

DAHLBERG'S BILINEAR ESTIMATE FOR SOLUTIONS OF DIVERGENCE FORM COMPLEX ELLIPTIC EQUATIONS

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ABSTRACT. We consider divergence form elliptic operators $L = -\operatorname{div} A(x)\nabla$, defined in $\mathbb{R}^{n+1} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}\}$, $n \geq 2$, where the L^∞ coefficient matrix A is $(n+1) \times (n+1)$, uniformly elliptic, complex and t -independent. Using recently obtained results concerning the boundedness and invertibility of layer potentials associated to such operators, we show that if $Lu = 0$ in \mathbb{R}_+^{n+1} , then for any vector-valued $\mathbf{v} \in W_{loc}^{1,2}$, we have the bilinear estimate

$$\left| \iint_{\mathbb{R}_+^{n+1}} \nabla u \cdot \bar{\mathbf{v}} dx dt \right| \leq C \sup_{t>0} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \left(\|t\nabla \mathbf{v}\| + \|N_* \mathbf{v}\|_{L^2(\mathbb{R}^n)} \right),$$

where $\|F\| \equiv \left(\iint_{\mathbb{R}_+^{n+1}} |F(x, t)|^2 t^{-1} dx dt \right)^{1/2}$, and where N_* is the usual non-tangential maximal operator. The result is new even in the case of real symmetric coefficients, and generalizes the analogous result of Dahlberg for harmonic functions on Lipschitz graph domains.

1. INTRODUCTION

In [6], B. Dahlberg considered the bilinear singular integral form

$$(1.1) \quad \int_{\Omega} \nabla u \cdot \bar{\mathbf{v}},$$

where u is harmonic in the domain $\Omega \equiv \{(x, t) \in \mathbb{R}^{n+1} : t > \varphi(x)\}$, with φ Lipschitz, and where $\mathbf{v} \in W_{loc}^{1,2}$ is vector valued. He showed that the bilinear form (1.1) is bounded by the L^2 norm of the square function plus the non-tangential maximal of u , times the same expression for \mathbf{v} . In the present note, we generalize Dahlberg's Theorem to variable coefficient divergence form elliptic operators. To be precise, let

$$L = -\operatorname{div} A \nabla \equiv - \sum_{i,j=1}^{n+1} \frac{\partial}{\partial x_i} \left(A_{i,j} \frac{\partial}{\partial x_j} \right)$$

be defined in $\mathbb{R}^{n+1} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}\}$, $n \geq 2$, (we use the convention that $x_{n+1} = t$), where $A = A(x)$ is an $(n+1) \times (n+1)$ matrix of complex-valued L^∞ coefficients, defined on \mathbb{R}^n (i.e., independent of the t variable), and satisfying the uniform ellipticity (accretivity) condition

$$(1.2) \quad \lambda |\xi|^2 \leq \Re \langle A(x)\xi, \xi \rangle, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda,$$

for some $\lambda > 0$, $\Lambda < \infty$, and for all $\xi \in \mathbb{C}^{n+1}$, $x \in \mathbb{R}^n$. Here, $\langle \cdot, \cdot \rangle$ denotes the usual hermitian inner product in \mathbb{C}^{n+1} , so that

$$\langle A(x)\xi, \xi \rangle \equiv \sum_{i,j=1}^{n+1} A_{ij}(x) \xi_j \bar{\xi}_i$$

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In order to state our theorem, we first recall that the non-tangential maximal operator N_* (and a variant \tilde{N}_*) are defined as follows. Given $x_0 \in \mathbb{R}^n$, let

$$\gamma(x_0) = \{(x, t) \in \mathbb{R}_+^{n+1} : |x_0 - x| < t\}$$

denote the cone with vertex at x_0 . Then for U defined in \mathbb{R}_+^{n+1} ,

$$N_*U(x_0) \equiv \sup_{(x,t) \in \gamma(x_0)} |U(x, t)|, \quad \tilde{N}_*U(x_0) \equiv \sup_{(x,t) \in \gamma(x_0)} \left(\iint_{\substack{|x-y| < t \\ |t-s| < t/2}} |U(y, s)|^2 dy ds \right)^{\frac{1}{2}}.$$

Our main result is the following:

Theorem 1.3. *Suppose that L is an operator of the type described above, with*

$$(1.4) \quad \|A - A_0\|_\infty \leq \epsilon,$$

for some real, symmetric, L^∞ , elliptic, and t -independent matrix A_0 . Suppose also that $Lu = 0$, and that $\mathbf{v} \in W_{loc}^{1,2}(\mathbb{R}^{n+1}, \mathbb{C}^{n+1})$. If $\epsilon \leq \epsilon_0$, with ϵ_0 sufficiently small, depending only on dimension and ellipticity, then we have the bilinear estimate

$$\left| \iint_{\mathbb{R}_+^{n+1}} \nabla u \cdot \bar{\mathbf{v}} dx dt \right| \leq C \sup_{t>0} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \left(\|t \nabla \mathbf{v}\| + \|N_* \mathbf{v}\|_{L^2(\mathbb{R}^n)} \right),$$

where $C = C(n, \lambda, \Lambda)$ and

$$\|F\| \equiv \left(\iint_{\mathbb{R}_+^{n+1}} |F(x, t)|^2 t^{-1} dx dt \right)^{1/2}.$$

We remark that the result is new even in the case of real, symmetric coefficients. The analogous result was proved by Dahlberg for harmonic functions in Lipschitz graph domains, using a special change of variable found by Kenig and Stein, and independently by Maz'ya. Our theorem includes that of Dahlberg, as may be seen by pulling back under the mapping $(x, t) \rightarrow (x, \varphi(x) + t)$. Dahlberg's original method seems inapplicable to the variable coefficient case, unless the coefficients are differentiable and satisfy an appropriate sort of Carleson condition as in the work of Kenig and Pipher [13]. In the present setting, in lieu of the special change of variable, we use recently obtained results of [1] concerning the boundedness and invertibility of layer potentials associated to variable coefficient t -independent operators.

The paper is organized as follows. In the next section, we recall some of the aforementioned results of [1]. In Section 3, we prove Theorem 1.3, and in Section 4 we discuss the analogue of another result of [6] concerning the domain of the infinitesimal generator of the Poisson semigroup for the equation $Lu = 0$ in \mathbb{R}_+^{n+1} .

Let us now set some notation and terminology that we shall use in the sequel. We shall employ the standard convention that the generic constant C is allowed to vary from one instance to the next, and may depend upon dimension and ellipticity. The symbol $\int_E f$ denotes the mean value, i.e., $\int_E f \equiv |E|^{-1} \int_E f$. We shall use the notation

$$D_j \equiv \partial_{x_j}, \quad 1 \leq j \leq n+1,$$

bearing in mind that $x_{n+1} = t$, and we use e_j , $1 \leq j \leq n+1$, to indicate the standard unit basis vector in the x_j direction. The symbol ∇ denotes the full $(n+1)$ -dimensional gradient, acting in both x and t , and we use ∇_x or ∇_{\parallel} to indicate the n -dimensional gradient acting only in x . We use adj to denote the hermitian adjoint of an operator acting on functions defined on \mathbb{R}^n . We define the homogeneous Sobolev space \dot{L}_1^2 to be the completion of C_0^∞ with respect to the seminorm $\|\nabla F\|_2$. As is well known, for $n \geq 2$, this space can be

identified (modulo constants) with the space $I_1(L^2) \equiv \Delta^{-1/2}(L^2)$. We write $F \rightarrow f$ n.t. to mean that for a.e. $x \in \mathbb{R}^n$, $F(y, t) \rightarrow f(x)$, as $(y, t) \rightarrow (x, 0)$, with $(y, t) \in \gamma(x)$.

2. RESULTS FOR VARIABLE COEFFICIENT LAYER POTENTIALS

We now recall the definitions of the layer potentials. We first note that by (1.4), the stability result of [2], and the classical De Giorgi-Nash Theorem [7, 15], solutions of $Lu = 0$ are locally Hölder continuous. Let $\Gamma(x, t, y, s)$ denote the fundamental solution for L (we refer the reader to [9] for the construction of, and estimates for, Γ in the case of complex coefficients, assuming De Giorgi-Nash bounds). By t -independence,

$$(2.1) \quad \Gamma(x, t, y, s) \equiv \Gamma(x, t - s, y, 0).$$

We define the single and double layer potentials, respectively, in the usual way:

$$(2.2) \quad \begin{aligned} S_t f(x) &\equiv \int_{\mathbb{R}^n} \Gamma(x, t, y, 0) f(y) dy, \quad t \in \mathbb{R} \\ \mathcal{D}_t f(x) &\equiv \int_{\mathbb{R}^n} \overline{\partial_{\nu^*(y)} \Gamma^*(y, 0, x, t)} f(y) dy, \quad t \neq 0, \end{aligned}$$

where ∂_{ν^*} is the adjoint exterior conormal derivative; i.e., if A^* denotes the hermitian adjoint of A , then

$$(2.3) \quad \partial_{\nu^*(y)} \Gamma^*(y, 0, x, t) = - \sum_{j=1}^{n+1} A_{n+1,j}^*(y) \frac{\partial \Gamma^*}{\partial y_j}(y, 0, x, t) = -e_{n+1} \cdot A^*(y) \nabla_{y,s} \Gamma^*(y, s, x, t) |_{s=0}$$

(recall that $y_{n+1} = s$), where $e_{n+1} \equiv (0, \dots, 0, 1)$ is the unit basis vector in the t direction. Here, Γ^* is the fundamental solution for L^* , the hermitian adjoint of L . Thus, Γ^* is the conjugate transpose of Γ ; i.e.,

$$\overline{\Gamma^*(y, s, x, t)} = \Gamma(x, t, y, s).$$

We also define (formally) the boundary singular integral

$$(2.4) \quad Kf(x) \equiv \text{“p.v.”} \int_{\mathbb{R}^n} \overline{\partial_{\nu^*} \Gamma^*(y, 0, x, 0)} f(y) dy.$$

(For the precise definition of the latter operator in the case of non-smooth coefficients, see [1], Section 4). In a departure from tradition impelled by the context of complex coefficients, K^* , S^* and \mathcal{D}^* will denote the analogues of K , S and \mathcal{D} corresponding to L^* .

In order to prove our Theorem, we shall require some of the main results of [1], which we summarize as follows:

Theorem 2.5. [1]. *Suppose that L satisfies the hypotheses of Theorem 1.3. There exists a small constant $\epsilon_0 = \epsilon_0(n, \lambda, \Lambda)$ such that if ϵ in (1.4) satisfies $\epsilon \leq \epsilon_0$, then the layer potential operators $\pm \frac{1}{2}I + K$, $\pm \frac{1}{2}I + K^*$ are isomorphisms on $L^2(\mathbb{R}^n)$, with the implicit constants depending only upon dimension and ellipticity. In addition,*

$$(2.6) \quad \sup_{t>0} \|\mathcal{D}_t f\|_2 + \|t \nabla \partial_t S_{\pm t} f\| + \sup_{1 \leq j \leq n} \|t \partial_t S_{\pm t} \mathcal{D}_j f\| \leq C \|f\|_2,$$

and $\mathcal{D}_{\pm t} f \rightarrow (\pm \frac{1}{2}I + K)f$ n.t. and in L^2 , for $f \in L^2$. Moreover, the corresponding statements hold also for \mathcal{D}_t^* , K^* and S_t^* . Finally, the solution to the Dirichlet problem

$$(D2) \quad \begin{cases} Lu = 0 \text{ in } \mathbb{R}_+^{n+1} = \{(x, t) \in \mathbb{R}^n \times (0, \infty)\} \\ \lim_{t \rightarrow 0} u(\cdot, t) = f \text{ in } L^2(\mathbb{R}^n) \text{ and n.t.} \\ \sup_{t>0} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty, \end{cases}$$

which exists by virtue of the aforementioned facts about layer potentials, is unique.

We shall also require the following technical facts.

Lemma 2.7. [1] (Lemma 2.2) *Suppose that L satisfies the hypotheses of Theorem 1.3, with $\epsilon \leq \epsilon_0$. Set $K_t(x, y) = t\partial_t^2 \Gamma(x, t, y, 0)$. Then*

$$(2.8) \quad |K_t(x, y)| \leq C \frac{t}{(t + |x - y|)^{n+1}}.$$

Lemma 2.9. [1] (Lemma 2.8) *Suppose that L satisfies the hypotheses of Theorem 1.3, with $\epsilon \leq \epsilon_0$. Then*

$$\|t^2 \nabla \partial_t^2 S_{\pm t}^* f\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_2.$$

Lemma 2.10. . *Suppose that $\{R_t\}_{t>0}$ is a family of operators defined by*

$$R_t f(x) \equiv \int_{\mathbb{R}^n} K_t(x, y) f(y) dy,$$

where the kernel K_t satisfies (2.8). Suppose also that $R_t 1 = 0$ for all $t \in \mathbb{R}$. Then for $h \in \dot{L}_1^2(\mathbb{R}^n)$,

$$(2.11) \quad \int_{\mathbb{R}^n} |R_t h|^2 \leq C t^2 \int_{\mathbb{R}^n} |\nabla_x h|^2.$$

The proof of the last lemma is a standard exercise in the use of Poincaré's inequality. We omit the details, but see, e.g. [1] (Lemma 3.5), for a more general result.

Finally, we shall use the following special case of the ‘‘Fatou Theorem’’ of [1].

Lemma 2.12. [1] (Corollary 4.41) *Suppose that L satisfies the hypotheses of Theorem 1.3, with ϵ sufficiently small, and that $Lu = 0$ in \mathbb{R}^{n+1} . Suppose also that*

$$\sup_{t>0} \|u(\cdot, t)\|_2 < \infty.$$

Then $u(\cdot, t)$ converges n.t. and in L^2 as $t \rightarrow 0$.

3. PROOF OF THEOREM 1.3

The proof of the theorem will use the following

Lemma 3.1. *Suppose that L satisfies the hypotheses of Theorem 1.3, with ϵ sufficiently small. Suppose also that $Lu = 0$ in \mathbb{R}^{n+1} , with $\sup_{t>0} \|u(\cdot, t)\|_2 < \infty$. Then*

$$\|t \nabla u\| \leq C \sup_{t>0} \|u(\cdot, t)\|_2.$$

Proof. It is enough to show that for each fixed $\eta \in (0, 10^{-10})$,

$$\int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} |\nabla u|^2 t dx dt d\delta \leq C \sup_{t>0} \|u(\cdot, t)\|_2.$$

Integrating by parts in t on the left side of the last inequality, we obtain

$$(3.2) \quad -\Re e \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \langle \partial_t \nabla u, \nabla u \rangle t^2 dt d\delta + \text{boundary},$$

where the boundary terms are dominated by

$$(3.3) \quad \sup_{r>0} \int_r^{2r} \int_{\mathbb{R}^n} r^2 |\nabla u(x, t)|^2 dx dt \leq C \sup_{t>0} \|u(\cdot, t)\|_2^2,$$

as desired, and in the last step we have split \mathbb{R}^n into cubes of side length $\approx r$ and used Caccioppoli's inequality. By Cauchy's inequality, the main term in (3.2) is no larger than

$$\frac{\varepsilon}{2} \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} |\nabla u|^2 t dx dt d\delta + \frac{1}{2\varepsilon} \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} |\nabla \partial_t u|^2 t^3 dx dt d\delta \equiv I + II,$$

where $\varepsilon > 0$ is at our disposal. Choosing ε small, we may hide term I . Having fixed ε , and applying Caccioppoli's inequality in Whitney boxes, we obtain that

$$II \leq C \iint_{\mathbb{R}_+^{n+1}} |\partial_t u|^2 dx dt.$$

By the Fatou Theorem of [1], Section 4, u converges in $L^2(\mathbb{R}^n)$ to some f , with

$$\|f\|_2 \leq \sup_{t>0} \|u(\cdot, t)\|_2.$$

Thus, by Theorem 2.5, $u(\cdot, t) = \mathcal{D}_t \left(-\frac{1}{2}I + K \right)^{-1} f$. By (2.6), the bijectivity of $\left(-\frac{1}{2}I + K \right)$ and the definition of \mathcal{D}_t , we obtain that

$$\iint_{\mathbb{R}_+^{n+1}} |\partial_t u|^2 dx dt \leq C \|f\|_2 \leq C \sup_{t>0} \|u(\cdot, t)\|_2.$$

□

Proof of Theorem 1.3. By the previous lemma, it is enough to establish the bound

$$(3.4) \quad \sup_{0 < \eta < 10^{-10}} \int_{\eta}^{2\eta} \left| \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} \nabla u \cdot \bar{\nabla} dx dt \right| d\delta \\ \leq C \left(\|t \nabla u\| + \sup_{t>0} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \right) \left(\|t \nabla \mathbf{v}\| + \sup_{t>0} \|\mathbf{v}(\cdot, t)\|_{L^2(\mathbb{R}^n)} \right).$$

We may suppose that the right hand side of (3.4) is finite, otherwise there is nothing to prove. On the left hand side of (3.4), for each fixed η , we integrate by parts in t to obtain the bound

$$(3.5) \quad \left| \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} \nabla u \cdot \bar{\partial}_t \bar{\nabla} dx dt d\delta \right| + \left| \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} \nabla \partial_t u \cdot \bar{\nabla} dx dt d\delta \right| + \text{boundary},$$

where the boundary terms are dominated by

$$C \left(\sup_{r>0} \int_r^{2r} \int_{\mathbb{R}^n} r^2 |\nabla u(x, t)|^2 dx dt \right)^{1/2} \left(\sup_{t>0} \|\mathbf{v}(\cdot, t)\|_2 \right) \\ \leq C \left(\sup_{t>0} \|u(\cdot, t)\|_2 \right) \left(\sup_{t>0} \|\mathbf{v}(\cdot, t)\|_2 \right),$$

and we have used (3.3) in the last step. Moreover, by Cauchy-Schwarz, the first term in (3.5) is no larger than $\|t \nabla u\| \|t \nabla \mathbf{v}\|$.

It therefore remains to treat the middle term in (3.5). To this end, we write $\nabla = \nabla_x + \partial_t e_{n+1}$, and $\mathbf{v} = v_{\parallel} + v_{n+1} e_{n+1}$, where $v_{n+1} \equiv \mathbf{v} \cdot e_{n+1}$. Now,

$$\left| \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} \nabla_x \partial_t u \cdot \bar{\nabla}_{\parallel} dx dt d\delta \right| \\ = \left| \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} \partial_t u \operatorname{div}_x \bar{\nabla}_{\parallel} dx dt d\delta \right| \leq C \|t \nabla u\| \|t \nabla \mathbf{v}\|,$$

as desired. Thus, it is enough to consider

$$(3.6) \quad \left| \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} \partial_t^2 u \overline{v_{n+1}} dx t dt d\delta \right| \leq \frac{1}{2} \left| \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} \partial_t^3 u \overline{v_{n+1}} dx t^2 dt d\delta \right| \\ + \frac{1}{2} \left| \int_{\eta}^{2\eta} \int_{\delta}^{1/\delta} \int_{\mathbb{R}^n} \partial_t^2 u \overline{\partial_t v_{n+1}} dx t^2 dt d\delta \right| + |\mathcal{B}| \\ \equiv |I| + |II| + |\mathcal{B}|,$$

where we have again integrated by parts in t , and \mathcal{B} denotes boundary terms which satisfy

$$|\mathcal{B}| \leq C \left(\sup_{r>0} \int_r^{2r} \int_{\mathbb{R}^n} r^4 |\partial_t^2 u(x, t)|^2 dx dt \right)^{1/2} \left(\sup_{t>0} \|\mathbf{v}(\cdot, t)\|_2 \right) \\ \leq C \left(\sup_{t>0} \|u(\cdot, t)\|_2 \right) \left(\sup_{t>0} \|\mathbf{v}(\cdot, t)\|_2 \right),$$

by a double application of Caccioppoli's inequality. Moreover,

$$|II| \leq C \|t^2 \partial_t^2 u\| \|t \partial_t \mathbf{v}\| \leq C \|t \partial_t u\| \|t \partial_t \mathbf{v}\|,$$

where we have used Caccioppoli in Whitney boxes to bound the first factor. Turning to the main term, we have that

$$(3.7) \quad |I| = C \left| \int_{\eta}^{2\eta} \int_{\delta/2}^{1/(2\delta)} \int_{\mathbb{R}^n} \partial_t^3 u(x, 2t) \overline{v_{n+1}(x, 2t)} dx t^2 dt d\delta \right|,$$

where we have made the change of variable $t \rightarrow 2t$. For t momentarily fixed, set

$$g_t(x) \equiv \partial_t u(x, t).$$

By Theorem 2.5 (i.e., the result of [1]), we have that

$$\tilde{u}_t(\cdot, s) \equiv \mathcal{D}_s \left(-\frac{1}{2}I + K \right)^{-1} g_t$$

is the unique solution of (D2) with data g_t . Hence, by t -independence,

$$\tilde{u}_t(\cdot, s) = \partial_t u(\cdot, t + s).$$

Setting $s = t$, we therefore obtain that

$$(D_{n+1}^3 u)(\cdot, 2t) = (\partial_t^2 \mathcal{D}_t) \left(-\frac{1}{2}I + K \right)^{-1} g_t.$$

We observe that by (2.1), (2.2) and (2.3),

$$\text{adj}(\partial_t^2 \mathcal{D}_t) = \partial_{v^*} \partial_t^2 S_{-t}^*.$$

Consequently, (3.7) becomes

$$|I| = C \left| \int_{\eta}^{2\eta} \int_{\delta/2}^{1/(2\delta)} \int_{\mathbb{R}^n} \left(-\frac{1}{2}I + K \right)^{-1} \partial_t u(\cdot, t) \overline{\partial_{v^*} D_{n+1}^2 S_{-t}^* v_{n+1}(\cdot, 2t)} dx t^2 dt d\delta \right|,$$

so by Cauchy-Schwarz and Theorem 2.5, it suffices to prove

$$(3.8) \quad \|t^2 \nabla D_{n+1}^2 S_{-t}^* \mathbf{v}(\cdot, 2t)\| \leq C (\|t \nabla \mathbf{v}\| + \|N_* \mathbf{v}\|_2).$$

The left hand side of (3.8) equals

$$\begin{aligned}
 (3.9) \quad & \left(\sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^n} |t^2 \nabla D_{n+1}^2 S_{-t}^* \mathbf{v}(\cdot, 2t)|^2 \frac{dx dt}{t} \right)^{1/2} \\
 & \leq \left(\sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^n} |t^2 \nabla D_{n+1}^2 S_{-t}^* (\mathbf{v}(\cdot, 2t) - \mathbf{v}(\cdot, 2t_k))|^2 \frac{dx dt}{t} \right)^{1/2} \\
 & \quad + \left(\sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \int_{\mathbb{R}^n} |t^2 \nabla D_{n+1}^2 S_{-t}^* \mathbf{v}(\cdot, 2t_k)|^2 \frac{dx dt}{t} \right)^{1/2} \equiv III + IV
 \end{aligned}$$

where $t_k = 2^{k-1}$. We consider term IV first. Dividing \mathbb{R}^n into cubes of side length 2^k , and using Caccioppoli's inequality, we deduce that

$$\begin{aligned}
 IV & \leq C \left(\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^{k+2}} \int_{\mathbb{R}^n} |t D_{n+1}^2 S_{-t}^* \mathbf{v}(\cdot, 2t_k)|^2 \frac{dx dt}{t} \right)^{1/2} \\
 & \leq C \left(\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^{k+2}} \int_{\mathbb{R}^n} |t D_{n+1}^2 S_{-t}^* (\mathbf{v}(\cdot, 2t_k) - \mathbf{v}(\cdot, 2t))|^2 \frac{dx dt}{t} \right)^{1/2} \\
 & \quad + \|\| (t D_{n+1}^2 S_{-t}^* 1)(P_t \mathbf{v}(\cdot, 2t)) \|\| + \|R_t \mathbf{v}(\cdot, 2t)\| \equiv IV_1 + IV_2 + IV_3,
 \end{aligned}$$

where

$$R_t \equiv t D_{n+1}^2 S_{-t}^* - (t D_{n+1}^2 S_{-t}^* 1) P_t,$$

and P_t is a nice approximate identity with a smooth, compactly supported kernel. By Lemma 2.10,

$$IV_3 \leq \|\| t \nabla \mathbf{v} \|\|.$$

By (2.6), Lemma 2.7, and a well known argument of Fefferman and Stein [8], we have that $|t \partial_t^2 S_{-t}^* 1|^2 \frac{dx dt}{t}$ is a Carleson measure, whence

$$IV_2 \leq C \|N_* \mathbf{v}\|_2.$$

By Lemma 2.7, the operator $f \rightarrow t D_{n+1}^2 S_{-t}^* f$ is bounded on $L^2(\mathbb{R}^n)$, so that

$$\begin{aligned}
 IV_1 & \leq C \left(\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^{k+2}} \int_{\mathbb{R}^n} \left| \frac{1}{\sqrt{t}} \int_{2t_k}^{2t} \partial_s \mathbf{v}(x, s) ds \right|^2 dx dt \right)^{1/2} \\
 & \leq C \left(\sum_{k=-\infty}^{\infty} \iint_{\mathbb{R}^{n+1}} \left| \int_{2t_k}^{2t} 1_{\{2^k \leq s \leq 2^{k+3}\}} \sqrt{s} \partial_s \mathbf{v}(x, s) ds \right|^2 dx dt \right)^{1/2} \leq C \|\| t \partial_t \mathbf{v} \|\|,
 \end{aligned}$$

where in the last step we have used the boundedness of the Hardy-Littlewood Maximal function.

Finally, we consider term III in (3.9). By Lemma 2.9, III may be handled like IV_1 above. We omit the details. \square

4. THE DOMAIN OF THE GENERATOR OF THE POISSON SEMIGROUP

In this section we generalize to our setting a result of [6] concerning the domain of the generator of the Poisson semigroup. We continue to suppose that the hypotheses of Theorem 1.3 hold. By Theorem 2.5, if $\epsilon \leq \epsilon_0$ is sufficiently small, then the Dirichlet

problem (D2) has a unique solution. Consequently, the solution operator $f \rightarrow \mathcal{P}(t)f \equiv u(\cdot, t)$, where u solves (D2) with data f , satisfies

$$(4.1) \quad \sup_{t>0} \|\mathcal{P}(t)\|_{2 \rightarrow 2} \leq C, \quad \lim_{t \rightarrow 0} \|\mathcal{P}(t)f - f\|_2 = 0,$$

and

$$(4.2) \quad \mathcal{P}(t+s) = \mathcal{P}(t)\mathcal{P}(s),$$

where the last identity uses also t -independence of the coefficients. Standard semigroup theory therefore implies that the semigroup $\{\mathcal{P}(t)\}$ has a densely defined infinitesimal generator on $L^2(\mathbb{R}^n)$, which we denote by \mathcal{A} . We will show that the domain $D(\mathcal{A})$ of this generator is the Sobolev space $L_1^2 \equiv L^2 \cap \dot{L}_1^2$. More precisely, we have the following

Theorem 4.3. *Suppose that L is as above. Then $D(\mathcal{A}) = L_1^2$, and*

$$\|\mathcal{A}f\|_2 \approx \|\nabla_x f\|_2.$$

We remark that this last theorem can be viewed as an extension of the Kato square root problem ([5],[11], [4],[10] and [3]) to the case that the coefficient matrix A is a full $(n+1) \times (n+1)$ matrix. Indeed, the Kato problem corresponds to the case that the coefficient matrix has the special ‘‘block’’ structure

$$(4.4) \quad \left[\begin{array}{c|c} & 0 \\ & \vdots \\ B & 0 \\ \hline 0 \dots 0 & 1 \end{array} \right]$$

where $B = B(x)$ is a $n \times n$ matrix. In the latter case the generator of the Poisson semigroup is

$$-\sqrt{-\operatorname{div}_x B \nabla_x},$$

and the conclusion of Theorem 4.3 is the (now established) Kato conjecture.

We also note that by Theorem 2.5, we have the representation

$$(4.5) \quad \mathcal{P}(t) = \mathcal{D}_t \left(-\frac{1}{2}I + K \right)^{-1}.$$

In order to prove the theorem we shall require the following result from [1].

Theorem 4.6. [1] *Suppose that L satisfies the hypotheses of Theorem 1.3. There exists a small constant $\epsilon_0 = \epsilon_0(n, \lambda, \Lambda)$ such that if ϵ in (1.4) satisfies $\epsilon \leq \epsilon_0$, then the single layer potential satisfies*

$$(4.7) \quad \sup_{t \in \mathbb{R}} \|\nabla S_t\|_{2 \rightarrow 2} \leq C,$$

and $S_0 \equiv S_t|_{t=0} : L^2(\mathbb{R}^n) \rightarrow \dot{L}_1^2(\mathbb{R}^n)$ is a bijection. Moreover, there is a unique solution to the Regularity problem

$$(R2) \quad \begin{cases} Lu = 0 \text{ in } \mathbb{R}_+^{n+1} \\ u(\cdot, t) \rightarrow f \in \dot{L}_1^2(\mathbb{R}^n) \text{ n.t.} \\ \tilde{N}_*(\nabla u) \in L^2(\mathbb{R}^n), \end{cases}$$

which has the representation

$$(4.8) \quad u(\cdot, t) \equiv S_t(S_0^{-1}f),$$

and $\partial_t u(\cdot, t)$ converges n.t. and in $L^2(\mathbb{R}^n)$ as $t \rightarrow 0$. Finally,

$$(4.9) \quad (\nabla S_t)|_{t=\pm s} f \rightarrow \mp \frac{1}{2} \cdot \frac{f(x)}{A_{n+1, n+1}(x)} e_{n+1} + \mathcal{T}f$$

weakly in $L^2(\mathbb{R}^n)$, where $\mathcal{T} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n, \mathbb{C}^{n+1})$ (see [1], Lemma 4.18 for a precise definition of \mathcal{T}).

Proof of Theorem 4.3. The deep results underlying Theorem 4.3 are Theorems 2.5 and 4.6, and we shall deduce the first as a straightforward corollary of the latter two. We observe that if u solves (R2) with data $f \in L^2_1$, then

$$\lim_{t \rightarrow 0} \partial_t u(\cdot, t) = \lim_{t \rightarrow 0} \partial_t \mathcal{P}(t)f \equiv \mathcal{A}f.$$

Thus, by (4.7), (4.8) and the bijectivity of S_0 ,

$$\|\mathcal{A}f\|_2 \leq C \|\nabla_x f\|_2.$$

The proof of the opposite inequality is only a bit harder, and we sketch the details briefly here. We modify slightly the strategy of Verchota [16]. By the well known Rellich identity (see, e.g., [12]), and the case $\epsilon = 0$ of Theorem 4.6, for A_0 real and symmetric we have that

$$(4.10) \quad \|\nabla_x u_0(\cdot, t)\|_2 \approx \|\partial_t u_0(\cdot, t)\|_2,$$

uniformly in $t \geq 0$, when $u_0(\cdot, t) \equiv S_t^0 f$, and S_t^0 is the single layer potential associated to $L_0 \equiv -\operatorname{div} A_0 \nabla$. By Theorem 4.6 and analytic perturbation theory,

$$(4.11) \quad \|(\nabla S_t^0 - \nabla S_t) f\| \leq C \|A_0 - A\|_\infty \|f\|_2.$$

The latter estimate, combined with (4.10) yields, uniformly in $t \geq 0$,

$$C^{-1} \|\partial_t S_t f\|_2 - C \epsilon_0 \|f\|_2 \leq \|\nabla_x S_t f\|_2 \leq C \|\partial_t S_t f\|_2 + C \epsilon_0 \|f\|_2.$$

Since the tangential derivatives $\nabla_x S_t f$ do not jump across the boundary, the latter bound, plus its analogue for the lower half space, and (4.9) imply

$$\left\| \frac{1}{2} (A_{n+1, n+1})^{-1} f + \mathcal{T}_{n+1} f \right\|_2 \leq \left\| \frac{1}{2} (A_{n+1, n+1})^{-1} f - \mathcal{T}_{n+1} f \right\|_2 + C \epsilon_0 \|f\|_2,$$

where $\mathcal{T}_{n+1} \equiv \mathcal{T} \cdot e_{n+1}$. Thus, by the accretivity of $A_{n+1, n+1}$ we have

$$\begin{aligned} \|f\|_2 &\leq \left\| \frac{1}{2} (A_{n+1, n+1})^{-1} f + \mathcal{T}_{n+1} f \right\|_2 + \left\| \frac{1}{2} (A_{n+1, n+1})^{-1} f - \mathcal{T}_{n+1} f \right\|_2 \\ &\leq C \left\| \frac{1}{2} (A_{n+1, n+1})^{-1} f - \mathcal{T}_{n+1} f \right\|_2 + C \epsilon_0 \|f\|_2. \end{aligned}$$

For ϵ_0 small enough, we may first hide the small term, and then obtain invertibility on L^2 of $-\frac{1}{2} (A_{n+1, n+1})^{-1} I + \mathcal{T}_{n+1}$ using (4.11) and the method of continuity as in [1]. Now, given $f \in \dot{D}(\mathcal{A})$, we set

$$\tilde{u}(\cdot, t) \equiv \mathcal{P}(t)f, \quad u(\cdot, t) \equiv S_t \left(-\frac{1}{2} (A_{n+1, n+1})^{-1} I + \mathcal{T} \right)^{-1} \mathcal{A}f,$$

so that $\partial_t \tilde{u}(\cdot, t)$, $\partial_t u(\cdot, t) \rightarrow \mathcal{A}f$ n.t. and in L^2 as $t \rightarrow 0$. By uniqueness in (D2), $\partial_t \tilde{u} = \partial_t u$, hence $\tilde{u} - u$ depends only on x , and therefore, since $L(\tilde{u} - u) = 0$, and $\tilde{u}(\cdot, t) - u(\cdot, t) \in \dot{L}^2_1$, for each fixed $t > 0$ (for \tilde{u} , this is a consequence of the representation (4.5)), we deduce that $\tilde{u} - u = \text{constant}$. Thus, we have that

$$\|\nabla_x f\|_2 \leq \sup_{t \geq 0} \|\nabla_x \tilde{u}(\cdot, t)\|_2 = \sup_{t \geq 0} \|\nabla_x u(\cdot, t)\|_2 \leq C \|\mathcal{A}f\|_2,$$

where in the last step we have used (4.7) and the bijectivity of $-\frac{1}{2} (A_{n+1, n+1})^{-1} I + \mathcal{T}_{n+1}$. \square

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