

A PROOF OF THE LOCAL Tb THEOREM FOR STANDARD CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. We give a proof of a so-called “local Tb ” Theorem for singular integrals whose kernels satisfy the standard Calderón-Zygmund conditions. The present theorem, which extends an earlier result of M. Christ [Ch], was proved in [AHMTT] for “perfect dyadic” Calderón-Zygmund operators. The proof in [AHMTT] essentially carries over to the case considered here, with some technical adjustments.

1. INTRODUCTION

Following Coifman and Meyer, we say that an operator T , initially defined as a mapping from test functions $C_0^\infty(\mathbb{R}^n)$ to distributions, is a singular integral operator if it is associated to a kernel $K(x, y)$ in the sense that for all $\phi, \psi \in C_0^\infty$ with disjoint supports, we have

$$\langle T\phi, \psi \rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y)\phi(y)\psi(x)dydx,$$

and if the kernel satisfies the standard “Calderón-Zygmund” bounds

$$(1.1a) \quad |K(x, y)| \leq \frac{C}{|x - y|^n}$$

$$(1.1b) \quad |K(x, y + h) - K(x, y)| + |K(x + h, y) - K(x, y)| \leq C \frac{|h|^\alpha}{|x - y|^{n+\alpha}},$$

where the later inequality holds for some $\alpha > 0$ whenever $|x - y| > 2|h|$.

For future reference, we note that, for any kernel $K(x, y)$ satisfying (1.1)(a), and for $1 < p < \infty$, we have

$$(1.2) \quad \int_Q \left| \int_{6Q \setminus Q} K(x, y) 1_{6Q \setminus Q}(y) f(y) dy \right|^p dx \leq C_p \int_{6Q \setminus Q} |f|^p.$$

We omit the proof.

The following theorem is an extension of a local Tb Theorem for singular integrals introduced by M. Christ [Ch] in connection with the theory of analytic capacity. See also [NTV], where a non-doubling versions of Christ’s local Tb Theorem is given. A 1-dimensional version of the present result, valid for “perfect dyadic” Calderón-Zygmund kernels, appears in [AHMTT]. In the sequel, we use the notation T^t to denote the transpose of the operator T .

Theorem 1.3. *Let T be a singular integral operator associated to a kernel K satisfying (1.1), and suppose that K satisfies the generalized truncation condition $K(x, y) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose also that there exist pseudo-accretive systems $\{b_Q^1\}, \{b_Q^2\}$ such that b_Q^1 and b_Q^2 are supported in Q , and*

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- (i) $\int_Q (|b_Q^1|^q + |b_Q^2|^q) \leq C|Q|$, for some $q > 2$
- (ii) $\int_Q (|Tb_Q^1|^2 + |T^u b_Q^2|^2) \leq C|Q|$
- (iii) $\frac{1}{C}|Q| \leq \min(\Re \int_Q b_Q^1, \Re \int_Q b_Q^2)$.

Then $T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$, with bound independent of $\|K\|_\infty$.

The theorem in [Ch] is similar, except that the L^2 (or $L^{2+\epsilon}$) control in conditions (i) and (ii) is replaced by L^∞ control. The proof of the present theorem follows that of [AHMTT], except for some technical adjustments related to the presence of the Calderón-Zygmund tails in condition (1.1b). These tails do not appear in the perfect dyadic setting considered in [AHMTT], and their absence allows one to take $q = 2$ in condition (i); moreover, Auscher and Yang [AY] have extended the present result to the case $q = 2$, by reducing to [AHMTT]. At present, we do not know a direct proof of our theorem without taking $q > 2$, nor (in contrast to the perfect dyadic case) any proof with $q < 2$.

The present version of the theorem has been applied in [AAAHK] to establish L^2 boundedness of layer potentials associated to certain divergence form elliptic operators with bounded measurable coefficients.

2. PRELIMINARIES

We begin by setting some notation, and recalling some familiar facts. In particular, we discuss adapted averages and difference operators following [CJS]. We define the standard dyadic conditional expectation and martingale difference operators

$$E_k f(x) = \sum_{Q \in \mathbb{D}_k} 1_Q(x) \frac{1}{|Q|} \int_Q f,$$

where \mathbb{D}_k , $k \in \mathbb{Z}$, denotes the standard grid of dyadic cubes in \mathbb{R}^n having side length 2^{-k} , and

$$\Delta_k \equiv E_{k+1} - E_k.$$

Then

$$E_j E_k = E_k, \quad j \geq k$$

and thus also

$$(2.1) \quad \begin{aligned} \Delta_j \Delta_k &= 0, \quad j \neq k \\ \Delta_k^2 &= \Delta_k \end{aligned}$$

Moreover, the operators E_k and Δ_k are self-adjoint. Consequently, we have the square function identity

$$(2.2) \quad \int_{\mathbb{R}^n} \sum_{k=-\infty}^{\infty} |\Delta_k f|^2 = \|f\|_2^2,$$

as well as the discrete Calderón reproducing formula

$$(2.3) \quad \sum \Delta_k^2 = \sum \Delta_k = I,$$

where the convergence is in the strong operator topology on L^2 , as well as point-wise *a.e.* for $f \in L^2$, as may be seen by the telescoping nature of the sum, and the fact that

$$(2.4) \quad \lim_{k \rightarrow \infty} E_k f = f \text{ a.e.}, \quad f \in L_{\text{loc}}^p, \quad 1 \leq p \leq \infty$$

(by Lebesgue's Differentiation Theorem), and

$$(2.5) \quad \lim_{k \rightarrow -\infty} E_k f = 0, \quad f \in L^p, 1 \leq p < \infty.$$

Details may be found in [St]. As a consequence of (2.2), we have the standard dyadic Carleson measure estimate.

Proposition 2.6. *There exists a constant C such that for every dyadic cube Q ,*

$$\frac{1}{|Q|} \int_Q \sum_{k: 2^{-k} \leq \ell(Q)} |\Delta_k h(x)|^2 dx \leq C \|h\|_{BMO}^2.$$

Remark. The well-known proof is the same as that in the continuous parameter case [FS], and is omitted.

Suppose now that b is dyadically pseudo-accretive, i.e.

$$(D\psi A) \quad b \in L^\infty, \quad |E_k b| \geq \delta,$$

for some $\delta > 0$, and for all $k \in \mathbb{Z}$, or more generally that

$$(2.7) \quad \left| \frac{1}{|Q|} \int_Q b \right| \geq \delta, \quad \int_Q |b|^2 \leq C|Q|$$

for all Q in some "good" subset of \mathbb{D}_k . Then we can define the adapted expectation operators

$$E_k^b f = \frac{E_k(fb)}{E_k(b)}$$

(at least on the good cubes), and we can also define the martingale difference operators

$$\Delta_k^b = E_{k+1}^b - E_k^b,$$

at least on cubes $Q \in \mathbb{D}_k$ which are not only "good", but whose dyadic children are also "good" (in the sense of (2.7)). The following result is well known (see, e.g. [Ch2, p. 45])

Proposition 2.8. *Suppose $b \in D\psi A$. Then we have the following square function estimate*

$$\int_{\mathbb{R}^n} \sum |\Delta_k^b f|^2 \leq C \|f\|_2^2.$$

We omit the proof.

It is routine to check that for $b \in D\psi A$, E_k^b, Δ_k^b also satisfy

$$(2.9) \quad \begin{aligned} \text{a)} & \quad E_k^b E_j^b = E_j^b E_k^b = E_k^b, \quad j \geq k \\ \text{b)} & \quad \Delta_j^b \Delta_k^b = 0 \quad j \neq k \\ \text{c)} & \quad (\Delta_k^b)^2 = \Delta_k^b \\ \text{d)} & \quad \lim_{k \rightarrow \infty} E_k^b f = f \text{ a.e.}, \quad f \in L_{\text{loc}}^p, \quad p \geq 1 \\ \text{e)} & \quad \lim_{k \rightarrow \infty} E_k^b f = 0, \quad f \in L^p, \quad 1 \leq p < \infty \\ \text{f)} & \quad \sum (\Delta_k^b)^2 = \sum \Delta_k^b = I. \end{aligned}$$

We shall also find it useful to consider the transposes of the operators E_k^b, Δ_k^b , which we denote as follows:

$$A_k^b \equiv (E_k^b)^{\text{tr}} = b \frac{E_k}{E_k(b)}, \quad D_k^b = A_{k+1}^b - A_k^b = (\Delta_k^b)^{\text{tr}}.$$

One may readily verify that for $b \in D\psi A$ the operators A_k^b, D_k^b satisfy the properties enjoyed by E_k^b, Δ_k^b in (2.9). Moreover, we have

Proposition 2.10. *If $b \in D\psi A$ then*

$$\sum_k \|D_k^b f\|_2^2 \leq C \|f\|_2^2.$$

Proof. Observe that $A_k^b f = b E_k f / E_k b$. Hence

$$|D_k^b f| \leq |b| \left| \frac{E_{k+1} f}{E_{k+1} b} - \frac{E_k f}{E_k b} \right| \leq \|b\|_\infty \left(\frac{|\Delta_k f|}{|E_{k+1} b|} + \frac{|E_k f| |\Delta_k b|}{|E_{k+1} b| |E_k b|} \right).$$

The conclusion of the proposition now follows from (2.2), Proposition 2.6, dyadic pseudo-accretivity, and the dyadic version of Carleson's Lemma. We omit the details. \square

Next, we introduce some further terminology.

Definition 2.11. Given a dyadic cube $Q \subseteq \mathbb{R}^n$, a “discrete Carleson region” is the collection

$$R_Q \equiv \{\text{dyadic } Q' \text{ such that } Q' \subseteq Q\}.$$

We shall refer to Q as the “top” of R_Q . We remark that in using the term “discrete Carleson region” in this fashion, we are implicitly identifying a cube Q' with its associated “Whitney box” $Q' \times [\ell(Q')/2, \ell(Q')]$.

Definition 2.12. Given a dyadic cube $Q \subseteq \mathbb{R}^n$, a “discrete sawtooth region” is the collection

$$\Omega \equiv R_Q \setminus (\cup R_{P_j}),$$

where $\{P_j\}$ is a family of non-overlapping dyadic sub-cubes of Q .

Definition 2.13. We say that b is “ q -dyadically pseudo accretive on a sawtooth domain Ω ” ($b \in q - D\psi A(\Omega)$), if there exist constants $\delta > 0$ and $C_0 < \infty$ such that for every $Q' \in \Omega$

- (i) $\left| \frac{1}{|Q'|} \int_{Q'} b \right| \geq \delta$
- (ii) $\frac{1}{|Q'|} \int_{Q'} |b|^q \leq C_0$.

We now introduce some alternative notation, which we shall find useful when working with discrete sawtooth regions. For $Q \in \mathbb{D}_k$, we set

$$D_Q^b f(x) \equiv 1_Q(x) D_k^b f(x)$$

and we adapt the analogous convention for $A_k^b(A_Q^b)$, $\Delta_k^b(\Delta_Q^b)$ and $E_k^b(E_Q^b)$. Since the cubes in a given dyadic scale are non-overlapping, we have, for example

$$\sum_Q \|D_Q^b f\|_2^2 = \sum_{k=-\infty}^{\infty} \|D_k^b f\|_2^2,$$

where the first sum runs over all dyadic cubes.

We also describe a convenient splitting of a discrete sawtooth region as follows. Given a dyadic cube Q_1 , and a discrete sawtooth

$$\Omega \equiv R_{Q_1} \setminus (\cup R_{P_j}),$$

we split

$$\Omega \equiv \Omega_1 \cup \Omega_{\text{buffer}},$$

where

$$\Omega_{\text{buffer}} \equiv \{Q \in \Omega : Q \text{ has at least one child not in } \Omega\}.$$

Thus, if $Q \in \Omega_1$, then every child of Q belongs to Ω . We have the following extension of Proposition 2.10:

Lemma 2.14. *Let $\Omega \equiv R_{Q_1} \setminus (\cup R_{P_j})$ be a discrete sawtooth region corresponding to a dyadic cube Q_1 , and let $\Omega_1 \cup \Omega_{\text{buffer}}$ be the splitting of Ω described above. Suppose also that $b \in 2 - D\psi A(\Omega)$. Then*

$$\sum_{Q \in \Omega_1} \|D_Q^b f\|_2^2 \leq C \|f\|_{L^2(Q_1)}^2.$$

Proof. Fix $Q \in \mathbb{D}_k \cap \Omega_1$. By definition,

$$\begin{aligned} \|D_Q^b f\|_2^2 &= \int_Q |D_k^b f|^2 = \int_Q \left| b \left(\frac{E_{k+1} f}{E_{k+1} b} - \frac{E_k f}{E_k b} \right) \right|^2 \\ &= \sum_{\substack{Q' \in \mathbb{D}_{k+1} \\ Q' \subseteq Q}} \int_{Q'} \left| b \left(\frac{E_{Q'} f}{E_{Q'} b} - \frac{E_Q f}{E_Q b} \right) \right|^2 = \sum_{\substack{Q' \in \mathbb{D}_{k+1} \\ Q' \subseteq Q}} \left| \frac{E_{Q'} f}{E_{Q'} b} - \frac{E_Q f}{E_Q b} \right|^2 \int_{Q'} |b|^2 \\ &= \sum_{\substack{Q' \in \mathbb{D}_{k+1} \\ Q' \subseteq Q}} \int_{Q'} \left| \frac{\Delta_{Q'} f}{E_{Q'} b} - \frac{E_{Q'} f \Delta_Q b}{E_{Q'} b E_Q b} \right|^2 \frac{1}{|Q'|} \int_{Q'} |b|^2, \end{aligned}$$

where in the last two steps we have used that $E_{Q'}$, E_Q are constant on Q' . But if $Q \in \Omega_1$, then its children Q' all belong to Ω . Since $b \in 2 - D\psi A(\Omega)$, the last expression is therefore bounded by

$$C \int_Q (|\Delta_Q f|^2 + |E_Q f|^2 |\Delta_Q b|^2).$$

Summing over $Q \in \Omega_1$ yields the desired estimate, once we have proved the following analogue of the discrete Fefferman-Stein Carleson measure estimate Proposition 2.6. \square

Lemma 2.15. *Let Q_1 , $\Omega = \Omega_1 \cup \Omega_{\text{buffer}}$ be as in the previous Lemma, and suppose that $b \in 2 - D\psi A(\Omega)$. Then*

$$\sup \frac{1}{|Q|} \sum_{Q \in \Omega_1, Q \subseteq \bar{Q}} \|\Delta_Q b\|_2^2 \leq C C_0,$$

where C_0 is the constant in Definition 2.13, and where the supreme runs over all dyadic $\bar{Q} \subseteq Q_1$.

Proof. We observe that

$$\sum_{\substack{Q \in \Omega_1 \\ Q \subseteq \bar{Q} \subseteq Q_1}} \|\Delta_Q b\|_2^2 = \sum_{\substack{Q \in \Omega_1 \\ Q \subseteq \bar{Q} \subseteq Q_1}} \|\Delta_Q (1_{\bar{Q}} b)\|_2^2$$

is non-zero only if $\bar{Q} \in \Omega$. But $b \in 2 - D\psi A(\Omega)$, so by (2.2) we have that

$$\sum_{Q \text{ dyadic}} \|\Delta_Q (1_{\bar{Q}} b)\|_2^2 \leq C \int_{\bar{Q}} |b|^2 \leq C C_0 |\bar{Q}|.$$

This concludes the proof of Lemma 2.15 and hence also that of Lemma 2.14. \square

3. PROOF OF THEOREM 1.3 (LOCAL Tb THEOREM FOR SINGULAR INTEGRALS)

We now proceed to give the proof of Theorem 1.3. The proof follows that of Theorem 6.8 in [AHMTT], which for the sake of expository simplicity treated only the case of ‘perfect dyadic’ Calderón-Zygmund kernels in one dimension. The more general version given here, in which the ‘perfect dyadic’ cancellation condition is replaced by (1.1)(b),

will entail dealing with a moderate amount of purely technical complication, but the gist of the proof is unchanged.

By the T1 theorem, plus a localization argument, it is enough to show that there is a constant C , depending only on dimension, the kernel bounds in (1.1), and the constants in hypotheses (i), (ii) and (iii) of the Theorem, such that for every dyadic cube Q ,

$$(T1_{loc}) \quad \begin{aligned} (a) \quad & \|T1_Q\|_{L^1(Q)} \leq C|Q| \\ (b) \quad & \|T^{tr}1_Q\|_{L^1(Q)} \leq C|Q| \end{aligned}$$

Indeed, it is well known that one may deduce both the weak boundedness property, and that $T1, T^{tr}1 \in BMO$, from $(T1_{loc})$, (1.1) and (1.2). We omit the details. In the sequel we shall use the generic C to denote a constant depending only on the benign parameters listed above.

Now, by the symmetry of our hypotheses, it will suffice to establish only $(T1_{loc})(b)$, and we do this for Q contained in some fixed cube Q_{big} . Since Q_{big} is arbitrary, the general case follows, as long as our constants are independent of Q_{big} (as they will be).

We thus fix Q_{big} , and define

$$B_1 \equiv \sup \frac{1}{|Q|} \|T^{tr}1_Q\|_{L^1(Q)},$$

where the supremum runs over all dyadic $Q \subseteq Q_{big}$. By our qualitative hypothesis that $K \in L^\infty$, we see that $B_1 < \infty$, although apparently it may depend on $\|K\|_\infty$ and Q_{big} . However, we shall show that there exists $\epsilon > 0$, depending only on the allowable parameters, such that for every $Q \subseteq Q_{big}$, and for every $f \in L^\infty(Q)$ with $\|f\|_\infty \leq 1$, we have the estimate

$$(3.1) \quad \left| \int_Q T f \right| \leq (1 - \epsilon)B_1|Q| + C|Q|.$$

By duality, this proves that $B_1 \leq (1 - \epsilon)B_1 + C$, and $(T1_{loc})(b)$ follows.

In the sequel, we shall use the following convenient notational convention:

$$\frac{1}{|Q|} \int_Q f = [f]_Q.$$

By renormalizing, we may assume that hypothesis (iii) of the Theorem reads

$$(3.2) \quad [b_Q^1]_Q = 1 = [b_Q^2]_Q.$$

Lemma 3.3. *Suppose that $\{b_Q\}$ satisfies (as in the hypotheses of Theorem 1.3)*

- (i) $\int_Q |b_Q|^q \leq C|Q|$, for some $q > 2$
- (ii) $\int_Q |Tb_Q|^2 \leq C|Q|$
- (iii) $[b_Q]_Q = 1$,

and that $\text{supp } b_Q \subseteq Q$. Then there exists $\epsilon > 0$, and for each fixed Q_1 a partition of R_{Q_1} into

$$R_{Q_1} = \Omega_1 \cup \Omega_{buffer} \cup (\cup R_{P_j}),$$

where the tops $\{P_j\}$ are non-overlapping dyadic sub-cubes of Q_1 , such that if $b \equiv b_{Q_1}$, then

$$(3.4) \quad \sum |P_j| \leq (1 - \epsilon)|Q_1|$$

$$(3.5) \quad b \in q - D\psi A(\Omega_1 \cup \Omega_{buffer})$$

$$(3.6) \quad \sup_{Q \subseteq \tilde{Q} \subseteq 2Q} [(Mb)^2]_{\tilde{Q}} \leq C,$$

for all $Q \in \Omega_1 \cup \Omega_{\text{buffer}}$ (here, $2Q$ denotes the concentric double of Q);

$$(3.7) \quad [Tb]^2_Q \leq C, \quad \forall Q \in \Omega_1 \cup \Omega_{\text{buffer}}$$

$$(3.8) \quad \sum_{Q \in \Omega_{\text{buffer}}} |Q| \leq C|Q_1|$$

$$(3.9) \quad f = [f]_{Q_1} b + \sum_{Q \in \Omega_1} D_Q^b f + \sum_j (f 1_{P_j} - [f]_{P_j} b_{P_j}) + \sum_{Q \in \Omega_{\text{buffer}}} \zeta_Q,$$

where

$$\zeta_Q \equiv S_Q^b f + \sum_{P_j \text{ children of } Q} [f]_{P_j} b_{P_j},$$

and, for $x \in Q'$, and Q' a child of $Q \in \Omega_{\text{buffer}}$,

$$S_Q^b f(x) \equiv \begin{cases} D_Q^b f(x), & x \in Q' \in (\Omega_1 \cup \Omega_{\text{buffer}}) \\ -A_Q^b f(x), & x \in Q' \notin (\Omega_1 \cup \Omega_{\text{buffer}}) \end{cases}.$$

Furthermore $\int \zeta_Q = 0$, and $\|\zeta_Q\|_2 \leq C|Q|^{1/2}$.

Proof of the lemma. We begin by verifying the claimed properties of ζ_Q , for $Q \in \Omega_{\text{buffer}}$, assuming (3.5). By definition of S_Q^b ,

$$\zeta_Q = \sum_{Q' \text{ child of } Q} \frac{b}{[b]_{Q'}} [f]_{Q'} 1_{Q'} + \sum_{\substack{Q' \notin \Omega \\ Q' \text{ child of } Q}} [f]_{Q'} b_{Q'} - \frac{b}{[b]_Q} [f]_Q 1_Q,$$

where in the middle term we have used that $[b_{Q'}]_{Q'} = 1$, and that if Q' is a child of $Q \in \Omega_{\text{buffer}}$, with $Q' \notin \Omega \equiv \Omega_1 \cup \Omega_{\text{buffer}}$, then $Q' = P_j$ for some j . It is now routine to verify that $\int \zeta_Q = 0$, since $[b_{Q'}]_{Q'} = 1$. Clearly, $\text{supp } \zeta_Q \subseteq Q$. Also, the bound

$$\|\zeta_Q\|_2 \leq C\|f\|_\infty |Q|^{\frac{1}{2}} \leq C|Q|^{\frac{1}{2}}$$

follows from (3.5) and Hölder's inequality.

We now turn to the main part of the proof. By hypothesis (i) of the Lemma, applied to b in Q_1 , and by the L^q boundedness of the maximal function, we have that

$$(3.10) \quad \int_{\mathbb{R}^n} (Mb)^q \leq C \int_{Q_1} |b|^q \leq C|Q_1|$$

where we have used that b is supported in Q_1 . We now perform a standard stopping time argument, subdividing Q_1 dyadically to extract a collection of sub-cubes $\{P_j\}$ which are maximal with respect to the property that for some $\delta > 0$ to be chosen, at least one of the following holds:

$$(3.11) \quad \begin{aligned} (1) \quad & |[b]_{P_j}| \leq \delta \\ (2) \quad & \sup_{\tilde{Q}: P_j \subseteq \tilde{Q} \subseteq 2P_j} [(Mb)^q]_{\tilde{Q}} + [Tb]^2_{P_j} \geq \frac{C}{\delta^2} \end{aligned}$$

As usual, we then set $\Omega \equiv R_{Q_1} \setminus (\cup R_{P_j})$, and we further decompose $\Omega = \Omega_1 \cup \Omega_{\text{buffer}}$, where as above

$$\Omega_{\text{buffer}} = \{Q \in \Omega : Q \text{ has at least one child not in } \Omega\}.$$

Then (3.5), (3.6) and (3.7) hold by construction. The representation (3.9) holds by definition of D_Q^b and S_Q^b , by the normalization $[b_Q]_Q = 1$, and by the telescoping nature of sums

involving the D_k^b operator. Furthermore, since each $Q \in \Omega_{\text{buffer}}$ contains at least one bad child P_j , we have that

$$\sum_{Q \in \Omega_{\text{buffer}}} |Q| \leq 2^n \sum |P_j| \leq 2^n |Q_1|,$$

which is (3.8). It therefore remains only to verify (3.4). To this end, we assign each ‘bad’ cube P_j to a family S_1 or S_2 , according to whether P_j satisfies property (1) or (2) of (3.11). If it happens to satisfy both of these inequalities, then we assign it arbitrarily to S_1 . We then define

$$\text{Bad}_1 = \cup_{P_j \in S_1} P_j, \quad \text{Bad}_2 = \cup_{P_j \in S_2} P_j$$

and

$$\text{Good} = Q_1 \setminus (\text{Bad}_1 \cup \text{Bad}_2).$$

Then by hypothesis (iii) of the lemma,

$$\begin{aligned} |Q_1| &= \int_{Q_1} b = \int_{\text{Good}} b + \int_{\text{Bad}_1} b + \int_{\text{Bad}_2} b \\ &\leq |\text{Good}|^{\frac{1}{2}} \|b\|_{L^2(Q_1)} + \delta \sum |P_j| + |\text{Bad}_2|^{\frac{1}{2}} \|b\|_{L^2(Q_1)}, \end{aligned}$$

where we have used (3.11)(1) to control the middle term. Now, by hypothesis (i) of the Lemma and Hölder’s inequality, we have that $\|b\|_2 \leq C|Q_1|^{\frac{1}{2}}$, whence

$$(3.12) \quad (1 - \delta)|Q_1| \leq C|\text{Good}|^{\frac{1}{2}}|Q_1|^{\frac{1}{2}} + |\text{Bad}_2|^{\frac{1}{2}}|Q_1|^{\frac{1}{2}}.$$

Choosing $\delta > 0$ sufficiently small, we will obtain the conclusion of the Lemma once we show that

$$|\text{Bad}_2| \leq C\delta^2|Q_1|.$$

To this end, we observe that by (3.11)(2) and the Hardy-Littlewood Theorem,

$$\begin{aligned} |\text{Bad}_2| &\leq \left| \left\{ M(Mb)^q > \frac{C}{2\delta^2} \right\} \right| + \left| \left\{ M(|Tb|^2 1_{Q_1}) > \frac{C}{2\delta^2} \right\} \right| \\ &\leq C\delta^2 \left(\int_{\mathbb{R}^n} (Mb)^q + \int_{Q_1} |Tb|^2 \right) \leq C\delta^2|Q_1|, \end{aligned}$$

as desired. This concludes the proof of Lemma 3.3. \square

We now return to the proof of (3.1). Fix a cube Q_1 , and let f be supported in Q_1 , with $\|f\|_{\infty} \leq 1$. We apply Lemma 3.3 in the cube Q_1 , with $b_Q = b_{Q_1}^1$, $b = b_{Q_1}^1 \equiv b_1$, so that we have a decomposition $R_{Q_1} = \Omega_1 \cup \Omega_{\text{buffer}} \cup (\cup P_j)$, for which (3.4)-(3.8) are satisfied, and furthermore f may be decomposed as in (3.9). We need to estimate $\left| \int_{Q_1} Tf \right|$, so by (3.9) it is enough to consider

$$\begin{aligned} |[f]_{Q_1}| &\int_{Q_1} |Tb_1| + \left| \sum_{Q \in \Omega_1} TD_Q^{b_1} f \right| + \left| \sum_j \int_{Q_1} T(f1_{P_j} - [f]_{P_j} b_{P_j}^1) \right| \\ &+ \left| \sum_{Q \in \Omega_{\text{buffer}}} \int_{Q_1} T\zeta_Q \right| \equiv |\text{I}| + |\text{II}| + |\text{III}| + |\text{IV}|. \end{aligned}$$

By hypothesis (ii) of Theorem 1.3 and Cauchy-Schwarz, we have that

$$|\text{I}| \leq C\|f\|_{\infty}|Q_1| \leq C|Q_1|.$$

Term II is the main term, and we defer its treatment momentarily. Next, we consider term III. For notational convenience, we set

$$f_j \equiv f1_{P_j} - [f]_{P_j} b_{P_j}^1.$$

Since $[b_{P_j}^1]_{P_j} = 1$, we have that $\int f_j = 0$. Moreover, $\text{supp } f_j \subseteq P_j$, and

$$(3.13) \quad \|f_j\|_2 \leq C\|f\|_\infty |P_j|^{1/2}.$$

We now claim that

$$(3.14) \quad \text{III} = \sum_j \int_{P_j} T f_j + O(\|f\|_\infty |Q_1|).$$

Indeed,

$$(3.15) \quad \int_{Q_1 \setminus P_j} T f_j = \int_{Q_1 \setminus 2P_j} T f_j + \int_{(Q_1 \cap 2P_j) \setminus P_j} T f_j.$$

The second term is dominated in absolute value by

$$C|P_j|^{1/2} \left(\int_{2P_j \setminus P_j} |T f_j|^2 \right)^{1/2} \leq C|P_j|^{1/2} \|f_j\|_2 \leq C\|f\|_\infty |P_j|,$$

where the first inequality is essentially dual to (1.2), by the kernel condition (1.1)(a) and the fact that $\text{supp } f_j \subseteq P_j$, and the second inequality is just (3.13). The first term in (3.15) may be handled by the classical Calderón-Zygmund estimate, using (1.1)(b) and the fact that $\int f_j = 0$, and we obtain the bound

$$\begin{aligned} & C \iint_{|x-y| > C\ell(P_j)} \frac{\ell(P_j)^\alpha}{|x-y|^{n+\alpha}} |f_j(y)| dx dy \\ & \leq C\|f_j\|_1 \leq C|P_j|^{1/2} \|f_j\|_2 \leq C\|f\|_\infty |P_j|. \end{aligned}$$

Summing in j , we obtain (3.14).

Thus, to finish our treatment of term III, we need only observe that

$$\begin{aligned} \left| \sum_j \int_{P_j} T f_j \right| & \leq \left| \sum_j \int_{P_j} T(f1_{P_j}) \right| + \left| \sum_j \left(\int_{P_j} T b_{P_j}^1 \right) [f]_{P_j} \right| \\ & \leq B_1 \|f\|_\infty \sum_j |P_j| + C\|f\|_\infty \sum_j |P_j|, \end{aligned}$$

where we have used the definition of B_1 and hypothesis (ii) of Theorem 1.3. From (3.4) and the normalization $\|f\|_\infty \leq 1$, we obtain the bound

$$|\text{III}| \leq B_1(1 - \epsilon)|Q_1| + C|Q_1|.$$

We now consider term IV. By Lemma 3.3 and the definition of ζ_Q , we have that

$$\text{supp } \zeta_Q \subseteq Q, \quad \int \zeta_Q = 0, \quad \text{and } \|\zeta_Q\|_2 \leq C|Q|^{1/2}.$$

Thus, from the same argument used to establish (3.14), we obtain

$$(3.16) \quad \text{IV} = \sum_{Q \in \Omega_{\text{buffer}}} \int_Q T \zeta_Q + O(|Q_1|),$$

where in the ‘big O ’ term we have used (3.8). We recall that

$$\zeta_Q = S_Q^{b_1} f + \sum_{P_j \text{ children of } Q} [f]_{P_j} b_{P_j}^1,$$

where for $x \in Q'$, with Q' a child of $Q \in \Omega_{\text{buffer}}$, we have either that

$$S_Q^{b_1} f(x) = -b_1(x) \frac{\int_Q f}{\int_Q b_1},$$

if $Q' \notin \Omega \equiv (\Omega_1 \cup \Omega_{\text{buffer}})$ (in which case we say that Q' is a ‘bad’ child of Q) or

$$S_Q^{b_1} f(x) = b_1(x) \left[\frac{\int_{Q'} f}{\int_{Q'} b_1} - \frac{\int_Q f}{\int_Q b_1} \right],$$

if $Q' \in \Omega$ (Q' is a ‘good’ child of Q).

Now, by (3.5), $b_1 \in q - D\psi A(\Omega)$ (Definition 2.13), so that

$$(3.17) \quad \left| \int_Q T\zeta_Q \right| \leq \frac{C}{\delta} \left(\sum_{Q' \text{ good child of } Q} \left| \int_Q T(b_1 1_{Q'}) \right| + \left| \int_Q T(b_1 1_Q) \right| \right) + \sum_{Q' \text{ bad child of } Q} \left| \int_Q T b_{Q'}^1 \right|,$$

where in the last term we have used that the bad children of Q are precisely those P_j which are children of Q .

We shall estimate this last expression via the following

Lemma 3.18. *Suppose that $Q \subseteq Q_1$. Then with $b_1 \equiv b_{Q_1}^1$, we have*

$$\int_{3Q} |T(b_1 1_Q)|^2 \leq C \int_Q |T b_1|^2 + \int_{2Q} |b_1|^2 + \int_Q (M(b_1))^2$$

and similarly for $b_{Q_1}^2, T^w$.

Let us take the lemma for granted momentarily. In (3.17), Q' is a child of Q , hence the concentric triple $3Q'$ contains Q . Moreover, the ‘good’ children, being in Ω , satisfy (3.6) and (3.7), with $b = b_1$. Consequently, we may apply the lemma to $Q' \subseteq Q_1$ or to $Q \subseteq Q_1$ in the first two terms on the right side of (3.17) to obtain the bound

$$\frac{C}{\delta} \left(\int_Q |T b_1|^2 + \int_{2Q} |b_1|^2 + \int_Q (M(b_1))^2 \right) \leq \frac{C}{\delta} |Q|.$$

In addition, the last term in (3.17) is no larger than

$$\sum_{Q'} \left(\left| \int_{Q \setminus Q'} T b_{Q'}^1 \right| + \left| \int_{Q'} T b_{Q'}^1 \right| \right) \leq C \sum_{Q'} |Q'| \leq C |Q|,$$

by the dual estimate to (1.2), plus hypotheses (i) and (ii) of Theorem 1.3. Since $\delta > 0$ is fixed, summing over Q in Ω_{buffer} yields that

$$|\text{IV}| \leq C |Q_1|,$$

by (3.8).

Combining our estimates for I, III and IV, we have therefore proved that

$$(3.19) \quad \left| \int_{Q_1} T f \right| \leq |\text{II}| + C |Q_1| + B_1 (1 - \epsilon) |Q_1|,$$

modulo the proof of Lemma 3.18, which we shall give now, before embarking on our treatment of the math term II.

Proof of Lemma 3.18. The proof is based on another Lemma.

Lemma 3.20. *For all dyadic Q , and for every $f \in L^2(Q)$, we have that*

$$\|f\|_{L^2(Q)} \leq C \left(\|f - [f]_Q\|_{L^2(Q)} + |Q|^{-\frac{1}{2}} |\langle f, b_Q^2 \rangle| \right),$$

and similarly for b_Q^1 .

We first show that this lemma yields Lemma 3.18. By the dual estimate to (1.2), we have that

$$\int_{3Q \setminus Q} |T(b_1 1_Q)|^2 \leq C \int_Q |b_1|^2.$$

Thus, it suffices to show that $\int_Q |T(b_1 1_Q)|^2 \leq \beta$, where

$$\beta \equiv \int_Q |Tb_1|^2 + \int_{2Q} |b_1|^2 + \int_Q M(b_1)^2.$$

We note that

$$|\langle T(b_1 1_Q), b_Q^2 \rangle| = |\langle b_1 1_Q, T^u b_Q^2 \rangle| \leq \|b_1\|_{L^2(Q)} \|T^u b_Q^2\|_{L^2(Q)} \leq C|Q|^{1/2} \|b_1\|_{L^2(Q)},$$

by hypothesis (ii) of Theorem 1.3. Thus, by Lemma 3.20, with $f = T(b_1 1_Q)$, it suffices to show that

$$(3.21) \quad \|f - [f]_Q\|_{L^2(Q)} \leq C \sqrt{\beta}.$$

In turn, (3.21) will follow if we can show that, for all $h \in L^2(Q)$ with $\int_Q h = 0$, we have

$$|\langle f, h \rangle| \leq C \|h\|_2 \sqrt{\beta}.$$

But

$$\langle f, h \rangle = \langle b_1 1_Q, T^u h \rangle = \langle b_1, T^u h \rangle - \langle b_1 1_{(2Q) \setminus Q}, T^u h \rangle - \langle b_1 1_{(2Q)^c}, T^u h \rangle \equiv U + V + W.$$

Now

$$|U| \leq \|Tb_1\|_{L^2(Q)} \|h\|_{L^2(Q)} \leq C \sqrt{\beta} \|h\|_2.$$

Moreover, we have that

$$|V| \leq \|b_1\|_{L^2(2Q)} \|T^u h\|_{L^2(2Q \setminus Q)} \leq C \sqrt{\beta} \|h\|_2,$$

where we have used the dual estimate to (1.2) in the last step. Finally, since $\int h = 0$, we have by the standard Calderón-Zygmund estimate that

$$|W| \leq \int_{(2Q)^c} |b_1(y)| \int_Q |h(x)| \frac{\ell(Q)^\alpha}{|x-y|^{n+\alpha}} dx dy \leq \int_Q |h(x)| M(b_1)(x) dx \leq C \|h\|_2 \sqrt{\beta}.$$

Thus, Lemma 3.20 implies Lemma 3.18. \square

We now give the

Proof of Lemma 3.20. Let $h \in L^2(Q)$, with $\|h\|_2 = 1$. Then

$$\langle f, h \rangle = \langle f, h - [h]_Q b_Q^2 \rangle + [h]_Q \langle f, b_Q^2 \rangle = \langle f - [f]_Q, h - [h]_Q b_Q^2 \rangle + [h]_Q \langle f, b_Q^2 \rangle,$$

where we have used that $\int_Q (h - [h]_Q b_Q^2) = 0$, since $[b_Q^2]_Q = 1$. Thus, by Cauchy-Schwarz,

$$|\langle f, h \rangle| \leq \|f - [f]_Q\|_{L^2(Q)} \left(1 + \|[h]_Q\| \|b_Q^2\|_2 \right) + |[h]_Q| |\langle f, b_Q^2 \rangle|.$$

But

$$\|h\|_{Q_1} \leq \left(\frac{1}{|Q_1|} \int_{Q_1} |h|^2 \right)^{1/2} \leq |Q_1|^{-\frac{1}{2}},$$

and by hypothesis (i), $\|b_Q^2\|_2 \leq C|Q|^{1/2}$. The conclusion of the lemma now follows readily. \square

Next, we return to (3.19), and more precisely, to the term

$$\Pi = \sum_{Q \in \Omega_1} \int_{Q_1} T D_Q^{b_1} f,$$

where f is supported in Q_1 , and $\|f\|_\infty \leq 1$. Having established (3.19), we must now show that $|\Pi| \leq C|Q_1|$, whence (3.1) follows, since Q_1 is arbitrary. But

$$\Pi = \sum_{Q \in \Omega_1} \langle \Delta_Q^{b_1} T^{\text{tr}} 1_{Q_1}, D_Q^{b_1} f \rangle,$$

because $(D_Q^{b_1})^2 = D_Q^{b_1}$, and $(D_Q^{b_1})^{\text{tr}} = \Delta_Q^{b_1}$. Thus

$$(3.22) \quad |\Pi| \leq \left(\sum_{Q \in \Omega_1} \|D_Q^{b_1} f\|_{L^2(Q)}^2 \right)^{\frac{1}{2}} \left(\sum_{Q \in \Omega_1} \|\Delta_Q^{b_1} T^{\text{tr}} 1_{Q_1}\|_{L^2(Q)}^2 \right)^{\frac{1}{2}}.$$

Since b_1 satisfies (3.5), we have by Lemma 2.14 that the first factor on the right side of (3.22) is bounded by $C\|f\|_{L^2(Q_1)} \leq C|Q_1|^{1/2}$. It is therefore enough to show that the second factor is also dominated by $C|Q_1|^{1/2}$. More generally, setting

$$B_2 \equiv \sup_{Q_2 \subseteq Q_1} \frac{1}{|Q_2|} \sum_{Q \in \Omega_1 \cap R_{Q_2}} \|\Delta_Q^{b_1} T^{\text{tr}} 1_{Q_1}\|_{L^2(Q)}^2,$$

we shall show that $B_2 \leq C$. More precisely, for $Q_2 \subseteq Q_1$ now fixed, we shall show that

$$(3.23) \quad \sum_{Q \in \Omega_1 \cap R_{Q_2}} \|\Delta_Q^{b_1} T^{\text{tr}} 1_{Q_1}\|_{L^2(Q)}^2 \leq (1 - \epsilon)B_2|Q_2| + C|Q_2|.$$

Once (3.23) is established, we shall be done. To this end, we decompose R_{Q_2} as in Lemma 3.3, with respect to $b = b_Q^2 \equiv b_2$. In particular, $R_{Q_2} = \Omega_2 \cup \Omega_{2,\text{buffer}} \cup (\cup R_{P_i^2})$, where

$$\sum |P_i^2| \leq (1 - \epsilon)|Q_2|, \quad \sum_{Q \in \Omega_{2,\text{buffer}}} |Q| \leq C|Q_2|,$$

and $b_2 \in q - D\psi A$ on $\Omega_2 \cup \Omega_{2,\text{buffer}}$. The left hand side of (3.23) then splits into

$$\sum_{Q \in \Omega_1 \cap \Omega_2} + \sum_{Q \in \Omega_1 \cap \Omega_{2,\text{buffer}}} + \sum_i \sum_{Q \in \Omega_1 \cap R_{P_i^2}} \equiv \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Now, by definition of B_2 ,

$$\Sigma_3 \equiv \sum_i \sum_{Q \in \Omega_1 \cap R_{P_i^2}} \|\Delta_Q^{b_1} T^{\text{tr}} 1_{Q_1}\|_{L^2(P_i^2)}^2 \leq B_2 \sum_i |P_i^2| \leq B_2(1 - \epsilon)|Q_2|.$$

Next, we consider Σ_2 . For $Q \in \Omega_1 \cap \Omega_{2,\text{buffer}}$, we write $1_{Q_1} = 1_{Q_1 \setminus 2Q} + 1_{(Q_1 \cap 2Q) \setminus Q} + 1_Q$. Since $b_1 \in q - D\psi A(\Omega_1 \cup \Omega_{\text{buffer}})$, we have that $\Delta_Q^{b_1} : L^2(Q) \rightarrow L^2(Q)$. Thus, using also (1.2), we obtain

$$\|\Delta_Q^{b_1} T^{\text{tr}} 1_{(Q_1 \cap 2Q) \setminus Q}\|_2^2 \leq C\|1_{2Q \setminus Q}\|_2^2 \leq C|Q|.$$

Summing this term over $Q \in \Omega_{2,\text{buffer}}$ yields the bound $C|Q_2|$ as desired. Also $\Delta_Q^{b_1}1 = 0$. Thus, if we denote by $\varphi_Q^{b_1}(x, y)$ the kernel of $\Delta_Q^{b_1}$, we have by (1.1)(b) that

$$|\Delta_Q^{b_1} T^{\text{tr}} 1_{Q_1 \setminus 2Q}(x)| \leq C \int |\varphi_Q^{b_1}(x, y)| dy \int_{|z-y_Q| > c\ell(Q)} \frac{\ell(Q)^\alpha}{|z-y_Q|^{n+\alpha}} dz \leq C,$$

where y_Q is the center of Q . Therefore

$$\|\Delta_Q^{b_1} T^{\text{tr}} 1_{Q_1 \setminus 2Q}\|_{L^2(Q)}^2 \leq C|Q|,$$

and we can again sum over $Q \in \Omega_{2,\text{buffer}}$ to obtain the bound $C|Q_2|$.

To finish our treatment of Σ_2 , it remains to consider the contribution of 1_Q . By definition,

$$\begin{aligned} \varphi_Q^{b_1}(x, y) &= -1_Q(x)1_Q(y) \frac{1}{|Q|} \frac{b_1(y)}{[b_1]_Q} + \sum_{Q' \text{ children of } Q} 1_{Q'}(x)1_{Q'}(y) \frac{1}{|Q'|} \frac{b_1(y)}{[b_1]_{Q'}} \\ &\equiv \lambda_Q^{b_1}(x, y) b_1(y). \end{aligned}$$

Then,

$$\Delta_Q^{b_1} T^{\text{tr}} 1_Q(x) = \langle \lambda_Q^{b_1}(x, \cdot) b_1, T^{\text{tr}} 1_Q \rangle = \int_Q T(b_1 \lambda_Q^{b_1}(x, \cdot)).$$

Since $x \in Q$ (otherwise $\lambda_Q^{b_1} = 0$), we have that by definition of $\lambda_Q^{b_1}$, the last expression equals

$$\sum_{Q' \text{ children of } Q} 1_{Q'}(x) \left(\int_Q T(b_1 1_{Q'}) \right) \frac{1}{|Q'|} \frac{1}{[b_1]_{Q'}} - \left(\int_Q T(b_1 1_Q) \right) \frac{1}{|Q|} \frac{1}{[b_1]_Q}.$$

Since $Q \in \Omega_1$, we have that $b_1 \in q - D\psi A$ on Q and all of its children, so that

$$|[b_1]_Q|, |[b_1]_{Q'}| \geq \delta.$$

Consequently,

$$\begin{aligned} \|\Delta_Q^{b_1} T^{\text{tr}} 1_Q\|_{L^\infty(Q)} &\leq C \sum_{Q' \text{ children of } Q} \left(\frac{1}{|Q'|} \int_Q |T(b_1 1_{Q'})|^2 \right)^{\frac{1}{2}} + C \left(\frac{1}{|Q|} \int_Q |T(b_1 1_Q)|^2 \right)^{\frac{1}{2}} \\ &\leq C|Q|^{-\frac{1}{2}} \left(\|Tb_1\|_{L^2(Q)} + \|b_1\|_{L^2(2Q)} + \|Mb_1\|_{L^2(Q)} \right) \leq C, \end{aligned}$$

where we have used Lemma 3.18, and then estimates (3.6) and (3.7), in the last two inequalities. Thus, $\|\Delta_Q^{b_1} T^{\text{tr}} 1_Q\|_{L^2(Q)}^2 \leq C|Q|$, and summation over $Q \in \Omega_{2,\text{buffer}}$ completes the estimate

$$\Sigma_2 \leq C|Q_2|.$$

This leaves Σ_1 . That is, we need to prove

$$(3.24) \quad \sum_{Q \in \Omega_1 \cap \Omega_2} \|\Delta_Q^{b_1} T^{\text{tr}} 1\|_2^2 \leq C|Q_2|,$$

where we have replaced 1_{Q_1} by 1 in the definition of Σ_1 . Indeed, the error may be controlled by a well-known argument of Fefferman and Stein [FS], since $\Delta_Q^{b_1} 1 = 0$, and the kernel of T^{tr} obeys (1.1). Combining (3.24) with our estimates for Σ_2 and Σ_3 , we obtain (3.23), and thus also the conclusion of Theorem 1.3.

We now proceed to prove (3.24). We fix k such that $Q \in \mathbb{D}_k$. We begin by observing that for $Q \in \Omega_2 \cap \mathbb{D}_k$, we have that

$$|\Delta_Q^{b_1} T^{\text{tr}} 1| \leq \frac{1}{\delta} |(\Delta_Q^{b_1} T^{\text{tr}} 1)[b_2]_Q| = \frac{1}{\delta} |\Delta_Q^{b_1} T^{\text{tr}} 1 E_k b_2|$$

where in the last step we have used that $\Delta_Q^{b_1} T^{\text{tr}} 1$ is supported in Q , by definition of $\Delta_Q^{b_1}$. We now use a variant of a trick of Coifman and Meyer [CM], to write

$$(3.25) \quad \begin{aligned} (\Delta_Q^{b_1} T^{\text{tr}} 1) E_k &= \{(\Delta_Q^{b_1} T^{\text{tr}} 1) E_k - \Delta_Q^{b_1} T^{\text{tr}} E_k\} + \Delta_Q^{b_1} T^{\text{tr}} (E_k - I) + \Delta_Q^{b_1} T^{\text{tr}} \\ &\equiv T_{Q,1} + T_{Q,2} + T_{Q,3}. \end{aligned}$$

It is therefore enough to establish (3.24) with $\Delta_Q^{b_1} T^{\text{tr}} 1$ replaced by each of $T_{Q,1} b_2$, $T_{Q,2} b_2$ and $T_{Q,3} b_2$.

The contribution of the latter term is easy to handle. To this end, we define an operator $\Lambda_Q^{b_1}$ by the relationship

$$\Lambda_Q^{b_1}(b_1 g) \equiv \Delta_Q^{b_1} g,$$

i.e. if $\varphi_Q^{b_1}(x, y)$ denotes the kernel of $\Delta_Q^{b_1}$, and, as above

$$\varphi_Q^{b_1}(x, y) = \lambda_Q^{b_1}(x, y) b_1(y),$$

then

$$\Lambda_Q^{b_1} g(x) = \int \lambda_Q^{b_1}(x, y) g(y) dy.$$

We shall prove the following.

Lemma 3.26. *Suppose that $Q_2 \subseteq Q_1$. Let $b_1, b_2 \in q - D\psi A$ on dyadic sawtooth regions $\Omega_1 \cup \Omega_{\text{buffer}}$, $\Omega_2 \cup \Omega_{2,\text{buffer}}$, respectively. Define $C_2 \equiv \sup_{Q \in \Omega_2 \cup \Omega_{2,\text{buffer}}} [|b_2|^2]_Q$. Then*

$$\sum_{Q \in \Omega_2 \cap \Omega_1} \|\Lambda_Q^{b_1}(b_2 g)\|_2^2 \leq C C_2 \|g\|_{L^2(Q_2)}^2.$$

We momentarily defer the proof of Lemma 3.26.

Applying this lemma with $b_1 = b_2$, $\Omega_1 = \Omega_2$, we obtain

$$(3.27) \quad \sum_{Q \in \Omega_1} \|\Delta_Q^{b_1} g\|_2^2 \leq C \|g\|_2^2.$$

Thus,

$$\sum_{Q \in \Omega_1 \cap \Omega_2} \|T_{Q,3} b_2\|_2^2 \leq \sum_{Q \in \Omega_1} \|\Delta_Q^{b_1}(1_{Q_2} T^{\text{tr}} b_2)\|_2^2 \leq C \int_{Q_2} |T^{\text{tr}} b_2|^2 \leq C |Q_2|,$$

as desired, where in the last step we have used hypothesis (ii) of Theorem 1.3.

Let us now prove Lemma 3.26. By (3.5), and Lebesgue's Differentiation Theorem, $b_2 \in L^\infty(F_2)$, where $F_2 \equiv Q_2 \setminus (\cup P_i^2)$, with

$$\|b_2\|_{L^\infty(F_2)}^2 \leq C_2.$$

We decompose

$$(3.28) \quad b_2 g = b_2 g 1_{F_2} + \sum_i (b_2 g) 1_{P_i^2}.$$

By definition, for $Q \in \mathbb{D}_k$, and $x \in Q$,

$$\Lambda_Q^{b_1} h(x) = \frac{E_{k+1} h(x)}{E_{k+1} b_1(x)} - \frac{E_k h(x)}{E_k b_1(x)} = \frac{\Delta_k h(x)}{E_{k+1} b_1(x)} - \frac{E_k h(x) \Delta_k b(x)}{E_{k+1} b_1(x) E_k b_1(x)}.$$

Since $Q \in \Omega_1$, we have that $|E_{k+1} b_1(x)|, |E_k b_1(x)| \geq \delta$, so by a familiar argument involving (2.2), Carleson's Lemma and Lemma 2.15, we have that

$$(3.29) \quad \sum_{Q \in \Omega_1} \|\Lambda_Q^{b_1} h\|_2^2 \leq C \|h\|_2^2.$$

Consequently

$$\sum_{Q \in \Omega_2 \cap \Omega_1} \|\Lambda_Q^{b_1}(b_2 g 1_{F_2})\|_2^2 \leq C \int_{F_2} |b_2 g|^2 \leq CC_2 \|g\|_{L^2(Q_2)}^2.$$

To treat the second term in (3.28), we note that if $P_i^2 \subseteq Q \in \Omega_2$, then $P_i^2 \subseteq Q$, so that $\lambda_Q^{b_1}(x, y)$ is constant on P_i^2 . Also,

$$\int_{P_i^2} (b_2 g - [b_2 g]_{P_i^2}) = 0,$$

so therefore we may replace $\sum_i (b_2 g) 1_{P_i^2}$ by $\sum_i [b_2 g]_{P_i^2} 1_{P_i^2}$. This leads to

$$\begin{aligned} \sum_{Q \in \Omega_2 \cap \Omega_1} \left\| \Lambda_Q^{b_1} \left(\sum_i 1_{P_i^2} [b_2 g]_{P_i^2} \right) \right\|_{L^2}^2 &\leq C \left\| \sum_i 1_{P_i^2} [b_2 g]_{P_i^2} \right\|_{L^2(Q_2)}^2 \\ &= C \sum_i |P_i^2| [b_2 g]_{P_i^2}^2 \leq CC_2 \sum_i \int_{P_i^2} |g|^2, \end{aligned}$$

where we have used (3.29) and then Cauchy-Schwarz and the estimate

$$\frac{1}{|P_i^2|} \int_{P_i^2} |b_2|^2 \leq \frac{C}{|2_D P_i^2|} \int_{2_D P_i^2} |b_2|^2 \leq CC_2.$$

In turn, the latter bound holds because $2_D P_i^2$, the dyadic double of P_i^2 , belongs to $\Omega_2 \cup \Omega_{2, \text{buffer}}$. This concludes the proof of Lemma 3.26, and hence also our treatment of the term $T_{Q,3}$ in (3.25).

Next, we consider the term $T_{Q,1}$ in (3.25). By definition, for $Q \in \mathbb{D}_k$

$$1_Q E_k b_2 = 1_Q [b_2]_Q.$$

Thus, since for any g , $\Delta_Q^{b_1} g$ is supported in Q , we have

$$T_{Q,1} b_2 = \Delta_Q^{b_1} T^{\text{tr}} (1_{Q^c} ([b_2]_Q - E_k b_2)) \equiv T'_{Q,1} b_2 + T''_{Q,1} b_2,$$

where

$$T'_{Q,1} b_2 = \Delta_Q^{b_1} T^{\text{tr}} (1_{3Q \setminus Q} ([b_2]_Q - E_k b_2)), \quad T''_{Q,1} b_2 = \Delta_Q^{b_1} T^{\text{tr}} (1_{(3Q)^c} ([b_2]_Q - E_k b_2)).$$

Now, for $Q \in \Omega_1$, $\Delta_Q^{b_1} : L^2(Q) \rightarrow L^2(Q)$. Moreover, $T^{\text{tr}} : L^2(3Q \setminus Q) \rightarrow L^2(Q)$, by (1.2). Thus

$$\|T'_{Q,1} b_2\|_2 \leq C \| [b_2]_Q - E_k b_2 \|_{L^2(3Q \setminus Q)}^2 \leq C \sum_{m=1}^{3^n-1} \|\tilde{\Delta}_k^m b_2\|_{L^2(Q)}^2,$$

where $\tilde{\Delta}_k^m$ is defined as follows. Given $Q \in \mathbb{D}_k$, we enumerate the $3^n - 1$ cubes in \mathbb{D}_k which are adjacent to Q (i.e., which are contained in $3Q \setminus Q$), and we do this in some canonical fashion so that the enumeration does not depend upon Q , but only on position relative to Q . Then for any $x \in Q$, and for Q^m one of these enumerated neighbors of Q , we set

$$\tilde{\Delta}_k^m g(x) = [g]_Q - [g]_{Q^m} \equiv \tilde{\Delta}_Q^m g(x)$$

We leave it to the reader to verify that for each $m = 1, 2, 3, \dots, 3^n - 1$, we have the square function estimate

$$\sum_{Q \text{ dyadic}} \|\tilde{\Delta}_Q^m g\|_2^2 = \sum_{k=-\infty}^{\infty} \|\tilde{\Delta}_k^m g\|_2^2 \leq C \|g\|_2^2.$$

Consequently,

$$\sum_{Q \in \Omega_1 \cap \Omega_2} \|T''_{Q,1} b_2\|_2^2 \leq C \|b_2\|_2^2 = C \|b_2\|_{L^2(Q_2)}^2 \leq C |Q_2|.$$

We now turn to the term $T''_{Q,1} b_2$. Let $\psi_Q(x, z)$ denote the kernel of $\Delta_Q^{b_1} T^{\text{tr}}$. Since $\Delta_Q^{b_1} 1 = 0$, we have that for $Q \in \Omega_1$ and $z \in (3Q)^c$,

$$\begin{aligned} |\psi_Q(x, z)| &= \left| \int \varphi_Q^{b_1}(x, y) [K^{\text{tr}}(y, z) - K^{\text{tr}}(x, z)] dy \right| \\ &\leq C 1_Q(x) 1_{(3Q)^c}(z) \frac{(\ell(Q))^\alpha}{|x - z|^{n+\alpha}} \frac{1}{|Q|} \int_Q |b_1| \leq C 1_Q \sum_{i=1}^{\infty} 2^{-i\alpha} (2^i \ell(Q))^{-n} 1_{2^i Q \setminus 2^{i-1} Q}(z), \end{aligned}$$

so that

$$(3.30) \quad |T''_{Q,1} b_2| \leq C 1_Q \sum_{i=1}^{\infty} 2^{-i\alpha} \frac{1}{|2^i Q|} \int_{2^i Q} |[b_2]_Q - E_k b_2|.$$

We note that the concentric dilate $2^i Q$ is covered by a purely dimensional number of dyadic cubes of the same side length $2^i \ell(Q) = 2^{i-k}$, namely the dyadic ancestor $(2_D)^i Q$ (here $2_D Q$ denotes the dyadic double of Q), along with its neighbors of the same generation \mathbb{D}_{k-i} . Enumerating these neighbors in the same canonical fashion as above (i.e., as in the definition of $\tilde{\Delta}_k^m$), we denote them by $Q^m(i)$, $1 \leq m \leq 3^n - 1$. We then write

$$(3.31) \quad \begin{aligned} [b_2]_Q &= [b_2]_Q - [b_2]_{2_D Q} + [b_2]_{2_D Q} - [b_2]_{(2_D)^2 Q} + \cdots - [b_2]_{(2_D)^j Q} + [b_2]_{(2_D)^j Q} \\ &= \sum_{\ell=1}^i \Delta_{k-\ell} b_2(x) + [b_2]_{(2_D)^j Q}, \end{aligned}$$

for any $x \in Q$. Similarly,

$$(3.32) \quad E_k b_2 = E_k b_2 - E_{k-1} b_2 + \cdots - E_{k-i} b_2 + E_{k-i} b_2 = \sum_{\ell=1}^i \Delta_{k-\ell} b_2 + E_{k-i} b_2.$$

By definition, $E_{k-i} b_2(x) = [b_2]_{Q^m(i)}$, if $x \in Q^m(i)$, and $E_{k-i} b_2(x) = [b_2]_{(2_D)^j Q}$, if $x \in (2_D)^j Q$. Thus, plugging (3.31) and (3.32) into (3.30), we obtain that

$$|T''_{Q,1} b_2| \leq C 1_Q \sum_{i=1}^{\infty} 2^{-i\alpha} \left[\sum_{\ell=1}^i (|\Delta_{k-\ell} b_2| + M(\Delta_{k-\ell} b_2)) + \sum_{m=1}^{3^n-1} |\tilde{\Delta}_{k-i}^m b_2| \right].$$

Consequently,

$$\begin{aligned} \left(\sum_{Q \in \Omega_1 \cap \Omega_2} \|T''_{Q,1} b_2\|_2^2 \right)^{\frac{1}{2}} &\leq C \sum_{i=1}^{\infty} 2^{-i\alpha} \sum_{\ell=1}^i \left(\sum_k \|\Delta_{k-\ell} b_2\|_2^2 \right)^{\frac{1}{2}} \\ &\quad + C \sum_{m=1}^{3^n-1} \sum_{i=1}^{\infty} 2^{-i\alpha} \left(\sum_k \|\tilde{\Delta}_{k-i}^m b_2\|_2^2 \right)^{\frac{1}{2}} \leq C \|b_2\|_2 \leq C |Q_2|^{\frac{1}{2}}. \end{aligned}$$

This completes our treatment of $T_{Q,1}$ in (3.25).

It remains now to consider the term $T_{Q,2}$, and this will be a more delicate matter. We note that by (2.4) and the definition of Δ_j ,

$$E_k - I = - \sum_{j=k}^{\infty} \Delta_j$$

We therefore have that

$$(3.33) \quad T_{Q,2}b_2 = -\Delta_Q^{b_1} T^{\text{tr}} \left[1_{Q^c} \sum_{j \geq k} \Delta_j b_2 \right] \\ + \left\{ \Delta_Q^{b_1} T^{\text{tr}} (1_Q([b_2]_Q - b_2)) - \Lambda_Q^{b_1} (([b_2]_Q - b_2) T b_1) \right\} \\ + \Lambda_Q^{b_1} (([b_2]_Q - b_2) T b_1) \equiv \text{Error}_1 + G_Q + \Phi_Q.$$

where we have used that $1_Q E_k b_2 = 1_Q [b_2]_Q$.

We first turn our attention to Error_1 . We fix

$$\delta \equiv C 2^{-j\epsilon} 2^{-k(1-\epsilon)},$$

with C a fixed large number and $\epsilon > 0$ to be chosen. For each $\mu > 0$, we let Q_μ denote the “ μ -neighborhood of Q ”, i.e.

$$Q_\mu \equiv \{x : \text{dist}(x, Q) < \mu\}.$$

We also define the μ -ring around Q by

$$R_\mu \equiv Q_\mu \setminus Q.$$

We choose a smooth cut-off function $\eta_\delta \in C_0^\infty(Q_{2\delta})$, with $\eta_\delta \equiv 1$ on Q_δ , $\|\nabla \eta_\delta\|_\infty \leq C/\delta$, and $\text{supp } \nabla \eta_\delta \subseteq R_{2\delta} \setminus R_\delta$. We write

$$1_{Q^c} = 1 - \eta_\delta + \eta_\delta - 1_Q.$$

We treat the contribution of $1 - \eta_\delta$ first; that is, we consider

$$\text{Error}'_1 \equiv - \sum_{j \geq k} \Delta_Q^{b_1} T^{\text{tr}} \left((1 - \eta_\delta) \Delta_j^2 b_2 \right),$$

where we have used that $\Delta_j \equiv \Delta_j^2$. We denote by $h(y, v)$ the kernel of the operator $H = T^{\text{tr}}(1 - \eta_\delta) \Delta_j$; i.e.

$$h(y, v) = \int K^{\text{tr}}(y, z) (1 - \eta_\delta(z)) \varphi_j(z, v) dz,$$

where the kernel $\varphi_j(z, v)$ of Δ_j satisfies $\int \varphi_j(z, v) dz = 0$ and $\int |\varphi_j(z, v)| dz \leq C$ for every v . We set

$$K_\delta^{\text{tr}}(y, z) = K^{\text{tr}}(y, z) (1 - \eta_\delta(z)).$$

Then for $y \in Q$, we have

$$|h(y, v)| \leq \int_{|y-z| > c\delta, |z-v| \leq C 2^{-j} < \delta} |K_\delta^{\text{tr}}(y, z) - K_\delta^{\text{tr}}(y, v)| |\varphi_j(z, v)| dz \\ \leq C \frac{2^{-j\alpha}}{|y-v|^{n+\alpha}} 1_{\{|y-v| > c\delta\}} + C \frac{2^{-j}}{\delta} \frac{1}{|y-v|^n} 1_{\{c\delta < |y-v| < C\ell(Q)\}} \\ \equiv h'(y, v) + h''(y, v).$$

We define operators H', H'' by

$$H' g(y) \equiv \int h'(y, v) g(v) dv, \quad H'' g(y) \equiv \int h''(y, v) g(v) dv.$$

Recall that $j \geq k$ and that $\delta \equiv C 2^{-j\epsilon} 2^{-k(1-\epsilon)} = C 2^{-j\epsilon} \ell(Q)^{1-\epsilon}$, so that

$$|h'(y, v)| \leq C 2^{-(j-k)\alpha(1-\epsilon)} \frac{\delta^\alpha}{(\delta + |y-v|)^{n+\alpha}}.$$

Furthermore,

$$\begin{aligned} |H''g(y)| &\leq C2^{-(j-k)(1-\epsilon)}\delta^{-n} \int_{|y-v|\leq C\ell(Q)} |g|dv \\ &\leq C2^{-(j-k)(1-\epsilon)} \left(\frac{\ell(Q)}{\delta}\right)^n Mg(y) = C2^{-(j-k)(1-\epsilon-\epsilon n)}Mg(y). \end{aligned}$$

Combining these estimates, we have that for ϵ chosen small enough, depending only on n , that

$$Hg(y) \leq C2^{-(j-k)\beta}Mg(y),$$

for some $\beta > 0$. Now, for $Q \in \Omega_1$, we have that $\Delta_Q^{b_1} : L^2(Q) \rightarrow L^2(Q)$. Consequently,

$$\|\Delta_Q^{b_1} T^{\text{tr}}((1-\eta_\delta)\Delta_j^2 b_2)\|_2 = \|\Delta_Q^{b_1} H\Delta_j b_2\|_2 \leq C2^{-(j-k)\beta} \|M\Delta_j b_2\|_{L^2(Q)}.$$

Moreover, summing over $Q \in \mathbb{D}_h \cap \Omega_1 \cap \Omega_2$, for each fixed k we obtain

$$\sum_{Q \in \mathbb{D}_k \cap \Omega_1 \cap \Omega_2} \|\Delta_Q^{b_1} H\Delta_j b_2\|_2^2 \leq C2^{-2(j-k)\beta} \|M\Delta_j b_2\|_{L^2(\mathbb{R}^n)}^2.$$

Therefore, by a variant of Schur's Lemma, we obtain

$$\sum_{Q \in \Omega_1 \cap \Omega_2} \|\text{Error}'_1\|_2^2 \leq C\|b_2\|_2^2 \leq C|Q_2|,$$

as desired.

We now consider the rest of Error_1 , namely,

$$\text{Error}'_1 \equiv - \sum_{j \geq k} \Delta_Q^{b_1} T^{\text{tr}}((\eta_\delta - 1_Q)\Delta_j b_2).$$

By (1.2), $T^{\text{tr}} : L^p(6Q \setminus Q) \rightarrow L^p(Q)$, $1 < p < \infty$. We choose p so that $\frac{1}{p} + \frac{1}{q} = 1$, where q is the exponent in hypothesis (i) of Theorem 1.3. Then, by definition of $\Delta_Q^{b_1}$, we have that for $Q \in \Omega_1$,

$$\begin{aligned} |\Delta_Q^{b_1} T^{\text{tr}}((\eta_\delta - 1_Q)\Delta_j b_2)| &\leq C \frac{1}{|Q|} \int_Q |b_1| |T^{\text{tr}}((\eta_\delta - 1_Q)\Delta_j b_2)| \\ &\leq C \| |b_1|^q \|_Q^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q |T^{\text{tr}}((\eta_\delta - 1_Q)\Delta_j b_2)|^p \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{1}{|Q|} \int_{R_{2\delta}} |\Delta_j b_2|^p \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{|R_{2\delta}|}{|Q|} \right)^{\frac{r-p}{pr}} \left(\frac{1}{|Q|} \int_{2Q} |\Delta_j b_2|^r \right)^{\frac{1}{r}} \\ &\leq C2^{-(j-k)\beta} \left(M(|\Delta_j b_2|^r) \right)^{\frac{1}{r}}(x), \end{aligned}$$

for some $\beta > 0$, and for all $x \in Q$, where we have used (3.5) in the third inequality, and where $p < r < 2$. Thus,

$$\|\Delta_Q^{b_1} T^{\text{tr}}((\eta_\delta - 1_Q)\Delta_j b_2)\|_2 \leq C2^{-(j-k)\beta} \left\| \left(M(|\Delta_j b_2|^r) \right)^{\frac{1}{r}} \right\|_{L^2(Q)},$$

so as above we obtain via Schur's Lemma that

$$\sum_{Q \in \Omega_1 \cap \Omega_2} \|\text{Error}'_1\|_2^2 \leq C\|b_2\|_2^2 \leq C|Q_2|.$$

This completes our treatment of Error_1 .

Next, we discuss Φ_Q in (3.33). Since $[b_2]_Q \leq C_2$, for all $Q \in \Omega_2$, we have by (3.29) that

$$\sum_{Q \in \Omega_1 \cap \Omega_2} \|\Lambda_Q^{b_1}([b_2]_Q T b_1)\|_2^2 \leq CC_2 \int_{Q_2} |T b_1|^2 \leq CC_2 |Q_2|,$$

where in the last step we have used that the left hand side is zero unless $Q_2 \in \Omega_1 \cup \Omega_{\text{buffer}}$, so that (3.7) applies to $T b_1$ in Q_2 . Moreover, the remaining part of Φ_Q , namely $-\Lambda_Q^{b_1}(b_2 T b_1)$, may be handled similarly via Lemma 3.26. We omit the routine details.

It remains now to treat G_Q in (3.33). To this end, we set

$$g_Q \equiv 1_Q([b_2]_Q - b_2),$$

so that

$$G_Q = \Delta_Q^{b_1} T^{\text{tr}} g_Q - \Lambda_Q^{b_1}(g_Q T b_1).$$

Suppose that $Q \in \mathbb{D}_k$. We write

$$\begin{aligned} G_Q &= \left\{ \Delta_Q^{b_1} T^{\text{tr}} E_{k+1} g_Q - \Lambda_Q^{b_1}(E_{k+1} g_Q T b_1) \right\} \\ &\quad + \left\{ \Delta_Q^{b_1} T^{\text{tr}}(g_Q - E_{k+1} g_Q) - \Lambda_Q^{b_1}((g_Q - E_{k+1} g_Q) T b_1) \right\} \\ &\equiv G'_Q + \text{Error}_2. \end{aligned}$$

We consider G'_Q first. Since $Q \in \mathbb{D}_k$, we have that $E_k g_Q = 0$. Thus,

$$E_{k+1} g_Q = (E_{k+1} - E_k) g_Q = \Delta_k g_Q = -\Delta_Q b_2,$$

because g_Q is supported in Q , and $\Delta_Q 1 = 0$. We therefore have that

$$\begin{aligned} G'_Q &= -\Delta_Q^{b_1} T^{\text{tr}} \Delta_Q b_2 + \Lambda_Q^{b_1}((\Delta_Q b_2)(T b_1)) \\ &= \text{I}_Q + \text{II}_Q, \end{aligned}$$

and we treat these terms separately. Since $\Delta_Q^{b_1} f = \Lambda_Q^{b_1}(b_1 f) = \langle \lambda_Q^{b_1} b_1, f \rangle$, we have that

$$\text{I}_Q(x) = \langle \lambda_Q^{b_1}(x, \cdot) b_1, T^{\text{tr}}(\Delta_Q b_2) \rangle = \langle T(\lambda_Q^{b_1}(x, \cdot) b_1), \Delta_Q b_2 \rangle.$$

We recall that by definition

$$\lambda_Q^{b_1}(x, y) = \sum_{Q'} \frac{1}{[b_1]_{Q'}} \frac{1}{|Q'|} 1_{Q'}(x) 1_{Q'}(y) - \frac{1}{[b_1]_Q} \frac{1}{|Q|} 1_Q(x) 1_Q(y),$$

where the sum runs over the children Q' of Q . Thus,

$$|\text{I}_Q(x)| \leq C 1_Q(x) \left(|Q|^{-1} |\langle T(1_Q b_1), \Delta_Q b_2 \rangle| + \sum_{Q'} |Q'|^{-1} |\langle T(1_{Q'} b_1), \Delta_Q b_2 \rangle| \right),$$

where we have used that $Q \in \Omega_1$ to control $[b_1]_Q$ and $[b_1]_{Q'}$ from below (again, the sum runs over the children Q' of Q). But by Cauchy-Schwarz, Lemma 3.18, and (3.6) and (3.7), this last expression is no longer than

$$C \left(\frac{1}{|Q|} \int_Q |\Delta_Q b_2|^2 \right)^{\frac{1}{2}}.$$

Similarly, but more simply, the term $\text{II}_Q(x)$ is dominated by

$$C \left(\frac{1}{|Q|} \int_Q |\Delta_Q b_2|^2 \right)^{\frac{1}{2}} \left(\frac{1}{|Q|} \int_Q |T b_1|^2 \right)^{\frac{1}{2}} \leq C \left(\frac{1}{|Q|} \int_Q |\Delta_Q b_2|^2 \right)^{\frac{1}{2}},$$

by (3.7). Altogether then,

$$\sum_{Q \in \Omega_1 \cap \Omega_2} \|G'_Q\|_2^2 \leq \sum_Q \|\Delta_Q b_2\|_2^2 \leq C \|b_2\|_2^2 \leq C |\mathcal{Q}_2|,$$

as desired.

Finally, we consider the term Error_2 . For $Q \in \mathbb{D}_k$, the children Q' of Q belong to \mathbb{D}_{k+1} , so that for each such child Q' ,

$$\int_{Q'} (g_Q - E_{k+1} g_Q) = 0.$$

We set $g'_Q \equiv g_Q - E_{k+1} g_Q$. Now $\Delta_Q^{b_1} f = \Lambda_Q^{b_1}(b_1 f)$, so that for $x \in Q$, we have

$$\begin{aligned} \text{Error}_2(x) &= \Lambda_Q^{b_1}(b_1 T^{\text{tr}} g'_Q)(x) - \Lambda_Q^{b_1}(g'_Q T b_1)(x) \\ &= \langle \lambda_Q^{b_1}(x, \cdot) b_1, T^{\text{tr}} g'_Q \rangle - \langle \lambda_Q^{b_1}(x, \cdot), g'_Q T b_1 \rangle \\ &= \langle T(\lambda_Q^{b_1}(x, \cdot) b_1), g'_Q \rangle - \langle T b_1, \lambda_Q^{b_1}(x, \cdot) g'_Q \rangle \\ &= \sum_{Q'} \left(\langle T(\lambda_Q^{b_1}(x, \cdot) b_1), g'_Q 1_{Q'} \rangle - \langle T b_1, \lambda_Q^{b_1}(x, \cdot) g'_Q 1_{Q'} \rangle \right) \\ &= \sum_{Q'} \left(\langle T(\lambda_Q^{b_1}(x, \cdot) b_1), g'_Q 1_{Q'} \rangle - \langle T(1_{Q'} \lambda_Q^{b_1}(x, \cdot) b_1), g'_Q 1_{Q'} \rangle \right), \end{aligned}$$

where the sum runs over the children Q' of Q , and where, in the last step, we have used that $\lambda_Q^{b_1}(x, \cdot)$ is constant on each child Q' of Q . Thus,

$$\begin{aligned} \text{Error}'_2(x) &= \sum_{Q'} \langle 1_{(Q')^c} \lambda_Q^{b_1}(x, \cdot) b_1, T^{\text{tr}}(g'_Q 1_{Q'}) \rangle \\ &= \sum_{Q'} \Delta_Q^{b_1} (1_{Q \setminus Q'} T^{\text{tr}}(g'_Q 1_{Q'}))(x). \end{aligned}$$

Now, by definition,

$$\begin{aligned} g'_Q 1_{Q'} &= (g_Q - E_{k+1} g_Q) 1_{Q'} \\ &= \{1_{Q'}([b_2]_Q - b_2) - E_{k+1} (1_{Q'}([b_2]_Q - b_2))\} 1_{Q'} \\ &= (E_{k+1} b_2 - b_2) 1_{Q'}, \end{aligned}$$

since $1_{Q'} E_{k+1} (1_Q [b_2]_Q) = [b_2]_Q 1_{Q'}$, for each child of Q' of Q . We expand

$$b_2 = \sum_{j \geq k} \Delta_j b_2 + E_k b_2,$$

and note that since $E_{k+1} E_k - E_k = 0$, we have that

$$E_{k+1} b_2 - b_2 = \sum_{j=k}^{\infty} (E_{k+1} \Delta_j b_2 - \Delta_j b_2).$$

Moreover,

$$E_{k+1} \Delta_j = E_{k+1} (E_{j+1} - E_j) = \begin{cases} E_{k+1} - E_{k+1} = 0, & \text{if } j \geq k+1 \\ E_{k+1} - E_k = \Delta_k, & \text{if } j = k. \end{cases}$$

Thus,

$$E_{k+1} b_2 - b_2 = - \sum_{j=k+1}^{\infty} \Delta_j b_2$$

and consequently,

$$\text{Error}_2 = - \sum_{Q' \text{ children of } Q} \Delta_Q^{b_1} \left(1_{Q \setminus Q'} T^{\text{tr}} \left(\sum_{j=k+1}^{\infty} \Delta_j b_2 1_Q \right) \right).$$

Since $\Delta_j = \Delta_j^2$, it again suffices to show that, for some $\beta > 0$, we have

$$(3.34) \quad \|\Delta_Q^{b_1} (1_{Q \setminus Q'} T^{\text{tr}}(1_{Q'} \Delta_j h))\|_2 \leq C 2^{-\beta(j-k)} \|h\|_{L^2(Q')},$$

for every $j > k$ and each child Q' of Q . Now, the kernel of $1_{Q'} \Delta_j$ is a sum

$$\sum_{\ell} \varphi_{Q_\ell^j}(z, v),$$

where $\ell(Q_\ell^j) = 2^{-j}$, $Q_\ell^j \subseteq Q'$,

$$(3.35) \quad |\varphi_{Q_\ell^j}(z, v)| \leq \frac{C}{|Q_\ell^j|} 1_{Q_\ell^j}(z) 1_{Q_\ell^j}(v),$$

and

$$(3.36) \quad \int \varphi_{Q_\ell^j}(z, v) dz = 0,$$

for each fixed v . We split

$$1_{Q \setminus Q'} = 1_{Q \setminus Q'_\delta} + 1_{R'_\delta \cap Q},$$

where as before Q'_δ is the δ neighborhood of Q' , and $R'_\delta = Q'_\delta \setminus Q'$. In the present situation, we choose $\delta = 2^{-j/2} 2^{-k/2}$. We let $J(y, v)$ denote the kernel of $T^{\text{tr}} 1_{Q'} \Delta_j$, and observe that for $y \in Q \setminus Q'_\delta$, and by (3.35) and (3.36), we have

$$\begin{aligned} |J(y, v)| &= \left| \sum_{\ell} 1_{Q_\ell^j}(v) \int (K^{\text{tr}}(y, z) - K^{\text{tr}}(y, v)) \varphi_{Q_\ell^j}(z, v) dz \right| \\ &\leq C \sum_{\ell} 1_{Q_\ell^j}(v) \int \frac{2^{-j\alpha}}{|y-v|^{n+\alpha}} |\varphi_{Q_\ell^j}(z, v)| dz 1_{\{|y-v|>C\delta\}} \\ &\leq C 1_{Q'}(v) 2^{-(j-k)\alpha/2} \frac{\delta^\alpha}{(\delta + |y-v|)^{n+\alpha}}. \end{aligned}$$

Since $\Delta_Q^{b_1} : L^2(Q) \rightarrow L^2(Q)$, for $Q \in \Omega_1$, we have that (3.34) holds for the contribution of $1_{Q \setminus Q'_\delta}$.

It remains now only to treat the contribution of $1_{R'_\delta \cap Q}$. To this end, we recall that $\varphi_Q^{b_1}(x, y)$, the kernel of $\Delta_Q^{b_1}$, satisfies

$$|\varphi_Q^{b_1}(x, y)| \leq \frac{C}{|Q|} 1_Q(x) 1_Q(y) b_1(y).$$

Then for $x \in Q \in \Omega_1 \cap \mathbb{D}_k$, q as in hypothesis (i) of Theorem 1.3 (and also (3.5)), $\frac{1}{p} + \frac{1}{q} = 1$, and $p < r < 2$, we have

$$\begin{aligned} |\Delta_Q^{b_1} 1_{R'_\delta \cap Q} T^{\text{tr}}(1_{Q'} \Delta_j h)(x)| &\leq C \frac{1}{|Q|} \int_{R'_\delta \cap Q} |b_1| |T^{\text{tr}}(1_{Q'} \Delta_j h)| \\ &\leq C \left(\frac{1}{|Q|} \int_Q |b_1|^q \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_{R'_\delta \cap Q} |T^{\text{tr}}(1_{Q'} \Delta_j h)|^p \right)^{\frac{1}{p}} \\ &\leq C \left(\frac{|R'_\delta \cap Q|}{|Q|} \right)^{\frac{r-p}{rp}} \left(\frac{1}{|Q|} \int_{Q'_\delta \setminus Q'} |T^{\text{tr}}(1_{Q'} \Delta_j h)|^r \right)^{\frac{1}{r}} \\ &\leq C 2^{-(j-k)\beta} \left(\frac{1}{|Q|} \int_Q |\Delta_j h|^r \right)^{\frac{1}{r}}, \end{aligned}$$

for some $\beta > 0$, where in the last step we have used the dual of the L^r version of (1.2). Since Q' is a child of Q , the last expression is bounded by

$$C 2^{-(j-k)\beta} \left(M(|1_{Q'} \Delta_j h|^r) \right)^{\frac{1}{r}}(x),$$

for every $x \in Q$. Since $1_{Q'} \Delta_j h = 1_{Q'} \Delta_j(1_{Q'} h)$, for $j \geq k+1$, (3.34) follows.

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