THE GREEN FUNCTION ESTIMATES FOR STRONGLY ELLIPTIC SYSTEMS OF SECOND ORDER

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Abstract. We establish existence and pointwise estimates of fundamental solutions and Green's matrices for divergence form, second order strongly elliptic systems in a domain $\Omega \subseteq \mathbb{R}^n, n \geq 3$, under the assumption that solutions of the system satisfy De Giorgi-Nash type local Hölder continuity estimates. In particular, our results apply to perturbations of diagonal systems, and thus especially to complex perturbations of a single real equation.

1. Introduction

In this article, we study Green's functions (or Green's matrices) of second order, strongly elliptic systems of divergence type in a domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$. In particular, we treat the Green matrix in the entire space, usually called the fundamental solution. We shall prove that if a given elliptic system has the property that all weak solutions of the system are locally Hölder continuous, then it has the Green's matrix in Ω. (For example, if coefficients of the system belong to the space of VMO introduced by Sarason [19], then it will enjoy such a property). For such elliptic systems, we study standard properties of the Green's matrix including pointwise bounds, L^p and weak L^p estimates for Green's matrix and its derivatives, etc.

For the scalar case, i.e., a single elliptic equation, the existence and properties of Green's function was studied by Littman, Stampacchia, and Weinberger [14] and Grüter and Widman [11]. In this article, we follow the approach of Grüter and Widman in constructing Green's matrix. The main technical difficulties arise from lack of Harnack type inequalities and the maximum principle for the systems. The key observation on which this article is based is that even in the scalar case, one can get around Moser's Harnack inequality [18] or maximum principle but instead rely solely on De Giorgi-Nash type oscillation estimates [5] in constructing and studying properties of Green's functions. From this point of view, this article provides a unified approach in studying Green's function for both scalar and systems of equations. We should point out that there has been some study of Green's matrix for systems with continuous coefficients, notably by Fuchs [7] and Dolzmann-Müller [6]. Our existence results and interior estimates of Green's function will include theirs, since as is well known, weak solutions of systems with uniformly continuous (or VMO) coefficients enjoy local Hölder estimates. On the other hand, we have not attempted to replicate their boundary estimates, which depend in particular on having a C^1 boundary. Our method does not require boundedness of the domain nor regularity of the boundary in constructing Green's matrices, while the methods

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of Fuchs [7] and Dolzmann-Müller [6] require both boundedness and regularity of the domain at the very beginning. We note that a scalar elliptic equation with complex coefficients can be identified as an elliptic system with real coefficients satisfying a special structure, and thus our results apply in particular to complex perturbations of a scalar real equation. In the complex coefficients setting, the main results of Section 3 in our paper can be also obtained by following the method of Auscher [2]. The estimates of the present paper will be applied to the development of the layer potential method for equations with complex coefficients in [1].

The organization of this paper is as follows. In Section 2, we define the property (H), which is essentially equivalent to De Giorgi's oscillation estimates in the scalar case, and introduce a function space $Y^{1,2}_0(\Omega)$ which substitutes $W^{1,2}_0(\Omega)$ in constructing Green's functions; they are identical if Ω is bounded but in general, $Y^{1,2}_0(\Omega)$ is a larger space and is more suitable for our purpose. In Section 3, we study Green's functions defined in the entire space, which are usually referred to as the fundamental solutions. The main result is that for a system whose coefficients are close to those of a diagonal system, the fundamental solution behaves very much like that of a single equation. In Section 4, we study Green's matrices in general domains, including unbounded ones. We also study the boundary behavior of Green's matrices when the boundary of domain satisfies a measure theoretic exterior cone condition, called the condition (S). We prove in particular that if the coefficients of the system are close to those of a diagonal system, then again the boundary behavior of its Green's function is much like that of a single equation. In section 5, we discuss the Green's matrices of the strongly elliptic systems with VMO coefficients. By following the same techniques already developed in the previous two sections, we construct the Green's matrix in general domains including the entire space. One subtle difference is that in this VMO coefficients case, one should play with a localized version of property (H) since basically, the regularity of weak solutions of the systems with VMO coefficients is inherited from the systems with constant coefficients when the scale is made small enough. Therefore, all the estimates for the Green's matrix stated in this section are only meaningful near a pole.

Finally, we would like to mention that when $n = 2$, the method used in this article breaks down in several places and for that reason we plan to treat the two dimensional case in a separate paper.

2. Preliminaries

2.1. **Strongly elliptic systems.** Throughout this article, the summation convention over repeated indices shall be assumed. Let L be a second order elliptic operator of divergence type acting on vector valued functions $\boldsymbol{u} = (u^1, \ldots, u^N)^T$ defined on $\mathbb{R}^n$ $(n \geq 3)$ in the following way:

$$L\boldsymbol{u} = -D_\alpha(\boldsymbol{A}^{\alpha\beta}\, D_\beta \boldsymbol{u}), \tag{2.1}$$

where $\boldsymbol{A}^{\alpha\beta} = \boldsymbol{A}^{\alpha\beta}(x)$ $(\alpha, \beta = 1, \ldots, n)$ are N by N matrices satisfying the strong ellipticity condition, i.e., there is a number $\lambda > 0$ such that

$$A^{\alpha\beta}_{ij}(x)\xi^j_\beta \xi^i_\alpha \geq \lambda \left|\boldsymbol{\xi}\right|^2 := \lambda \sum_{i=1}^{N}\sum_{\alpha=1}^{n} |\xi^i_\alpha|^2, \quad \forall x \in \mathbb{R}^n \tag{2.2}$$

We also assume that $A_{ij}^{\alpha\beta}$ are bounded, i.e., there is a number $\Lambda > 0$ such that

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} |A_{ij}^{\alpha\beta}(x)|^2 \le \Lambda^2, \quad \forall x \in \mathbb{R}^n. \tag{2.3}$$

If we write (2.1) component-wise, then we have

$$(L\boldsymbol{u})^i = -D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j), \quad \forall i = 1, \ldots, N. \tag{2.4}$$

The transpose operator of ${}^tL$ of L is defined by

$${}^tL\boldsymbol{u} = -D_\alpha({}^t\boldsymbol{A}^{\alpha\beta} D_\beta \boldsymbol{u}), \tag{2.5}$$

where ${}^t\boldsymbol{A}^{\alpha\beta} = (\boldsymbol{A}^{\beta\alpha})^T$ (i.e., ${}^tA_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$). Note that the coefficients ${}^tA_{ij}^{\alpha\beta}$ satisfy (2.2), (2.3) with the same constants λ, Λ.

In the sequel, we shall use the notation $\fint_S f := \frac{1}{|S|}\int_S f$ (assuming $0 < |S| < \infty$), where S is a measurable subset of $\mathbb{R}^n$ and $|S|$ denotes the Lebesgue measure of measurable S.

Definition 2.1. We say that the operator L satisfies the property (H) if there exist $\mu_0, H_0 > 0$ such that all weak solutions $\boldsymbol{u}$ of $L\boldsymbol{u} = 0$ in $B_R = B_R(x_0)$ satisfy

$$\int_{B_r} |D\boldsymbol{u}|^2 \le H_0 \left(\frac{r}{s}\right)^{n-2+2\mu_0} \int_{B_s} |D\boldsymbol{u}|^2, \quad 0 < r < s \le R. \tag{2.6}$$

Similarly, we say that the transpose operator ${}^tL$ satisfies the property (H) if corresponding estimates hold for all weak solutions $\boldsymbol{u}$ of ${}^tL\boldsymbol{u} = 0$ in B_R.

Lemma 2.2. *Let $(a^{\alpha\beta}(x))_{\alpha,\beta=1}^n$ be coefficients satisfying the following conditions: There are constants $\lambda_0, \Lambda_0 > 0$ such that for all $x \in \mathbb{R}^n$*

$$a^{\alpha\beta}(x)\xi_\beta\xi_\alpha \ge \lambda_0 |\xi|^2, \quad \forall \xi \in \mathbb{R}^n; \quad \sum \left|a^{\alpha\beta}(x)\right|^2 \le \Lambda_0^2. \tag{2.7}$$

Then, there exists $\epsilon_0 = \epsilon_0(n, \lambda_0, \Lambda_0)$ such that if

$$\epsilon^2(x) := \sum \left|A_{ij}^{\alpha\beta}(x) - a^{\alpha\beta}(x)\delta_{ij}\right|^2 < \epsilon_0^2, \quad \forall x \in \mathbb{R}^n, \tag{2.8}$$

then the operator L associated with the coefficients $A_{ij}^{\alpha\beta}$ satisfies the condition (H) with $\mu_0 = \mu_0(n, \lambda_0, \Lambda_0), H_0 = H_0(n, N, \lambda_0, \Lambda_0) > 0$.

Proof. See e.g., [12, Proposition 2.1]. □

Lemma 2.3. *Suppose that the operator L satisfies the following Hölder property for weak solutions: There are constants $\mu_0, C_0 > 0$ such that all weak solutions $\boldsymbol{u}$ of $L\boldsymbol{u} = 0$ in $B_{2R} = B_{2R}(x_0)$ satisfy the estimate*

$$[\boldsymbol{u}]_{C^{\mu_0}(B_R)} \le C_0 R^{-\mu_0} \left(\fint_{B_{2R}} |\boldsymbol{u}|^2\right)^{1/2}, \tag{2.9}$$

where $[f]_{C^\mu(\Omega)}$ denotes the usual $C^\mu(\Omega)$ semi-norm of f; see [10] *for the definition. Then, the operator L satisfies the property (H) with μ_0 and $H_0 = H_0(n, N, \lambda, \Lambda, C_0)$.*

Proof. We may assume that $r < s/4$; otherwise, (2.6) is trivial. Denote $\overline{\boldsymbol{u}}_r = \fint_{B_r} \boldsymbol{u}$. We may assume, by replacing $\boldsymbol{u}$ by $\boldsymbol{u} - \overline{\boldsymbol{u}}_s$, if necessary, that $\overline{\boldsymbol{u}}_s = 0$. From the Caccioppoli inequality, (2.9), and then the Poincaré inequality, it follows

$$\begin{aligned}\int_{B_r} |D\boldsymbol{u}|^2 &\le Cr^{-2} \int_{B_{2r}} |\boldsymbol{u} - \overline{\boldsymbol{u}}_{2r}|^2 \le Cr^{-2} \int_{B_{2r}} \fint_{B_{2r}} |\boldsymbol{u}(x) - \boldsymbol{u}(y)|^2 \, dy\, dx \\ &\le Cr^{-2} [\boldsymbol{u}]^2_{C^{\mu_0}(B_{2r})} (2r)^{2\mu_0} |B_{2r}| \le Cr^{n-2+2\mu_0} [\boldsymbol{u}]^2_{C^{\mu_0}(B_{s/2})} \\ &\le C(r/s)^{n-2+2\mu_0} s^{-2} \int_{B_s} |\boldsymbol{u}|^2 \le C(r/s)^{n-2+2\mu_0} \int_{B_s} |D\boldsymbol{u}|^2 .\end{aligned}$$

The proof is complete. □

Lemma 2.4. *Assume that the operator L satisfies the property (H). Then, the operator L satisfies the Hölder property* (2.9). *Moreover, for any $p > 0$, there exists $C_p = C_p(n, N, \lambda, \Lambda, \mu_0, H_0, p) > 0$ such that all weak solutions $\boldsymbol{u}$ of $L\boldsymbol{u} = 0$ in $B_R = B_R(x_0)$ satisfy*

$$\|\boldsymbol{u}\|_{L^\infty(B_r)} \le \frac{C_p}{(R - r)^{n/p}} \|\boldsymbol{u}\|_{L^p(B_R)}, \quad \forall r \in (0, R). \tag{2.10}$$

Proof. From a theorem of Morrey [17, Thoerem 3.5.2], the property (H), and the Caccioppoli inequality, it follows that

$$[\boldsymbol{u}]^2_{C^{\mu_0}(B_R)} \le CR^{2-n-2\mu_0} \|D\boldsymbol{u}\|^2_{L^2(B_{3R/2})} \le CR^{-n-2\mu_0} \|\boldsymbol{u}\|^2_{L^2(B_{2R})} . \tag{2.11}$$

Then, by a well known averaging argument (see e.g., [12]) we derive

$$\|\boldsymbol{u}\|_{L^\infty(B_{R/2})} \le C \left(\fint_{B_{2R}} |\boldsymbol{u}|^2 \right)^{1/2}, \tag{2.12}$$

where $C = C(n, N, \lambda, \Lambda, \mu_0, H_0) > 0$. For the proof that (2.12) implies (2.10), we refer to [9, pp. 80–82]. □

2.2. Function spaces $Y^{1,2}(\Omega)$ and $Y^{1,2}_0(\Omega)$.

Definition 2.5. For an open set $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$, the space $Y^{1,2}(\Omega)$ is defined as the family of all weakly differentiable functions $u \in L^{2^*}(\Omega)$, where $2^* = \frac{2n}{n-2}$, whose weak derivatives are functions in $L^2(\Omega)$. The space $Y^{1,2}(\Omega)$ is endowed with the norm

$$\|u\|_{Y^{1,2}(\Omega)} := \|u\|_{L^{2^*}(\Omega)} + \|Du\|_{L^2(\Omega)} .$$

We define $Y^{1,2}_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ in $Y^{1,2}(\Omega)$, where $C^\infty_c(\Omega)$ is the set of all infinitely differentiable functions with compact supports in Ω.

We note that in the case $\Omega = \mathbb{R}^n$, it is well known that $Y^{1,2}(\mathbb{R}^n) = Y^{1,2}_0(\mathbb{R}^n)$; see e.g., [15, p. 46]. By the Sobolev inequality, it follows that

$$\|u\|_{L^{2^*}(\Omega)} \le C(n) \|Du\|_{L^2(\Omega)}, \quad \forall u \in Y^{1,2}_0(\Omega). \tag{2.13}$$

Therefore, we have $W^{1,2}_0(\Omega) \subset Y^{1,2}_0(\Omega)$ and $W^{1,2}_0(\Omega) = Y^{1,2}_0(\Omega)$ when Ω has finite Lebesgue measure.

From (2.13), it follows that the bilinear form

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathbf{H}} := \int_\Omega D_\alpha u^i D_\alpha v^i \tag{2.14}$$

defines an inner product on $\mathbf{H} := Y_0^{1,2}(\Omega)^N$. Also, it is routine to check that $\mathbf{H}$ equipped with the inner product (2.14) is a Hilbert space.

Definition 2.6. We shall denote by $\mathbf{H}$ the Hilbert space $Y_0^{1,2}(\Omega)^N$ with the inner product (2.14). We denote

$$\|\boldsymbol{u}\|_{\mathbf{H}} := \langle \boldsymbol{u}, \boldsymbol{u}\rangle_{\mathbf{H}}^{1/2} = \|D\boldsymbol{u}\|_{L^2(\Omega)}.$$

We also define the bilinear form associated to the operator L as

$$B(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha v^i.$$

By the strong ellipticity (2.2), it follows that the bilinear form B is coercive; i.e,

$$B(\boldsymbol{u}, \boldsymbol{u}) \geq \lambda \langle \boldsymbol{u}, \boldsymbol{u}\rangle_{\mathbf{H}}. \tag{2.15}$$

3. Fundamental matrix in $\mathbb{R}^n$

Throughout this section, we assume that the operators L and ${}^tL$ satisfy the property (H). The main goal of this section is to construct the fundamental matrix of the the operator L in the entire $\mathbb{R}^n$, where $n \geq 3$. Since $Y^{1,2}(\mathbb{R}^n) = Y_0^{1,2}(\mathbb{R}^n)$, we have, as in Definition 2.5,

$$\|u\|_{L^{2^*}(\mathbb{R}^n)} \leq C(n) \|Du\|_{L^2(\mathbb{R}^n)}, \quad \forall u \in Y^{1,2}(\mathbb{R}^n).$$

We note that $W^{1,2}(\mathbb{R}^n) \subset Y^{1,2}(\mathbb{R}^n) \subset W^{1,2}_{loc}(\mathbb{R}^n)$. Unless otherwise stated, we employ the letter C to denote a constant depending on n, N, λ, Λ, μ_0, H_0, and sometimes on an exponent p characterizing Lebesgue classes. It should be understood that C may vary from line to line.

3.1. **Averaged fundamental matrix.** Our approach here is based on that in [11]. Let $y \in \mathbb{R}^n$ and $1 \leq k \leq N$ be fixed. For $\rho > 0$, consider the linear functional $\boldsymbol{u} \mapsto \fint_{B_\rho(y)} u^k$. Since

$$\left| \fint_{B_\rho(y)} u^k \right| \leq C\rho^{(2-n)/2} \|\boldsymbol{u}\|_{L^{2^*}(\mathbb{R}^n)} \leq C\rho^{(2-n)/2} \|\boldsymbol{u}\|_{\mathbf{H}}, \tag{3.1}$$

Lax-Milgram lemma implies that there exists a unique $\boldsymbol{v}_\rho = \boldsymbol{v}_{\rho;y,k} \in \mathbf{H}$ such that

$$\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta} D_\beta v_\rho^j D_\alpha u^i = \fint_{B_\rho(y)} u^k, \quad \forall \boldsymbol{u} \in \mathbf{H}. \tag{3.2}$$

Note that (2.15), (3.2), and (3.1) imply that

$$\lambda \|\boldsymbol{v}_\rho\|_{\mathbf{H}}^2 \leq B(\boldsymbol{v}_\rho, \boldsymbol{v}_\rho) \leq C\rho^{(2-n)/2} \|\boldsymbol{v}_\rho\|_{\mathbf{H}},$$

and thus we have

$$\|D\boldsymbol{v}_\rho\|_{L^2(\mathbb{R}^n)} = \|\boldsymbol{v}_\rho\|_{\mathbf{H}} \leq C\rho^{(2-n)/2}. \tag{3.3}$$

We define the "averaged fundamental matrix" $\boldsymbol{\Gamma}^\rho(\,\cdot\,, y) = (\Gamma^\rho_{jk}(\,\cdot\,, y))_{j,k=1}^N$ by

$$\Gamma^\rho_{jk}(\,\cdot\,, y) = v_\rho^j = v_{\rho;y,k}^j. \tag{3.4}$$

Note that we have

$$\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta} D_\beta \Gamma^\rho_{jk}(\,\cdot\,, y) D_\alpha u^i = \fint_{B_\rho(y)} u^k, \quad \forall \boldsymbol{u} \in \mathbf{H}, \tag{3.5}$$

and equivalently ($\alpha \leftrightarrow \beta$, $i \leftrightarrow j$).

$$\int_{\mathbb{R}^n} {}^t\!A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \Gamma_{ik}^\rho(\,\cdot\,,y) = \fint_{B_\rho(y)} u^k, \quad \forall \boldsymbol{u} \in \mathbf{H}. \tag{3.6}$$

In the sequel, we shall denote by $L_c^\infty(\Omega)$ the family of all L^∞ functions with compact supports in Ω. For a given $\boldsymbol{f} \in L_c^\infty(\mathbb{R}^n)^N$ consider a linear functional

$$\boldsymbol{w} \mapsto \int_{\mathbb{R}^n} \boldsymbol{f}\cdot\boldsymbol{w}, \tag{3.7}$$

which is bounded on $\mathbf{H}$ since

$$\left|\int_{\mathbb{R}^n} \boldsymbol{f}\cdot\boldsymbol{w}\right| \le \|\boldsymbol{f}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \|\boldsymbol{w}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \le C\|\boldsymbol{f}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \|\boldsymbol{w}\|_{\mathbf{H}}. \tag{3.8}$$

Therefore, by Lax-Milgram lemma, there exists $\boldsymbol{u} \in \mathbf{H}$ such that

$$\int_{\mathbb{R}^n} {}^t\!A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha w^i = \int_{\mathbb{R}^n} f^i w^i, \quad \forall \boldsymbol{w} \in \mathbf{H}. \tag{3.9}$$

In particular, if we set $\boldsymbol{w} = \boldsymbol{v}_\rho$ in (3.9), then by (3.6), we have

$$\int_{\mathbb{R}^n} \Gamma_{ik}^\rho(\,\cdot\,,y) f^i = \fint_{B_\rho(y)} u^k. \tag{3.10}$$

Moreover, by setting $\boldsymbol{w} = \boldsymbol{u}$ in (3.9), it follows from (3.8) that

$$\|D\boldsymbol{u}\|_{L^2(\mathbb{R}^n)} \le C\|\boldsymbol{f}\|_{L^{2n/(n+2)}(\mathbb{R}^n)}. \tag{3.11}$$

3.2. **L^∞ estimates for averaged fundamental matrix.** Let $\boldsymbol{u} \in \mathbf{H}$ be given as in (3.9). We will obtain local L^∞ estimates for $\boldsymbol{u}$ in $B_R(x_0)$, where $x_0 \in \mathbb{R}^n$ and $R > 0$ are fixed but arbitrary.

Fix $x \in B_R(x_0)$ and $0 < s \le R$. We decompose $\boldsymbol{u}$ as $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where $\boldsymbol{u}_1 \in W^{1,2}(B_s(x))^N$ is the weak solution of ${}^tL\boldsymbol{u}_1 = 0$ in $B_s(x)$ satisfying $\boldsymbol{u}_1 = \boldsymbol{u}$ on $\partial B_s(x)$; i.e., $\boldsymbol{u}_1 - \boldsymbol{u} \in W_0^{1,2}(B_s(x))$. Then, for $0 < r < s$, we have

$$\begin{aligned}
\int_{B_r(x)} |D\boldsymbol{u}|^2 &\le 2\int_{B_r(x)} |D\boldsymbol{u}_1|^2 + 2\int_{B_r(x)} |D\boldsymbol{u}_2|^2 \\
&\le C\left(\frac{r}{s}\right)^{n-2+2\mu_0} \int_{B_s(x)} |D\boldsymbol{u}_1|^2 + 2\int_{B_s(x)} |D\boldsymbol{u}_2|^2 \\
&\le C\left(\frac{r}{s}\right)^{n-2+2\mu_0} \int_{B_s(x)} |D\boldsymbol{u}|^2 + C\int_{B_s(x)} |D\boldsymbol{u}_2|^2.
\end{aligned}$$

Since $\boldsymbol{u}_2 \in W_0^{1,2}(B_s(x))^N$ is a weak solution of ${}^tL\boldsymbol{u}_2 = \boldsymbol{f}$ in $B_s(x)$, we have

$$\int_{B_s(x)} |D\boldsymbol{u}_2|^2 \le C\|\boldsymbol{f}\|^2_{L^{2n/(n+2)}(B_s(x))}.$$

For given $p > n/2$, choose $p_0 \in (n/2, p)$ such that $\mu_1 := 2 - n/p_0 < \mu_0$. Then

$$\|\boldsymbol{f}\|^2_{L^{\frac{2n}{n+2}}(B_s(x))} \le \|\boldsymbol{f}\|^2_{L^{p_0}(B_s(x))} |B_s|^{1+2/n-2/p_0} \le C\|\boldsymbol{f}\|^2_{L^{p_0}(\mathbb{R}^n)} s^{n-2+2\mu_1}. \tag{3.12}$$

Therefore, after combining the above inequalities, we have for all $r < s \le R$

$$\int_{B_r(x)} |D\boldsymbol{u}|^2 \le C\left(\frac{r}{s}\right)^{n-2+2\mu_0} \int_{B_s(x)} |D\boldsymbol{u}|^2 + Cs^{n-2+2\mu_1}\|\boldsymbol{f}\|^2_{L^{p_0}(\mathbb{R}^n)}.$$

By a well known iteration argument (see e.g., [8, Lemma 2.1, p. 86]), we have

$$
\begin{aligned}
\int_{B_r(x)} |D\boldsymbol{u}|^2 &\le C\left(\frac{r}{R}\right)^{n-2+2\mu_1} \int_{B_R(x)} |D\boldsymbol{u}|^2 + Cr^{n-2+2\mu_1} \|\boldsymbol{f}\|^2_{L^{p_0}(\mathbb{R}^n)} \\
&\le C\left(\frac{r}{R}\right)^{n-2+2\mu_1} \int_{\mathbb{R}^n} |D\boldsymbol{u}|^2 + Cr^{n-2+2\mu_1} \|\boldsymbol{f}\|^2_{L^{p_0}(\mathbb{R}^n)},
\end{aligned} \tag{3.13}
$$

for all $0 < r < R$ and $x \in B_R(x_0)$. From (3.13) it follows (see, e.g. [12])

$$
[\boldsymbol{u}]^2_{C^{\mu_1}(B_R(x_0))} \le C\left(R^{-(n-2+2\mu_1)} \|D\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)} + \|\boldsymbol{f}\|^2_{L^{p_0}(\mathbb{R}^n)}\right). \tag{3.14}
$$

Note that since $\boldsymbol{u} \in \mathbf{H}$, we have

$$
\|\boldsymbol{u}\|^2_{L^2(B_R(x_0))} \le \|\boldsymbol{u}\|^2_{L^{2^*}(B_R(x_0))} |B_R|^{2/n} \le CR^2 \|D\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)}.
$$

Consequently, we have

$$
\begin{aligned}
\|\boldsymbol{u}\|^2_{L^\infty(B_{R/2}(x_0))} &\le CR^{2\mu_1} [\boldsymbol{u}]^2_{C^{\mu_1}(B_R(x_0))} + CR^{-n} \|\boldsymbol{u}\|^2_{L^2(B_R(x_0))} \\
&\le C\left(R^{2-n} \|D\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)} + R^{2\mu_1} \|\boldsymbol{f}\|^2_{L^{p_0}(\mathbb{R}^n)}\right) + CR^{2-n} \|D\boldsymbol{u}\|^2_{L^2(\mathbb{R}^n)} \\
&\le CR^{2-n} \|\boldsymbol{f}\|^2_{L^{2n/(n+2)}(\mathbb{R}^n)} + CR^{2\mu_1} \|\boldsymbol{f}\|^2_{L^{p_0}(\mathbb{R}^n)},
\end{aligned}
$$

where we used the inequality (3.11) in the last step.

Therefore, if $\boldsymbol{f}$ is supported in $B_R(x_0)$, then (3.12) yields (recall $\mu_1 = 2 - n/p_0$)

$$
\|\boldsymbol{u}\|_{L^\infty(B_{R/2}(x_0))} \le CR^{2-n/p_0} \|\boldsymbol{f}\|_{L^{p_0}(B_R(x_0))} \le CR^{2-n/p} \|\boldsymbol{f}\|_{L^p(B_R(x_0))}. \tag{3.15}
$$

Now, (3.10) implies that for $\rho < R/2$, we have, by setting $x_0 = y$ in (3.15),

$$
\left|\int_{B_R(y)} \Gamma^\rho_{ik}(\,\cdot\,, y) f^i\right| \le \fint_{B_\rho(y)} |\boldsymbol{u}| \le CR^{2-n/p} \|\boldsymbol{f}\|_{L^p(B_R(y))}, \quad \forall p > n/2 \tag{3.16}
$$

provided that $\boldsymbol{f}$ is supported in $B_R(y)$. Therefore, by duality, we see that

$$
\|\boldsymbol{v}_\rho\|_{L^q(B_R(y))} \le CR^{2-n+n/q}, \quad \forall q \in [1, \tfrac{n}{n-2}), \quad \forall \rho \in (0, R/2), \tag{3.17}
$$

where $\boldsymbol{v}_\rho = \boldsymbol{v}_{\rho;y,k}$ is as in (3.4).

Fix $x \ne y$ and let $r := \frac{2}{3}|x-y|$. If $\rho < r/2$, then since $\boldsymbol{v}_\rho \in W^{1,2}(B_r(x))^N$ and satisfies $L\boldsymbol{v}_\rho = 0$ weakly in $B_r(x)$, it follows from Lemma 2.4 that

$$
|\boldsymbol{v}_\rho(x)| \le Cr^{-n} \|\boldsymbol{v}_\rho\|_{L^1(B_r(x))} \le Cr^{-n} \|\boldsymbol{v}_\rho\|_{L^1(B_{3r}(y))} \le Cr^{2-n}. \tag{3.18}
$$

Since ρ, y, k are arbitrary, we have obtained the following estimates.

$$
|\boldsymbol{\Gamma}^\rho(x,y)| \le C|x-y|^{2-n}, \quad \forall \rho < |x-y|/3. \tag{3.19}
$$

3.3. **Uniform weak-$L^{\frac{n}{n-2}}$ estimates for $\boldsymbol{\Gamma}^\rho(\,\cdot\,, y)$.** We claim that the following estimate holds:

$$
\int_{\mathbb{R}^n \setminus B_R(y)} |\boldsymbol{\Gamma}^\rho(\,\cdot\,, y)|^{\frac{2n}{n-2}} \le CR^{-n}, \quad \forall R > 0, \quad \forall \rho > 0. \tag{3.20}
$$

If $R > 3\rho$, then by (3.19) we have

$$
\int_{\mathbb{R}^n \setminus B_R(y)} |\boldsymbol{\Gamma}^\rho(x,y)|^{\frac{2n}{n-2}}\, dx \le C\int_{\mathbb{R}^n \setminus B_R(y)} |x-y|^{-2n}\, dx \le CR^{-n}.
$$

Next, we consider the case $R \leq 3\rho$. Let $\boldsymbol{v}_\rho^T$ be the k-th column of the averaged fundamental matrix $\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)$ as in (3.4). From (3.3), we see that

$$\|\boldsymbol{v}_\rho\|_{L^{2^*}(\mathbb{R}^n\setminus B_R(y))} \leq \|\boldsymbol{v}_\rho\|_{L^{2^*}(\mathbb{R}^n)} \leq \|D\boldsymbol{v}_\rho\|_{L^2(\mathbb{R}^n)} \leq C\rho^{(2-n)/2}.$$

and thus (3.20) also follows in the case when $R \leq 3\rho$.

Now, let $A_t = \{x \in \mathbb{R}^n : |\boldsymbol{\Gamma}^\rho(x,y)| > t\}$ and choose $R = t^{-1/(n-2)}$. Then,

$$|A_t \setminus B_R(y)| \leq t^{-\frac{2n}{n-2}} \int_{A_t\setminus B_R(y)} |\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)|^{\frac{2n}{n-2}} \leq Ct^{-\frac{2n}{n-2}}\, t^{\frac{n}{n-2}} = Ct^{-\frac{n}{n-2}}.$$

Obviously, $|A_t \cap B_R(y)| \leq CR^n = Ct^{-\frac{n}{n-2}}$. Therefore, we obtained that for all $t>0$, we have

$$|\{x \in \mathbb{R}^n : |\boldsymbol{\Gamma}^\rho(x,y)| > t\}| \leq Ct^{-\frac{n}{n-2}}, \quad \forall \rho > 0. \tag{3.21}$$

3.4. **Uniform weak-$L^{\frac{n}{n-1}}$ estimates for $D\boldsymbol{\Gamma}^\rho(\,\cdot\,y)$.** Let $\boldsymbol{v}_\rho$ be as before. Fix a cut-off function $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta \equiv 0$ on $B_{R/2}(y)$, $\eta \equiv 1$ outside $B_R(y)$, and $|D\eta| \leq C/R$. If we set $\boldsymbol{u} := \eta^2 \boldsymbol{v}_\rho$, then by (3.2)

$$0 = \int_{\mathbb{R}^n} \eta^2 A_{ij}^{\alpha\beta} D_\beta v_\rho^j D_\alpha v_\rho^i + \int_{\mathbb{R}^n} 2\eta A_{ij}^{\alpha\beta} D_\beta v_\rho^j v_\rho^i D_\alpha \eta,$$

which together with (3.19) implies that if $R > 6\rho$, then

$$\int_{\mathbb{R}^n\setminus B_R(y)} |D\boldsymbol{v}_\rho|^2 \leq CR^{-2} \int_{B_R(y)\setminus B_{R/2}(y)} |\boldsymbol{v}_\rho|^2 \leq CR^{2-n}.$$

On the other hand, if $R \leq 6\rho$, then (3.3) again implies

$$\int_{\mathbb{R}^n\setminus B_R(y)} |D\boldsymbol{v}_\rho|^2 \leq \int_{\mathbb{R}^n} |D\boldsymbol{v}_\rho|^2 \leq C\rho^{2-n} \leq CR^{2-n}.$$

Therefore, we have

$$\int_{\mathbb{R}^n\setminus B_R(y)} |D\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)|^2 \leq CR^{2-n}, \quad \forall R>0, \quad \forall \rho > 0. \tag{3.22}$$

Next, let $A_t = \{x \in \mathbb{R}^n : |D_x\boldsymbol{\Gamma}^\rho(x,y)| > t\}$ and choose $R = t^{-1/(n-1)}$. Then

$$|A_t \setminus B_R(y)| \leq t^{-2} \int_{A_t\setminus B_R(y)} |D\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)|^2 \leq Ct^{-\frac{n}{n-1}}$$

and $|A_t \cap B_R(y)| \leq CR^n = Ct^{-\frac{n}{n-1}}$. We have thus find that for all $t>0$, we have

$$|\{x \in \mathbb{R}^n : |D_x\boldsymbol{\Gamma}^\rho(x,y)| > t\}| \leq Ct^{-\frac{n}{n-1}}, \quad \forall \rho > 0. \tag{3.23}$$

3.5. **Construction of the fundamental matrix.** First, we claim

$$\|D\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)\|_{L^p(B_R(y))} \leq C_p R^{1-n+n/p}, \quad \forall \rho > 0, \quad \forall p \in (0, \tfrac{n}{n-1}). \tag{3.24}$$

Let $\boldsymbol{v}_\rho$ be as before. Note that

$$\begin{aligned}\int_{B_R(y)} |D\boldsymbol{v}_\rho|^p &= \int_{B_R(y)\cap\{|D\boldsymbol{v}^\rho|\leq\tau\}} |D\boldsymbol{v}_\rho|^p + \int_{B_R(y)\cap\{|D\boldsymbol{v}^\rho|>\tau\}} |D\boldsymbol{v}_\rho|^p \\ &\leq \tau^p\,|B_R| + \int_{\{|D\boldsymbol{v}^\rho|>\tau\}} |D\boldsymbol{v}_\rho|^p\,.\end{aligned}$$

By using (3.23), we estimate

$$\begin{aligned}\int_{\{|D\boldsymbol{v}^\rho|>\tau\}} |D\boldsymbol{v}_\rho|^p &= \int_0^\infty pt^{p-1}\,|\{|D\boldsymbol{v}^\rho| > \max(t,\tau)\}|\,dt \\ &\le C\tau^{-\frac{n}{n-1}}\int_0^\tau pt^{p-1}\,dt + C\int_\tau^\infty pt^{p-1-n/(n-1)}\,dt \\ &= C\left(1 - p/(p-\tfrac{n}{n-1})\right)\tau^{p-n/(n-1)}.\end{aligned}$$

By optimizing over τ, we get

$$\int_{B_R(y)} |D\boldsymbol{v}_\rho|^p \le CR^{(1-n)p+n}, \tag{3.25}$$

from which (3.24) follows.

If we utilize (3.21) instead of (3.23), we obtain a similar estimates for $\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)$

$$\|\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)\|_{L^p(B_R(y))} \le C_p R^{2-n+n/p}, \quad \forall\rho>0, \quad \forall p\in(0,\tfrac{n}{n-2}). \tag{3.26}$$

Let us fix $q\in(1,\frac{n}{n-1})$. We have seen that for all $R>0$, there exists some $C(R)<\infty$ such that

$$\|\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)\|_{W^{1,q}(B_R(y))} \le C(R), \quad \forall\rho>0.$$

Therefore, by a diagonalization process, we obtain a sequence $\{\rho_\mu\}_{\mu=1}^\infty$ and $\boldsymbol{\Gamma}(\,\cdot\,,y)$ in $W^{1,q}_{loc}(\mathbb{R}^n)^{N\times N}$ such that $\lim_{\mu\to\infty}\rho_\mu = 0$ and that

$$\boldsymbol{\Gamma}^{\rho_\mu}(\,\cdot\,,y) \rightharpoonup \boldsymbol{\Gamma}(\,\cdot\,,y) \text{ in } W^{1,q}(B_R(y))^{N\times N}, \quad \forall R>0, \tag{3.27}$$

where we recall that $\rightharpoonup$ denotes weak convergence. Then, for any $\boldsymbol{\phi}\in C_c^\infty(\mathbb{R}^n)^N$, it follows from (3.5)

$$\begin{aligned}\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta} D_\beta\Gamma_{jk}(\,\cdot\,,y)D_\alpha\phi^i &= \lim_{\mu\to\infty}\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta} D_\beta\Gamma_{jk}^{\rho_\mu}(\,\cdot\,,y)D_\alpha\phi^i \\ &= \lim_{\mu\to\infty} \fint_{B_{\rho_\mu}(y)} \phi^k = \phi^k(y).\end{aligned} \tag{3.28}$$

Let $\boldsymbol{v}_\rho^T$ be the k-th column of $\boldsymbol{\Gamma}^\rho(\,\cdot\,,y)$ as before, and let $\boldsymbol{v}^T$ be the corresponding k-th column of $\boldsymbol{\Gamma}(\,\cdot\,,y)$. Then, for any $\boldsymbol{g}\in L_c^\infty(B_R(y))^N$, (3.26) yields

$$\left|\int_{\mathbb{R}^n}\boldsymbol{v}\cdot\boldsymbol{g}\right| = \lim_{\mu\to\infty}\left|\int_{\mathbb{R}^n}\boldsymbol{v}_{\rho_\mu}\cdot\boldsymbol{g}\right| \le C_pR^{2-n+n/p}\,\|\boldsymbol{g}\|_{L^{p'}(B_R(y))}\,, \tag{3.29}$$

where p' denotes the conjugate exponent of $p\in[1,\frac{n}{n-2})$. Therefore, we obtain

$$\|\boldsymbol{\Gamma}(\,\cdot\,,y)\|_{L^p(B_R(y))} \le C_p\,R^{2-n+n/p}, \quad \forall p\in[1,\tfrac{n}{n-2}). \tag{3.30}$$

By a similar reasoning, we also have by (3.24)

$$\|D\boldsymbol{\Gamma}(\,\cdot\,,y)\|_{L^p(B_R(y))} \le C_p\,R^{1-n+n/p}, \quad \forall p\in[1,\tfrac{n}{n-1}). \tag{3.31}$$

Also, with the aid of (3.20) and (3.22), we obtain

$$\int_{\mathbb{R}^n\setminus B_R(y)} |\boldsymbol{\Gamma}(\,\cdot\,,y)|^{2^*} \le CR^{-n}, \tag{3.32}$$

$$\int_{\mathbb{R}^n\setminus B_R(y)} |D\boldsymbol{\Gamma}(\,\cdot\,,y)|^{2} \le CR^{2-n}. \tag{3.33}$$

In particular, (3.32), (3.33) imply that

$$\|\mathbf{\Gamma}(\,\cdot\,,y)\|_{Y^{1,2}(\mathbb{R}^n\setminus B_r(y))}\le Cr^{1-n/2},\quad \forall r>0. \tag{3.34}$$

Moreover, arguing as before, we see that the estimates (3.32) and (3.33) imply

$$\left|\{x\in\mathbb{R}^n:|\mathbf{\Gamma}(x,y)|>t\}\right|\le Ct^{-\frac{n}{n-2}},\quad \forall t>0 \tag{3.35}$$

$$\left|\{x\in\mathbb{R}^n:|D_x\mathbf{\Gamma}(x,y)|>t\}\right|\le Ct^{-\frac{n}{n-1}}\quad \forall t>0. \tag{3.36}$$

Next, we turn to pointwise bounds for $\mathbf{\Gamma}(\,\cdot\,,y)$. Let $\boldsymbol{v}^T$ be the k-th column of $\mathbf{\Gamma}(\,\cdot\,,y)$. For each $x\neq y$, denote $r=\frac{2}{3}|x-y|$. Then, it follows from (3.34) and (3.28) that $\boldsymbol{v}$ is a weak solution of $L\boldsymbol{v}=0$ in $B_r(x)$. Therefore, by Lemma 2.4 and (3.30) we find

$$|\boldsymbol{v}(x)|\le Cr^{-n}\,\|\boldsymbol{v}\|_{L^1(B_r(x))}\le Cr^{-n}\,\|\boldsymbol{v}\|_{L^1(B_{3r}(y))}\le Cr^{2-n}, \tag{3.37}$$

from which it follows

$$|\mathbf{\Gamma}(x,y)|\le C\,|x-y|^{2-n},\quad \forall x\neq y. \tag{3.38}$$

3.6. **Continuity of the fundamental matrix.** From the property (H), it follows that $\mathbf{\Gamma}(\,\cdot\,,y)$ is Hölder continuous in $\mathbb{R}^n\setminus\{y\}$. In fact, (2.11) together with (3.28) and (3.33) implies

$$|\mathbf{\Gamma}(x,y)-\mathbf{\Gamma}(z,y)|\le C\,|x-z|^{\mu_0}\,|x-y|^{2-n-\mu_0}\quad \text{if } |x-z|<|x-y|/2. \tag{3.39}$$

Moreover, by the same reasoning, it follows from (2.11) and (3.22) that for any given compact set $K\Subset\mathbb{R}^n\setminus\{y\}$, the sequence $\{\mathbf{\Gamma}^{\rho_\mu}(\,\cdot\,,y)\}_{\mu=1}^{\infty}$ is equicontinuous on K. Also, by Lemma 2.4 and (3.20), we find that there are $C_K<\infty$ and $\rho_K>0$ such that

$$\|\mathbf{\Gamma}^{\rho}(\,\cdot\,,y)\|_{L^\infty(K)}\le C_K\quad \forall\rho<\rho_K\quad \text{for any compact } K\Subset\mathbb{R}^n\setminus\{y\}\,. \tag{3.40}$$

Therefore, we may assume, by passing if necessary to a subsequence, that

$$\mathbf{\Gamma}^{\rho_\mu}(\,\cdot\,,y)\to\mathbf{\Gamma}(\,\cdot\,,y)\ \text{uniformly on } K, \text{ for any compact } K\Subset\mathbb{R}^n\setminus\{y\}\,. \tag{3.41}$$

We will now show that $\mathbf{\Gamma}(x,\,\cdot\,)$ is also Hölder continuous in $\mathbb{R}^n\setminus\{x\}$. Denote by ${}^t\mathbf{\Gamma}^{\sigma}(\,\cdot\,,x)$ the averaged fundamental matrix associated to ${}^tL$, the transpose of L. Since each column of $\mathbf{\Gamma}^{\rho}(\,\cdot\,,y)$ and ${}^t\mathbf{\Gamma}^{\sigma}(\,\cdot\,,x)$ belongs to $\mathbf{H}$, we have by (3.5),

$$\begin{aligned}\fint_{B_\rho(y)}{}^t\Gamma^{\sigma}_{kl}(\,\cdot\,,x)&=\int_{\mathbb{R}^n}A^{\alpha\beta}_{ij}D_\beta\Gamma^{\rho}_{jk}(\,\cdot\,,y)D_\alpha{}^t\Gamma^{\sigma}_{il}(\,\cdot\,,x)\\&=\int_{\mathbb{R}^n}{}^tA^{\beta\alpha}_{ji}D_\alpha{}^t\Gamma^{\sigma}_{il}(\,\cdot\,,x)D_\beta\Gamma^{\rho}_{jk}(\,\cdot\,,y)=\fint_{B_\sigma(x)}\Gamma^{\rho}_{lk}(\,\cdot\,,y).\end{aligned} \tag{3.42}$$

By the same argument as appears in Sec. 3.5, we obtain a sequence $\{\sigma_\nu\}_{\nu=1}^{\infty}$ tending to 0 such that ${}^t\mathbf{\Gamma}^{\sigma_\nu}(\,\cdot\,,x)$ converges to ${}^t\mathbf{\Gamma}(\,\cdot\,,x)$ uniformly on any compact subset of $\mathbb{R}^n\setminus\{x\}$, where ${}^t\mathbf{\Gamma}(\,\cdot\,,x)$ is a fundamental matrix for ${}^tL$ satisfying all properties stated in Sec. 3.5. By (3.42), we find that

$$g^{kl}_{\mu\nu}:=\fint_{B_{\rho_\mu}(y)}{}^t\Gamma^{\sigma_\nu}_{kl}(\,\cdot\,,x)=\fint_{B_{\sigma_\nu}(x)}\Gamma^{\rho_\mu}_{lk}(\,\cdot\,,y).$$

From the continuity of $\Gamma^{\rho_\mu}_{lk}(\,\cdot\,,y)$, it follows that for $x,y\in\mathbb{R}^n$ with $x\neq y$, we have

$$\lim_{\nu\to\infty}g^{kl}_{\mu\nu}=\lim_{\nu\to\infty}\fint_{B_{\sigma_\nu}(x)}\Gamma^{\rho_\mu}_{lk}(\,\cdot\,,y)=\Gamma^{\rho_\mu}_{lk}(x,y)$$

and thus by (3.41) we obtain

$$\lim_{\mu\to\infty}\lim_{\nu\to\infty} g_{\mu\nu}^{kl} = \lim_{\mu\to\infty}\Gamma_{lk}^{\rho_\mu}(x,y) = \Gamma_{lk}(x,y).$$

On the other hand, (3.27) yields

$$\lim_{\nu\to\infty} g_{\mu\nu}^{kl} = \lim_{\nu\to\infty}\fint_{B_{\rho_\mu}(y)} {}^t\Gamma_{kl}^{\sigma_\nu}(\,\cdot\,,x) = \fint_{B_{\rho_\mu}(y)} {}^t\Gamma_{kl}(\,\cdot\,,x)$$

and thus it follows from the continuity of ${}^t\Gamma_{kl}(\,\cdot\,,x)$ that

$$\lim_{\mu\to\infty}\lim_{\nu\to\infty} g_{\mu\nu}^{kl} = \lim_{\mu\to\infty}\fint_{B_{\rho_\mu}(y)} {}^t\Gamma_{kl}(\,\cdot\,,x) = {}^t\Gamma_{kl}(y,x).$$

We have thus shown that

$$\Gamma_{lk}(x,y) = {}^t\Gamma_{kl}(y,x), \quad \forall k,l=1,\ldots,N, \quad \forall x\neq y,$$

which is equivalent to say

$$\mathbf{\Gamma}(x,y) = {}^t\mathbf{\Gamma}(y,x)^T, \quad \forall x\neq y. \tag{3.43}$$

Therefore, we have proved the claim that $\mathbf{\Gamma}(x,\,\cdot\,)$ is Hölder continuous in $\mathbb{R}^n\setminus\{x\}$.

So far, we have seen that there is a sequence $\{\rho_\mu\}_{\mu=1}^\infty$ tending to 0 such that $\mathbf{\Gamma}^{\rho_\mu}(\,\cdot\,,y)\to\mathbf{\Gamma}(\,\cdot\,,y)$ in $\mathbb{R}^n\setminus\{y\}$. However, by (3.42), we obtain

$$\begin{aligned}\Gamma_{lk}^{\rho}(x,y) &= \lim_{\nu\to\infty}\fint_{B_{\sigma_\nu}(x)}\Gamma_{lk}^{\rho}(\,\cdot\,,y) = \lim_{\nu\to\infty}\fint_{B_\rho(y)} {}^t\Gamma_{kl}^{\sigma_\nu}(\,\cdot\,,x)\\ &= \fint_{B_\rho(y)} {}^t\Gamma_{kl}(\,\cdot\,,x) = \fint_{B_\rho(y)}\Gamma_{lk}(x,\,\cdot\,),\end{aligned} \tag{3.44}$$

i.e., we have the following representation for the averaged fundamental matrix:

$$\mathbf{\Gamma}^\rho(x,y) = \fint_{B_\rho(y)}\mathbf{\Gamma}(x,z)\,dz. \tag{3.45}$$

Therefore, by the continuity, we obtain

$$\lim_{\rho\to 0}\mathbf{\Gamma}^\rho(x,y) = \mathbf{\Gamma}(x,y), \quad x\neq y. \tag{3.46}$$

3.7. **Properties of fundamental matrix.** We record what we obtained so far in the following theorem:

Theorem 3.1. *Assume that operators L and ${}^tL$ satisfy the property (H). Then, there exists a unique fundamental matrix $\mathbf{\Gamma}(x,y)=(\Gamma_{ij}(x,y))_{i,j=1}^N$ ($x\neq y$) which is continuous in $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x\neq y\}$ and such that $\mathbf{\Gamma}(x,\,\cdot\,)$ is locally integrable in $\mathbb{R}^n$ for all $x\in\mathbb{R}^n$ and that for all $\boldsymbol{f}=(f^1,\ldots,f^N)^T\in C_c^\infty(\mathbb{R}^n)^N$, the function $\boldsymbol{u}=(u^1,\ldots,u^N)^T$ given by*

$$\boldsymbol{u}(x) := \int_{\mathbb{R}^n}\mathbf{\Gamma}(x,y)\boldsymbol{f}(y)\,dy \tag{3.47}$$

belongs to $Y^{1,2}(\mathbb{R}^n)^N$ and satisfies $L\boldsymbol{u}=\boldsymbol{f}$ in the sense

$$\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta}D_\beta u^j D_\alpha\phi^i = \int_{\mathbb{R}^n} f^i\phi^i, \quad \forall\boldsymbol{\phi}\in C_c^\infty(\mathbb{R}^n)^N. \tag{3.48}$$

Moreover, $\mathbf{\Gamma}(x,y)$ has the property

$$\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta}D_\beta\Gamma_{jk}(\,\cdot\,,y)D_\alpha\phi^i = \phi^k(y), \quad \forall\boldsymbol{\phi}\in C_c^\infty(\mathbb{R}^n)^N. \tag{3.49}$$

Furthermore, $\mathbf{\Gamma}(x,y)$ *satisfies the following estimates:*

$$\|\mathbf{\Gamma}(\,\cdot\,,y)\|_{Y^{1,2}(\mathbb{R}^n\setminus B_r(y))}+\|\mathbf{\Gamma}(x,\,\cdot\,)\|_{Y^{1,2}(\mathbb{R}^n\setminus B_r(x))}\le Cr^{1-\frac{n}{2}},\quad \forall r>0, \tag{3.50}$$

$$\|\mathbf{\Gamma}(\,\cdot\,,y)\|_{L^p(B_r(y))}+\|\mathbf{\Gamma}(x,\,\cdot\,)\|_{L^p(B_r(x))}\le C_p r^{2-n+\frac{n}{p}},\quad \forall p\in[1,\tfrac{n}{n-2}), \tag{3.51}$$

$$\|D\mathbf{\Gamma}(\,\cdot\,,y)\|_{L^p(B_r(y))}+\|D\mathbf{\Gamma}(x,\,\cdot\,)\|_{L^p(B_r(x))}\le C_p r^{1-n+\frac{n}{p}},\quad \forall p\in[1,\tfrac{n}{n-1}), \tag{3.52}$$

$$|\{x\in\mathbb{R}^n:|\mathbf{\Gamma}(x,y)|>t\}|+|\{y\in\mathbb{R}^n:|\mathbf{\Gamma}(x,y)|>t\}|\le Ct^{-\frac{n}{n-2}}, \tag{3.53}$$

$$|\{x\in\mathbb{R}^n:|D_x\mathbf{\Gamma}(x,y)|>t\}|+|\{y\in\mathbb{R}^n:|D_y\mathbf{\Gamma}(x,y)|>t\}|\le Ct^{-\frac{n}{n-1}}, \tag{3.54}$$

$$|\mathbf{\Gamma}(x,y)|\le C\,|x-y|^{2-n},\quad \forall x\ne y, \tag{3.55}$$

$$|\mathbf{\Gamma}(x,y)-\mathbf{\Gamma}(z,y)|\le C\,|x-z|^{\mu_0}\,|x-y|^{2-n-\mu_0}\quad \textit{if } |x-z|<|x-y|/2, \tag{3.56}$$

$$|\mathbf{\Gamma}(x,y)-\mathbf{\Gamma}(x,z)|\le C\,|y-z|^{\mu_0}\,|x-y|^{2-n-\mu_0}\quad \textit{if } |y-z|<|x-y|/2, \tag{3.57}$$

where $C=C(n,N,\lambda,\Lambda,\mu_0,H_0)>0$ *and* $C_p=C_p(n,N,\lambda,\Lambda,\mu_0,H_0,p)>0$.

Proof. Let $\mathbf{\Gamma}^\rho(x,y)$ and $\mathbf{\Gamma}(x,y)$ be constructed as above. We have already seen that $\mathbf{\Gamma}$ is continuous in $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x\ne y\}$ and satisfies all the properties (3.49) – (3.57). By using the Lax-Milgram lemma as in Sec. 3.1, we find that for all $\boldsymbol{f}\in C_c^\infty(\mathbb{R}^n)^N$, there is a unique $\boldsymbol{u}\in Y^{1,2}(\mathbb{R}^n)^N$ satisfying

$$\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta}D_\beta u^j D_\alpha v^i=\int_{\mathbb{R}^n} f^i v^i,\quad \forall \boldsymbol{v}\in Y^{1,2}(\mathbb{R}^n)^N.$$

If we set $v^i=\Gamma_{ki}^\rho(x,\,\cdot\,)$ above, then (3.5) together with (3.43) implies that

$$\int_{\mathbb{R}^n}\Gamma_{ki}^\rho(x,\,\cdot\,)f^i=\int_{\mathbb{R}^n}{}^tA_{ji}^{\beta\alpha}D_\alpha{}^t\Gamma_{ik}^\rho(\,\cdot\,,x)D_\beta u^j=\fint_{B_\rho(x)}u^k. \tag{3.58}$$

Assume that $\boldsymbol{f}$ is supported in $B_R(x)$ for some $R>0$. Then, by (3.27) and (3.43) we have

$$\lim_{\rho\to0}\int_{\mathbb{R}^n}\Gamma_{ki}^\rho(x,\,\cdot\,)f^i=\lim_{\rho\to0}\int_{B_R(x)}\Gamma_{ki}^\rho(x,\,\cdot\,)f^i=\int_{\mathbb{R}^n}\Gamma_{ki}(x,\,\cdot\,)f^i.$$

By the same argument which lead to (3.14) in Section 3.2, we find that $\boldsymbol{u}$ is Hölder continuous. Therefore, (3.47) follows by taking the limits in (3.58).

Now, it only remains to prove the uniqueness. Assume that $\tilde{\mathbf{\Gamma}}(x,y)$ is another matrix such that $\tilde{\mathbf{\Gamma}}$ is continuous on $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x\ne y\}$ and such that $\Gamma(x,\,\cdot\,)$ is locally integrable in $\mathbb{R}^n$ for all $x\in\mathbb{R}^n$ and that for all $\boldsymbol{f}\in C_c^\infty(\mathbb{R}^n)^N$,

$$\tilde{\boldsymbol{u}}(x):=\int_{\mathbb{R}^n}\tilde{\mathbf{\Gamma}}(x,y)\boldsymbol{f}(y)\,dy$$

belongs to $Y^{1,2}(\mathbb{R}^n)$ and satisfies $L\boldsymbol{u}=\boldsymbol{f}$ in the sense of (3.48). Then by the uniqueness in $\mathbf{H}=Y^{1,2}(\mathbb{R}^n)^N$, we must have $\boldsymbol{u}=\tilde{\boldsymbol{u}}$. Therefore, for all $x\in\mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n}(\mathbf{\Gamma}-\tilde{\mathbf{\Gamma}})(x,\cdot)\boldsymbol{f}=0,\quad \forall \boldsymbol{f}\in C_c^\infty(\mathbb{R}^n)^N,$$

and thus we have $\mathbf{\Gamma}\equiv\tilde{\mathbf{\Gamma}}$ in $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : x\ne y\}$. □

Theorem 3.2. *Assume that the operators* L *and* ${}^tL$ *satisfy the property (H). If* $\boldsymbol{f}\in(L^{\frac{2n}{n+2}}(\mathbb{R}^n)\cap L^p_{loc}(\mathbb{R}^n))^N$ *for some* $p>n/2$*, then there exists a unique* $\boldsymbol{u}$ *in*

$Y^{1,2}(\mathbb{R}^n)^N$ such that

$$\int_{\mathbb{R}^n} A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha v^i = \int_{\mathbb{R}^n} f^i v^i, \quad \forall \boldsymbol{v} \in Y^{1,2}(\mathbb{R}^n)^N. \tag{3.59}$$

Moreover, $\boldsymbol{u}$ is continuous and has the following representation:

$$u^k(x) = \int_{\mathbb{R}^n} \Gamma_{ki}(x,y) f^i(y)\, dy, \quad k = 1, \ldots, N, \tag{3.60}$$

where $(\Gamma_{ki}(x,y))_{k,i=1}^N$ is the fundamental matrix of L.

Proof. Since $\boldsymbol{f} \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)^N$, the same argument as appears in Sec. 3.1 implies that there is $\boldsymbol{u} \in Y^{1,2}(\mathbb{R}^n)^N$ satisfying (3.59). If we set $v^i = \Gamma_{ki}^\rho(x,\cdot)$ in (3.59), then (3.2) implies that

$$\int_{\mathbb{R}^n} \Gamma_{ki}^\rho(x,\cdot) f^i = \int_{\mathbb{R}^n} {}^t\!A_{ji}^{\beta\alpha} D_\alpha {}^t\Gamma_{ik}^\rho(\cdot,x) D_\beta u^j = \fint_{B_\rho(x)} u^k. \tag{3.61}$$

Next, note that (3.20), (3.26), and the assumption $f^i \in L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^p_{loc}(\mathbb{R}^n)$ for some $p > n/2$, imply that

$$\begin{aligned} \lim_{\rho\to 0} \int_{\mathbb{R}^n} \Gamma_{ki}^\rho(x,\cdot) f^i &= \lim_{\rho \to 0} \left(\int_{B_1(x)} \Gamma_{ki}^\rho(x,\cdot) f^i + \int_{\mathbb{R}^n \setminus B_1(x)} \Gamma_{ki}^\rho(x,\cdot) f^i \right) \\ &= \int_{B_1(x)} \Gamma_{ki}(x,\cdot) f^i + \int_{\mathbb{R}^n\setminus B_1(x)} \Gamma_{ki}(x,\cdot) f^i \\ &= \int_{\mathbb{R}^n} \Gamma_{ki}(x,\cdot) f^i. \end{aligned} \tag{3.62}$$

Finally, by the same argument which lead to (3.14) in Sec. 3.2, we find that $\boldsymbol{u}$ is Hölder continuous, and thus (3.60) follows from (3.61) and (3.62). □

Corollary 3.3. *Suppose that $\boldsymbol{f} = (f^1, \ldots, f^N)^T$ has a bound*

$$|\boldsymbol{f}(x)| \le C(1+|x|)^{-(1+n/2+\epsilon)} \quad \forall x \in \mathbb{R}^n \text{ for some } \epsilon > 0. \tag{3.63}$$

Then, $\boldsymbol{u} = (u^1, \ldots, u^N)^T$ given by (3.60) is a unique $Y^{1,2}(\mathbb{R}^n)^N$ solution of $L\boldsymbol{u} = \boldsymbol{f}$ in $\mathbb{R}^n$ in the sense of (3.59).

Proof. Note that (3.63) implies $\boldsymbol{f} \in (L^{\frac{2n}{n+2}}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n))^N$. □

Theorem 3.4. *Assume that L and ${}^tL$ satisfy the property (H). If $\boldsymbol{f} \in Y^{1,2}(\mathbb{R}^n)^N$ satisfies $D\boldsymbol{f} \in L^p_{loc}(\mathbb{R}^n)^{N\times n}$ for some $p > n$, then*

$$f^k(x) = \int_{\mathbb{R}^n} D_\alpha \Gamma_{ki}(x,\cdot) A_{ij}^{\alpha\beta} D_\beta f^j, \quad k = 1, \ldots, N, \tag{3.64}$$

where $(\Gamma_{ki}(x,y))_{k,i=1}^N$ is the fundamental matrix of L.

Proof. We denote by ${}^t\mathbf{\Gamma}^\rho$ the averaged fundamental matrix of ${}^tL$. Recall that columns of ${}^t\mathbf{\Gamma}^\rho$ belong to $\mathbf{H}$. Then, by (3.5) we have

$$\int_{\mathbb{R}^n} {}^t\!A_{ji}^{\beta\alpha} D_\alpha {}^t\Gamma_{ik}^\rho(\cdot,x) D_\beta f^j = \fint_{B_\rho(x)} f^k.$$

As in (3.62), the assumption $D\boldsymbol{f} \in L^p_{loc}(\mathbb{R}^n)^N$ for $p > n$, together with (3.22) and (3.24) yields

$$\begin{aligned}\lim_{\rho\to 0} \fint_{B_\rho(x)} f^k &= \lim_{\rho\to 0}\left(\int_{B_1(x)} + \int_{\mathbb{R}^n\setminus B_1(x)} {}^t\!A^{\beta\alpha}_{ji} D_\alpha {}^t\Gamma^\rho_{ik}(\,\cdot\,,x) D_\beta f^j\right)\\ &= \int_{B_1(x)} + \int_{\mathbb{R}^n\setminus B_1(x)} {}^t\!A^{\beta\alpha}_{ji} D_\alpha {}^t\Gamma_{ik}(\,\cdot\,,x) D_\beta f^j\\ &= \int_{\mathbb{R}^n} A^{\alpha\beta}_{ij} D_\alpha \Gamma_{ki}(x,\,\cdot\,) D_\beta f^j.\end{aligned} \tag{3.65}$$

where we used (3.43) in the last step. By the Morrey's inequality [17], $\boldsymbol{f}$ is continuous and thus (3.64) follows from (3.65). □

Corollary 3.5. *Assume that L, ${}^tL$, $\tilde L$, and ${}^t\tilde L$ satisfy the property (H). Denote by $\boldsymbol{\Gamma}$ and $\tilde{\boldsymbol{\Gamma}}$ the fundamental matrices of L and $\tilde L$, respectively. If the coefficients $A^{\alpha\beta}_{ij}$ of L and $\tilde A^{\alpha\beta}_{ij}$ of $\tilde L$ are Hölder continuous, then*

$$\tilde\Gamma_{lm}(x,y) = \Gamma_{lm}(x,y) + \int_{\mathbb{R}^n} D_\alpha \Gamma_{li}(x,\,\cdot\,)(A^{\alpha\beta}_{ij} - \tilde A^{\alpha\beta}_{ij}) D_\beta \tilde\Gamma_{jm}(\,\cdot\,,y), \quad x\neq y. \tag{3.66}$$

Proof. We denote by $\boldsymbol{\Gamma}^\rho$ and $\tilde{\boldsymbol{\Gamma}}^\rho$ ($\rho < |x-y|/4$) the averaged fundamental matrices of L and $\tilde L$ respectively. Recall that columns of $\boldsymbol{\Gamma}^\rho$ and $\tilde{\boldsymbol{\Gamma}}^\rho$ belong to $\mathbf{H}$. Moreover, since we assume that the coefficients are Hölder continuous, the standard elliptic theory, (3.38), and (3.45) implies that $D\boldsymbol{\Gamma}^\rho(x,\,\cdot\,)$ and $D\tilde{\boldsymbol{\Gamma}}^\rho(\,\cdot\,,y)$ are locally bounded. Therefore, by setting $f^j = \tilde\Gamma^\rho_{jm}(\,\cdot\,,y)$ in (3.64) we have

$$\tilde\Gamma^\rho_{lm}(x,y) = \int_{\mathbb{R}^n} D_\alpha \Gamma_{li}(x,\,\cdot\,) A^{\alpha\beta}_{ij} D_\beta \tilde\Gamma^\rho_{jm}(\,\cdot\,,y), \tag{3.67}$$

Next, set $f^j = \Gamma^\rho_{lj}(x,\,\cdot\,)$ and apply (3.64) with L replaced by ${}^t\tilde L$ to get

$$\Gamma^\rho_{lm}(x,y) = \int_{\mathbb{R}^n} D_\alpha {}^t\tilde\Gamma_{mi}(y,\,\cdot\,) {}^t\!\tilde A^{\alpha\beta}_{ij} D_\beta \Gamma^\rho_{lj}(x,\,\cdot\,).$$

By using (3.43) and interchanging indices ($\alpha\leftrightarrow\beta$, $i\leftrightarrow j$), we obtain

$$\Gamma^\rho_{lm}(x,y) = \int_{\mathbb{R}^n} D_\alpha \Gamma^\rho_{li}(x,\,\cdot\,) \tilde A^{\alpha\beta}_{ij} D_\beta \tilde\Gamma_{jm}(\,\cdot\,,y). \tag{3.68}$$

Now, set $r = |x-y|/4$ and split the integral (3.67) into three pieces (recall $\rho < r$)

$$\int_{B_r(x)} + \int_{B_r(y)} + \int_{\mathbb{R}^n\setminus(B_r(x)\cup B_r(x))} D_\alpha \Gamma_{li}(x,\,\cdot\,) A^{\alpha\beta}_{ij} D_\beta \tilde\Gamma^\rho_{jm}(\,\cdot\,,y).$$

Since we assume that the coefficients are Hölder continuous, it follows from the standard elliptic theory that $D\boldsymbol{\Gamma}(x,\,\cdot\,)$ and $D\tilde{\boldsymbol{\Gamma}}(\,\cdot\,,y)$ are continuous (and thus bounded) on $B_r(y)$ and $B_r(x)$ respectively. Moreover, (3.45) implies

$$D\tilde{\boldsymbol{\Gamma}}^\rho(\,\cdot\,,y) \to D\tilde{\boldsymbol{\Gamma}}(\,\cdot\,,y) \quad \text{uniformly on } B_r(x) \text{ as } \rho\to 0.$$

Therefore, as in (3.65), we may take the limit $\rho\to 0$ in (3.67) to get

$$\tilde\Gamma_{lm}(x,y) = \int_{\mathbb{R}^n} D_\alpha \Gamma_{li}(x,\,\cdot\,) A^{\alpha\beta}_{ij} D_\beta \tilde\Gamma_{jm}(\,\cdot\,,y),$$

Similarly, by taking the limit $\rho \to 0$ in (3.68), we obtain

$$\Gamma_{lm}(x,y) = \int_{\mathbb{R}^n} D_\alpha \Gamma_{li}(x,\,\cdot\,)\tilde{A}^{\alpha\beta}_{ij} D_\beta \tilde{\Gamma}_{jm}(\,\cdot\,,y).$$

The proof is complete. □

Remark 3.6. We note that in terms of matrix multiplication (3.60) is written as

$$\boldsymbol{u}(x) = \int_{\mathbb{R}^n} \boldsymbol{\Gamma}(x,y)\boldsymbol{f}(y)\,dy,$$

where both $\boldsymbol{u}, \boldsymbol{f}$ are understood as column vectors. Also, (3.66) reads

$$\tilde{\boldsymbol{\Gamma}}(x,y) = \boldsymbol{\Gamma}(x,y) + \int_{\mathbb{R}^n} D_\alpha \boldsymbol{\Gamma}(x,\,\cdot\,)(\boldsymbol{A}^{\alpha\beta} - \tilde{\boldsymbol{A}}^{\alpha\beta}) D_\beta \boldsymbol{\Gamma}(\,\cdot\,,y).$$

4. Green's matrix in general domains

4.1. **Construction of Green's matrix.** In this section, we shall construct the Green's matrix in any open, connected set $\Omega \subset \mathbb{R}^n$, where $n \geq 3$. To construct the Green's matrix in Ω, we need to adjust arguments in Section 3.

Henceforth, we shall denote $\Omega_r(y) := \Omega \cap B_r(y)$ and $d_y := \operatorname{dist}(y, \partial\Omega)$. Also, as in Section 3, we use the letter C to denote a constant depending on n, N, λ, Λ, μ_0, H_0, and sometimes on an exponent p characterizing Lebesgue classes.

It is routine to check that for any given $y \in \Omega$ and $1 \leq k \leq N$, the linear functional $\boldsymbol{u} \mapsto \fint_{\Omega_\rho(y)} u^k$ is bounded on $\mathbf{H} = Y^{1,2}_0(\Omega)^N$. Therefore, by Lax-Milgram lemma, there exists a unique $\boldsymbol{v}_\rho = \boldsymbol{v}_{\rho;y,k} \in \mathbf{H}$ such that

$$\int_\Omega A^{\alpha\beta}_{ij} D_\beta v^j_\rho\, D_\alpha u^i = \fint_{\Omega_\rho(y)} u^k, \quad \forall \boldsymbol{u} \in \mathbf{H}. \tag{4.1}$$

Note that as in (3.3), we have

$$\|D\boldsymbol{v}_\rho\|_{L^2(\Omega)} = \|\boldsymbol{v}_\rho\|_{\mathbf{H}} \leq C\,|\Omega_\rho(y)|^{\frac{2-n}{2n}}. \tag{4.2}$$

We define the "averaged Green's matrix" $\boldsymbol{G}^\rho(\,\cdot\,,y) = (G^\rho_{jk}(\,\cdot\,,y))^N_{j,k=1}$ by

$$G^\rho_{jk}(\,\cdot\,,y) = v^j_\rho = v^j_{\rho;y,k}.$$

Note that as in (3.5), we have

$$\int_\Omega A^{\alpha\beta}_{ij} D_\beta G^\rho_{jk}(\,\cdot\,,y) D_\alpha u^i = \fint_{\Omega_\rho(y)} u^k, \quad \forall \boldsymbol{u} \in \mathbf{H}. \tag{4.3}$$

Next, observe that as in (3.7)–(3.10), for any given $\boldsymbol{f} \in L^\infty_c(\Omega)^N$, there exists a unique $\boldsymbol{u} \in \mathbf{H}$ such that

$$\int_\Omega G^\rho_{ik}(\,\cdot\,,y) f^i = \fint_{\Omega_\rho(y)} u^k.$$

Moreover, as in (3.11), we have

$$\|D\boldsymbol{u}\|_{L^2(\Omega)} \leq C\,\|\boldsymbol{f}\|_{L^{2n/(n+2)}(\Omega)}.$$

Also, by following the argument as appears in Section 3.2, we find that if $\boldsymbol{f}$ is supported in $B_R(y)$, then we have

$$\|\boldsymbol{u}\|_{L^\infty(B_{R/4}(y))} \leq CR^{2-n/p}\,\|\boldsymbol{f}\|_{L^p(B_R(y))}, \quad \forall R < d_y, \quad \forall p > n/2.$$

Therefore, as in (3.16), for any $\boldsymbol{f} \in L^\infty_c(B_R(y))$, $R < d_y$, we have

$$\left| \int_{B_R(y)} G^\rho_{ik}(\,\cdot\,,y) f^i \right| \leq C R^{2-n/p} \|\boldsymbol{f}\|_{L^p(B_R(y))}, \quad \forall \rho < R/4, \quad \forall p > n/2.$$

Therefore, as in (3.17), we see that if $R < d_y$, then

$$\|\boldsymbol{G}^\rho(\,\cdot\,,y)\|_{L^q(B_R(y))} \leq C R^{2-n+n/q}, \quad \forall \rho < R/4, \quad \forall q \in [1, \tfrac{n}{n-2}).$$

Then, by following the lines in (3.18)–(3.19), we obtain

$$|\boldsymbol{G}^\rho(x,y)| \leq C\,|x-y|^{2-n} \quad \text{if } |x-y| < d_y/2, \quad \forall \rho < |x-y|/3.$$

Next, we shall derive an estimate corresponding to (3.22). Let $\eta \in C^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ outside $B_{R/2}(y)$, $\eta \equiv 0$ on $B_{R/4}(y)$, and $|D\eta| \leq C/R$, where $R \leq d_y$. By setting $\boldsymbol{u} = \eta^2 \boldsymbol{v}_\rho \in \mathbf{H}$ in (4.1), we obtain

$$\begin{aligned} \int_\Omega \eta^2 |D\boldsymbol{v}_\rho|^2 &\leq C \int_\Omega |D\eta|^2 |\boldsymbol{v}_\rho|^2 \leq C R^{-2} \int_{B_{R/2}(y)\setminus B_{R/4}(y)} |\boldsymbol{v}_\rho|^2 \\ &\leq C R^{-2} \int_{B_{R/2}(y)\setminus B_{R/4}(y)} |x-y|^{2(2-n)}\,dx \\ &= C R^{-2} R^{4-n} = C R^{2-n}, \quad \forall \rho < R/12. \end{aligned} \tag{4.4}$$

Therefore, we have ($r = R/2$)

$$\int_{\Omega\setminus B_r(y)} |D\boldsymbol{G}^\rho(\,\cdot\,,y)|^2 \leq C r^{2-n}, \quad \forall \rho < r/6, \quad \forall r < d_y/2. \tag{4.5}$$

On the other hand, (4.2) implies that if $\rho \geq r/6$, then

$$\int_{\Omega\setminus B_r(y)} |D\boldsymbol{G}^\rho(\,\cdot\,,y)|^2 \leq \int_\Omega |D\boldsymbol{G}^\rho(\,\cdot\,,y)|^2 \leq C\,|\Omega_\rho(y)|^{\frac{2-n}{n}} \leq C r^{2-n}. \tag{4.6}$$

Therefore, by combining (4.5) and (4.6), we obtain

$$\int_{\Omega\setminus B_r(y)} |D\boldsymbol{G}^\rho(\,\cdot\,,y)|^2 \leq C r^{2-n}, \quad \forall r < d_y/2, \quad \forall \rho > 0. \tag{4.7}$$

From the estimate (4.7), which corresponds to (3.22), we can derive an estimate corresponding to (3.24) as follows. By following the lines between (3.22) and (3.23), we obtain

$$|\{x \in \Omega : |D_x \boldsymbol{G}^\rho(x,y)| > t\}| \leq C t^{-\frac{n}{n-1}}, \quad \forall \rho > 0 \quad \text{if } t > (d_y/2)^{1-n}. \tag{4.8}$$

Then, by following lines (3.24)–(3.25), we find (set $\tau = (R/2)^{1-n}$)

$$\int_{B_R(y)} |D\boldsymbol{G}^\rho(\,\cdot\,,y)|^p \leq C R^{p(1-n)+n}, \quad \forall R < d_y, \quad \forall \rho > 0, \quad \forall p \in (0, \tfrac{n}{n-1}). \tag{4.9}$$

Now, we will derive estimates corresponding (3.20) and (3.26). Let η be the same as in (4.4). Note that (4.4) and (4.7) implies that for $R < d_y$,

$$\int_\Omega |D(\eta \boldsymbol{v}_\rho)|^2 \leq 2 \int_\Omega \eta^2 |D\boldsymbol{v}_\rho|^2 + 2 \int_\Omega |D\eta|^2 |\boldsymbol{v}_\rho|^2 \leq C R^{2-n}, \quad \forall \rho < R/12. \tag{4.10}$$

Since $\eta \boldsymbol{v}_\rho \in \mathbf{H} = Y^{1,2}_0(\Omega)$, it follows from (4.10) and (2.13) that

$$\int_{\Omega\setminus B_r(y)} |\boldsymbol{v}_\rho|^{2^*} \leq C r^{-n}, \quad \forall r < d_y/2, \quad \forall \rho < r/6. \tag{4.11}$$

On the other hand, if $\rho \geq r/6$, then (4.2) implies

$$\int_{\Omega\setminus B_r(y)} |\boldsymbol{v}_\rho|^{\,2^*} \leq \int_\Omega |\boldsymbol{v}_\rho|^{\,2^*} \leq C\left(\int_\Omega |D\boldsymbol{v}_\rho|^{\,2}\right)^{2^*/2} \leq C\,|\Omega_\rho|^{-1} \leq Cr^{-n}. \tag{4.12}$$

Therefore, by combining (4.11) and (4.12), we obtain

$$\int_{\Omega\setminus B_r(y)} |\boldsymbol{G}^\rho(\,\cdot\,,y)|^{\,2^*} \leq Cr^{-n}, \quad \forall r < d_y/2, \quad \forall \rho > 0. \tag{4.13}$$

As in Section 3.3, the above estimate (4.13) yields

$$|\{x\in\Omega : |\boldsymbol{G}^\rho(x,y)| > t\}| \leq Ct^{-\frac{n}{n-2}}, \quad \forall \rho > 0 \quad \text{if } t > (d_y/2)^{2-n}. \tag{4.14}$$

Then, as we argued in (4.9), we find (set $\tau = (R/2)^{2-n}$)

$$\int_{B_R(y)} |\boldsymbol{G}^\rho(\,\cdot\,,y)|^p \leq CR^{p(2-n)+n}, \quad \forall R < d_y, \quad \forall \rho > 0, \quad \forall p \in (0, \tfrac{n}{n-2}). \tag{4.15}$$

Now, observe that (4.9) and (4.15) in particular imply that

$$\|\boldsymbol{G}^\rho(\,\cdot\,,y)\|_{W^{1,p}(B_{d_y}(y))} \leq C(d_y) \text{ for some } p \in (1, \tfrac{n}{n-1}), \text{ uniformly in } \rho. \tag{4.16}$$

Therefore, from (4.16) together with (4.7) and (4.13), it follows that there exist a sequence $\{\rho_\mu\}_{\mu=1}^\infty$ tending to 0 and functions $\boldsymbol{G}(\,\cdot\,,y)$ and $\tilde{\boldsymbol{G}}(\,\cdot\,,y)$ such that

$$\boldsymbol{G}^{\rho_\mu}(\,\cdot\,,y) \rightharpoonup \boldsymbol{G}(\,\cdot\,,y) \quad \text{in} \quad W^{1,p}(B_{d_y}(y))^{N\times N} \quad \text{and} \tag{4.17}$$

$$\boldsymbol{G}^{\rho_\mu}(\,\cdot\,,y) \rightharpoonup \tilde{\boldsymbol{G}}(\,\cdot\,,y) \quad \text{in} \quad Y^{1,2}(\Omega\setminus B_{d_y/2}(y))^{N\times N} \text{ as } \mu\to\infty. \tag{4.18}$$

Since $\boldsymbol{G}(\,\cdot\,,y) \equiv \tilde{\boldsymbol{G}}(\,\cdot\,,y)$ on $B_{d_y}(y)\setminus B_{d_y/2}(y)$, we shall extend $\boldsymbol{G}(\,\cdot\,,y)$ to entire Ω by setting $\boldsymbol{G}(\,\cdot\,,y) = \tilde{\boldsymbol{G}}(\,\cdot\,,y)$ on $\Omega\setminus B_{d_y}(y)$ but still call it $\boldsymbol{G}(\,\cdot\,,y)$ in the sequel. Moreover, by applying a diagonalization process and passing to a subsequence, if necessary, we may assume that

$$\boldsymbol{G}^{\rho_\mu}(\,\cdot\,,y) \rightharpoonup \boldsymbol{G}(\,\cdot\,,y) \quad \text{in } Y^{1,2}(\Omega\setminus B_r(y))^{N\times N} \text{ as } \mu\to\infty, \quad \forall r < d_y. \tag{4.19}$$

We claim that the following holds:

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta G_{jk}(\,\cdot\,,y) D_\alpha \phi^i = \phi^k(y), \quad \forall \boldsymbol{\phi} \in C_c^\infty(\Omega)^N. \tag{4.20}$$

To see (4.20), write $\boldsymbol{\phi} = \eta\boldsymbol{\phi} + (1-\eta)\boldsymbol{\phi}$, where $\eta \in C_c^\infty(B_{d_y}(y))$ is a cut-off function satisfying $\eta \equiv 1$ on $B_{d_y/2}(y)$. Then, (4.3), (4.17), and (4.19) yield

$$\begin{aligned}
\phi^k(y) &= \lim_{\mu\to\infty} \fint_{\Omega_{\rho_\mu}(y)} \eta\phi^k + \lim_{\mu\to\infty} \fint_{\Omega_{\rho_\mu}(y)} (1-\eta)\phi^k \\
&= \lim_{\mu\to\infty} \int_\Omega A_{ij}^{\alpha\beta} D_\beta G_{jk}^{\rho_\mu}(\cdot,y) D_\alpha(\eta\phi^i) + \lim_{\mu\to\infty} \int_\Omega A_{ij}^{\alpha\beta} D_\beta G_{jk}^{\rho_\mu}(\cdot,y) D_\alpha((1-\eta)\phi^i) \\
&= \int_\Omega A_{ij}^{\alpha\beta} D_\beta G_{jk}(\,\cdot\,,y) D_\alpha(\eta\phi^i) + \int_\Omega A_{ij}^{\alpha\beta} D_\beta G_{jk}(\,\cdot\,,y) D_\alpha((1-\eta)\phi^i) \\
&= \int_\Omega A_{ij}^{\alpha\beta} D_\beta G_{jk}(\,\cdot\,,y) D_\alpha \phi^i \quad \text{as desired.}
\end{aligned}$$

Next, we claim that $\boldsymbol{G}(\,\cdot\,,y) = 0$ on $\partial\Omega$ in the sense that for all $\eta \in C_c^\infty(\Omega)$ satisfying $\eta \equiv 1$ on $B_r(y)$ for some $r < d_y$, we have

$$(1-\eta)\boldsymbol{G}(\,\cdot\,,y) \in Y_0^{1,2}(\Omega)^{N\times N}.$$

To see this, it is enough to show that

$$(1-\eta)\boldsymbol{G}^{\rho_\mu}(\,\cdot\,,y) \rightharpoonup (1-\eta)\boldsymbol{G}(\,\cdot\,,y) \quad \text{in } Y^{1,2}(\Omega)^{N\times N} \text{ as } \mu\to\infty, \tag{4.21}$$

for $(1-\eta)\boldsymbol{G}^{\rho_\mu}(\,\cdot\,,y) \in Y^{1,2}_0(\Omega)^{N\times N}$ for all $\mu \ge 1$ and $Y^{1,2}_0(\Omega)$ is weakly closed in $Y^{1,2}(\Omega)$ by Mazur's theorem. To show (4.21), we note that (4.19) yields

$$\begin{aligned}\int_\Omega (1-\eta)G_{kl}(\,\cdot\,,y)\phi &= \int_\Omega G_{kl}(\,\cdot\,,y)(1-\eta)\phi = \lim_{\mu\to\infty}\int_\Omega G^{\rho_\mu}_{kl}(\,\cdot\,,y)(1-\eta)\phi\\ &= \lim_{\mu\to\infty}\int_\Omega (1-\eta)G^{\rho_\mu}_{kl}(\,\cdot\,,y)\phi, \quad \forall \phi \in L^{\frac{2n}{n+2}}(\Omega),\end{aligned}$$

$$\begin{aligned}\int_\Omega D((1-\eta)G_{kl}(\,\cdot\,,y))\cdot\boldsymbol{\psi} &= -\int_\Omega G_{kl}(\,\cdot\,,y)D\eta\cdot\boldsymbol{\psi} + \int_\Omega DG_{kl}(\,\cdot\,,y)\cdot(1-\eta)\boldsymbol{\psi}\\ &= -\lim_{\mu\to\infty}\int_\Omega G^{\rho_\mu}_{kl}(\,\cdot\,,y)D\eta\cdot\boldsymbol{\psi} + \lim_{\mu\to\infty}\int_\Omega DG^{\rho_\mu}_{kl}(\,\cdot\,,y)\cdot(1-\eta)\boldsymbol{\psi}\\ &= \lim_{\mu\to\infty}\int_\Omega D((1-\eta)G^{\rho_\mu}_{kl}(\,\cdot\,,y))\cdot\boldsymbol{\psi}, \quad \forall\boldsymbol{\psi}\in L^2(\Omega)^N.\end{aligned}$$

By using the same duality argument as in (3.29), we derive the following estimates that correspond to (3.30)–(3.36):

$$\|\boldsymbol{G}(\,\cdot\,,y)\|_{L^p(B_r(y))} \le C_p\, r^{2-n+n/p}, \quad \forall r<d_y, \quad \forall p\in[1,\tfrac{n}{n-2}), \tag{4.22}$$

$$\|D\boldsymbol{G}(\,\cdot\,,y)\|_{L^p(B_r(y))} \le C_p\, r^{1-n+n/p}, \quad \forall r<d_y, \quad \forall p\in[1,\tfrac{n}{n-1}), \tag{4.23}$$

$$\|\boldsymbol{G}(\,\cdot\,,y)\|_{Y^{1,2}(\Omega\setminus B_r(y))} \le Cr^{1-n/2}, \quad \forall r<d_y/2, \tag{4.24}$$

$$|\{x\in\Omega : |\boldsymbol{G}(x,y)|>t\}| \le Ct^{-\frac{n}{n-2}}, \quad \forall t>(d_y/2)^{2-n}, \tag{4.25}$$

$$|\{x\in\Omega : |D_x\boldsymbol{G}(x,y)|>t\}| \le Ct^{-\frac{n}{n-1}}, \quad \forall t>(d_y/2)^{1-n}. \tag{4.26}$$

Also, we obtain pointwise bound and Hölder continuity estimate for $\boldsymbol{G}(\,\cdot\,,y)$ corresponding to (3.38) and (3.39), respectively, as follows. Denote by $\boldsymbol{v}^T$ the k-th column of $\boldsymbol{G}(\,\cdot\,,y)$ and set $R := \bar{d}_{x,y}/2$, where

$$\bar{d}_{x,y} := \min(d_x, d_y, |x-y|). \tag{4.27}$$

Since $\boldsymbol{v}$ is a weak solution of $L\boldsymbol{u}=0$ in $B_{3R/2}(x)\subset\Omega\setminus B_{R/2}(y)$, it follows from (2.10) and (4.24) that

$$|\boldsymbol{v}(x)| \le CR^{(2-n)/2}\,\|\boldsymbol{v}\|_{L^{2^*}(\Omega\setminus B_{R/2}(y))} \le CR^{2-n},$$

which in turn implies that

$$|\boldsymbol{G}(x,y)| \le C\bar{d}^{\,2-n}_{x,y}, \quad \text{where } \bar{d}_{x,y} := \min(d_x,d_y,|x-y|). \tag{4.28}$$

In particular, we have

$$|\boldsymbol{G}(x,y)| \le C\,|x-y|^{2-n} \quad \text{if } |x-y|<d_x/2 \text{ or } |x-y|<d_y/2. \tag{4.29}$$

Similarly, it follows from (2.11) and (4.24) that

$$[\boldsymbol{v}]^2_{C^{\mu_0}(B_R(x))} \le CR^{2-n-2\mu_0}\int_{B_{3R/2}(x)}|D\boldsymbol{v}|^2 \le CR^{2(2-n-\mu_0)}. \tag{4.30}$$

Therefore, we find that

$$|\boldsymbol{G}(x,y)-\boldsymbol{G}(z,y)| \le C\,|x-z|^{\mu_0}\,\bar{d}^{\,2-n-\mu_0}_{x,y} \quad \text{if } |x-z|<\bar{d}_{x,y}/2, \tag{4.31}$$

where $\bar{d}_{x,y}$ is given by (4.27).

Denote by ${}^t\boldsymbol{G}^\sigma(\,\cdot\,,x)$ the averaged Green's matrix of ${}^tL$ in Ω with a pole at $x\in\Omega$. Observe that we have an identity corresponding to (3.42).

$$\fint_{\Omega_\rho(y)} {}^tG^\sigma_{kl}(\,\cdot\,,x) = \fint_{\Omega_\sigma(x)} G^\rho_{lk}(\,\cdot\,,y). \tag{4.32}$$

Let ${}^t\boldsymbol{G}(\,\cdot\,,x)$ be a Green's matrix of ${}^tL$ in Ω with a pole at $x\in\Omega$ that is obtained by a sequence $\{\sigma_\nu\}_{\nu=1}^\infty$ tending to 0. Then, by a similar argument as appears in Section 3.6, we obtain

$$G_{lk}(x,y) = {}^tG_{kl}(y,x),\quad \forall k,l=1,\ldots,N,\quad \forall x,y\in\Omega,\quad x\neq y, \tag{4.33}$$

which is equivalent to say

$$\boldsymbol{G}(x,y) = {}^t\boldsymbol{G}(y,x)^T,\quad \forall x,y\in\Omega,\quad x\neq y. \tag{4.34}$$

Using (4.34), we find that $\boldsymbol{G}(x,\,\cdot\,)$ satisfies the estimates corresponding to (4.22)–(4.26) and (4.31). Moreover, by following the lines (3.44)–(3.45) and using (4.32) we obtain

$$\boldsymbol{G}^\rho(x,y) = \fint_{\Omega_\rho(y)} \boldsymbol{G}(x,z)\,dz. \tag{4.35}$$

Therefore, by the continuity, we find

$$\lim_{\rho\to 0}\boldsymbol{G}^\rho(x,y) = \boldsymbol{G}(x,y),\quad \forall x,y\in\Omega,\quad x\neq y. \tag{4.36}$$

Finally, we summarize what we obtained so far in the following theorem.

Theorem 4.1. *Let Ω be an open connected set in $\mathbb{R}^n$. Denote $d_x := \operatorname{dist}(x,\partial\Omega)$ for $x\in\Omega$; we set $d_x=\infty$ if $\Omega=\mathbb{R}^n$. Assume that operators L and ${}^tL$ satisfy the property (H). Then, there exists a unique Green's matrix $\boldsymbol{G}(x,y) = (G_{ij}(x,y))_{i,j=1}^N$ ($x,y\in\Omega, x\neq y$) which is continuous in $\{(x,y)\in\Omega\times\Omega : x\neq y\}$ and such that $\boldsymbol{G}(x,\,\cdot\,)$ is locally integrable in Ω for all $x\in\Omega$ and that for all $\boldsymbol{f}=(f^1,\ldots,f^N)^T\in C^\infty_c(\Omega)^N$, the function $\boldsymbol{u}=(u^1,\ldots,u^N)^T$ given by*

$$\boldsymbol{u}(x) := \int_\Omega \boldsymbol{G}(x,y)\boldsymbol{f}(y)\,dy \tag{4.37}$$

belongs to $Y^{1,2}_0(\Omega)^N$ and satisfies $L\boldsymbol{u}=\boldsymbol{f}$ in the sense

$$\int_\Omega A^{\alpha\beta}_{ij}D_\beta u^j D_\alpha\phi^i = \int_\Omega f^i\phi^i,\quad \forall\boldsymbol{\phi}\in C^\infty_c(\Omega)^N. \tag{4.38}$$

Moreover, $\boldsymbol{G}(x,y)$ has the properties that

$$\int_\Omega A^{\alpha\beta}_{ij}D_\beta G_{jk}(\,\cdot\,,y)D_\alpha\phi^i = \phi^k(y),\quad \forall\boldsymbol{\phi}\in C^\infty_c(\Omega)^N \tag{4.39}$$

and that for all $\eta\in C^\infty_c(\Omega)$ satisfying $\eta\equiv 1$ on $B_r(y)$ for some $r<d_y$,

$$(1-\eta)\boldsymbol{G}(\,\cdot\,,y)\in Y^{1,2}_0(\Omega)^{N\times N}. \tag{4.40}$$

Furthermore, $\boldsymbol{G}(x,y)$ satisfies the following estimates:

$$\|\boldsymbol{G}(\,\cdot\,,y)\|_{L^p(B_r(y))}\le C_p\,r^{2-n+n/p},\quad \forall r<d_y,\quad \forall p\in[1,\tfrac{n}{n-2}), \tag{4.41}$$

$$\|\boldsymbol{G}(x,\,\cdot\,)\|_{L^p(B_r(x))}\le C_p\,r^{2-n+n/p},\quad \forall r<d_x,\quad \forall p\in[1,\tfrac{n}{n-2}), \tag{4.42}$$

$$\|D\boldsymbol{G}(\,\cdot\,,y)\|_{L^p(B_r(y))}\le C_p\,r^{1-n+n/p},\quad \forall r<d_y,\quad \forall p\in[1,\tfrac{n}{n-1}), \tag{4.43}$$

$$\|D\boldsymbol{G}(x,\,\cdot\,)\|_{L^p(B_r(x))}\le C_p\,r^{1-n+n/p},\quad \forall r<d_x,\quad \forall p\in[1,\tfrac{n}{n-1}), \tag{4.44}$$

$$\|\boldsymbol{G}(\,\cdot\,,y)\|_{Y^{1,2}(\Omega\setminus B_r(y))} \le Cr^{1-n/2}, \quad \forall r < d_y/2, \tag{4.45}$$

$$\|\boldsymbol{G}(x,\,\cdot\,)\|_{Y^{1,2}(\Omega\setminus B_r(x))} \le Cr^{1-n/2}, \quad \forall r < d_x/2, \tag{4.46}$$

$$|\{x\in\Omega : |\boldsymbol{G}(x,y)| > t\}| \le Ct^{-\frac{n}{n-2}}, \quad \forall t > (d_y/2)^{2-n}, \tag{4.47}$$

$$|\{y\in\Omega : |\boldsymbol{G}(x,y)| > t\}| \le Ct^{-\frac{n}{n-2}}, \quad \forall t > (d_x/2)^{2-n}, \tag{4.48}$$

$$|\{x\in\Omega : |D_x\boldsymbol{G}(x,y)| > t\}| \le Ct^{-\frac{n}{n-1}}, \quad \forall t > (d_y/2)^{1-n}. \tag{4.49}$$

$$|\{y\in\Omega : |D_y\boldsymbol{G}(x,y)| > t\}| \le Ct^{-\frac{n}{n-1}}, \quad \forall t > (d_x/2)^{1-n}, \tag{4.50}$$

$$|\boldsymbol{G}(x,y)| \le C\bar{d}_{x,y}^{2-n}, \quad \textit{where } \bar{d}_{x,y} := \min(d_x,d_y,|x-y|), \tag{4.51}$$

$$|\boldsymbol{G}(x,y)-\boldsymbol{G}(z,y)| \le C\,|x-z|^{\mu_0}\,\bar{d}_{x,y}^{2-n-\mu_0} \quad \textit{if } |x-z| < \bar{d}_{x,y}/2, \tag{4.52}$$

$$|\boldsymbol{G}(x,y)-\boldsymbol{G}(x,z)| \le C\,|y-z|^{\mu_0}\,\bar{d}_{x,y}^{2-n-\mu_0} \quad \textit{if } |y-z| < \bar{d}_{x,y}/2, \tag{4.53}$$

where $C=C(n,N,\lambda,\Lambda,\mu_0,H_0)>0$ *and* $C_p=C_p(n,N,\lambda,\Lambda,\mu_0,H_0,p)>0$.

Proof. Let $\boldsymbol{G}^\rho(x,y)$ and $\boldsymbol{G}(x,y)$ be constructed as above. We have already seen that $\boldsymbol{G}$ is continuous on $\{(x,y)\in\Omega\times\Omega : x\neq y\}$ and satisfies all the properties (4.39) – (4.53). Also, as in the proof of Theorem 3.1, we find that for all $\boldsymbol{f}\in C_c^\infty(\Omega)^N$, there is a unique $\boldsymbol{u}\in(Y_0^{1,2}(\Omega)\cap C(\Omega))^N$ satisfying

$$\int_\Omega A_{ij}^{\alpha\beta}D_\beta u^j D_\alpha v^i = \int_\Omega f^i v^i, \quad \forall \boldsymbol{v}\in Y_0^{1,2}(\Omega)^N.$$

If we set $v^i = G_{ki}^\rho(x,\,\cdot\,)$ above, then by (4.3) and (4.34), we find

$$\int_\Omega G_{ki}^\rho(x,\,\cdot\,)f^i = \int_\Omega {}^tA_{ji}^{\beta\alpha}D_\alpha {}^tG_{ik}^\rho(\,\cdot\,,x)D_\beta u^j = \fint_{\Omega_\rho(x)} u^k. \tag{4.54}$$

Fix $r<d_x/2$. By (4.17), (4.18), and (4.34), we have

$$\begin{aligned}\lim_{\mu\to\infty}\int_\Omega G_{ki}^{\rho_\mu}(x,\,\cdot\,)f^i &= \lim_{\mu\to\infty}\left(\int_{B_r(x)} G_{ki}^{\rho_\mu}(x,\,\cdot\,)f^i + \int_{\Omega\setminus B_r(x)} G_{ki}^{\rho_\mu}(x,\,\cdot\,)f^i\right)\\ &= \int_{B_1(x)} G_{ki}(x,\,\cdot\,)f^i + \int_{\Omega\setminus B_1(x)} G_{ki}(x,\,\cdot\,)f^i\\ &= \int_\Omega G_{ki}(x,\,\cdot\,)f^i.\end{aligned}$$

Therefore, (4.37) follows by taking the limits in (4.54). By proceeding as in the proof of Theorem 3.1, we also derive the uniqueness of Green's matrix in Ω. □

4.2. **Boundary regularity.** Let Σ be any subset of $\overline{\Omega}$ and u be a $W^{1,2}(\Omega)$ function. Then we shall say $u=0$ on Σ (in the sense of $W^{1,2}(\Omega)$) if u is a limit in $W^{1,2}(\Omega)$ of a sequence of functions in $C_c^\infty(\overline{\Omega}\setminus\Sigma)$.

We shall denote $\Sigma_R(x) := \partial\Omega\cap B_R(x)$ for any $R>0$. We shall abbreviate $\Omega_R=\Omega_R(x)$ and $\Sigma_R=\Sigma_R(x)$ if the point x is well understood in the context.

Lemma 4.2 (Boundary Poincaré inequality). *Assume that* $|B_R\setminus\Omega|\ge\theta\,|B_R|$ *for some* $\theta>0$. *Then, for any* $u\in W^{1,2}(\Omega_R)$ *satisfying* $u=0$ *on* Σ_R, *we have the following estimate:*

$$\|u\|_{L^2(\Omega_R)} \le \frac{1}{\theta}R\,\|Du\|_{L^2(\Omega_R)}\,. \tag{4.55}$$

Proof. Since $u = 0$ in Σ_R, we may extend u to a $W^{1,2}(B_R)$ function by setting $u = 0$ in $S := B_R \setminus \Omega$. Note that $Du = 0$ in S. Then the lemma follows from (7.45) in [10, p. 164]. □

Lemma 4.3 (Boundary Caccioppoli inequality)**.** *Let the operator L satisfy conditions* (2.2), (2.3). *Suppose $\boldsymbol{u}$ is a $W^{1,2}(\Omega_R)^N$ solutions of $L\boldsymbol{u} = 0$ in Ω_R satisfying $\boldsymbol{u} = 0$ on Σ_R. Then, we have*

$$\|D\boldsymbol{u}\|_{L^2(\Omega_r)} \le \frac{C}{R-r} \|u\|_{L^2(\Omega_R)}, \quad \forall 0 < r < R, \tag{4.56}$$

where $C = C(n, N, \lambda, \Lambda) > 0$.

Proof. It is well known. □

Definition 4.4. We say that Ω satisfies the condition (S) at a point $\bar{x} \in \partial\Omega$ if there exist $\theta > 0$ and $R_a \in (0, \infty]$ such that

$$|B_R(\bar{x}) \setminus \Omega| \ge \theta \, |B_R(\bar{x})|, \quad \forall R < R_a. \tag{4.57}$$

We say that Ω satisfies the condition (S) uniformly on $\Sigma \subset \partial\Omega$ if there exist $\theta > 0$ and R_a such that (4.57) holds for all $\bar{x} \in \Sigma$.

Definition 4.5. Let Ω satisfy the condition (S) at $\bar{x} \in \partial\Omega$. We shall say that an operator L satisfies the property (BH) if there exist $\mu_1, H_1 > 0$ such that if $\boldsymbol{u} \in W^{1,2}(\Omega_R(\bar{x}))^N$ is a weak solution of the problem, $L\boldsymbol{u} = 0$ in $\Omega_R(\bar{x})$ and $\boldsymbol{u} = 0$ on $\Sigma_R(\bar{x})$, where $R < R_a$, then $\boldsymbol{u}$ satisfies the following estimates:

$$\int_{\Omega_r(\bar{x})} |D\boldsymbol{u}|^2 \le H_1 \left(\frac{r}{s}\right)^{n-2+2\mu_1} \int_{\Omega_s(\bar{x})} |D\boldsymbol{u}|^2, \quad \forall 0 < r < s \le R. \tag{4.58}$$

Lemma 4.6. *There exists $\epsilon_0 = \epsilon_0(n, \lambda_0, \Lambda_0) > 0$ such that if the coefficients of the operator L in* (2.1) *satisfies* (2.8) *in Lemma 2.2, then L satisfies the property (BH) with $\mu_1 = \mu_1(n, \lambda_0, \Lambda_0, \theta) > 0$ and $H_1 = H_1(n, N, \lambda_0, \Lambda_0, \theta) > 0$.*

Proof. Throughout the proof, we shall abbreviate $\Omega_r = \Omega_r(\bar{x})$ for any $r > 0$, $\Sigma_r = \Sigma_r(\bar{x})$, the point $\bar{x} \in \partial\Omega$ to be understood. For any $s \le R < R_a$, let v^i $(i = 1, \ldots, N)$ be a unique $W^{1,2}(\Omega_s)$ solution of $L_0 v^i = 0$ in Ω_s satisfying $v^i - u^i \in W_0^{1,2}(\Omega_s)$, where $L_0 v^i = -D_\alpha(a^{\alpha\beta} D_\beta v^i)$.

We claim that there exist $\mu_2(n, \lambda_0, \Lambda_0, \theta) > 0$ and $C(n, \lambda_0, \Lambda_0, \theta) > 0$ such that the following estimate holds:

$$\int_{\Omega_r} |D\boldsymbol{v}|^2 \le C \left(\frac{r}{s}\right)^{n-2+2\mu_2} \int_{\Omega_s} |D\boldsymbol{v}|^2, \quad \forall 0 < r < s. \tag{4.59}$$

We first note that we may assume that $r \le s/8$; otherwise (4.59) becomes trivial. Since each v^i satisfies $v^i = 0$ on Σ_s, it follows from Theorem 8.27 [10, pp. 203–204] and Theorem 8.25 [10, pp. 202–203] that there is $\mu_2 = \mu_2(n, \lambda_0, \Lambda_0, \theta) > 0$ and $C = C(n, \lambda_0, \Lambda_0, \theta) > 0$ such that

$$\operatorname*{osc}_{\Omega_{2r}} v^i \le C r^{\mu_2} s^{-\mu_2} \sup_{\Omega_{s/4}} |v^i| \le C r^{\mu_2} s^{-\mu_2 - n/2} \|v^i\|_{L^2(\Omega_{s/2})}. \tag{4.60}$$

In particular, the estimate (4.60) implies $v^i(\bar{x}) = \lim_{x\to\bar{x}} v^i(x) = 0$. Then, Lemma 4.3 and Lemma 4.2 imply that for all $i = 1,\ldots,N$ (recall $r < s/8$)

$$\begin{aligned}\int_{\Omega_r} \left|Dv^i\right|^2 &\le Cr^{-2}\int_{\Omega_{2r}} |v^i|^2 = Cr^{-2}\int_{\Omega_{2r}} |v^i - v^i(\bar{x})|^2\\ &\le Cr^{n-2}\left(\operatorname*{osc}_{\Omega_{2r}} v^i\right)^2 \le C\left(\frac{r}{s}\right)^{n-2+2\mu_2} s^{-2}\int_{\Omega_{s/2}} |v^i|^2\\ &\le C\left(\frac{r}{s}\right)^{n-2+2\mu_2}\int_{\Omega_s} |Dv^i|^2,\end{aligned}$$

and thus we have proved the claim.

Next, note that $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$ belongs to $W^{1,2}_0(\Omega_s)^N$ and thus it satisfies

$$\lambda\int_{\Omega_s} |D\boldsymbol{w}|^2 \le \int_{\Omega_s} a^{\alpha\beta} D_\beta w^i D_\alpha w^i = \int_{\Omega_s} (a^{\alpha\beta}\delta_{ij} - A^{\alpha\beta}_{ij}) D_\beta u^j D_\alpha w^i.$$

Therefore, we have

$$\int_{\Omega_s} |D\boldsymbol{w}|^2 \le (\lambda^{-1}\left\|\epsilon\right\|_{L^\infty})^2 \int_{\Omega_s} |D\boldsymbol{u}|^2\,, \tag{4.61}$$

where $\epsilon(x)$ is as defined in (2.8). By combining (4.59) and (4.61), we obtain

$$\int_{\Omega_r} |D\boldsymbol{u}|^2 \le C\left(\frac{r}{s}\right)^{n-2+2\mu_2}\int_{\Omega_s} |D\boldsymbol{u}|^2 + C_0\left\|\epsilon\right\|^2_{L^\infty}\int_{\Omega_s} |D\boldsymbol{u}|^2\,,\quad \forall 0 < r < s.$$

Now, choose a $\mu_1 \in (0,\mu_2)$. Then, from a well known iteration argument (see, e.g., [8, Lemma 2.1, p. 86]), it follows that there is ϵ_0 such that if $\left\|\epsilon\right\|_{L^\infty} < \epsilon_0$, then (4.58) holds. □

Theorem 4.7. *Let the operator L satisfy the properties (H) and (BH). Assume that Ω satisfies the condition (S) at $\bar{x} \in \partial\Omega$ with parameters θ, R_a. Let $x \in \Omega$ such that $|x - \bar{x}| = d_x \le R/2$, where $R < R_a$ is given. Then, any weak solution $\boldsymbol{u}$ of $L\boldsymbol{u} = 0$ in $\Omega_R(\bar{x})$ satisfying $\boldsymbol{u} = 0$ on $\Sigma_R(\bar{x})$, we have*

$$|\boldsymbol{u}(x)| \le C d_x^{\mu} R^{1-n/2-\mu}\left\|D\boldsymbol{u}\right\|_{L^2(\Omega_R(\bar{x}))}\,,\quad d_x := \operatorname{dist}(x,\partial\Omega), \tag{4.62}$$

where $C = C(n, N, \lambda, \Lambda, \theta, \mu_0, \mu_1, H_0, H_1) > 0$ and $\mu = \min(\mu_0, \mu_1)$.

Proof. The proof is an adaptation of a technique due to Campanato [4]. In this proof, we shall use the notation $\boldsymbol{u}_{x,r} := \fint_{\Omega_r(x)} \boldsymbol{u}$. Also, we shall abbreviate $d = d_x$. Observe that

$$\Omega_d(x) = B_d(x) \subset \Omega_{2d}(x) \cap \Omega_{2d}(\bar{x}). \tag{4.63}$$

We may assume that $R > 3d$ so that $\Omega_{2d}(x) \subset \Omega_R(\bar{x})$; otherwise $2d \le R \le 3d$ and (4.62) follows from Lemma 2.4. We estimate $\boldsymbol{u}(x)$ by

$$|\boldsymbol{u}(x)| \le |\boldsymbol{u}(x) - \boldsymbol{u}_{x,2d}| + |\boldsymbol{u}_{x,2d} - \boldsymbol{u}_{\bar{x},2d}| + |\boldsymbol{u}_{\bar{x},2d}| := I + II + III.$$

We shall estimate I first. For any $r_1 < r_2 \le 2d$, we estimate

$$\left|\boldsymbol{u}_{x,r_1} - \boldsymbol{u}_{x,r_2}\right|^2 \le 2\left|\boldsymbol{u}(z) - \boldsymbol{u}_{x,r_1}\right|^2 + 2\left|\boldsymbol{u}(z) - \boldsymbol{u}_{x,r_2}\right|^2. \tag{4.64}$$

Note that since $B_d(x) \subset \Omega$, we have

$$|\Omega_r(x)| \ge Cr^n,\quad \forall r \le 2d.$$

Therefore, by integrating (4.64) over $\Omega_{r_1}(x)$ with respect to z, we estimates

$$|\boldsymbol{u}_{x,r_1} - \boldsymbol{u}_{x,r_2}|^2 \le C r_1^{-n} \left(\int_{\Omega_{r_1}} |\boldsymbol{u} - \boldsymbol{u}_{x,r_1}|^2 + \int_{\Omega_{r_2}} |\boldsymbol{u} - \boldsymbol{u}_{x,r_2}|^2 \right). \tag{4.65}$$

Since $\boldsymbol{u} = 0$ on $\Sigma_R(\bar{x})$, we may extend u to $B_R(\bar{x})$ as a $W^{1,2}$ function by setting $\boldsymbol{u} = 0$ on $B_R(\bar{x}) \setminus \Omega$. Therefore, by a version of Poincaré inequality (see, e.g. (7.45) in [10, p. 164]), we have for all $r \le 2d$,

$$\int_{\Omega_r} |\boldsymbol{u} - \boldsymbol{u}_{x,r}|^2 \le \int_{B_r} |\boldsymbol{u} - \boldsymbol{u}_{x,r}|^2 \le C r^2 \int_{B_r} |D\boldsymbol{u}|^2 = C r^2 \int_{\Omega_r} |D\boldsymbol{u}|^2. \tag{4.66}$$

Therefore, by (4.65) and (4.66), we obtain

$$|\boldsymbol{u}_{x,r_1} - \boldsymbol{u}_{x,r_2}|^2 \le C r_1^{-n} \left(r_1^2 \int_{\Omega_{r_1}(x)} |D\boldsymbol{u}|^2 + r_2^2 \int_{\Omega_{r_2}(x)} |D\boldsymbol{u}|^2 \right). \tag{4.67}$$

Next, we claim that the following estimate holds:

$$\int_{\Omega_r(x)} |D\boldsymbol{u}|^2 \le C \left(\frac{r}{R} \right)^{n-2+2\mu} \int_{\Omega_R(\bar{x})} |D\boldsymbol{u}|^2, \quad \forall r \le 2d. \tag{4.68}$$

We first consider the case when $r \le d$. Note that in this case, we have $\Omega_r(x) = B_r(x)$ and $\Omega_d(x) = B_d(x)$. Since L satisfies (H), it follows from (4.63) that

$$\int_{\Omega_r(x)} |D\boldsymbol{u}|^2 \le C \left(\frac{r}{d} \right)^{n-2+2\mu} \int_{\Omega_d(x)} |D\boldsymbol{u}|^2 \le C \left(\frac{r}{d} \right)^{n-2+2\mu} \int_{\Omega_{2d}(\bar{x})} |D\boldsymbol{u}|^2. \tag{4.69}$$

On the other hand, since L satisfies (BH), it follows from (4.58) that

$$\int_{\Omega_{2d}(\bar{x})} |D\boldsymbol{u}|^2 \le C \left(\frac{d}{R} \right)^{n-2+2\mu} \int_{\Omega_R(\bar{x})} |D\boldsymbol{u}|^2. \tag{4.70}$$

By combining (4.69) and (4.70), we obtain (4.68). Next, consider the case when $d < r$. In this case, we have $\Omega_r(x) \subset \Omega_{2r}(\bar{x})$, and thus it follows from (4.58)

$$\int_{\Omega_r(x)} |D\boldsymbol{u}|^2 \le \int_{\Omega_{2r}(\bar{x})} |D\boldsymbol{u}|^2 \le C \left(\frac{r}{R} \right)^{n-2+2\mu} \int_{\Omega_R(\bar{x})} |D\boldsymbol{u}|^2.$$

We proved the claim (4.68).

Now, by using (4.68), we estimates (4.67) as follows (recall $r_1 < r_2 \le 2d$):

$$|\boldsymbol{u}_{x,r_1} - \boldsymbol{u}_{x,r_2}|^2 \le C r_1^{-n} (r_1^{n+2\mu} + r_2^{n+2\mu}) R^{2-n-2\mu} \int_{\Omega_R(\bar{x})} |D\boldsymbol{u}|^2. \tag{4.71}$$

For any $r \le 2d$, set $r_1 = r 2^{-(i+1)}$ and $r_2 = r 2^{-i}$ in (4.71) to get

$$\left| \boldsymbol{u}_{x,r2^{-(i+1)}} - \boldsymbol{u}_{x,r2^{-i}} \right|^2 \le C r^{2\mu} 2^{-2\mu(i+1)} R^{2-n-2\mu} \int_{\Omega_R(\bar{x})} |D\boldsymbol{u}|^2.$$

Therefore, for $0 \le j < k$, we obtain

$$\begin{aligned} \left| \boldsymbol{u}_{x,r2^{-k}} - \boldsymbol{u}_{x,r2^{-j}} \right| &\le \sum_{i=j}^{k-1} \left| \boldsymbol{u}_{x,r2^{-(i+1)}} - \boldsymbol{u}_{x,r2^{-i}} \right| \\ &\le C r^{\mu} \left(\sum_{i=j}^{\infty} 2^{-\mu(i+1)} \right) R^{1-n/2-\mu} \|D\boldsymbol{u}\|_{L^2(\Omega_R(\bar{x}))} \\ &= C 2^{-j\mu} r^{\mu} R^{1-n/2-\mu} \|D\boldsymbol{u}\|_{L^2(\Omega_R(\bar{x}))}. \end{aligned} \tag{4.72}$$

By setting $r = 2d$, $j = 0$, and letting $k \to \infty$ in (4.72), we obtain

$$I = |\boldsymbol{u}(x) - \boldsymbol{u}_{x,2d}| \leq C d^{\mu} R^{1-n/2-\mu} \|D\boldsymbol{u}\|_{L^2(\Omega_R(\bar{x}))}. \tag{4.73}$$

Next, we estimate III. Since $|B_r(\bar{x}) \cap B_d(x)| \geq C r^n$ for $r \leq 2d$, we have

$$|\Omega_r(\bar{x})| \geq C r^n, \quad \forall r \leq 2d. \tag{4.74}$$

Also, as in (4.66), we have for all $r \leq 2d$ (recall $\boldsymbol{u} \equiv 0$ on $B_R(\bar{x}) \setminus \Omega$)

$$\int_{\Omega_r} |\boldsymbol{u} - \boldsymbol{u}_{\bar{x},r}|^2 \leq \int_{B_r} |\boldsymbol{u} - \boldsymbol{u}_{\bar{x},r}|^2 \leq C r^2 \int_{B_r} |D\boldsymbol{u}|^2 = C r^2 \int_{\Omega_r} |D\boldsymbol{u}|^2. \tag{4.75}$$

Therefore, as in (4.67) we have for $r_1 < r_2 \leq 2d$,

$$|\boldsymbol{u}_{\bar{x},r_1} - \boldsymbol{u}_{\bar{x},r_2}|^2 \leq C r_1^{-n} \left(r_1^2 \int_{\Omega_{r_1}(\bar{x})} |D\boldsymbol{u}|^2 + r_2^2 \int_{\Omega_{r_2}(\bar{x})} |D\boldsymbol{u}|^2 \right).$$

Then, by using the property (BH), we obtain (c.f. (4.72), (4.73))

$$|\hat{\boldsymbol{u}}(\bar{x}) - \boldsymbol{u}_{\bar{x},2d}| \leq C d^{\mu} R^{1-n/2-\mu} \|D\boldsymbol{u}\|_{L^2(\Omega_R(\bar{x}))}, \tag{4.76}$$

where $\hat{\boldsymbol{u}}(\bar{x}) := \lim_{k\to\infty} \boldsymbol{u}_{\bar{x},2^{-k}r}$. (note that (4.72) implies $\hat{\boldsymbol{u}}(\bar{x})$ exists). It follows from (4.74), (4.55), and (4.58) that for any $r \leq 2d$, we have

$$\begin{aligned}
|\boldsymbol{u}_{\bar{x},r}|^2 &\leq \fint_{\Omega_r(\bar{x})} |\boldsymbol{u}|^2 \leq C r^{-n} \int_{\Omega_r(\bar{x})} |\boldsymbol{u}|^2 \\
&\leq C r^{2-n} \int_{\Omega_r(\bar{x})} |D\boldsymbol{u}|^2 \leq C r^{2-n} \left(\frac{r}{R}\right)^{n-2+2\mu} \int_{\Omega_R(\bar{x})} |D\boldsymbol{u}|^2 \\
&= C r^{2\mu} R^{2-n-2\mu} \int_{\Omega_R(\bar{x})} |D\boldsymbol{u}|^2,
\end{aligned}$$

and thus that $\hat{\boldsymbol{u}}(\bar{x}) = 0$. Therefore, by (4.76) we obtain

$$III = |\boldsymbol{u}_{\bar{x},2d}| = |\hat{\boldsymbol{u}}(\bar{x}) - \boldsymbol{u}_{\bar{x},2d}| \leq C d^{\mu} R^{1-n/2-\mu} \|D\boldsymbol{u}\|_{L^2(\Omega_R(\bar{x}))}. \tag{4.77}$$

Finally, we estimate II.

$$|\boldsymbol{u}_{x,2d} - \boldsymbol{u}_{\bar{x},2d}|^2 \leq 2\,|\boldsymbol{u}(z) - \boldsymbol{u}_{x,2d}|^2 + 2\,|\boldsymbol{u}(z) - \boldsymbol{u}_{\bar{x},2d}|^2. \tag{4.78}$$

By integrating (4.78) over $B_d(x) \subset \Omega_{2d}(x) \cap \Omega_{2d}(\bar{x})$ with respect to z, we estimate

$$\begin{aligned}
|\boldsymbol{u}_{x,2d} - \boldsymbol{u}_{\bar{x},2d}|^2 &\leq C d^{-n} \left(\int_{\Omega_{2d}(x)} |\boldsymbol{u} - \boldsymbol{u}_{x,2d}|^2 + \int_{\Omega_{2d}(\bar{x})} |\boldsymbol{u} - \boldsymbol{u}_{\bar{x},2d}|^2 \right) \\
&\leq C d^{2-n} \left(\int_{\Omega_{2d}(x)} |D\boldsymbol{u}|^2 + \int_{\Omega_{2d}(\bar{x})} |D\boldsymbol{u}|^2 \right) \\
&\leq C d^{2\mu} R^{2-n-2\mu} \int_{\Omega_R(\bar{x})} |D\boldsymbol{u}|^2,
\end{aligned} \tag{4.79}$$

where we have used (4.66), (4.75), (4.68), and (4.58). Therefore, by combining (4.73), (4.77), and (4.79), we obtain (4.62). □

Theorem 4.8. *Let the operators L, ${}^tL$ satisfy the properties (H) and (BH). Assume that Ω satisfies the condition (S) uniformly on $\partial\Omega$ with parameters θ, R_a. Denote*

$$R_{x,y} := \min(|x - y|, 4R_a).$$

Then the Green matrix $\boldsymbol{G}(x,y)$ *satisfies*

$$|\boldsymbol{G}(x,y)| \le C d_x^{\mu} R_{x,y}^{1-n/2-\mu} d_y^{1-n/2} \quad \textit{if } d_x \le R_{x,y}/8, \tag{4.80}$$

$$|\boldsymbol{G}(x,y)| \le C d_y^{\mu} R_{x,y}^{1-n/2-\mu} d_x^{1-n/2} \quad \textit{if } d_y \le R_{x,y}/8, \tag{4.81}$$

where $C = C(n, N, \lambda, \Lambda, \theta, \mu_0, \mu_1, H_0, H_1) > 0$ *and* $\mu = \min(\mu_0, \mu_1)$. *As a consequence, we have* $\boldsymbol{G}(\,\cdot\,, y) = 0$, $\boldsymbol{G}(x, \,\cdot\,) = 0$ *on* $\partial\Omega$ *in the usual sense.*

Proof. We only need to prove (4.80), for (4.81) will then follow from (4.34). Set $R = R_{x,y}/4$, $r = d_y/2$, and choose $\bar{x} \in \partial\Omega$ such that $|x - \bar{x}| = d_x$. Then, since

$$d_y \le |x-y| + d_x \le \tfrac{9}{8}\,|x-y|\,,$$

we have

$$|y - \bar{x}| \ge |x-y| - d_x \ge \tfrac{7}{8}\,|x-y| \ge R + r,$$

and thus, $\Omega_R(\bar{x}) \subset \Omega \setminus B_r(y)$. Now, we apply Theorem 4.7 with $\boldsymbol{u} = \boldsymbol{G}(\,\cdot\,, y)$. Then, by (4.62) and (4.24), we obtain

$$|\boldsymbol{G}(x,y)| \le C d_x^{\mu} R^{1-n/2-\mu} \, \|D\boldsymbol{G}(\,\cdot\,, y)\|_{L^2(\Omega\setminus B_r(y))} \le C d_x^{\mu} R_{x,y}^{1-n/2-\mu} d_y^{1-n/2}.$$

The proof is complete. ☐

Remark 4.9. We note that in the scalar case, the maximum principle yields (see [11, Theorem 1.1])

$$G(x,y) \le C\,|x-y|^{2-n}\,, \quad \forall x \ne y \in \Omega. \tag{4.82}$$

Then, by the boundary Caccioppoli inequality, we have (c.f. (4.4)–(4.7))

$$\int_{\Omega\setminus B_r(y)} |DG(\,\cdot\,, y)|^2 \le C r^{2-n}, \quad \forall r > 0.$$

Therefore, in the scalar case we don't need to require that $r < d_y/2$ (or $r < d_x/2$) in the proof of Theorem 4.8 and we may as well set $r = |x-y|/2$ to get

$$G(x,y) \le C d_x^{\mu} R_{x,y}^{1-n/2-\mu}\,|x-y|^{1-n/2} \quad \text{if } d_x \le R_{x,y}/8,$$
$$G(x,y) \le C d_y^{\mu} R_{x,y}^{1-n/2-\mu}\,|x-y|^{1-n/2} \quad \text{if } d_y \le R_{x,y}/8.$$

In particular, if $G(x,y)$ is the Green's function on $\mathbb{R}^n_+$, then we obtain

$$G(x,y) \le C d_x^{\mu}\,|x-y|^{2-n-\mu} \quad \text{if } d_x \le |x-y|/8,$$
$$G(x,y) \le C d_y^{\mu}\,|x-y|^{2-n-\mu} \quad \text{if } d_y \le |x-y|/8,$$

for $\partial\mathbb{R}^n_+$ satisfies the condition (S) with $\theta = 1/2$ and $R_a = \infty$.

5. Remarks on VMO coefficients case

Definition 5.1 (Sarason [19]). For a measurable function f defined on $\mathbb{R}^n$, we shall denote $\overline{f}_{x,r} = \fint_{B_r(x)} f$ and for $0 < \delta < \infty$ we define

$$M_\delta(f) := \sup_{x\in\mathbb{R}^n} \sup_{r\le\delta} \fint_{B_r(x)} \left|f - \overline{f}_{x,r}\right|; \quad M_0(f) := \lim_{\delta\to 0} M_\delta(f). \tag{5.1}$$

We shall say that f belongs to VMO if $M_0(f) = 0$.

Definition 5.2. We say that the operator L satisfies the property $(\mathrm{H})_{loc}$ if there exist $\mu_0, H_0, R_c > 0$ such that all weak solutions $\boldsymbol{u}$ of $L\boldsymbol{u} = 0$ in $B_R = B_R(x_0)$ with $R \le R_c$ satisfy

$$\int_{B_r} |D\boldsymbol{u}|^2 \le H_0 \left(\frac{r}{s}\right)^{n-2+2\mu_0} \int_{B_s} |D\boldsymbol{u}|^2, \quad 0 < r < s \le R. \tag{5.2}$$

Similarly, we say that the transpose operator ${}^t\!L$ satisfies the property $(\mathrm{H})_{loc}$ if corresponding estimates hold for all weak solutions $\boldsymbol{u}$ of ${}^t\!L\boldsymbol{u} = 0$ in B_R with $R \le R_c$.

Lemma 5.3. *Let the coefficients of the operator L in* (2.1) *satisfy the conditions* (2.2) *and* (2.3). *If the coefficients belong to* VMO *in addition, then L satisfies the property* $(\mathrm{H})_{loc}$.

Proof. It is well known that if the coefficients are uniformly continuous, then L satisfies the property $(\mathrm{H})_{loc}$; see e.g. [8, pp. 87–89]. Essentially, the same proof carries over to the VMO coefficients case. One only needs to make a note of the following two facts. First, a theorem of Meyers [16] implies that there is some $p = p(n, N, \lambda, \Lambda) > 2$ such that if $\boldsymbol{u}$ is a weak solution of $L\boldsymbol{u} = 0$ in $B_R(x)$, then

$$\left(\fint_{B_r(x)} |D\boldsymbol{u}|^p\right)^{1/p} \le C\left(\fint_{B_{2r(x)}} |D\boldsymbol{u}|^2\right)^{1/2}, \quad \forall r < R/2.$$

Secondly, note that the John-Nirenberg theorem [13] implies that

$$\left(\fint_{B_r(x)} \left|f - \overline{f}_{r,x}\right|^q\right)^{1/q} \le C(n,q) M_\delta(f), \quad \forall r < c(n)\delta, \quad \forall q \in (0,\infty),$$

where $M_\delta(f)$ is defined as in (5.1). For the details, we refer to [3, pp. 47–48]. □

In the rest of this section, we shall assume that the operators L and ${}^t\!L$ satisfy the property $(\mathrm{H})_{loc}$ with parameters μ_0, H_0, R_c. We shall denote

$$r_x := \min(d_x, R_c), \quad \bar{r}_{x,y} := \min(\bar{d}_{x,y}, R_c), \tag{5.3}$$

where $d_x = \operatorname{dist}(x, \partial\Omega)$ and $\bar{d}_{x,y}$ is as in (4.28). It is routine to check that all estimates appearing in Section 4.1 remain valid if d_x, $\bar{d}_{x,y}$ are replaced by r_x, $\bar{r}_{x,y}$, respectively. Therefore, we have the following theorem:

Theorem 5.4. *Let Ω be an open connected set in $\mathbb{R}^n$. Denote $d_x := \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$; we set $d_x = \infty$ if $\Omega = \mathbb{R}^n$. Assume that operators L and ${}^t\!L$ satisfy the property* $(\mathrm{H})_{loc}$. *Then, there exists a unique Green's matrix $\boldsymbol{G}(x,y) = (G_{ij}(x,y))_{i,j=1}^N$ ($x, y \in \Omega, x \ne y$) which is continuous in $\{(x,y) \in \Omega \times \Omega : x \ne y\}$ and such that $\boldsymbol{G}(x, \cdot)$ is locally integrable in Ω for all $x \in \Omega$ and that for all $\boldsymbol{f} = (f^1, \ldots, f^N)^T \in C_c^\infty(\Omega)^N$, the function $\boldsymbol{u} = (u^1, \ldots, u^N)^T$ given by*

$$\boldsymbol{u}(x) := \int_\Omega \boldsymbol{G}(x,y)\boldsymbol{f}(y)\,dy \tag{5.4}$$

belongs to $Y_0^{1,2}(\Omega)^N$ and satisfies $L\boldsymbol{u} = \boldsymbol{f}$ in the sense

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta u^j D_\alpha \phi^i = \int_\Omega f^i \phi^i, \quad \forall \boldsymbol{\phi} \in C_c^\infty(\Omega)^N. \tag{5.5}$$

Moreover, $\boldsymbol{G}(x,y)$ has the properties that

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta G_{jk}(\cdot, y) D_\alpha \phi^i = \phi^k(y), \quad \forall \boldsymbol{\phi} \in C_c^\infty(\Omega)^N \tag{5.6}$$

and that for all $\eta \in C_c^\infty(\Omega)$ *satisfying* $\eta \equiv 1$ *on* $B_r(y)$ *for some* $r < d_y$,

$$(1-\eta)\boldsymbol{G}(\,\cdot\,, y) \in Y_0^{1,2}(\Omega)^{N\times N}. \tag{5.7}$$

Furthermore, $\boldsymbol{G}(x,y)$ *satisfies the following estimates: For* $r_x, r_y, \bar{r}_{x,y}$ *as in* (5.3),

$$\|\boldsymbol{G}(\,\cdot\,,y)\|_{L^p(B_r(y))} \le C_p\, r^{2-n+n/p}, \quad \forall r < r_y, \quad \forall p \in [1, \tfrac{n}{n-2}), \tag{5.8}$$

$$\|\boldsymbol{G}(x,\,\cdot\,)\|_{L^p(B_r(x))} \le C_p\, r^{2-n+n/p}, \quad \forall r < r_x, \quad \forall p \in [1, \tfrac{n}{n-2}), \tag{5.9}$$

$$\|D\boldsymbol{G}(\,\cdot\,,y)\|_{L^p(B_r(y))} \le C_p\, r^{1-n+n/p}, \quad \forall r < r_y, \quad \forall p \in [1, \tfrac{n}{n-1}), \tag{5.10}$$

$$\|D\boldsymbol{G}(x,\,\cdot\,)\|_{L^p(B_r(x))} \le C_p\, r^{1-n+n/p}, \quad \forall r < r_x, \quad \forall p \in [1, \tfrac{n}{n-1}), \tag{5.11}$$

$$\|\boldsymbol{G}(\,\cdot\,,y)\|_{Y^{1,2}(\Omega\setminus B_r(y))} \le C r^{1-n/2}, \quad \forall r < r_y/2, \tag{5.12}$$

$$\|\boldsymbol{G}(x,\,\cdot\,)\|_{Y^{1,2}(\Omega\setminus B_r(x))} \le C r^{1-n/2}, \quad \forall r < r_x/2, \tag{5.13}$$

$$|\{x \in \Omega : |\boldsymbol{G}(x,y)| > t\}| \le C t^{-\frac{n}{n-2}}, \quad \forall t > (r_y/2)^{2-n}, \tag{5.14}$$

$$|\{y \in \Omega : |\boldsymbol{G}(x,y)| > t\}| \le C t^{-\frac{n}{n-2}}, \quad \forall t > (r_x/2)^{2-n}, \tag{5.15}$$

$$|\{x \in \Omega : |D_x\boldsymbol{G}(x,y)| > t\}| \le C t^{-\frac{n}{n-1}}, \quad \forall t > (r_y/2)^{1-n}, \tag{5.16}$$

$$|\{y \in \Omega : |D_y\boldsymbol{G}(x,y)| > t\}| \le C t^{-\frac{n}{n-1}}, \quad \forall t > (r_x/2)^{1-n}, \tag{5.17}$$

$$|\boldsymbol{G}(x,y)| \le C \bar{r}_{x,y}^{2-n}, \quad \forall x, y \in \Omega, \tag{5.18}$$

$$|\boldsymbol{G}(x,y) - \boldsymbol{G}(z,y)| \le C\,|x-z|^{\mu_0}\, \bar{r}_{x,y}^{2-n-\mu_0} \quad \text{if } |x-z| < \bar{r}_{x,y}/2, \tag{5.19}$$

$$|\boldsymbol{G}(x,y) - \boldsymbol{G}(x,z)| \le C\,|y-z|^{\mu_0}\, \bar{r}_{x,y}^{2-n-\mu_0} \quad \text{if } |y-z| < \bar{r}_{x,y}/2, \tag{5.20}$$

where $C = C(n, N, \lambda, \Lambda, \mu_0, H_0) > 0$ *and* $C_p = C_p(n, N, \lambda, \Lambda, \mu_0, H_0, p) > 0$.

Remark 5.5. Dolzmann-Müller [6] derived a global estimate

$$|\boldsymbol{G}(x,y)| \le C\,|x-y|^{2-n} \quad \forall x, y \in \Omega, \quad x \ne y, \tag{5.21}$$

assuming that Ω is a bounded C^1 domain. We have not attempted to derive the corresponding estimate here. However, we would like to point out that the constant C in their estimate depends on the domain (e.g., the diameter of the domain and also some characteristics of $\partial\Omega$) while our interior estimate (5.18) does not.

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