Local Tb Theorems and applications in PDE

Steve Hofmann*

Abstract. A Tb Theorem is a boundedness criterion for singular integrals, which allows the L^2 boundedness of a singular integral operator T to be deduced from sufficiently good behavior of T on some suitable non-degenerate test function b. However, in some PDE applications, including, for example, the solution of the Kato problem for square roots of divergence form elliptic operators, it may be easier to test the operator T locally (say on any given dyadic cube Q), on a test function b_Q that depends upon Q, rather than on a single, globally defined b. Or to be more precise, in the applications, it may be easier to find a family of b_Q's for which Tb_Q is locally well behaved, than it is to find a single b for which Tb is nice globally. In this lecture, we'll discuss some versions of local Tb theorems, as well as some applications to PDE.

Mathematics Subject Classification (2000). Primary 42B20, 42B25; Secondary 35J25, 35J55, 47F05, 47B44.

Keywords. Tb Theorem, singular integrals, square functions, boundary value problems, Kato problem, layer potentials.

1. Introduction

The Tb Theorem, and its predecessor, the $T1$ Theorem, were introduced in large part to better understand the Cauchy integral operator on a Lipschitz curve, and the related Calderón commutators. In this note, we shall discuss more recent "local" versions of the Tb Theorem, as well as the application of such theorems to some questions in PDE.

We begin by recalling the statements of the original $T1$ and Tb theorems. To this end, we require a few definitions.

We say that T is a singular integral operator (in the generalized sense of Coifman and Meyer), if T is a mapping from test functions $\mathcal{D}(\mathbb{R}^n)$ into distributions $\mathcal{D}'(\mathbb{R}^n)$, which is associated to a "Calderón-Zygmund kernel" $K(x, y)$, in the sense that

$$\langle T\varphi, \psi\rangle = \int\int \psi(x)K(x, y)\varphi(y)dxdy,$$

whenever $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ with disjoint supports (the theory can be extended to settings other than Euclidean space, and there are worthwhile reasons for doing so,

*supported by NSF

but we shall not insist much on that point, for the sake of simplicity of exposition). A Calderón-Zygmund kernel is one which satisfies the "standard" bounds

$$|K(x,y)| \leq C|x-y|^{-n} \tag{1.1}$$

and

$$|K(x,y+h)-K(x,y)| + |K(x+h,y)-K(x,y)| \leq \frac{C|h|^\alpha}{|x-y|^{n+\alpha}}, \tag{1.2}$$

for some $\alpha \in (0,1]$, whenever $|x-y| \geq 2|h|$. A singular integral operator T is said to satisfy the "Weak Boundedness Property" (WBP), if

$$\sup \left(R^{-n}|\langle T\varphi,\psi\rangle|\right) \leq C < \infty, \tag{1.3}$$

where the supremum runs over all $R > 0$, over all balls $B(x,R)$ of radius R and arbitrary center x, and over all test functions φ, ψ supported in $B(x,R)$, and normalized so that $\|\varphi\|_\infty + R\|\nabla\varphi\|_\infty \leq 1$ and $\|\psi\|_\infty + R\|\nabla\psi\|_\infty \leq 1$. In order to demystify this condition, we note that it holds automatically for any L^2 bounded operator (just apply Cauchy-Schwarz). Moreover, with just a small amount of work, it can be shown that, given an anti-symmetric kernel (i.e., one for which $K(x,y) = -K(y,x)$), which in addition satisfies the size condition (1), there is an associated "principal value" type singular integral operator for which WBP holds.

We recall that BMO is the space of locally integrable functions modulo constants for whom the norm

$$\|b\|_* = \sup |Q|^{-1} \int_Q |b(x) - [b]_Q| dx$$

is finite. Here, the supremum runs over all cubes (balls work just as well) with sides parallel to the co-ordinate axes, and

$$[b]_Q \equiv |Q|^{-1} \int_Q b(x)dx.$$

The $T1$ Theorem of David and Journé [DJ] is the following:

Theorem 1.1. *Suppose that T is a singular integral operator associated to a standard kernel $K(x,y)$ satisfying (1) and (2). Then T extends to a bounded operator on L^2 if and only if T satisfies WBP, and $T1, T^*1 \in BMO$.*

Here, T^* is the formal transpose of T. It is of course associated to the kernel $K^*(x,y) = K(y,x)$. One might ask whether T and T^* are well defined on constant functions, but it is not hard to show, using the kernel condition (2), that $T1$ and T^*1 exist as distributions modulo constants. This result may be understood as follows. If T is bounded on L^2, and its kernel satisfies the smoothness condition (1.2), then by a result obtained independently by Peetre [P], Spanne [Sp] and Stein [St] it follows that $T : L^\infty \to BMO$, and similarly for T^*. Conversely, if both T and T^* are bounded from $L^\infty \to BMO$, then by duality and interpolation (using results of Fefferman and Stein [FS]), we have that T is bounded on L^2. The $T1$ theorem says that in order to obtain the latter conclusion, one needn't test T on all of L^∞, but rather, only on a very special element of L^∞, namely the constant function 1.

The Tb Theorem is an extension of the $T1$ theorem, in which the function 1 is replaced by a suitable function $b \in L^\infty$ (or, more generally, by two such functions b_1 and b_2: one for T, and one for T^*). One supposes that $b_2 T b_1$ is a mapping from test functions to distributions which satisfies WBP (in particular, principal value operators associated to anti-symmetric kernels have this property) and that $Tb_1, T^* b_2 \in BMO$. Then, if b_1 and b_2 are sufficiently non-degenerate, one again deduces that T extends to a bounded operator on L^2. In the original versions of this theorem, b was assumed to be essentially bounded and "accretive", i.e., for some $\delta > 0$,

$$\Re e \, b \geq \delta,$$

or merely "pseudo-accretive":

$$\inf_Q |[b]_Q| \geq \delta,$$

or even "para-accretive", a relaxed version of pseudo-accretivity in which nondegeneracy of the average over each given cube is replaced by nondegeneracy of the average over some sub-cube of comparable size. The special case that $Tb_1 = 0 = T^* b_2$ (here 0 is meant in the sense of BMO, i.e., modulo constants) and b_1, b_2 are accretive, is due to McIntosh and Meyer [McM], the general case to David, Journé and Semmes [DJS].

The special case treated in [McM] already had a spectacular application: as a direct corollary, one obtains an alternative proof of the Cauchy integral theorem of Coifman, McIntosh and Meyer. Indeed, for a Lipschitz function A, the kernel $(x - y + i(A(x) - A(y))^{-1}$ is anti-symmetric and standard, so that L^2 boundedness of

$$T_A f(x) = p.v. \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(y)}{x - y + i(A(x) - A(y)} dy$$

follows immediately from the theorem of [McM] and the observation that, at least formally, by the formula of Plemelj, $T_A b = 1/2$, where $b = 1 + iA'$. In practice, some care must be taken in interpreting the Plemelj formula on an infinite graph, but this can be managed.

In some applications, it may not be at all evident that there is an accretive (or pseudo-accretive) b for which Tb is well behaved. On the other hand, in such cases it is sometimes possible to find a family $\{b_Q\}$, indexed by dyadic cubes Q, such that Tb_Q behaves well locally on Q. This motivates the introduction of the notion of a "local Tb Theorem", in which good local control of a singular integral operator T, on each member of a family of suitably non-degenerate functions b_Q (one for each dyadic cube Q), still suffices to deduce L^2 boundedness of T. The appropriate version of non-degeneracy in this setting was introduced by M. Christ [Ch]: a "pseudo-accretive system" is a collection of functions $\{b_Q\}$, indexed on the dyadic cubes, with b_Q supported in Q and integrable, such that for some $\delta > 0$, we have that

$$\left| |Q|^{-1} \int_Q b_Q \right| \geq \delta. \tag{1.4}$$

The first local Tb Theorem was proved by Christ:

Theorem 1.2. *[Ch] Suppose that T is a singular integral operator associated to a standard kernel K(x,y), which in addition we assume to be in L^∞. Suppose also that there are constants $\delta > 0$ and $C_0 < \infty$, and pseudo-accretive systems $\{b_Q^1\}, \{b_Q^2\}$, with supp $b_Q^i \subseteq Q$, $i = 1, 2$, such that for each dyadic cube Q,*

(i) $\|b_Q^1\|_{L^\infty(Q)} + \|b_Q^2\|_{L^\infty(Q)} \leq C_0$

(ii) $\|Tb_Q^1\|_{L^\infty(Q)} + \|T^* b_Q^2\|_{L^\infty(Q)} \leq C_0$

(iii) $\delta|Q| \leq \min(\left|\int_Q b_Q^1\right|, \left|\int_Q b_Q^2\right|)$.

Then T extends to a bounded operator on L^2, with bound depending only on n, δ, C_0 and the kernel constants in (1.1) and (1.2), but not on the L^∞ norm of $K(x, y)$.

A few remarks are in order. The assumption that $K \in L^\infty$ is merely qualitative, and is satisfied, e.g., by smooth truncations of a standard kernel. This assumption allows one to make certain formal manipulations with impunity, during the course of the proof. Christ actually proved this theorem in the setting of a space of homogeneous type (that is, a space endowed with a pseudo-metric and a doubling measure), which (as he demonstrated) possesses a suitable version of a "dyadic cube" structure. Christ's theorem and the technique of its proof are related to the solution of Painlevé's problem concerning the characterization of those compact sets $K \subset \mathbb{C}$ for which there exist non-constant bounded analytic functions on $\mathbb{C} \setminus K$. We will not discuss this aspect of the theory in detail, but we mention that extensions of either local or global Tb Theorems to the non-doubling setting have been obtained by G. David [D1] and by Nazarov, Treil and Volberg [NTV1, NTV2]; moreover, the circle of ideas involved in [Ch], [D1] and [NTV1, NTV2] have played a crucial role in the solution of the Painlevé problem, see Mattila, Melnikov and Verdera [MMV], G. David [D1, D2] and X. Tolsa [T], and also Volberg [Vo], where the higher dimensional version of this theory is treated.

Instead, in this note we shall concentrate on extensions of Christ's result in another direction, in which L^∞ control of b_Q and Tb_Q is replaced by local, scale invariant L^2 control. These extensions have been useful in certain applications in PDE, including the solution of the Kato problem. In the next two sections, we discuss local Tb theorems for square functions, and for singular integrals, respectively.

Acknowledgements. Various portions of the material discussed in this paper were joint work with M. Alfonseca, P. Auscher, A. Axelsson, M. Lacey, S. Kim, A. McIntosh, C. Muscalu, T. Tao, P. Tchamitchian and C. Thiele. To all of them I express my deepest appreciation.

2. Local *Tb* Theorems for Square Functions, and Applications

We begin with a local Tb theorem for square functions, which extends a global version due to Semmes [S]. Suppose that we have a family of kernels $\{\psi_t(x,y)\}_{t\in(0,\infty)}$, satisfying, for some exponent $\alpha > 0$,

$$\begin{aligned} |\psi_t(x,y)| &\leq C\frac{t^\alpha}{(t+|x-y|)^{n+\alpha}} \\ |\psi_t(x,y+h)-\psi_t(x,y)| &\leq C\frac{|h|^\alpha}{(t+|x-y|)^{n+\alpha}} \end{aligned} \tag{2.1}$$

whenever $|h| \leq \frac{1}{2}|x-y|$ or $|h| \leq |t|/2$.

Theorem 2.1. *Let $\theta_t f(x) \equiv \int \psi_t(x,y)f(y)dy$, where $\psi_t(x,y)$ satisfies* (2.1)*. Suppose also that there exist constants $\delta > 0$, $C_0 < \infty$, and a system $\{b_Q\}$ of functions indexed by dyadic cubes $Q \subseteq \mathbb{R}^n$ such that for each dyadic cube Q*

(i) $\int_{\mathbb{R}^n} |b_Q|^2 \leq C_0|Q|$

(ii) $\int_0^{\ell(Q)}\int_Q |\theta_t b_Q(x)|^2 \frac{dxdt}{t} \leq C_0|Q|$

(iii) $\delta|Q| \leq \left|\int_Q b_Q\right|$.

Then we have the square function bound

$$\iint_{\mathbb{R}^{n+1}_+} |\theta_t f(x)|^2 \,\frac{dxdt}{t} \leq C\|f\|_2^2.$$

Proof. The proof combines the ideas of [S] and [AT] with those of [Ch], [HMc], [HLMc] and [AHLMcT]. (Actually, the argument below preceded the subsequent matrix-valued versions used in [HMc], [HLMc] and [AHLMcT] to solve the Kato problem, but the author never published it in this scalar-valued form; see also Auscher's lecture notes on the Kato problem [A], where the present formulation is given explicitly). We begin by recalling the following well known fact, due explicitly to Christ and Journé [CJ], but also at least implicit in the work of Coifman and Meyer [CM].

Proposition 2.2. *[CJ] Let $\theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x,y)f(y)dy$, where $\psi_t(x,y)$ satisfies* (2.1)*. Suppose that we have the Carleson measure estimate*

$$\sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\theta_t 1(x)|^2\frac{dxdt}{t} \leq C. \tag{2.2}$$

Then we have the square function estimate

$$\iint_{\mathbb{R}^{n+1}_+} |\theta_t f(x)|^2 \,\frac{dxdt}{t} \leq C\|f\|_2^2. \tag{2.3}$$

Remark 2.3. The converse direction (i.e. that (2.3) implies (2.2)) is essentially due to Fefferman and Stein [FS].

Thus, to prove Theorem 2.1, it suffices to verify that $|\theta_t 1|^2 dxdt/t$ is a Carleson measure, given the existence of a family $\{b_Q\}$ satisfying hypotheses (i), (ii) and (iii) of the Theorem. To this end, we first observe that, as in [S] and [AT], it is enough to verify that for $\{b_Q\}$ as in the Theorem, we have the bound

$$\sup_Q \frac{1}{|Q|} \iint_{R_Q} |\theta_t 1|^2 \frac{dxdt}{t} \leq C \sup_Q \frac{1}{|Q|} \iint_{R_Q} |(\theta_t 1)(P_t b_Q)|^2 \frac{dxdt}{t} + C, \tag{2.4}$$

where $R_Q \equiv Q \times (0, \ell(Q))$ is the "Carleson box" above Q, and where P_t is a nice approximate identity, whose kernel satisfies, say, (2.1). Indeed, suppose momentarily that (2.4) holds. Then to obtain (2.2) (and thus also the conclusion of the Theorem), it suffices to show that the right hand side of (2.4) is bounded. Following [CM], we write

$$\begin{aligned} (\theta_t 1) P_t b_Q &= [(\theta_t 1) P_t b_Q - \theta_t b_Q] + \theta_t b_Q \\ &= R_t b_Q + \theta_t b_Q. \end{aligned}$$

The contribution of $\theta_t b_Q$ is bounded, by hypothesis (ii) of the Theorem. Moreover, by (2.1) and the fact that $R_t 1 = 0$, it follows by a well-known orthogonality argument that

$$\iint_{\mathbb{R}^{n+1}_+} |R_t f(x)|^2 \, \frac{dxdt}{t} \leq C \|f\|_2^2.$$

Thus, by (i), the contribution of $R_t b_Q$ is also bounded.

Therefore, to finish the proof of the Theorem, it remains to verify (2.4). In fact, it suffices to proof that (2.4) holds with P_t replaced by the dyadic averaging operator A_t, defined by

$$A_t f(x) \equiv A_t^Q f(x) \equiv \frac{1}{|Q(x,t)|} \int_{Q(x,t)} f(y) dy,$$

where $Q(x,t)$ denotes the minimal dyadic subcube of Q containing x, with side length at least t. Indeed, a standard orthogonality argument yields the fact that

$$\iint_{R_Q} |(A_t - P_t) f(x)|^2 \, \frac{dxdt}{t} \leq C \|f\|_2^2,$$

so that the error is bounded.

Now, by a well known "John-Nirenberg" type lemma for Carleson measures (see, e.g. [AHLT, Lemma 3.3]), in order to establish (2.6) (or rather its analogue with A_t in place of P_t), it suffices to show that there is a positive constant $\eta > 0$ such that for each Q, there is a dyadic sawtooth region

$$E_Q^* \equiv R_Q \setminus (\cup R_{Q_j}), \tag{2.5}$$

where $\{Q_j\}$ are non-overlapping dyadic sub-cubes of Q, with

$$|Q \setminus (\cup Q_j)| > \eta |Q|$$

and

$$\iint_{E_Q^*} |\theta_t 1(x)|^2 \frac{dxdt}{t} \leq C \iint_{E_Q^*} |(\theta_t 1(x))(A_t b_Q(x))|^2 \frac{dxdt}{t}. \tag{2.6}$$

We prove (2.6) via a stopping time argument as in [HMc], [HLMc] and [AHLMcT] (but see also [Ch], where a similar idea had previously appeared). Our starting point is (iii). Dividing by an appropriate complex constant, we may suppose that

$$\frac{1}{|Q|} \int_Q b_Q = 1. \tag{2.7}$$

We then sub-divide Q dyadically, to select a family of non-overlapping cubes $\{Q_j\}$ which are maximal with respect to the property that

$$\Re e \, \frac{1}{|Q_j|} \int_{Q_j} b_Q \leq 1/2. \tag{2.8}$$

If E_Q^* is defined as in (2.5) with respect to this family $\{Q_j\}$, then by construction, if $(x,t) \in E_Q^*$, it follows that

$$\frac{1}{2} \leq \Re e \, A_t b_Q(x),$$

so that (2.6) holds with $C = 4$. It remains only to verify that there exists $\eta > 0$ such that

$$|E| > \eta |Q|, \tag{2.9}$$

where $E \equiv Q \setminus (\cup Q_j)$. By (2.7) we have that

$$\begin{aligned} |Q| = \int_Q b_Q = \Re e \int_Q b_Q &= \Re e \int_E b_Q + \Re e \sum_j \int_{Q_j} b_Q \\ &\leq |E|^{\frac{1}{2}} \left(\int_Q |b_Q|^2 \right)^{\frac{1}{2}} + \frac{1}{2} \sum |Q_j|, \end{aligned}$$

when in the last step we have used (2.8). From hypothesis (i) of Theorem 2.1, we then obtain that

$$|Q| \leq C |E|^{\frac{1}{2}} |Q|^{\frac{1}{2}} + \frac{1}{2} |Q|,$$

and (2.9) now follows readily. This concludes the proof of Theorem 2.1.

□

As alluded to above, the previous Theorem has an extension to the matrix valued setting. We shall explain momentarily why this is interesting. Let $\mathbb{M}^N$ denote the space of $N \times N$ matrices with complex entries.

Theorem 2.4. *Suppose that* $\mathbf{\Psi_t} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^N$ *satisfies (2.1). Define, for* $\mathbf{f} : \mathbb{R}^n \to \mathbb{C}^N$, *the operator*

$$\mathbf{\Theta_t} \cdot \mathbf{f}(x) \equiv \int \mathbf{\Psi}_t(x,y) \cdot \mathbf{f}(y) dy. \tag{2.10}$$

Suppose also that there are constants $\delta > 0$, $C_0 < \infty$ *and a system of matrix valued functions* $\mathbf{b}_Q : \mathbb{R}^n \to \mathbb{M}^N$, *indexed on the dyadic cubes, such that*

(i) $\int_{\mathbb{R}^n} |\mathbf{b}_Q|^2 \leq C_0 |Q|$

(ii) $\int_0^{\ell(Q)} \int_Q |\mathbf{\Theta}_t \mathbf{b}_Q(x)|^2 \frac{dxdt}{t} \leq C_0 |Q|$

(iii) $\delta |\xi|^2 \leq \Re e \ \xi \cdot \left(|Q|^{-1} \int_Q \mathbf{b}_Q \right) \overline{\xi}.$

where the ellipticity condition (iii) holds for all $\xi \in \mathbb{C}^N$, *and where the action of* $\mathbf{\Theta}_t$ *on the matrix valued function* $\mathbf{b}_Q$ *is defined in the obvious way as in* (2.10) *by viewing the kernel* $\mathbf{\Psi}_t(x,y)$ *as a* $1 \times N$ *matrix which multiplies the* $N \times N$ *matrix* $\mathbf{b}_Q$. *Then*

$$\iint_{\mathbb{R}^{n+1}_+} |\mathbf{\Theta}_t \cdot \mathbf{f}|^2 \frac{dxdt}{t} \leq C \|\mathbf{f}\|_2^2.$$

It turns out that a variant of this theorem lies at the heart of the solution of the Kato problem [HMc], [HLMc],[AHLMcT] (see also [AT]). We now sketch the proof, which is essentially the same as the argument used to establish the Kato conjecture. Let $\mathbf{1}$ denote the $N \times N$ identity matrix. Since

$$\mathbf{\Theta}_t \mathbf{1} = (\theta_t^1 1, \theta_t^2 1, ..., \theta_t^N 1),$$

Proposition 2.3 therefore implies that it is enough to show that $|\mathbf{\Theta}_t \mathbf{1}|^2 \, t^{-1} dxdt$ is a Carleson measure. For ϵ small, but fixed, we cover $\mathbb{C}^N$ by cones of aperture ϵ. Enumerating these cones as $\Gamma_1^\epsilon, ..., \Gamma_K^\epsilon$,, where $K = K(\epsilon, N)$, we see that

$$\int_0^{\ell(Q)} \int_Q |\mathbf{\Theta}_t \mathbf{1}|^2 \frac{dxdt}{t} = \sum_{k=1}^K \int_0^{\ell(Q)} \int_Q |\mathbf{\Theta}_t \mathbf{1}|^2 \, 1_{\Gamma_k^\epsilon}(\mathbf{\Theta}_t \mathbf{1}) \frac{dxdt}{t}.$$

Thus, it is enough to show that there is a uniform constant

$$C_1 = C_1(\epsilon, \delta, C_0, n, N)$$

such that

$$\sup_Q |Q|^{-1} \int_0^{\ell(Q)} \int_Q |\mathbf{\Theta}_t \mathbf{1}|^2 \, 1_{\Gamma^\epsilon}(\mathbf{\Theta}_t \mathbf{1}) \frac{dxdt}{t} \leq C_1,$$

for each fixed cone Γ^ϵ with ϵ small enough.

To this end, normalizing so that $\delta = 1$, and fixing Q, we follow the stopping time argument of the previous theorem, in the present case extracting dyadic subcubes

$Q_j \subset Q$ which are maximal with respect to the property that at least one of the following holds:

$$\int_{Q_j} |\mathbf{b}_Q| \geq \frac{1}{4\epsilon} \tag{2.11}$$

or

$$\Re e\ \nu \cdot \left(|Q_j|^{-1} \int_{Q_j} \mathbf{b}_Q \right) \overline{\nu} \leq \frac{3}{4}, \tag{2.12}$$

where $\nu \in \mathbb{C}^N$ is the unit normal in the direction of the central axis of Γ^ϵ, i.e.,

$$\Gamma^\epsilon = \{z \in \mathbb{C}^N : \left| \frac{z}{|z|} - \nu \right| < \epsilon\}.$$

As in the proof of the previous theorem, one may check that

$$|E| \equiv |Q \setminus (\cup Q_j)| \geq \eta |Q|,$$

for some fixed $\eta > 0$. Moreover, for $(x,t) \in E_Q^* \equiv R_Q \setminus (\cup R_{Q_j})$, (we recall that $R_Q \equiv Q \times (0, \ell(Q)$ is the Carleson box above Q) and for $z \in \Gamma^\epsilon$, we claim that

$$|z \cdot A_t \mathbf{b}_Q(x) \overline{\nu}| \geq \frac{1}{2} |z|, \tag{2.13}$$

where again A_t denotes the dyadic averaging operator with respect to the dyadic grid of Q. Indeed, since the opposite inequalities to (2.14) and (2.15) hold in E_Q^*, we have that

$$|\omega \cdot A_t \mathbf{b}_Q(x) \overline{\nu}| \geq |\nu \cdot A_t \mathbf{b}_Q(x) \overline{\nu}| - |(\omega - \nu) \cdot A_t \mathbf{b}_Q(x) \overline{\nu}| \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

whenever $|\omega - \nu| < \epsilon$ and $(x,t) \in E_Q^*$. Taking $\omega = z/|z|$, with $z \in \Gamma^\epsilon$, we obtain (2.16).

Consequently, we have that

$$\iint_{E_Q^*} |\mathbf{\Theta}_t \mathbf{1}|^2 \, 1_{\Gamma^\epsilon}(\mathbf{\Theta}_t \mathbf{1}) \frac{dxdt}{t} \leq 4 \iint_{E_Q^*} |\mathbf{\Theta}_t \mathbf{1} \cdot A_t \mathbf{b}_Q \overline{\nu}|^2 \, \frac{dxdt}{t},$$

and the rest of the proof follows as in the previous theorem.

As mentioned above, a variant of this last theorem leads to the solution of the Kato problem. We recall the statement of the problem. Let B be an $n \times n$ matrix of complex, L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda |\xi|^2 \leq \Re e \, \langle B\xi, \xi \rangle \equiv \Re e \sum_{i,j} B_{ij}(x) \, \xi_j \, \overline{\xi}_i \,, \quad \|B\|_\infty \leq \Lambda, \tag{2.14}$$

for $\xi \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \leq \Lambda < \infty$. We define a divergence form operator

$$Ju \equiv -\operatorname{div}(B(x)\nabla u), \tag{2.15}$$

which we interpret in the usual weak sense via a sesquilinear form. The accretivity condition (2.14) enables one to define an accretive square root $\sqrt{J} \equiv J^{1/2}$ (see [K1, K2]), and the "Kato problem", or "square root problem", is to establish the estimate

$$\|\sqrt{J}f\|_{L^2(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^n)}, \tag{2.16}$$

with C depending only on n, λ and Λ. The latter estimate is connected with the question of the analyticity of the mapping $B \to J^{\frac{1}{2}}$, which in turn has applications to the perturbation theory for certain classes of hyperbolic equations (see [Mc]). We remark that (2.16) is equivalent to the opposite inequality for the square root of the adjoint operator J^* (which amounts to the L^2 boundedness of the Riesz transforms $\nabla(J^*)^{-1/2}$). In [HMc, HLMc, AHLMcT], (but see also [AT]), estimate (2.16) was deduced, in effect, from a variant of Theorem 2.12, with $N = n$, in which the matrix $\mathbf{b}_Q$ is the derivative matrix of a carefully chosen $\mathbb{C}^n$-valued solution F_Q of an appropriate PDE. For example, one can take F_Q to be a certain $W^{1,2}$ solution of the parabolic equation $\frac{\partial u}{\partial t} + Ju = 0$, with t frozen at the scale $t = (\epsilon\ell(Q))^2$ (ϵ chosen small, but fixed depending on n, λ and Λ), or it could be a solution of the resolvent equation $(1 + (\epsilon\ell(Q))^2 J)F_Q = x$. In the case of the Kato problem, the operators $\mathbf{\Theta}_t$ which arise do not satisfy the kernel conditions (2.1), but they do posses some extra structure inherited from the operator J, which suffices to carry through the same argument sketched above in the proof of Theorem 2.12.

3. Local *Tb* Theorems for Singular Integrals, and Applications

To help motivate our next application, we discuss the Kato problem from the perspective of elliptic boundary value problems. Consider the Dirichlet problem

$$\begin{cases} u_{tt} + \operatorname{div}_x B(x)\nabla_x u = 0 \text{ in } \mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^n \times (0,\infty)\} \\ u(x,0) = f(x) \in L^2(\mathbb{R}^n). \end{cases} \tag{D}$$

Then a solution u is given in terms of the Poisson semigroup: $u(x,t) = e^{-t\sqrt{J}}f(x)$. Note that the outward normal derivative is given by

$$\frac{\partial u}{\partial \nu} = -\frac{\partial u}{\partial t} = \sqrt{J}u,$$

and that the tangential gradient is simply

$$\nabla_{tan} u = \nabla_x u.$$

Thus, the Kato estimate (2.16), together with the reverse inequality for the Riesz transforms $\nabla J^{-1/2}$, can be thought of as a "Rellich identity"

$$\|\frac{\partial u}{\partial \nu}\|_2 \approx \|\nabla_{tan} u\|_2 \tag{3.1}$$

for solutions of the boundary value problem (D). The Rellich identity, in turn, plays a vital role in the solution of the Neumann and regularity problems with L^2 estimates (see, e.g., Jerison and Kenig [JK2], Verchota [V] and Kenig and Pipher [KP]); moreover, a local scale invariant Rellich identity can be used to establish L^2 estimates for solutions of the Dirichlet problem [JK1, JK3]. We observe that the equation

$$u_{tt} + \operatorname{div}_x B(x)\nabla_x u = 0$$

can be written in the form

$$\operatorname{div}_{x,t} A(x)\nabla_{x,t} u = 0, \tag{3.2}$$

where A is the $(n+1)\times(n+1)$ matrix

$$\left[\begin{array}{c|c} & 0 \\ B & \vdots \\ & 0 \\ \hline 0\cdots 0 & 1 \end{array}\right]. \tag{3.3}$$

The question then naturally arises whether the special "semi-group structure" (3.3) is needed to establish the "Rellich identity" (3.1) (and more generally, to obtain L^2 estimates for the Dirichlet, Neumann and regularity problems). Perhaps one might be able to consider equations of the type (3.2) with a "full" $(n+1)\times(n+1)$ elliptic coefficient matrix $A(x)$ (still independent of the t variable). Indeed, for real symmetric coefficients, this is the case [JK3], [KP], [Ke]. Unfortunately, the analogous statement fails for coefficients which are real, but non-symmetric (let alone complex): solvability with L^2 estimates does not hold in general if the non-symmetry is sufficiently severe [KKPT]. On the other hand, it turns out that the magnitude of the non-symmetry matters. Suppose that $A_1(x)$ is a complex, L^∞ elliptic matrix (in the sense of (2.17), but now with $\xi \in \mathbb{C}^{n+1}$), and that $\|A_1 - A_0\|_{L^\infty(\mathbb{R}^n)} \leq \epsilon$, where $A_0(x)$ is real, symmetric, L^∞ and elliptic, and where ϵ depends only on dimension and the ellipticity parameters for A_0. Then the Rellich identity (3.1) holds for solutions of the equation

$$L_1 u = -\operatorname{div} A_1(x)\nabla u = 0$$

(in (3.1), $\frac{\partial u}{\partial \nu}$ now denotes the "co-normal" derivative), and one has solvability with L^2 estimates for the Dirichlet, Neumann and Regularity problems [AAAHK]. The proof entails establishing an analytic perturbation result for the layer potentials associated to operators close to $L_0 = -\operatorname{div} A_0(x)\nabla$, and therefore the first step requires that we obtain L^2 boundedness (and invertibility) of the layer potentials associated to L_0. We remark here that since L_0 has real, symmetric, t-independent coefficients, the solvability, with L^2 estimates, of the Dirichlet [JK1, JK3], [Ke, pp 63-64] and Neumann and Regularity [KP] problems for the equation $L_0 u = 0$ was already known, but the layer potential theory is new, and is used to jump start the perturbation scheme. This first step brings us back to the subject of local Tb theorems, this time for singular integrals rather than for square functions.

(Actually, the subsequent analytic perturbation step also uses local Tb technology, in the spirit of the proof of the Kato problem, but this aspect of the theory is rather involved, and we shall not discuss it here).

The following theorem was (essentially) proved in [AHMTT].

Theorem 3.1. *Let T be a singular integral operator associated to a kernel K satisfying the Calderón-Zygmund kernel conditions (1.1) and (1.2), as well as the generalized truncation condition $K(x,y) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose also that there exist pseudo-accretive systems $\{b_Q\}$, $\{b_Q^*\}$ such that* $\operatorname{supp} b_Q$, $\operatorname{supp} b_Q^* \subseteq Q$, *and*

(i) $\int_Q (|b_Q|^{2+\epsilon} + |b_Q^*|^{2+\epsilon}) \leq C|Q|$, *for some* $\epsilon > 0$

(ii) $\int_Q (|Tb_Q|^2 + |T^* b_Q^*|^2) \leq C|Q|$

(iii) $\frac{1}{C}|Q| \leq \min\left(|\int_Q b_Q|, |\int_Q b_Q^*|\right)$.

Then $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, with bound independent of $\|K\|_\infty$.

Remark. This theorem was proved in [AHMTT] for so-called “perfect dyadic” singular integral operators, which are associated to a kernel for which the smoothness condition (1.2) is replaced by the condition that

$$K(x,y) - K(x',y) = 0, \tag{3.4}$$

whenever $x, x' \in Q$ and $y \in Q^c$, where Q is a dyadic cube. In that case, one can take ϵ to be 0 in condition (i). The proof of the present theorem is essentially the same as that in [AHMTT] except that one is forced to treat several error terms caused by the decaying tail in (1.2), which are absent when one has the perfect localization (3.5) (see [AAAHK] for details). We will not give the proof of this theorem here (it is a bit complicated), although we shall briefly discuss some of the ideas involved in its proof. First however, we describe how one may use this theorem to deduce boundedness of the layer potentials associated to a divergence form elliptic operator in $\mathbb{R}^{n+1}_+$, with real symmetric $t-$independent coefficients.

Suppose now that

$$Lu = -\operatorname{div} A(x)\nabla u,$$

is defined in $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times (-\infty, \infty)\}$ (so that $\operatorname{div}, \nabla$ are taken in all $n+1$ variables x and t), where $A(x)$ is real, symmetric, elliptic and L^∞. There is a global fundamental solution

$$\Gamma(x,t,y,s) = \Gamma(x,t-s,y,0)$$

associated to L, which by De Giorgi-Nash-Moser estimates satisfies

$$|\Gamma(x,t,y,0)| \leq C(|t| + |x-y|)^{1-n} \tag{3.5}$$

$$\left|\frac{\partial}{\partial t}\Gamma(x,t,y,0)\right| \leq C(|t| + |x-y|)^{-n}, \tag{3.6}$$

$$\begin{aligned} \left|\frac{\partial}{\partial t}\left(\Gamma(x+h,t,y,0) - \Gamma(x,t,y,0)\right)\right| + \left|\frac{\partial}{\partial t}\left(\Gamma(x,t,y+h,0) - \Gamma(x,t,y,0)\right)\right| \\ \leq C\frac{|h|^\alpha}{(|t| + |x-y|)^{n+\alpha}}, \end{aligned} \tag{3.7}$$

whenever $|h| \leq \frac{1}{2}|x-y|$ or $|h| \leq |t|/2$, for some $\alpha > 0$. We define as usual the single layer potential operator

$$S_t f(x) \equiv \int_{\mathbb{R}^n} \Gamma(x,t,y,0) f(y) dy,$$

and also singular integrals

$$T_t f(x) \equiv \frac{\partial}{\partial t} S_t f(x) = \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \Gamma(x,t,y,0) f(y) dy.$$

We observe that, by virtue of (3.7) and (3.8), the kernel of the latter operator satisfies the hypotheses of Theorem 3.4, uniformly in $t > 0$. Thus, the estimate

$$\sup_{t>0} \|T_t f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_2 \tag{3.8}$$

will follow immediately if we can produce pseudo-accretive systems $\{b_Q\}$, $\{b_Q^*\}$ satisfying the conditions (i),(ii) and (iii). Given (3.9), bounds for the tangential derivatives of $S_t f$ will follow from the estimate

$$\|\nabla_x S_t f\|_2 \leq C\|\frac{\partial}{\partial t} S_t f\|_2,$$

which in turn is easily obtained for real, symmetric coefficients via integration by parts (indeed, it is a consequence of the Rellich identity). The invertibility of the appropriate boundary integral operators is also then obtainable from the Rellich identity. Let us now indicate how one may deduce (3.9) from Theorem 3.4. We shall only produce a pseudo-accretive system for the operator T_t; one may treat T_t^* by a transparent variation of the same method. We recall the following fundamental result of Jerison and Kenig [JK3] (see also [Ke, pp 63-64]):

Theorem 3.2. *[JK3] Suppose that $L = -\operatorname{div} A\nabla$, where A is real, symmetric, $(n+1)\times(n+1)$, t-independent, L^∞ and uniformly elliptic. Then the elliptic-harmonic measure associated to L, in the lower half-space $\mathbb{R}^{n+1}_-$, is absolutely continuous with respect to n-dimensional Lebesgue measure on the boundary $\{t=0\}$. Moreover if $k^{A_Q^-}(y)$ denotes the Poisson kernel, at the point $A_Q^- = (x_Q, -\ell(Q))$, where Q is a cube on the boundary with center x_Q, then we have the scale invariant estimate*

$$\int_{\mathbb{R}^n} (k^{A_Q^-}(y))^{2+\epsilon} dy \leq C|Q|^{-1-\epsilon}, \tag{3.9}$$

for some $\epsilon > 0$ depending only on dimension and ellipticity.

We now set

$$b_Q \equiv |Q| 1_Q\, k^{A_Q^-}. \tag{3.10}$$

Observe that condition (i) of Theorem 3.4 follows immediately from Theorem 3.10. Moreover (iii) is an immediate consequence of the following well known estimate of Caffarelli, Fabes, Mortola and Salsa [CFMS] (see also [Ke, Lemma 1.3.2, p. 9]):

$$\int_Q k^{A_Q^-}(y) dy \geq \frac{1}{C}. \tag{3.11}$$

It remains to establish condition (ii). We consider first

$$\partial_t S_t \tilde{b}_Q(x) = |Q| \int_{\mathbb{R}^n} \partial_t \Gamma(x,t,y,0) k^{A_Q^-}(y) dy$$
$$= |Q| \partial_t \Gamma(x,t,A_Q^-),$$

where $\tilde{b}_Q$ is defined as in (3.10) (except that we have dropped the indicator function of the cube Q), and where we have used that for $(x,t) \in \mathbb{R}^{n+1}_+$ fixed, the function $\partial_t \Gamma(x,t,\cdot,\cdot)$ solves $Lu = 0$ in $\mathbb{R}^{n+1}_-$. Since $t > 0$, we then have by (3.5) and translation invariance in t that

$$|\partial_t S_t b_Q(x)| \leq C,$$

uniformly in $(x,t) \in \mathbb{R}^{n+1}_+$. It is not hard to use integration by parts, coupled again with the fact that $\partial_t \Gamma(x,t,\cdot,\cdot)$ is a solution in the lower half-space to obtain a similar estimate for $\partial_t S_t(\eta_Q \tilde{b}_Q)(x)$, where $\eta_Q \in C_0^\infty$, $\eta_Q \equiv 1$ on $5Q$, $\operatorname{supp} \eta_Q \leq 6Q$, with $\|\nabla \eta_Q\|_\infty \leq C/\ell(Q)$. One then obtains the L^2 bound (ii) for $\partial_t S_t b_Q$ by using (i) and the kernel estimate (3.7) to handle the error $\partial_t S_t \left((\eta_Q - 1_Q) b_Q \right)(x)$. We omit the details (the interested reader may consult [AAAHK]).

Let us conclude the article by sketching some of the ideas involved in the proof of Theorem 3.4. We begin by trying to mimic, as far as possible, the proof of Theorem 2.2. By the $T1$ Theorem, it is enough to show that $T1 \in BMO$ (we ignore the matter of establishing WBP - it turns out that there is a local version of the $T1$ condition, in which 1 is replaced by 1_Q, that yields weak boundedness also, and in practice, it is this local condition that one establishes). By [FS], it would be enough to verify the Carleson measure estimate

$$\sup_Q |Q|^{-1} \int_0^{\ell(Q)} \int_Q |\Delta_t T1(x)|^2 \frac{dxdt}{t} < \infty, \tag{3.12}$$

where

$$\Delta_t f(x) \equiv \int t^{-n} \psi \left(\frac{x-y}{t} \right) f(y) dy,$$

and $\psi \in C_0^\infty(\{|x| < 1\})$ is radial with

$$\int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} = 1.$$

We attempt to proceed as in the proof of Theorem 2.2, but now with $\Delta_t T$ in place of θ_t. As before, we use conditions (i) and (iii) to produce a dyadic sawtooth region E_Q^*, with $|\partial E_Q^* \cap Q| > \eta |Q|$, on which

$$|\Delta_t T1| \leq C |(\Delta_t T1)(P_t b_Q)|$$

(modulo acceptable errors). It is then enough to control

$$\iint_{E_Q^*} |(\Delta_t T1)(P_t b_Q)|^2 \frac{dxdt}{t}.$$

Again following the idea of [CM], we write

$$(\Delta_t T1)(P_t b_Q) = [(\Delta_t T1)(P_t b_Q) - \Delta_t T b_Q] + \Delta_t T b_Q \equiv \Lambda_t b_Q + \Delta_t T b_Q.$$

The contribution of the second summand can be handled using condition (ii) and the boundedness of the square function

$$f \to \left(\int_0^\infty |\Delta_t f|^2 \frac{dt}{t} \right)^{1/2}.$$

It is the first summand which causes problems. In contrast to the situation in Theorem 2.2, in which the kernel of the operator $R_t = (\theta_t 1)P_t - \theta_t$ gave rise to a bounded square function, the contribution of $\Lambda_t b_Q$ is now problematic, owing to the failure of the estimates (2.1) for the kernel of Λ_t. Let us try to isolate the difficulty, by writing

$$\Lambda_t = [(\Delta_t T1)P_t - \Delta_t T P_t] + \Delta_t T(P_t - I) \equiv \tilde{R}_t + \tilde{\Lambda}_t.$$

Then we can at least handle $\tilde{R}_t$ exactly as we did R_t in Theorem 2.2: it is not hard to show that its kernel satisfies (2.1) (I'm cheating a bit here - the bound for the kernel of $\Delta_t T P_t$ uses WBP) and clearly $\tilde{R}_t 1 = 0$. It is the term $\tilde{\Lambda}_t$ which now causes problems. If $T^*1 = 0$, or even if $T^*1 \in BMO$, then one can prove square function bounds for the contribution of $\tilde{\Lambda}_t$ (this is easiest to do when the Littlewood -Paley operators Δ_t have been discretized, and when T is of "perfect dyadic" type; see the "one-sided Tb Theorem" in [AHMTT]). In the absence of this "one-sided" condition, the idea is to build discretized Δ_t operators which are adapted to b_Q^* (as in [CJS]), to take advantage of the fact that we can control (locally) $T^* b_Q^*$ in place of T^*1. The difficulty is that these adapted Δ_t operators are now well behaved only in sawtooth regions on which b_Q^* is non-degenerate, and therefore there is a stopping time construction needed just to reach the analogue of (3.14), which now, for a given cube Q, becomes an estimate over a sawtooth adapted to b_Q^*. Morally speaking, one then proceeds more or less as I've described above. In practice, this is a bit delicate. We refer the interested reader to [AHMTT] and [AAAHK] for details.

References

[A] P. Auscher, Lectures on the Kato square root problem, *Surveys in analysis and operator theory* (Canberra, 2001), 1–18, Proc. Centre Math. Appl. Austral. Nat. Univ. **40**, Austral. Nat. Univ., Canberra, 2002.

[AAAHK] M. Alfonseca, P. Auscher, A. Axelsson, S. Hofmann and S. Kim, Stability of layer potentials and L^2 Solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients, preprint.

[AHLT] P. Auscher, S. Hofmann, J.L. Lewis, P. Tchamitchian, Extrapolation of Carleson measures and the analyticity of Kato's square root operators, *Acta Math.* **187** (2001), no. 2, 161–190.

[AHMTT] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, C. Thiele, Carleson measures, trees, extrapolation, and $T(b)$ theorems, *Publ. Mat.* **46** (2002), no. 2, 257–325.

[AT] P. Auscher and Ph. Tchamitchian, *Square root problem for divergence operators and related topics*, Astérisque Vol. 249 (1998), Société Mathématique de France.

[AHLMcT] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian, The solution of the Kato Square Root Problem for Second Order Elliptic operators on $\mathbb{R}^n$, *Annals of Math.* **156** (2002), 633–654.

[CFMS] L. Caffarelli, E. Fabes, S. Mortola and S. Salsa, Boundary behavior of nonnegative solutions of elliptic operators in divergence form. *Indiana Univ. Math. J.* **30** (1981), no. 4, 621–640.

[Ch] M. Christ, A T(b) theorem with remarks on analytic capacity and the Cauchy integral, *Colloquium Mathematicum* LX/LXI (1990) 601-628.

[CJ] M. Christ and J.-L. Journé, Polynomial growth estimates for multilinear singular integral operators. *Acta Math.* **159** (1987), no. 1-2, 51–80.

[CJS] R. Coifman, P. Jones and S. Semmes, Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves. *J. Amer. Math. Soc.* **2** (1989), no. 3, 553–564

[CMcM] R. Coifman, A. M^{c}Intosh, and Y. Meyer. L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes. *Annals of Math.* **116** 361–387, 1982.

[CM] R. Coifman and Y. Meyer. Non-linear harmonic analysis and PDE. In E. M. Stein, editor, *Beijing Lectures in Harmonic Analysis*, volume 112 of *Annals of Math. Studies*. Princeton Univ. Press, 1986.

[D1] G. David, Unrectifiable 1-sets have vanishing analytic capacity. *Rev. Mat. Iberoamericana* **14** (1998), no. 2, 369–479.

[D2] G. David, Analytic capacity, Caldern-Zygmund operators, and rectifiability. *Publ. Mat.* **43** (1999), no. 1, 3–25.

[DJ] G. David and J.-L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators. *Ann. Math.* **120** 371–398, 1984.

[DJS] G. David, J.-L. Journé, and S. Semmes, Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation. *Rev. Mat. Iberoamericana* **1** 1–56, 1985.

[FS] C. Fefferman, and E. M. Stein, H^p spaces of several variables. *Acta Math.* **129** (1972), no. 3-4, 137–193.

[HLMc] S. Hofmann, M. Lacey and A. M^{c}Intosh. The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds. *Annals of Math.* **156** (2002), pp 623-631.

[HMc] S. Hofmann and A. McIntosh, The solution of the Kato problem in two dimensions, Proceedings of the Conference on Harmonic Analysis and PDE held in El Escorial, Spain in July 2000, *Publ. Mat.* Vol. extra, 2002 pp. 143-160.

[JK1] D. Jerison and C. Kenig, An identity with applications to harmonic measure. *Bull. Amer. Math. Soc. (N.S.)* **2** (1980), no. 3, 447–451.

[JK2] D. Jerison and C. Kenig, The Neumann problem on Lipschitz domains, *Bull. Amer. Math. Soc. (N.S.)* **4** (1981), no. 2, 203–207.

[JK3] D. Jerison and C. Kenig, The Dirichlet problem in nonsmooth domains, *Ann. of Math. (2)* **113** (1981), no. 2, 367–382.

[K1] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York 1966.

[K2] T. Kato, *Fractional powers of dissipative operators*, J. Math. Soc. Japan **13** (1961), p. 246-274.

[Ke] C. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, 83. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994

[KKPT] C. Kenig, H. Koch, H. J. Pipher and T. Toro, A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations. *Adv. Math.* **153** (2000), no. 2, 231–298.

[KP] C. Kenig and J. Pipher, The Neumann problem for elliptic equations with nonsmooth coefficients, *Invent. Math.* **113** (1993), no. 3, 447–509.

[Mc] A.McIntosh, Square roots of operators and applications to hyperbolic $p.d.e.'s$, Miniconference on Operator Theory and PDE, Proceedings of the Centre for Mathematical Analysis, The Australian National University, Canberra, vol. 5 (1984), pp.124-136.

[McM] A. M^{c}Intosh and Y. Meyer. Algèbres d'opérateurs définis par des intégrales singulières. *C. R. Acad. Sci. Paris* **301** Série 1 395–397, 1985.

[MMV] P. Mattila, M. Melnikov and J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. of Math.* (2) **144** (1996), no. 1, 127–136.

[NTV1] F. Nazarov, S. Treil and A. Volberg, Accretive system Tb-theorems on nonhomogeneous spaces. *Duke Math. J.* **113** (2002), no. 2, 259–312.

[NTV2] F. Nazarov, S. Treil and A. Volberg, The Tb-theorem on non-homogeneous spaces. *Acta Math.* **190** (2003), no. 2, 151–239.

[P] J. Peetre, On convolution operators leaving $L^{p,\,\lambda}$ spaces invariant, *Ann. Mat. Pura Appl.* (4) **72** (1966) 295–304.

[S] S. Semmes. Square function estimates and the $T(b)$ Theorem. *Proc. Amer. Math. Soc.* **110** (1990), no. 3, 721–726.

[Sp] S. Spanne, Sur l'interpolation entre les espaces $\mathcal{L}_k{}^{p\Phi}$, *Ann. Scuola Norm. Sup. Pisa* (3) **20** (1966) 625–648.

[St] E. M. Stein, Singular integrals, harmonic functions, and differentiability properties of functions of several variables, Singular integrals *Proc. Sympos. Pure Math.*, Chicago, Ill., 1966, pp. 316–335, Amer. Math. Soc., Providence, R.I., 1967.

[T] X. Tolsa, Painlevé's problem and the semiadditivity of analytic capacity. *Acta Math.* **190** (2003), no. 1, 105–149.

[V] G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.* **59** (1984), no. 3, 572–611.

[Vo] A. Volberg, *Caldern-Zygmund capacities and operators on nonhomogeneous spaces*, CBMS Regional Conference Series in Mathematics **100**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2003.

Dept. of Mathematics
University of Missouri
Columbia, MO 65211
E-mail: hofmann@math.missouri.edu