MEAN SQUARE DISCREPANCY BOUNDS FOR THE NUMBER OF LATTICE POINTS IN LARGE CONVEX BODIES

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Dedicated to the memory of Thomas Wolff

1. Introduction

Let Ω be a convex domain in $\mathbb{R}^d$ containing the origin in its interior. We mostly assume that Ω has smooth boundary and that the Gaussian curvature of the boundary vanishes nowhere. Let

$$\mathcal{N}_\Omega(t) = \mathrm{card}(t\Omega \cap \mathbb{Z}^d),$$

the number of integer lattice point inside the dilated domain $t\Omega$. It is well known that $\mathcal{N}_\Omega(t)$ is asymptotic to $t^d \mathrm{vol}(\Omega)$ as $t \to \infty$. We denote by

$$\Delta_\Omega(t) = \frac{\mathcal{N}_\Omega(t) - t^d \mathrm{vol}(\Omega)}{t^d \mathrm{vol}(\Omega)} \tag{1.1}$$

the relative error, or the discrepancy function. It is conjectured that in dimensions $d \geq 5$ the relative error is $O(t^{-2})$ as $t \to \infty$. This conjecture is known to be true in the case of a ball centered at the origin, and for ellipsoids in various degrees of generality (see Landau [20], Walfisz [31], [32], Bentkus and Götze [2]). The error can be even smaller. For example, Jarník [14] established the bound $O(t^{-d/2+\varepsilon})$ for the relative error, with any $\varepsilon > 0$, for almost all ellipsoids with axes parallel to the coordinate axes. For general convex domains with non-vanishing curvature on the boundary, W. Müller [22] proved that $\Delta_\Omega(t) = O(t^{-2+\lambda(d)+\varepsilon})$, where $\lambda(d) = (d+4)/(d^2+d+2)$, if $d \geq 5$, $\lambda(4) = 6/17$ and $\lambda(3) = 20/43$, improving on earlier results by Krätzel and Nowak [19]. For planar domains, Huxley [11] obtained this estimate with $\lambda(2) = 46/73$, which implies the relative error $O(t^{-100/73}(\log t)^{315/146})$.

In this paper we study the *mean square discrepancy* of the lattice rest, the square function

$$\mathcal{G}_\Omega(R) = \Big(\frac{1}{R}\int_R^{2R} |\Delta_\Omega(t)|^2 dt\Big)^{1/2} \tag{1.2}$$

and related expressions. For the ball B_d in $\mathbb{R}^d$, centered at the origin, bounds and various asymptotics for mean square discrepancies have been obtained by Walfisz [32] for $d \geq 4$, Jarník [16] for $d = 3$ and Katai [17] for $d = 2$.

In the more general situation where the boundary of Ω is smooth and is assumed to have everywhere non-vanishing Gaussian curvature, Nowak [25] proved that $\mathcal{G}_\Omega(R) = O(R^{-3/2})$ for planar

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domains. This estimate is sharp by the results of Bleher [3] who investigated the limit of $R^{3/2}\mathcal{G}_\Omega(R)$ as $R \to \infty$. The higher dimensional case was considered by W. Müller [21], who proved a nearly sharp estimate for $d \geq 4$, namely that $\mathcal{G}_\Omega(R) \leq C_\varepsilon R^{-2+\varepsilon}$ for any $\varepsilon > 0$. The case $d = 3$ was left open.

The main purpose of this note is to show that the known endpoint bounds for the mean square discrepancy in the case of the ball remain valid in the general case, provided that $d \geq 4$. Moreover, we prove a nearly sharp estimate in dimension $d = 3$, where we are off by a factor of $\sqrt{\log R}$.

Theorem 1.1. *Let Ω be a convex domain in $\mathbb{R}^d$ containing the origin in its interior, and assume that Ω has smooth boundary with everywhere non-vanishing Gaussian curvature. Then there exists a constant $C(\Omega)$, such that for all $R \geq 2$,*

$$\mathcal{G}_\Omega(R) \leq C(\Omega) \begin{cases} R^{-2} & \text{if } d \geq 4 \\ R^{-2}\log R & \text{if } d = 3\ . \\ R^{-3/2} & \text{if } d = 2 \end{cases} \tag{1.3}$$

As we noted above, the sharp estimate $O(R^{-3/2})$ in the plane was already known for more general *planar* domains with the non-vanishing curvature assumption. In fact, it turns out that this estimate holds even if we replace the mean square discrepancy over $[R, 2R]$ by the mean square discrepancy over substantially smaller intervals $[R, R+h]$. A closely related result due to Huxley [10] says that $\big(\int_R^{R+1} |\Delta_\Omega(t)|^2 dt\big)^{1/2} \leq C_\Omega R^{-3/2}(\log R)^{1/2}$.

Theorem 1.2.1. *Let Ω be a convex domain in $\mathbb{R}^2$ containing the origin in its interior, and assume that Ω has C^∞ boundary with non-vanishing curvature. Then there is a constant $C = C(\Omega)$ so that for all $R \geq 2$,*

$$\Big(\frac{1}{h}\int_R^{R+h} |\Delta_\Omega(t)|^2 dt\Big)^{1/2} \leq C(\Omega)R^{-3/2} \quad \text{if} \quad \log R \leq h \leq R. \tag{1.4}$$

As an immediate consequence of Theorem 1.2.1, the mean square discrepancy over $[R, R+h]$ is dominated by $C(\Omega)R^{-3/2}(\log R)^{1/2}h^{-1/2}$ if $0 \leq h \leq \log R$. In particular, the aforementioned result of Huxley follows if we set $h = 1$.

We now consider more general domains in the plane. We say that a convex domain is of type at most $\mathfrak{m}$ if its boundary has order of contact at most $\mathfrak{m}$ with every tangent line. Thus if $\mathfrak{m} = 2$ we recover the case of everywhere non-vanishing curvature considered above. It is known that the analogue of Theorem 1.2.1 may fail if the order is greater than 2 (*cf.* [26], [5], and [23]). However for almost all rotations the estimate remains true for the rotated domain. More precisely we have the following result.

Theorem 1.2.2. *Let Ω be a convex domain in $\mathbb{R}^2$ containing the origin in its interior, and assume that the boundary is smooth and of finite type at most $\mathfrak{m}$, in the sense that the order of contact of $\partial\Omega$ with every tangent line is at most $\mathfrak{m}$. For $A \in SO(2)$, denote by $A\Omega$ the rotated domain $\{Ax : x \in \Omega\}$. Then for almost all rotations A, the inequality (1.4) holds for $A\Omega$, with the constant $C_{A\Omega}$ depending on A. More precisely, the following hold.*

(i) The maximal function

$$\mathcal{C}_\Omega(A) = \sup_{R\geq 2} R^{3/2} \sup_{\log R\leq h\leq R} \Big(\frac{1}{h}\int_R^{R+h} |\Delta_{A\Omega}(t)|^2 dt\Big)^{1/2} \tag{1.5}$$

belongs to the weak type space $L^{\frac{2\mathfrak{m}-2}{\mathfrak{m}-2},\infty}(SO(2))$.

(ii) Let Γ *be the set of all points* $P \in \partial\Omega$ *where the curvature vanishes, and for* $P \in \Gamma$ *assume that the curvature vanishes of order* $m_P - 2$ $(\leq \mathfrak{m} - 2)$. *Let* n_P *be the outer unit normal at* P *and* v_P *a unit tangent vector at* P. *Then* $\mathcal{C}_\Omega(A) < \infty$, *if* A *satisfies, for some* $\epsilon > 0$, *the Diophantine condition*

$$\max_{P\in\Gamma} \sup\{|k|^{\frac{m_P}{m_P-2}-\epsilon}|\langle k, A^* v_P\rangle| : \operatorname{dist}(k, \mathbb{R}n_P) \leq 1\} > 0. \tag{1.6}$$

In particular the set $\{A \in SO(2) : \mathcal{C}_\Omega(A) = \infty\}$ *is of Hausdorff dimension* $\leq \frac{\mathfrak{m}-2}{\mathfrak{m}-1}$.

It is likely that one can weaken the Diophantine condition and thus the estimate for the upper bound of the Hausdorff dimension is presumably not sharp. The latter theorem is related to the results by Colin de Verdiére [5] and Tarnopolska-Weiss [30] who proved similar statements about the maximal function $A \mapsto \sup_{t\geq 1} t^{4/3}\Delta_{A\Omega}(t)$; see also the improvements by Nowak [24] who obtained the van der Corput type bounds $|\Delta_{A\Omega}(t)| \leq C_A t^{-4/3-\delta}$ for suitable $\delta = \delta(\Omega) > 0$, again under appropriate Diophantine conditions on the rotation.

We remark that in Theorem 1.2.1 the smoothness assumption can be relaxed considerably; moreover a slightly weaker variant of Theorem 1.2.2 holds without any assumption on the boundary of the convex domain. These issues are taken up in the sequel [13] to this paper.

Notation: Given two quantities A, B we write $A \lesssim B$ if there is an absolute positive constant, depending only on the specific domain Ω, so that $A \leq CB$. We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

We denote by Ω^* the polar set of Ω,

$$\Omega^* = \{\xi : \langle x, \xi\rangle \leq 1 \text{ for all } x \in \Omega\}, \tag{2.1}$$

and let ρ^* be the Minkowski functional associated to Ω^*; i.e. ρ^* is homogeneous of degree 1 and satisfies $\rho^*(\xi) = 1$ if $\xi \in \partial\Omega^*$. Then, if $P_+(\xi)$ is the unique point in $\partial\Omega$ at which ξ is an outer normal to $\partial\Omega$, then

$$\rho^*(\xi) = \langle P_+(\xi), \xi\rangle. \tag{2.2.1}$$

Similary, if $P_-(\xi)$ is the unique point in $\partial\Omega$ at which ξ is an inner normal, then

$$\rho^*(-\xi) = -\langle P_-(\xi), \xi\rangle. \tag{2.2.2}$$

If $t \mapsto x(t)$ is a regular C^k parametrization of $\partial\Omega$ near a point $P_0 = x(t_0)$, and $t \mapsto n(t)$ denotes the outward unit normal vector, then $t \mapsto x^*(t) = \langle x(t), n(t)\rangle^{-1} n(t)$ parametrizes the boundary of

Ω^*, and x^* is of class C^{k-1}. If $\kappa(P_0)$ denotes the Gaussian curvature at P_0, and $\kappa(P_0) \neq 0$ then the parametrization $t \mapsto x^*(t)$ is regular near $P_0^* = x^*(t_0)$ and the curvature $\kappa^*(P_0^*)$ of $\partial\Omega^*$ at P_0^* satisfies

$$|\kappa(P_0^*)\kappa(P_0)| = (|P_0| \cdot |P_0^*|)^{-d-1}. \tag{2.3}$$

For these facts see e.g. Lemma 1 in [21].

We shall also need asymptotics for the indicator function of a convex domains. Suppose that Ω is of finite line type (in the sense that every tangent line has finite order of contact with $\partial\Omega$). Let $d\mu$ be a smooth density on the boundary of Ω. We define the Fourier transform by $\widehat{f}(\xi) = \int f(y)\exp(\imath\langle -y, \xi\rangle)dy$, and then a result by Bruna, Nagel and Wainger [4] says that

$$\widehat{d\mu}(\xi) = e^{-\imath\langle P_+(\xi),\xi\rangle}a_+(\xi) + e^{-\imath\langle P_-(\xi),\xi\rangle}a_-(\xi), \tag{2.4.1}$$

where $a_\pm$ is smooth and satisfies the symbol estimates

$$|\partial_\xi^\alpha a_\pm(\xi)| \leq C_\alpha \gamma_\pm(\xi)|\xi|^{-|\alpha|}, \quad |\xi| \geq 1 \tag{2.4.2}$$

for all multiindices α, and $\gamma_\pm$ is defined as follows. Let H_P be the (affine) tangent plane to Ω at P. Then $\gamma_\pm(\xi)$ is the surface measure of the cap

$$\gamma_\pm(\xi) = \sigma\big(\{y \in \partial\Omega : \mathrm{dist}(y, H_{P_\pm(\xi)}) \leq |\xi|^{-1}\}\big), \tag{2.5}$$

where σ denotes surface measure on $\partial\Omega$. By the divergence theorem, $\partial_{x_i}\chi_\Omega = -n_i d\sigma$, in the sense of distributions, where $n = (n_1, \dots, n_d)$ is the outward unit normal. Thus we get $\widehat{\chi_\Omega}(\xi) = -\imath\sum_{i=1}^d(\xi_i/|\xi|^2)\widehat{n_i d\sigma}(\xi)$. If one combines this with (2.2.1/2) and (2.4.1/2), one obtains

$$\widehat{\chi_\Omega}(\xi) = e^{-\imath\rho^*(\xi)}b_+(\xi) + e^{\imath\rho^*(-\xi)}b_-(\xi), \tag{2.7}$$

where

$$|\partial_\xi^\alpha b_\pm(\xi)| \leq C_\alpha \gamma_\pm(\xi)|\xi|^{-1-|\alpha|}, \quad |\xi| \geq 1. \tag{2.8}$$

In the case of non-vanishing curvature one has $\gamma_\pm(\xi) \lesssim |\xi|^{-(d-1)/2}$ but of course the above statement, and more precise asymptotics, follow from the method of stationary phase as in papers by Hlawka [8] (see also §7 in [9]). More generally, for finite type domains one has

$$\gamma_\pm(\xi) \lesssim |\kappa(x_\pm(\xi))|^{-1/2}|\xi|^{-(d-1)/2}. \tag{2.9}$$

This is proved in [29], and can also be deduced from the cap estimates (2.5) using an argument in [6]. However, it should be noted that these results are much easier in the two-dimensional case needed here. See [27] and also [1].

Definitions. Let $\delta_0 > 0$ be fixed so that the ball $B_{2\delta_0}(0)$ with center 0 and radius $2\delta_0$ is contained in Ω. Let ζ be a smooth nonnegative radial cutoff function supported in the ball $B_\delta(0)$ so that $\int \zeta(x)dx = 1$. Let $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$.

We set $N(t) = \mathcal{N}_\Omega(t)$,

$$E(t) = N(t) - t^d \mathrm{vol}(\Omega),$$

and

$$\begin{aligned} N_\varepsilon(t) &= \sum_{k\in\mathbb{Z}^d} \chi_{t\Omega} * \zeta_\varepsilon(k) \\ E_\varepsilon(t) &= N_\varepsilon(t) - t^d\mathrm{vol}(\Omega). \end{aligned} \tag{2.10}$$

We also denote by $N_\varepsilon^*(t)$ and $E_\varepsilon^*(t)$ the corresponding expressions for the polar domain Ω^*.

Three elementary Lemmas.

Lemma 2.1. *Suppose that Ω has C^1 boundary. Then there is a constant $C = C(\Omega)$ such that for $1 \le R \le t \le 2R$, $0 < \varepsilon \le 1$,*

$$|E_\varepsilon(t-\varepsilon)| - Ct^{d-1}\varepsilon \le |E(t)| \le |E_\varepsilon(t+\varepsilon)| + Ct^{d-1}\varepsilon \tag{2.11.1}$$

$$|E(t-\varepsilon)| - Ct^{d-1}\varepsilon \le |E_\varepsilon(t)| \le |E(t+\varepsilon)| + Ct^{d-1}\varepsilon \tag{2.11.2}$$

Proof. By the properties of the the cutoff ζ_ε we have

$$N_\varepsilon(t-\varepsilon) \le N(t) \le N_\varepsilon(t+\varepsilon),$$

and if we subtract $V(t) = t^d \mathrm{vol}(\Omega)$ throughout, we get

$$E_\varepsilon(t-\varepsilon) + [V(t-\varepsilon) - V(t)] \le E(t) \le E_\varepsilon(t+\varepsilon) + [V(t+\varepsilon) - V(t)].$$

Clearly $|V(t\pm\varepsilon) - V(t)| \lesssim t^{d-1}\varepsilon$ and (2.11.1) follows. (2.11.2) follows as well if we apply (2.11.1) with $t \pm \varepsilon$ in place of t. $\square$

Lemma 2.2. *Suppose that $\mu \in [0,1]$ and that the estimate*

$$\sup_{t>0} t^{-(d-1-\mu)}|E(t)| \le C_1 \tag{2.12}$$

holds. Then for $t \ge 1$

$$E_\varepsilon(t) \lesssim \max\{t^{d-1-\mu}, t^{d-1}\varepsilon\}. \tag{2.13}$$

Moreover there a constant C so that for $0 < \varepsilon \le h \le r$

$$\Big|\Big(\frac{1}{h}\int_r^{r+h} |E(t)|^2 dt\Big)^{1/2} - \Big(\frac{1}{h}\int_r^{r+h} |E_\varepsilon(t)|^2 dt\Big)^{1/2}\Big| \le Cr^{d-1}[h^{-1/2}\varepsilon^{1/2}r^{-\mu} + \varepsilon].$$

Proof. We first observe that (2.13) is immediate by Lemma 2.1. We integrate and obtain

$$\begin{aligned}\int_r^{r+h} |E(t)|^2 dt &\le \int_r^{r+h+\varepsilon} |E_\varepsilon(t)|^2 dt + Chr^{2d-2}\varepsilon^2 \\ &\le \int_r^{r+h} |E_\varepsilon(t)|^2 dt + Chr^{2d-2}\varepsilon^2 + C'\varepsilon r^{2(d-1-\mu)},\end{aligned}$$

which implies one of the desired inequalities, the other is obtained in the same way. $\square$

Lemma 2.3. *Let $0 < \varepsilon < 1$ and let for $\tau \ge 1$*

$$\mathfrak{S}(\tau,\varepsilon) = \mathrm{card}\{\ell \in \mathbb{Z}^d : \tau - \varepsilon \le \rho(\ell) \le \tau + \varepsilon\}. \tag{2.15}$$

Then

$$\mathfrak{S}(\tau,\varepsilon) \le C_1\tau^{d-1}\varepsilon + C_2\Big(\int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} E_\epsilon(t)^2 N'_\epsilon(t) dt\Big)^{1/3}, \qquad \epsilon = \frac{4\varepsilon}{\delta_0}. \tag{2.16}$$

Proof. Let $t \in (\tau - \varepsilon, \tau + \varepsilon)$. We use the elementary inequality

$$\int \chi_{(t+h)\Omega \setminus t\Omega}(x-y)\epsilon^{-d}\zeta(\epsilon^{-1}y)dy \geq c_0 h/\epsilon \quad \text{if } h \ll \varepsilon,\, x \in (\tau+\varepsilon)\Omega \setminus (\tau-\varepsilon)\Omega.$$

This implies

$$\begin{aligned} N_\epsilon(t+h) - N_\epsilon(t) &= \sum_k \int \chi_{(t+h)\Omega \setminus t\Omega}(k-y)\epsilon^{-d}\zeta(\epsilon^{-1}y)dy \\ &\geq c_0 \frac{h}{\epsilon}\mathfrak{S}(\tau,\varepsilon) \end{aligned}$$

and thus

$$N_\epsilon'(t) \geq c_0 \mathfrak{S}(\tau,\varepsilon)\varepsilon^{-1}, \quad |t-\tau| \leq \varepsilon, \quad \epsilon = \frac{4\varepsilon}{\delta_0}. \tag{2.17}$$

We now turn to the proof of (2.16). We may assume that $\mathfrak{S}(\tau,\varepsilon) \geq C_1 \varepsilon \tau^{d-1}$ with $C_1 = d2^{d+1}c_0^{-1}\mathrm{vol}(\Omega)$. Then by (2.17),

$$\begin{aligned} E_\epsilon'(t) &= N_\epsilon'(t) - d\,t^{d-1}\mathrm{vol}(\Omega) \\ &\geq N_\epsilon'(t) - d\,(2\tau)^{d-1}\mathrm{vol}(\Omega) \\ &\geq c_0\mathfrak{S}(\tau,\varepsilon)\varepsilon^{-1} - 2^d C_1^{-1} d\,\varepsilon^{-1}\mathrm{vol}(\Omega)\mathfrak{S}(\tau,\varepsilon) \\ &\geq c_0(2\varepsilon)^{-1}\mathfrak{S}(\tau,\varepsilon). \end{aligned} \tag{2.18}$$

Let $I_{\tau,\varepsilon} = [\tau - \varepsilon/2, \tau + \varepsilon/2]$ and pick $t_0 \in I_{\tau,\varepsilon}$ so that $\min_{t \in I_{\tau,\varepsilon}} |E_\epsilon(t)| = |E_\epsilon(t_0)|$; thus $|E_\epsilon(t)| \geq |E_\epsilon(t) - E_\epsilon(t_0)|/2$ and $|E_\epsilon(t)| \geq |\int_{t_0}^t E_\epsilon'(s)ds|/2 \geq c_0(4\epsilon)^{-1}|t-t_0|\mathfrak{S}(\tau,\varepsilon)$. We use also (2.17) and obtain that

$$\int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} E_\epsilon(t)^2 N_\epsilon'(t)dt \geq \int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} (\tfrac{c_0}{4\varepsilon}\mathfrak{S}(\tau,\varepsilon))^2|t-t_0|^2 \tfrac{c_0}{\varepsilon}\mathfrak{S}(\tau,\varepsilon)\,dt \geq c[\mathfrak{S}(\tau,\varepsilon)]^3$$

as asserted. □

3. Proof of Theorem 1.1

In this section we assume that Ω has a smooth boundary with everywhere non-vanishing curvature. This implies that Ω^* is also smooth and has everywhere non-vanishing Gaussian curvature. See (2.3) above. We estimate the square-function

$$G_\varepsilon(R) = \Big(\frac{1}{R}\int_R^{2R} |E_\varepsilon(t)|^2 dt\Big)^{1/2}$$

for $0 < \varepsilon \leq 1/2$ and $R \geq 2$, and set

$$w_d(R) = \begin{cases} R^{2-d} & \text{if } d \geq 4 \\ (R\log R)^{-1} & \text{if } d = 3 \\ R^{-1/2} & \text{if } d = 2 \end{cases} \tag{3.1}$$

and for $0 < s \le 1/2$ let

$$A_d(s) = \sup_{s<\varepsilon\le 1/2}\sup_{R\ge 2}(1+\varepsilon R)^{-d-1} w_d(R) G_\varepsilon(R). \tag{3.2}$$

Analogously, we denote by $A_d^*(s)$ the corresponding quantity associated to Ω^*. It is not hard to see that $A_d(s)$ is finite for every s since we have a trivial estimate $A_d(s) \lesssim \sup_{R\ge 2}(1+sR)^{-d-1}R \lesssim s^{-1}$, and, similarly, $A_d^*(s) \lesssim s^{-1}$ for every $s \le 1/2$. We shall see that $A_d(s)$ is bounded as $s \to 0$. Once this is established, the required bound for $\mathcal{G}_\Omega$ follows from

$$\mathcal{G}_\Omega(R) \lesssim R^{-d}\big(G_{1/R}(R) + R^{d-2}\big), \tag{3.3}$$

which is a consequence of Lemma 2.2.

The boundedness of $A_d(s)$ can be deduced from the following iterative procedure.

Proposition 3.1. *There is a constant C_Ω so that for $s \le 1/2$*

$$A_d(s)^2 \le C_\Omega\big(1 + A_d^*(s)\big). \tag{3.4}$$

Indeed, since $\Omega^{**} = \Omega$, (3.4) implies that $A_d^*(s)^2 \le C_{\Omega^*}\big(1 + A_d(s)\big)$, so

$$A_d(s)^2 \le C_\Omega(1 + \sqrt{C_{\Omega^*}(1 + A_d(s))})$$

from which the boundedness of A_d is immediate.

Proof of Proposition 3.1. We estimate $G_\varepsilon(R)$ assuming first that

$$R^{-1} \le \varepsilon \le 1/2.$$

We apply the Poisson summation formula $\sum_{k\in\mathbb{Z}^d} f(k) = (2\pi)^d \sum_{k\in\mathbb{Z}^d} \widehat{f}(2\pi k)$ to $f = \chi_\Omega(t\cdot) * \zeta_\varepsilon$. This yields

$$E_\varepsilon(t) = \sum_{k\neq 0}(2\pi t)^d \widehat{\chi_\Omega}(2\pi t k)\widehat{\zeta}(2\pi\varepsilon k). \tag{3.5.1}$$

We split $E_\varepsilon(t) = \sum_\pm E_\varepsilon^\pm(t)$ by using (2.7/8); here

$$\begin{aligned} E_\varepsilon^+(t) &= \sum_{k\neq 0}(2\pi t)^d b_+(2\pi t k)\exp(-2\pi\imath\rho^*(k)) \\ E_\varepsilon^-(t) &= \sum_{k\neq 0}(2\pi t)^d b_-(2\pi t k)\exp(2\pi\imath\rho^*(-k)). \end{aligned} \tag{3.5.2}$$

Now fix a nonnegative $\eta \in C^\infty(\mathbb{R})$ so that $\eta(t) = 1$ for $t \in [1,2]$ and η is supported in $(1/2, 3)$. Then

$$\begin{aligned} G_\varepsilon(R) &\le G_\varepsilon^+(R) + G_\varepsilon^-(R) \\ &:= \sum_\pm \Big(R^{-1}\int |E_\varepsilon^\pm(t)|^2 \eta(R^{-1}t)dt\Big)^{1/2}. \end{aligned}$$

We shall only consider estimates for $G_\varepsilon^+(R)$ because the estimates for $G_\varepsilon^-(R)$ are exactly analogous. Multiplying out the squared expression we get

$$G_\varepsilon^+(R)^2 = \sum_{\substack{k\neq 0\\ k'\neq 0}} \widehat{\zeta}(2\pi\varepsilon k)\overline{\widehat{\zeta}(2\pi\varepsilon k')} R^{-1} \int e^{2\pi\imath t(\rho^*(k)-\rho^*(k'))} q_{k,k'}(t)dt \tag{3.6}$$

where

$$q_{k,k'}(t) = b_+(2\pi t k)\overline{b_+(2\pi t k')} t^{2d}\eta(t/R). \tag{3.7}$$

Thus $q_{k,k'}$ is supported in $[R/2, 3R]$ and by (2.8) and $\gamma_\pm(\xi) = O(|\xi|^{-(d-1)/2})$ we have the symbol estimates

$$\Big|\Big(\frac{d}{dt}\Big)^m q_{k,k'}(t)\Big| \le C_m R^{d-1-m}|k|^{-(d+1)/2}|k'|^{-(d+1)/2}. \tag{3.8}$$

We now integrate by parts in t. We note that $|k|\approx \rho^*(k)$ and $|\widehat{\zeta}(2\pi k/R)| \le C_N(1+|k/R|)^{-N}$ and obtain the estimate

$$G_\varepsilon^+(R)^2 \le C_{M,N}\sum_{k\neq 0}\sum_{k'\neq 0} R^{d-1}(1+R|\rho^*(k)-\rho^*(k')|)^{-M}(1+\varepsilon|k|+\varepsilon|k'|)^{-N}[\rho^*(k)\rho^*(k')]^{-\frac{d+1}{2}}.$$

The terms with $|\rho^*(k)-\rho^*(k')| \ge R^{-1/2}$ give a contribution of $O(R^{d-1-M/2}\varepsilon^{-2d}) = O(R^{3d-1-M/2})$ and we may choose $M = 6d$.

Thus

$$\begin{aligned} G_\varepsilon^+(R)^2 &\le C_1 \sum_{-R^{1/2}\le n\le R^{1/2}} \sum_{k\neq 0}(1+\varepsilon\rho^*(k))^{-N} \sum_{\substack{|\rho^*(\ell)-\rho^*(k)|\\ \in[\frac{n-1}{R},\frac{n}{R}]}} \frac{R^{d-1}}{(1+n)^M}[\rho^*(k)]^{-d-1} + C_2R^{3d-1-M/2}\\ &\le C_1' R^{d-1}\sum_{-R^{1/2}\le n\le R^{1/2}} (1+n)^{-M}\sum_{k\neq 0}(1+\varepsilon\rho^*(k))^{-N}\frac{\mathfrak{S}^*(\rho^*(k),\frac{n+1}{R})}{\rho^*(k)^{d+1}} + C_2'R^{3d-1-M/2}; \end{aligned} \tag{3.9}$$

here recall that $\mathfrak{S}^*(\tau,\varepsilon) = \operatorname{card}\{\ell\in\mathbb{Z}^d : \tau-\varepsilon\le\rho^*(\ell)\le\tau+\varepsilon\}$. Now

$$\begin{aligned} &\sum_{k\neq 0}(1+\varepsilon\rho^*(k))^{-N}\frac{\mathfrak{S}^*(\rho^*(k),\frac{n+1}{R})}{\rho^*(k)^{d+1}}\\ &\lesssim \sum_{l=0}^{\infty} 2^{-l}(1+\varepsilon 2^l)^{-N}\Big(\frac{1}{2^{ld}}\sum_{2^l\le\rho^*(k)<2^{l+1}}[\mathfrak{S}^*(\rho^*(k),\tfrac{n+1}{R})]^2\Big)^{1/2}\\ &\lesssim \sum_{l=0}^{\infty} 2^{-l}(1+\varepsilon 2^l)^{-N}[(n+1)I_l + II_{n,l}] \end{aligned}$$

where

$$I_l = \Big(\frac{1}{2^{ld}}\sum_{2^l\le\rho^*(k)<2^{l+1}}\rho^*(k)^{2d-2}R^{-2}\Big)^{1/2},$$

$$II_{n,l} = \Big(\frac{1}{2^{ld}}\sum_{2^l\le\rho^*(k)<2^{l+1}}\int_{\rho^*(k)-(n+1)/2R}^{\rho^*(k)+(n+1)/2R}\frac{E_\epsilon^*(t)^2 {N_\epsilon^*}'(t)}{\mathfrak{S}^*(\rho^*(k),\frac{n+1}{R})}dt\Big)^{1/2},$$

with $\epsilon = 4\varepsilon/\delta_0$; here we used Lemma 2.3. Observe that for N large,

(3.10)
$$\sum_{l=0}^{\infty} 2^{-l}(1+\varepsilon 2^l)^{-N} I_l \lesssim R^{-1}\sum_{l=0}^{\infty} 2^{l(d-2)}(1+\varepsilon 2^l)^{-N} \lesssim \begin{cases} R^{-1}\varepsilon^{2-d} & \text{if } d\geq 3 \\ R^{-1}\log(2+\varepsilon^{-1}) & \text{if } d=2 \end{cases}$$

and thus, since we are assuming $\varepsilon \leq 1/R$,

$$R^{d-1}\sum_{l=0}^{\infty} 2^{-l}(1+\varepsilon 2^l)^{-N} I_l \lesssim R^{d-2}\max\{\varepsilon^{2-d}, \log(2+\varepsilon^{-1})\} \lesssim w_d(R)^{-2}.$$

We now estimate $II_{n,l}$ and set $J_{k,n} := [\rho^*(k)-(n+1)/2R, \rho^*(k)+(n+1)/2R]$. Observe that

$$\begin{aligned} \mathfrak{S}(\rho^*(k), \tfrac{n+1}{R}) &= \mathrm{card}\{\ell : \rho^*(k) - \tfrac{n+1}{R} \leq \rho^*(\ell) \leq \rho^*(k) + \tfrac{n+1}{R}\} \\ &\geq \mathrm{card}\{\ell : t - \tfrac{n+1}{2R} \leq \rho^*(\ell) \leq t + \tfrac{n+1}{2R}\} \qquad \text{if } |t-\rho^*(k)| \leq \tfrac{n+1}{2R}; \end{aligned}$$

which is saying that $\mathfrak{S}(\rho^*(k), \frac{n+1}{R}) \geq \mathfrak{S}(t, \frac{n+1}{2R})$ if $t \in J_{k,n}$. Thus

$$\sum_{2^l<\rho^*(k)\leq 2^{l+1}} \frac{\chi_{J_{k,n}(t)}}{\mathfrak{S}(\rho^*(k), \frac{n+1}{R})} \leq \frac{1}{\mathfrak{S}(t, \frac{n+1}{2R})}\sum_k \chi_{J_{k,n}}(t) = 1.$$

Therefore

$$\begin{aligned} II_{n,l}^2 &= 2^{-ld}\int \Big[\sum_{2^l\leq\rho^*(k)<2^{l+1}} \chi_{J_{k,n}}(t)\Big]\frac{E_\epsilon^*(t)^2 N_\epsilon^{*\prime}(t)}{\mathfrak{S}^*(\rho^*(k), \frac{n+1}{R})}dt \\ &\leq \frac{1}{2^{ld}}\int_{2^{l-1}}^{2^{l+2}} E_\epsilon^*(t)^2 N_\epsilon^{*\prime}(t)dt \\ &= \frac{1}{2^{ld}}\int_{2^{l-1}}^{2^{l+2}} E_\epsilon^*(t)^2 E_\epsilon^{*\prime}(t)dt + \frac{1}{2^{ld}}\int_{2^{l-1}}^{2^{l+2}} E_\epsilon^*(t)^2\frac{d}{dt}(vol(t\Omega))dt \\ &\leq \frac{1}{2^{ld}}\Big(\frac{[E_\epsilon^*(2^{l+2})]^3}{3} - \frac{[E_\epsilon^*(2^{l-1})]^3}{3}\Big) + \frac{C}{2^l}\int_{2^{l-1}}^{2^{l+2}} E_\epsilon^*(t)^2 dt \\ &\lesssim \Big(2^{l(2d-6+\frac{6}{d+1})} + 2^{l(2d-3)}\epsilon^3 + \frac{1}{2^l}\int_{2^{l-1}}^{2^{l+2}} E_\epsilon^*(t)^2 dt\Big); \end{aligned}$$

here we have used the estimate $|E_\epsilon^*(t)| \lesssim 2^{l(d-2+\frac{2}{d+1})} + 2^{l(d-1)}\epsilon$, $t \approx 2^l$, which by Lemma 2.1 is a consequence of the classical estimate $|E^*(t)| = O(t^{d-2+\frac{2}{d+1}})$, $d\geq 2$. Thus

$$\begin{aligned} &R^{d-1}\sum_{l=0}^{\infty} 2^{-l}(1+\varepsilon 2^l)^{-N} II_{n,l} \\ &\lesssim R^{d-1}\Big(\sum_{l=0}^{\infty}(1+\varepsilon 2^l)^{-N}\Big[2^{l(d-4+\frac{3}{d+1})} + 2^{l(d-5/2)}\epsilon^{3/2} + 2^{-l}\Big(\frac{1}{2^l}\int_{2^{l-1}}^{2^{l+2}} |E_\epsilon^*(t)|^2 dt\Big)^{1/2}\Big]\Big) \\ &\lesssim R^{d-1}\varepsilon^{3-d} + R^{d-1}\sum_{l=0}^{\infty} 2^{-l}(1+\varepsilon 2^{-l})^{-N}\sum_{i=-1}^{1} G_\epsilon(2^{l-i}) \end{aligned}$$

(3.11)
$$\lesssim R^{d-1}\varepsilon^{3-d} + R^{d-1}\sum_{l=0}^{\infty} 2^{-l}(1+\varepsilon 2^{-l})^{-N}\frac{(1+\epsilon 2^l)^{d+1}}{w_d(2^l)}\sup_{r\geq 0}\big\{(1+\epsilon 2^r)^{-d-1}G_\epsilon^*(2^r)w_d(2^r)\big\}.$$

Now since $R^{d-1}\varepsilon^{3-d} \lesssim w_d(R)^{-2}$ for $\varepsilon \geq R^{-1}$ we have

$$\begin{aligned}
&R^{d-1}\sum_{l=0}^{\infty} 2^{-l}\big(w_d(2^l)\big)^{-1}(1+\varepsilon 2^{-l})^{-N}(1+\epsilon 2^{-l})^{d+1}\\
&\lesssim R^{d-1}\sum_{l=0}^{\infty} 2^{-l}\big(w_d(2^l)\big)^{-1}(1+\epsilon 2^{-l})^{-N+d+1}\\
&\lesssim w_d(R)^{-2}(1+\epsilon R)^{d-2}, \qquad (3.12)
\end{aligned}$$

where the third inequality follows in a straightforward manner from the definition of w_d. It is precisely at this point where one needs to distinguish the cases $d=2$, $d=3$ and $d\geq 4$. Combining the previous estimates (3.10), (3.11) with (3.12) we obtain for $s \leq 1$ and $\max\{s, R^{-1}\} \leq \varepsilon \leq 1/2$

$$\begin{aligned}
&\big[(1+\varepsilon R)^{-d-1} w_d(R) G^+_\varepsilon(R)\big]^2\\
&\lesssim 1 + (1+\epsilon R)^{-2d-2} w_d(R)^2 R^{d-1} \sum_{l\geq 0}(1+\varepsilon 2^l)^{-N} \sum_{|n|\leq R^{1/2}} (1+n)^{-3}\big((n+1)I_l + II_{n,l}\big)\\
&\lesssim 1 + \sup_{r\geq 0}\big\{(1+\epsilon 2^r)^{-d-1} G^*_\epsilon(2^r) w_d(2^r)\big\} \qquad (3.13)
\end{aligned}$$

for $\epsilon = 4\varepsilon/\delta_0$ The same estimate holds for $G^-_\varepsilon(R)$ and thus for $G_\varepsilon(R)$. Consequently, since $\varepsilon \approx \epsilon$, we have

$$\big[(1+\varepsilon R)^{-d-1} w_d(R) G_\varepsilon(R)\big]^2 \leq C(1+A^*_d(s)) \qquad \text{if } R^{-1} \leq \varepsilon \leq 1/2 \qquad (3.14)$$

The required estimate for $\varepsilon \leq 1/R$ follows from a small modification. Namely we can use Lemma 2.2 to see that

$$\begin{aligned}
G_\varepsilon(R) &\leq C_1\Big[\Big(\frac{1}{R}\int_R^{2R} |E(t)|^2 dt\Big)^{1/2} + R^{d-2}\Big]\\
&\lesssim C_2\Big[\Big(\frac{1}{R}\int_R^{2R} |E_{1/R}(t)|^2 dt\Big)^{1/2} + R^{d-2}\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
&(1+\varepsilon R)^{-2(d+1)} w_d(R)^2 G_\varepsilon(R)^2\\
&\lesssim w_d(R)^2\big[R^{2d-4} + G_{1/R}(R)^2\big]\\
&\leq C(1+A^*_d(s)) \qquad \text{if} \quad s \leq \varepsilon \leq R^{-1}. \qquad (3.15)
\end{aligned}$$

The desired estimate (3.4) follows from (3.14), (3.15). □

4. Localized square functions in the plane

In this section we give the simple proof of Theorem 1.2.1. We assume that Ω is a convex domain in the plane, with smooth boundary, and that the curvature does not vanish at the boundary.

We may apply Lemma 2.2 with $\mu = 0$, say, and we let $1 \le h \le R$ and $\varepsilon = R^{-1}$. Then

$$\Big(\frac{1}{h}\int_R^{R+h} |E(t)|^2 dt\Big)^{1/2} \lesssim \Big[\Big(\frac{1}{h}\int_R^{R+h} |E_{1/R}(t)|^2 dt\Big)^{1/2} + (R/h)^{1/2}\Big]. \tag{4.1}$$

Let η_0 be a nonnegative C^∞ function supported in $(-1/2, 3/2)$ and which equals 1 on $[0,1]$. Then

$$\frac{1}{h}\int_R^{R+h} |E_{1/R}(t)|^2 dt \lesssim \sum_{\pm} \frac{1}{h}\int |E^{\pm}_{1/R}(t)|^2 \eta_0(\tfrac{t-R}{h}) dt \tag{4.2}$$

with $E^{\pm}$ as in (3.5.2). The expressions on the right hand side are estimated by integration by parts, as in the previous section. We square the series. The cutoff $\eta(t/R)$ is now replaced by $\eta_0(\frac{t-R}{h})$ and this affects the argument since in the symbol estimates for the modification of $q_{k,k'}$ the estimate R^{d-1-m} in (3.8) is now replaced by $R^{d-1}h^{-m}$.

As a result we obtain the estimate

$$\begin{aligned}&\frac{1}{h}\int |E^{\pm}_{1/R}(t)|^2 \eta_0(\tfrac{t-R}{h}) dt \\ &\quad\lesssim R\sum_{k\neq 0}\sum_{k'\neq 0} (1 + h|\rho^*(k) - \rho^*(k')|)^{-M}(1 + |k|/R + |k'|/R)^{-N}|\rho^*(k)\rho^*(k')|^{-3/2}\end{aligned}$$

and this term is estimated by a constant times

$$\sum_{|n|\le R^{1/2}}\sum_{k\neq 0} (1+|n|)^{-M}\rho^*(k)^{-3}(1+\rho^*(k)/R)^{-N}\mathfrak{S}^*(\rho^*(k) + \tfrac{n}{h}, \tfrac{1}{h}) + R^{1-M/2}, \tag{4.3}$$

where, as before, $\mathfrak{S}^*(\tau, \varepsilon) = \mathrm{card}\big(\{\ell \in \mathbb{Z}^2 : \tau - \varepsilon \le \rho^*(\ell) \le \tau + \varepsilon\}\big)$.

Now by the classical estimate for the remainder term $E(t)$ with $t = \rho^*(k) + (n \pm 1)/h \approx \rho^*(k)$ we have

$$\mathfrak{S}^*(\rho^*(k) + \tfrac{n}{h}, \tfrac{1}{h}) \lesssim h^{-1}\rho^*(k) + \rho^*(k)^{2/3}. \tag{4.4}$$

Putting the previous estimates together, we have

$$\begin{aligned}\frac{1}{h}\int |E^{\pm}_{1/R}(t)|^2\eta_0(\tfrac{t-R}{h})dt &\lesssim R\sum_{k\neq 0}(1 + R^{-1}\rho^*(k))^{-N}\min\{h^{-1}\rho^*(k)^{-2}, \rho^*(k)^{-7/3}\} + R^{1-M/2}\\ &\lesssim R\big(1 + h^{-1}\log R\big)\end{aligned}$$

which is $O(R)$ if $h \gtrsim \log R$. This finishes the proof of Theorem 1.2.1. □

5. Estimates for finite type domains in the plane

We shall give a proof of Theorem 1.2.2. Let Ω be a convex finite type domain in $\mathbb{R}^2$ which contains the origin in its interior. We first give a version of the standard lattice rest estimate for the polar set Ω^* which has a C^1 boundary.

Lemma 5.1. *We have the following estimate for the Fourier transform of the characteristic function of Ω^*,*

$$\big|\widehat{\chi_{\Omega^*}}(\xi)\big| \le C(1 + |\xi|)^{-3/2}. \tag{5.1}$$

Taken Lemma 5.1 for granted we obtain as a consequence

Corollary 5.2. *Let Ω be a convex set in $\mathbb{R}^2$, containing the origin in its interior and suppose that Ω has smooth finite type boundary. Let Ω^* be the polar set. Then*

$$\mathcal{N}_{\Omega^*}(t) = t^2\, area(\Omega^*) + O(t^{2/3}) \tag{5.2}$$

as $t \to \infty$.

Proof. This follows from Lemma 5.1 using the standard argument (see e.g. [8], or §7 of [9]).

The Corollary can be improved by using more sophisticated techniques which however are not needed here.

Before proving Lemma 5.1 we recall some terminology: We denote by Γ the set of all points in $\partial\Omega$ at which the curvature vanishes; these points are separated and thus Γ is finite. For every $P \in \Gamma$ let m_P be the type at P (i.e. the curvature vanishes of order $m_P - 2$ at P). For every $P \in \partial\Omega$ there is a unique $P^* \in \partial\Omega^*$ so that $\langle P, P^*\rangle = 1$ and we define $\Gamma^* = \{P^* : P \in \Gamma\}$.

Proof of Lemma 5.1.

The boundary $\partial\Omega^*$ is smooth away from Γ^* and it is C^1 everywhere. Thus surface measure $d\sigma$ is well defined and by an application of the divergence theorem as in §2 estimate (5.1) follows provided we can show that

$$|\widehat{\chi d\sigma}(\xi)| \lesssim (1 + |\xi|)^{-1/2} \tag{5.3}$$

for $\chi \in C_0^\infty$.

To see this we introduce a partition of unity $\chi d\sigma = \sum_\nu \chi_\nu d\sigma$ where each $P^* \in \Gamma^*$ lies in exactly one of the supports of the functions χ_ν. Clearly it suffices to prove the estimate $\widehat{d\sigma_\nu}(\xi) = O(|\xi|^{-1/2})$ for each $\sigma_\nu := \chi_\nu d\sigma$.

Fix ν and $P \in \Gamma$. If $P^* \notin \text{supp } d\sigma_\nu$ then $\widehat{d\sigma_\nu}(\xi) = O(|\xi|^{-1/2})$ by the standard stationary phase argument. Thus suppose $P \in \Gamma \cap \text{supp } d\sigma_\nu$. By a rotation we may assume that $n_P = (0,1)$ and by an additional translation we may also assume that P lies on the x_2-axis. Let $m = m_P$ be the type at P. Near P the boundary of Ω is parametrized by $(t, f(t))$ where

$$f(t) = a_0 - a_m \frac{t^m}{m} + t^{m+1} g_1(t)$$

with $a_0 > 0$, $a_m > 0$. Thus a parametrization of $\partial\Omega^*$ near $P^* = (a_0^{-1}, 1)$ is given by

$$t \mapsto \frac{n(t)}{\langle x(t), n(t)\rangle} = \frac{1}{f(t) - tf'(t)} \frac{(-f'(t), 1)}{\sqrt{1 + f'(t)^2}};$$

however this parametrization is not regular. Denote by $\omega(t)$ the first coordinate of $\langle x(t), n(t)\rangle^{-1} n(t)$. Then it is easy to see that

$$\omega(t) = (a_m/a_0) t^{m-1}(1 + t g_2(t)) = (c_0 s(t))^{m-1}$$

where $c_0 = (a_m/a_0)^{1/(m-1)}$ and $s(t) = t + O(t^2)$. Moreover

$$(f(t) - tf'(t))^{-1}(\sqrt{1 + f'(t)^2})^{-1} = a_0^{-1}\big(1 - \tfrac{m-1}{m}\tfrac{a_m}{a_0} t^m + t^{m+1} g_2(t)\big).$$

Thus setting $\tau = (c_0 s(t))^{m-1}$ we see after a short computation that near P^* the boundary is parametrized by $\tau \mapsto (\tau, h(\tau))$ with

$$h(\tau) = a_0^{-1}\big(1 - c_1\tau^{m/(m-1)} + \tau^{\frac{m+1}{m-1}} g_3(\tau^{\frac{1}{m-1}})\big)$$

where $c_1 = (m-1)m^{-1}(a_m/a_0)c_0^{-m} = (m-1)m^{-1}(a_m/a_0)^{-1/(m-1)}$ and g_3 is smooth. Thus we have to show that

$$J(\xi) = \int e^{-i(\xi_1\tau + \xi_2 h(\tau))}\eta(\tau)d\tau = O(|\xi|^{-1/2}) \tag{5.4}$$

as $|\xi| \to \infty$; here we may assume that the support of η_ν is contained in a small interval $(-\delta, \delta)$.

It suffices to estimate the analogous integral extended over the set $\{\tau : |\tau| \geq |\xi|^{-1/2}$ since the error is $O(|\xi|^{-1/2})$. Observe that for small τ we have $|h'(\tau)| \ll 1$ and $|h''(\tau)| \geq c\tau^{-(m-2)/(m-1)} \gtrsim 1$. Thus by van der Corput's lemma ([28], ch. VIII.1) we obtain for large $|\xi|$ the estimate $|J(\xi)| \lesssim |\xi_1|^{-1}$ if $|\xi_1| \geq |\xi_2|$ (using first derivatives of the phase function) and the estimate $|J(\xi)| \lesssim |\xi_2|^{-1/2}$ if $|\xi_2| \geq |\xi_1|$ (using second derivatives). This implies (5.4) and thus (5.3). $\square$

Proof of Theorem 1.2.2. We shall decompose the Fourier transform of χ_Ω as in [27], following rather closely [13]. Using the divergence theorem as above, we see that

$$\widehat{\chi_\Omega}(\xi) = \imath \sum_{i=1}^{d} \frac{\xi_i}{|\xi|^2} \int_\Sigma n_i(y) e^{-\imath\langle y,\xi\rangle} d\sigma(y) \tag{5.5}$$

where n_i denotes the i^{th} component of n_P.

For every $P \in \Gamma$ we choose a narrow conic symmetric neighborhood V_P of the normals $\{\pm n_P\}$, a small neighborhood U_P of P in Σ and a C_0^∞ function χ_P whose restriction to Σ vanishes off $\mathcal{U}$ and so that χ_P equals one in a neighborhood of P. We may arrange these neighborhoods so that the sets $\overline{V}_P \cap \{\xi : |\xi| \geq 1\}$, $P \in \Gamma$ are pairwise disjoint and that the normals to all points in a neighborhood of $\overline{U}_P$ are contained in V_P, so that the $\overline{U}_P$'s are disjoint too.

Define

$$F_{i,P}(\xi) = \int_\Sigma \chi_P(y) n_i(y) e^{-\imath\langle y,\xi\rangle} d\sigma(y)$$

Let v_P a unit tangent vector to $\partial\Omega$ at P. Then if the cones V_P are chosen sufficiently narrow, we have

$$\sum_{i=1}^{d} \frac{\xi_i}{|\xi|^2} F_{i,P}(\xi) = e^{-\imath\rho^*(\xi)} b_+(\xi) + e^{\imath\rho^*(-\xi)} b_-(\xi) \tag{5.5}$$

where

$$|\partial_\xi^\alpha b_\pm(\xi)| \leq C_\alpha \begin{cases} |\xi|^{-1-|\alpha|} \min\{|\xi|^{-\frac{1}{m_P}}, \xi^{-\frac{1}{2}}\Theta_P(\xi)\} & \text{if } \xi \in V_P \\ C_N|\xi|^{-N} & \text{if } \xi \notin V_P; \end{cases} \tag{5.6}$$

with

$$\Theta_P(\xi) = \left|\frac{\langle v_P, \xi\rangle}{\langle n_P, \xi\rangle}\right|^{-\frac{m_P-2}{2(m_P-1)}}. \tag{5.7}$$

This follows from (2.8) (with $\alpha = 0$) and (2.9) by a straightforward computation. Moreover

$$\sum_{i=1}^{d} \frac{\xi_i}{|\xi|^2}\big(F_i(\xi) - \sum_{P\in\Gamma} F_{i,P}(\xi)\big) = e^{-\imath\rho^*(\xi)}c_+(\xi) + e^{\imath\rho^*(-\xi)}c_-(\xi) \tag{5.8}$$

where

$$|\partial_\xi^\alpha c_\pm(\xi)| \le C_\alpha |\xi|^{-3/2-|\alpha|}. \tag{5.9}$$

The estimate for $\xi \in V_P$ follows from Proposition 1.2, and the estimate for $\xi \notin V_P$ follows by a simple integration by parts; namely if $t \mapsto \gamma(t)$ parametrizes Σ near P then $|\langle\gamma'(t), \xi\rangle| \approx |\xi|$ for $\gamma(t) \in U_P$ and $\xi \notin V_P$.

Moreover by the usual stationary phase or van der Corput estimate we have

$$|F_i(\xi) - \sum_{P\in\Gamma} F_{i,P}(\xi)| \lesssim (1 + |\xi|)^{-1/2} \tag{5.10}$$

here we used the definition of Γ and the fact that χ_P is equal to 1 near P.

Let $E_{1/R,A}(t)$ be the remainder term (2.10) with $\varepsilon = 1/R$, with Ω replaced by the rotated domain $A\Omega$; that is

$$\begin{aligned} E_{1/R,A}(t) &= \sum_{k\in\mathbb{Z}^2} \chi_{t\Omega} * \zeta_{1/R}(A^{-1}k) - t^2 \text{area}(\Omega) \\ &= \sum_{k\neq 0} t^2 \widehat{\zeta}(2\pi R^{-1}Ak) \sum_{i=1}^{d} \frac{2\pi t\langle Ak, e_i\rangle}{|2\pi tAk|^2} F_i(2\pi tAk) \end{aligned} \tag{5.11}$$

For $P \in \Gamma$, $A \in SO(2)$ let

$$\begin{aligned} \mathcal{Z}_I^P(A) &= \{k \in \mathbb{Z}^d : Ak \in V_P, k \neq 0, \text{dist}(Ak, \mathbb{R}n_P) < 1\} \\ \mathcal{Z}_{II}^P(A) &= \{k \in \mathbb{Z}^d : Ak \in V_P, k \neq 0, \text{dist}(Ak, \mathbb{R}n_P) \ge 1\} \end{aligned}$$

and let

$$\mathcal{Z}_{III}(A) = \{k \in \mathbb{Z}^d : k \neq 0, k \notin \cup_{P\in\Gamma} V_P\}.$$

We may use estimate (4.1) which does not depend on any curvature assumptions and see that it suffices to estimate the square function $(h^{-1}\int |E_{1/R,A}(t)|^2 \eta_0(\frac{t-R}{h})dt)^{1/2}$ (*cf.* (4.2)). We decompose for $R \le t \le 2R$

$$\begin{aligned} E_{1/R,A}(t) =& \Big(\sum_{P\in\Gamma}\sum_{k\in\mathcal{Z}_I^P(A)} + \sum_{P\in\Gamma}\sum_{k\in\mathcal{Z}_{II}^P(A)} t^2\widehat{\zeta}(2\pi R^{-1}Ak) \sum_{i=1}^{d} \frac{2\pi t\langle Ak, e_i\rangle}{|2\pi tAk|^2} F_{i,P}(2\pi tAk) \\ &+ \sum_{k\in\mathcal{Z}_{III}(A)} t^2\widehat{\zeta}(2\pi R^{-1}Ak) \sum_{i=1}^{d} \frac{2\pi t\langle Ak, e_i\rangle}{|2\pi tAk|^2}\big(F_i(2\pi tAk) - \sum_{P\in\Gamma} F_{i,P}(2\pi tAk)\big) \\ &+ \sum_{P}\sum_{k\notin V_P} t^2\widehat{\zeta}(2\pi R^{-1}Ak) \sum_{i=1}^{d} \frac{2\pi t\langle Ak, e_i\rangle}{|2\pi tAk|^2} F_{i,P}(2\pi tAk) \\ =& \sum_{\pm}\Big(\sum_{P\in\Gamma} I_P^\pm(t) + \sum_{P\in\Gamma} II_P^\pm(t) + III^\pm(t)\Big) + IV(t) \end{aligned}$$

where

$$|IV(t)| = O(t^{-N}) \tag{5.12}$$

and

$$\begin{aligned}
I_P^+(t,A) &= \sum_{k\in\mathcal{Z}_I^P(A)} \widehat{\zeta}(2\pi R^{-1}Ak)b_+(2\pi tAk)e^{-2\pi it\rho^*(Ak)} \\
II_P^+(t,A) &= \sum_{k\in\mathcal{Z}_{II}^P(A)} \widehat{\zeta}(2\pi R^{-1}Ak)b_+(2\pi tAk)e^{-2\pi it\rho^*(Ak)} \\
III^+(t,A) &= \sum_{k\in\mathcal{Z}_{III}^P(A)} \widehat{\zeta}(2\pi R^{-1}Ak)c_+(2\pi tAk)e^{-2\pi it\rho^*(Ak)},
\end{aligned}$$

and the expressions I_P^-, II_P^- and III_P^- are defined by replacing b_+ by b_-, c_+ by c_-, and $e^{-2\pi it\rho^*(Ak)}$ by $e^{2\pi it\rho^*(-Ak)}$.

The argument in the previous section applies to the square functions associated to $III^\pm(t,A)$ and we obtain the bound

$$\frac{1}{h}\int |III^\pm(t,A)|^2\eta_0(\tfrac{t-R}{h})dt \lesssim R(1+h^{-1}\log R), \tag{5.13}$$

uniformly in A.

A small variation of this argument also applies to the square function associated to $II_P^\pm(t,A)$. Namely, arguing as in §3 and using (5.6/7) we see that

$$\begin{aligned}
&\frac{1}{h}\int |II_P^+(t,A)|^2\eta_0(\tfrac{t-R}{h})dt \\
&\lesssim 2 \sum_{\substack{k\in\mathcal{Z}_{II}^P(A)\\ k'\in\mathcal{Z}_{II}^P(A)\\ \Theta_P(Ak)\geq\Theta_P(Ak')}} R(1+h|\rho^*(Ak)-\rho^*(Ak')|)^{-N}\big(1+\frac{|k|+|k'|}{R}\big)^{-N}\frac{\Theta_P(Ak)\Theta_P(Ak')}{\rho^*(Ak)^{3/2}\rho^*(Ak')^{3/2}} \\
&\lesssim R\sum_{k\in\mathcal{Z}_{II}^P(A)}\frac{\Theta_P(Ak)^2}{\rho^*(Ak)^3}\sum_{\substack{n\in\mathbb{Z}\\ |n|\leq 2^{k-5}}}(1+n)^{-N}(1+|k|/R)^{-N}\mathfrak{S}_A^*(\rho_*(Ak)+\tfrac{n}{h},\tfrac{1}{h})+R^{4-N}
\end{aligned}$$

where now $\mathfrak{S}_A^*(\tau,\varepsilon) = \operatorname{card}\big(\{\ell\in\mathbb{Z}^2 : \tau-\varepsilon\leq\rho^*(A\ell)\leq\tau+\varepsilon\}\big)$.

Observe that $\operatorname{dist}(Ak,\mathbb{R}n_P)\geq 1$ and $\operatorname{dist}(A\xi,Ak)\leq 1/2$ implies that $\Theta_P(Ak)\approx\Theta_P(A\xi)$. Thus we can use the argument in §3 and Lemma 5.1 and estimate

$$\begin{aligned}
&\frac{1}{h}\int_R^{R+h}|II_P^\pm(t,A)|^2\eta_0(\tfrac{t-R}{h})dt \\
&\quad\lesssim R\int_{V_P}(1+|\rho^*(\xi)|)^{-3}\Theta_P^2(\xi)(1+R^{-1}\rho^*(\xi))^{-N}\min\{h^{-1}\rho^*(\xi)^{-2},\rho^*(k)^{-7/3}\}d\xi + R^{1-M/2}.
\end{aligned}$$

Since Θ_P^2 is homogeneous of degree 0 and integrable over the sphere $\{\rho^*(\eta) = 1\}$ it is easy to see that the former expression is bounded by $R(1 + h^{-1} \log R)$, thus

$$\frac{1}{h} \int_R^{R+h} |II_P^+(t, A)|^2 dt \lesssim R(1 + h^{-1} \log R), \tag{5.14}$$

for $|h| \le R$, uniformly in A. The same estimate holds true with II_P^+ replaced by II_P^-- the proof only requires changes in the notation.

In order to estimate the square function involving I_P^+, we let $S_P(A)$ be the set of all $k \in \mathbb{Z}^2 \setminus \{0\}$ with $\mathrm{dist}(k, \mathbb{R}A^* n_P) < 1$, and define

$$M_{P,\varepsilon}(A) = \sup \big\{ |k|^{-1+\varepsilon} \Theta_P(k) : k \in S_P(A) \big\}.$$

Then

$$\begin{aligned}
&\frac{1}{h} \int |I_P^+(t)|^2 \eta_0(\tfrac{t-R}{h}) dt \\
&\lesssim \sum_{k \in S_P(A)} \sum_{k' \in S_P(A)} R(1 + h|\rho^*(Ak) - \rho^*(Ak')|)^{-N} (1 + \tfrac{|k|+|k'|}{R})^{-N} \frac{\Theta_P(Ak)\Theta_P(Ak')}{|k|^{3/2}|k'|^{3/2}} \\
&\lesssim M_{P,\varepsilon}(A)^2 \sum_{k \in S_P(A)} R(1 + h|\rho^*(Ak) - \rho^*(Ak')|)^{-N} (1 + \tfrac{|k|+|k'|}{R})^{-N} |k|^{-\varepsilon-1/2} |k'|^{-\varepsilon-1/2} \\
&\lesssim M_{P,\varepsilon}(A)^2 \sum_{k \in S_P(A)} R(1 + |k|/R)^{-N} |k|^{-1-2\varepsilon},
\end{aligned}$$

and thus

$$\frac{1}{h} \int |I_P^+(t)|^2 \eta_0(\tfrac{t-R}{h}) dt \le C_\varepsilon M_{P,\varepsilon}(A)^2 R. \tag{5.15}$$

Again the same estimate remains true for $I_P^-(t)$.

For each $k \neq 0$ the function $A \mapsto \Theta_P(Ak)$ belongs to the space $L^{(2m_P-2)/(m_P-2),\infty}$. For $\alpha > 0$ the set $\{A \in SO(2) : M_{P,\varepsilon}(A) > \alpha\}$ is the union of the sets $E_k(\alpha) = \{A : \Theta_P(Ak) > |k|^{1-\varepsilon}\alpha\}$, $k \in \mathbb{Z}^2 \setminus \{0\}$ and the measure of $E_k(\alpha)$ is $\lesssim (k^{1-\varepsilon}\alpha)^{-(2m_P-2)/(m_p-2)}$. Since $(2m_P - 2)/(m_p - 2) > 2$ we may sum over all $k \in \mathbb{Z}^2 \setminus \{0\}$ and we see that $M_{P,\varepsilon} \in L^{(2m_P-2)/(m_P-2),\infty}(SO(2))$ provided that $\varepsilon \le 1/2$. Combining the estimates (5.12-5.15) this proves that $\mathcal{C}_\Omega \in L^{(2m_P-2)/(m_P-2),\infty}(SO(2))$.

The Diophantine condition (1.6) for some $\epsilon > 0$ is equivalent with the condition $M_{P,\varepsilon}(A) < \infty$, for some $\varepsilon > 0$. Fix P. The estimates (5.12-15) show that $\mathcal{C}_\Omega(A) = \infty$ also implies $M_{P,\varepsilon}(A) = \infty$ for at least one $P \in \Gamma$. Thus we can complete the proof if for any sufficiently small $\varepsilon > 0$ we demonstrate that the set $\{A \in SO(2) : M_{P,\varepsilon}(A) = \infty\}$ has Hausdorff dimension $\le (m_P - 2)(m_P - 1)^{-1}(1-\varepsilon)^{-1}$.

Set $\beta = (m_P - 2)/(2m_P - 2)$, thus $\beta < 1/2$. Now $M_{P,\varepsilon}(A) = \infty$ implies that there are infinitely many $k \in S_P(A)$ so that $|k|^{\varepsilon-1} |\langle k/|k|, v_P \rangle|^{-\beta} \ge 1$. If $A^* v_P = (\alpha_1, \alpha_2)$ this means $|k_1\alpha_1 + k_2\alpha_2| \le |k|^{(\beta-1+\varepsilon)/\beta}$. Now $|\alpha_1| \ge |\alpha_2|$ implies $|k_1| \lesssim |k_2|$ and $|\alpha_2| \ge |\alpha_1|$ implies $|k_2| \lesssim |k_1|$ (as $k \in S_P(A)$). Thus if $|\alpha_1| \ge |\alpha_2|$ the condition $M_{P,\varepsilon}(A) = \infty$ implies that for infinitely many k with $|k_2| \approx |k|$ we have that

$$|k_1/k_2 - \alpha_2/\alpha_1| \le C|k_2|^{(\varepsilon-1)/\beta} \quad \text{or} \quad |k_1/k_2 + \alpha_2/\alpha_1| \le C|k_2|^{(\varepsilon-1)/\beta}. \tag{5.16.1}$$

Likewise, if $|\alpha_2| \geq |\alpha_1|$ and $M_{P,\varepsilon}(A) = \infty$ then

$$|k_2/k_1 - \alpha_1/\alpha_2| \leq C|k_1|^{(\varepsilon-1)/\beta} \qquad \text{or} \qquad |k_2/k_1 + \alpha_1/\alpha_2| \leq C|k_1|^{(\varepsilon-1)/\beta} \tag{5.16.2}$$

for infinitely many k with $|k_1| \approx |k|$.

Let P_θ denote the set of all $x \in [-1, 1]$ for which there exists infinitely many rationals p/q such that $|x - p/q| \leq q^{-2-\theta}$. By a Theorem of Jarník [15] (see also [18]) the Hausdorff dimension of P_θ is equal to $2/(2+\theta)$ (and we need only the easy upper bound). Now choose in (5.16.1/2) a small $\varepsilon > 0$ (in particular so that $\beta < (1-\varepsilon)/2$) and we apply the last statement with $\theta = (1-\varepsilon)\beta^{-1} - 2$ and then $2/(2+\theta) = 2\beta(1-\varepsilon)^{-1} = (m_P - 2)(m_P - 1)^{-1}(1-\varepsilon)^{-1}$.

Consequently, with $\mathfrak{m}$ being the maximal type, the Hausdorff dimension of the set $\{A \in SO(2) : \mathcal{C}_\Omega(A) = \infty\}$ does not exceed $(\mathfrak{m} - 2)/(\mathfrak{m} - 1)$. $\square$

References

1. R. M. Beals, *L^p boundedness of Fourier integral operators*, Mem. Amer. Math. Soc. **264** (1982).
2. V. Bentkus and F. Götze, *On the lattice point problem for ellipsoids*, Acta Arithmetica (1997), 101–125.
3. P. Bleher, *On the distribution of the number of lattice points inside a family of convex ovals*, Duke Math. J. **67** (1992), 461–481.
4. J. Bruna, A. Nagel and S. Wainger, *Convex hypersurfaces and Fourier transform*, Ann. Math. **127** (1988), 333–365.
5. Y. Colin de Verdière, *Nombre de points entiers dans une famille homothétique de domaines de $\mathbb{R}^2$*, Ann. Scient. Ec. Norm. Sup. **10** (1977), 559–576.
6. M. Cowling, S. Disney, G. Mauceri and D. Müller, *Damping oscillatory integrals*, Invent. Math. **101** (1990), 237–260.
7. C. S. Herz, *On the number of lattice points in a convex set*, Amer. Math. J. **84** (1962), 126–133.
8. E. Hlawka, *Über Integrale auf konvexen Körpern I*, Monatshefte Math. **54** (1950), 1-36; *II*, Monatshefte Math. **54** (1950), 81-99.
9. L. Hörmander, *The analysis of linear partial differential operators Vol. I*, Springer-Verlag, New York, Berlin, 1983.
10. M. N. Huxley, *The mean lattice point discrepancy*, Proc. Edinburgh Math. Soc. **38** (1995), 523–531.
11. ______, *Area, Lattice Points, and Exponentials Sums*, London Mathematical Society Monographs New Series 13, Oxford Univ. PressI, 1996.
12. A. Iosevich, E. Sawyer and A. Seeger, *Two problems associated with convex finite type domains*, preprint.
13. ______, *Bounds for the mean square lattice point discrepancy associated with convex domains in the plane*, manuscript.
14. V. Jarník, *Über Gitterpunkte in mehrdimensionalen Kugeln*, Math. Ann. **100** (1928), 699-721.
15. ______, *Über die simultanen diophantischen Approximationen*, Math. Z. **33** (1931), 505–543.
16. ______, *Über die Mittelwertsätze in der Gitterpunktlehre, V*, Časopis Pěst. Mat. **69** (1940), 148–174; *VI*, Časopis Pěst. Mat. **70** (1941), 89–103.
17. I. Kátai, *The number of lattice points in a circle*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **8** (1965), 39–60.
18. R. Kaufman, *On the theorem of Jarnik and Besicovich*, Acta Arithmetica **39**, 265-267.
19. E. Krätzel and W. G. Nowak, *Lattice points in large convex bodies*, Monatshefte Math. **112** (1991), 61–72; *II*, Acta Arithmetica **62** (1992), 285-295.
20. E. Landau, *Über Gitterpunkte in mehrdimensionalen Ellipsoiden*, Math. Z. **21** (1924), 126–132.
21. W. Müller, *On the average order of the lattice rest of a convex body*, Acta Arithmetica **53** (1997), 89–100.
22. ______, *Lattice points in large convex bodies*, Monatshefte Math. **128** (1999), 315–330.
23. W. G. Nowak, *Zur Gitterpunktlehre in der Euklidischen Ebene*, Indag. Math **46** (1984), 209-223.
24. ______, *Zur Gitterpunktlehre in der Euklidischen Ebene, II*, Österreich. Akad. Wiss. Math. Natur. Kl. Sitzungsber. II **194** (1985), 31–37.

25. ______, *On the average order of the lattice rest of a convex planar domain*, Proc. Camb. Phil. Soc. **98** (1985), 1–4.

26. B. Randol, *A lattice point problem*, Trans. Amer. Math. Soc. **121** (1966), 257–268; *II*, Trans. Amer. Math. Soc. **125**, 101–113.

27. ______, *On the Fourier transform of the indicator function of a planar set*, Trans. Amer. Math. Soc. **139** (1969), 271–278.

28. E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, 1993.

29. I. Svensson, *Estimates for the Fourier transform of the characteristic function of a convex set*, Arkiv Mat. **9** (1971), 11–22.

30. M. Tarnopolska-Weiss, *On the number of lattice points in planar domains*, Proc. Amer. Math. Soc. **69** (1978), 308–311.

31. A. Walfisz, *Über Gitterpunkte in mehrdimensionalen Ellipsoiden*, Math. Z. **19** (1924), 300-307; Math. Z. **27** (1927), 245–268.

32. ______, *Über Gitterpunkte in mehrdimensionalen Kugeln*, Monografie Mat. 33, Polish Scientific Publ., Warszawa, 1957.

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