_3, we have that

$$d_1(d_1+d_2)(\alpha-2) = d_1(d_1+d_2)\alpha - 2d_1(d_1+d_2) \leq 3!d - 18,$$

which is true if $2d_1(d_1+d_2) \geq 18$. But $2d_1(d_1+d_2) \geq 2(2)(5) = 20$, so the bound holds.

If on the other hand $m_n = \alpha-1$, then it must be true that $d_1=d_2=d_3=2$. We know

$$\prod_{i=1}^{n} m_i \leq (\alpha-1)\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j,$$

and so it suffices to show

$$\alpha\prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j - \prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j \leq n!d - 3\,n!,$$

which would follow from

$$3\,n! \leq \prod_{i=1}^{n-1}\sum_{j=1}^{i} d_j.$$

Since $d_4 \geq 2$, we have that

$$5!3 = 3\cdot 2\cdot 3\cdot 4\cdot 5 \leq 2\cdot 4\cdot 6\cdot 8 \leq 2\cdot 4\cdot 6\cdot(6+d_4) = \prod_{i=1}^{4}\sum_{j=1}^{i} d_j,$$

and once n is at least 6, $\prod_{i=6}^{n} i \leq \prod_{i=5}^{n-1}\sum_{j=1}^{i} d_j$; hence the desired inequality follows if $n \geq 5$.

If $n=4$, then we check directly. We have that $m_1=2$, $m_2=4$, $m_3=6$ and $m_4=d_4+5$. A simple calculation shows that in fact $4!(8d_4-3) \geq (2)(4)(6)(d_4+5)$ since $d_4 \geq 2$.

If $n=3$, then again we may check directly. We have that $m_1=2$, $m_2=3$, and $m_3=5$. So we see that $30 = 3!(8-3) \geq (2)(3)(5) = 30$. □

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