3. Measure theory, partitions, and all that

> Natural language plays an immense role in the discovery, discussion, and preservation of scientific knowledge, but it is very poorly adapted to the exact transmission of the content of this knowledge, and especially to its processing, which forms an important part of scientific thought. Natural language performs different functions, and ascribes to other values. The language of modern set-theoretic mathematics can affect the role of such a language-intermediary thanks to its unique capacity to form geometric, spatial and kinematic forms, and at the same time to provide a precise formalism for transcribing their mathematical content.
>
> —Yu.I. Manin, *Mathematics and Physics* (1981)

> The set of all $\zeta(\Sigma)$-sets coincides with the set of all sets which can be obtained from the sets of the system Σ with the help of all possible set-theoretic operations, repeated as many times as desired.
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> —V. A. Rohlin, *On the Fundamental Ideas of Measure Theory* (1962)

First we review some fundamental notions from measure theory [57], [21], [58]. We shall follow the terminology of Mackey. Let M be a set. A *Borel structure* (or measure structure) on M is a family $\mathcal{A}$ of subsets of M which is closed under the operations of complementation and countable unions, *i.e.*,

$$A \in \mathcal{A} \Rightarrow M \backslash B \in \mathcal{A}, \quad A_n \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbf{Z}} A_n \in \mathcal{A}.$$

Such a family of subsets is often called a σ-field, as it can be given the algebraic structure of a field [34]. If M has a topology, the σ-field generated by the open sets is usually called the Borel σ-field. The following properties follow easily from the definition:

$$A, B \in \mathcal{A} \Rightarrow A \backslash B \in \mathcal{A}, \quad A_n \in \mathcal{A} \Rightarrow \bigcap_{n \in \mathbf{Z}} A_n \in \mathcal{A}.$$

We shall call the pair $M = (M, \mathcal{A})$ a Borel or a measure space, and the subsets in $\mathcal{A}$ Borel sets, or measurable sets. Given a family $\mathcal{C}$ of subsets of M, the unique smallest σ-field containing $\mathcal{C}$ is called the Borel structure *generated* by $\mathcal{F}$, and denoted by $\mathcal{F}(\mathcal{C})$. The Borel space $M = (M, \mathcal{A})$ is called *countably generated* if there exists a countable family which generates the Borel structure. A family $\mathcal{C}$ of Borel subsets *separates* M if for any two distinct points $x, y \in M$ there exists a

$C \in \mathcal{C}$ such that either $x \in C$ and $y \notin C$, or $x \notin C$ and $y \in C$. The Borel space M is called *countably separated* if there exist a countable family which separates M. It is clear that a countably generated space is countably separated. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be Borel spaces. A map $f : M \to N$ is called Borel, or measurable, if $f^{-1}(B) \in \mathcal{A}$ for each $B \in \mathcal{B}$. A Borel map $f : M \to N$ is a Borel isomorphism if f is an isomorphism of sets, and the inverse f^{-1} is a Borel map.

We now discuss standard Borel spaces. Let X be a complete separable metric space, with the Borel structure generated by the open (equivalently closed) sets.

Definition 3.1. A space $(M, \mathcal{A})$ is *standard* if it is Borel isomorphic to a Borel subset of X.

Proposition 3.2. [59], [58] Two standard spaces M_1 and M_2 are Borel isomorphic if and only if they have the same cardinality.

In particular, any standard space is Borel isomorphic to the unit interval of the real line $I = [0, 1] \subset \mathbf{R}$. We collect here a few related results which will be used in following sections.

Proposition 3.3. ([57]) Let M be standard, and A a subset of M. A is standard if and only if A is a Borel subset.

Proposition 3.4. ([57]) Let M be standard, and $\mathcal{C}$ a countable separating family of Borel sets of M. Then $\mathcal{C}$ is a generating family for the Borel structure of M.

The *product* of two measure spaces $(M, \mathcal{B})$ and $(M', \mathcal{B}')$ is the cartesian product $M \times M'$ with the Borel structure generated by sets of the form $B \times B'$, with $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$. We recall also that the product of two topological spaces X and X' is defined as the cartesian product $X \times X'$, with the topology generated by products of open sets.

Proposition 3.5. The product of two standard spaces $(M, \mathcal{B})$ and $(M', \mathcal{B}')$ is a standard space.

Proof: By definition 3.1, we can take M and M' to be Borel subsets of complete separable metric spaces X and X' respectively. The cartesian product of topological spaces $X \times X'$ is a complete separable metric space, and $M \times M'$ is a Borel subset. ∎

Proposition 3.6. (see [60], prop. 8.1.6.) Let X and Y be standard spaces, and let $f : X \to Y$ be a Borel map. Then the graph of f is a Borel subset of $X \times Y$.

We note also that a countably generated Borel space is not necessarily standard [57].

We briefly review the notion of a measure. A measure μ is a map from the Borel sets to the extended positive real numbers $\bar{\mathbf{R}}^+$, which is countably additive, *i.e.*,

$$\mu(\bigcup_{n\in\mathbf{Z}^+} B_n) = \sum_{n\in\mathbf{Z}^+} \mu(B_n),$$

for any countable family $\{B_n\}$ of disjoint Borel sets. A measure μ on M is σ-finite if M is the union of a countable number of pieces with finite measure. A measure μ on S is called standard if S is the union of a measurable subset which is standard, and subsets of sets of measure zero. A measure space $(M, \mathcal{A}, \mu)$ with standard measure μ satisfying $\mu(M) = 1$ is called a standard probability space, or a *Lebesgue space.* A Lebesgue space can have at most a countable number of points of positive measure. One usually restricts the term Lebesgue to spaces which are Lebesgue as above, and in addition have no points of positive measure. An alternate term for this is smooth Lebesgue. The notion of isomorphism of Borel spaces with a measure requires also that the map preserve the measure of all Borel sets.

Proposition 3.7. [21] Any two (smooth) Lebesgue spaces M_1 and M_2 are isomorphic.

We now study partitions of Borel spaces, or equivalently, quotient measure spaces. Let $(M, \mathcal{A})$ be a Borel space. A *partition* of M is a family $\xi = \{C\}$ of nonempty disjoint subsets of M, whose union is M, $\bigcup_{C\in\xi} C = M$. The notion an equivalence relation on M is the same. This can equivalently be described as a surjection $p : M \to M/\xi$, where the points of the space M/ξ correspond to the sets C of the partition. An element of the partition is the inverse image $p^{-1}(c)$ of a point $c \in M/\xi$. The quotient Borel structure on M/ξ is defined as the pushforward under p of the Borel structure on M, *i.e.*, a subset $A \subset M/\xi$ is measurable if $p^{-1}(A) \subset M$ is measurable.

We now suppose we have a finite standard measure μ on M, *i.e.*, M is Lebesgue, and a measurable partition ξ. The quotient measure μ' on M/ξ is defined as the pushforward of μ under the natural projection, *i.e.*, for a Borel subset $A \subset M/\xi$, define $\mu'(A) = \mu(p^{-1}(A))$. When we have a measure, we generally work modulo the sets of zero measure. A *partition (mod 0)* is a partition of a measurable subset of M of full measure. We shall generally not distinguish between partitions and partitions (mod 0). Subsets of M which are unions of elements of ξ are called ξ-sets. We now discuss the notion of a basis for a partition. A *basis* for the partition ξ is a countable separating family of measurable subsets of M/ξ. A partition is called *measurable* if a basis exists, *i.e.*, if M/ξ is countably separated. The significance of the measurability of a partition lies in the following result.

Proposition 3.8. [57,21] The quotient M/ξ of a Lebesgue space M by a measurable partition ξ is a Lebesgue space.

General references for these ideas are [21,45,8].

The following results will be of use in later chapters.

Definition 3.9. The *graph* of an equivalence relation on a space X is the subset of $X \times X$ consisting of pairs (x, y), where x is equivalent to y.

Proposition 3.10. Let ξ be a measurable partition of a standard measure space M. The graph R of the partition ξ (considered as an equivalence relation, definition 3.9) is a Borel subset of $M \times M$.

Proof: Using proposition 3.8, we see that M/ξ is standard, and the map $p : M \to M/\xi$ is Borel. The graph T of the Borel function p is a Borel subset of $X \times X/\xi$ by proposition 3.6. The map

$$\mathbf{1} \times p : X \times X \to X \times X/\xi$$

is clearly Borel. The graph $R \subset X \times X$ of the partition ξ is the inverse image of T under $\mathbf{1} \times p$, so R is a Borel subset of $X \times X$. ∎

The following notion is important for the entropy theory [21,45].

Definition 3.11. A *canonical system of conditional measures associated to the partition* ξ is a family of measures $\{\mu_C, C \in \xi\}$, with the properties

- (i) μ_C is a Lebesgue measure on C
- (ii) for any $A \in \mathcal{A}$, the set $A \cap C$ is measurable in C for almost all $C \in M/\xi$ and the function $\mu_C(A \cap C)$ is measurable on M/ξ, with

$$\mu(A) = \int_{M/\xi} \mu_C(A \cap C) d\mu'.$$

The measure μ' is the quotient measure on M/ξ.

Proposition 3.12. Each measurable partition has a canonical system of conditional measures which is unique (mod 0), i.e. any other system is the same for almost all $C \in M/\xi$. Conversely, the existence of a canonical system for a partition implies that the partition is measurable.

Proof: See [21,45] ∎

Example 3.13. We present a simple example of a canonical system of conditional measures. Consider the two-dimensional torus T^2, defined as the Cartesian product of two circles, $T_x \times T_y$. Let $x \pmod 1$ and $y \pmod 1$ be coordinates on T_x and T_y respectively, then (x, y) are coordinates on T^2. Consider the partition ξ of T^2 with elements $T_x \times y$, the circles on the torus with constant

y-coordinate. The space T^2/ξ is the circle T_y. ξ is a measurable partition, since as a measure space, $T^2 = T_x \times T_y$, the cartesian product of two circles, with the product measurable structure. Any basis of T_y is thus a basis for the partition ξ. It is easy to verify that the conditional measure μ_C on the element of the partition $C = T_x \times y$ is the normalised Lebesgue measure on T_x. An analogous construction is easily made for the partition of the torus into lines of rational slope, since such lines eventually close, and are thus circles. ∎

Example 3.14. Suppose we consider the partition ξ of the torus by the lines of irrational slope $\alpha \notin \mathbf{Q}$. It is a familiar fact that any such line passes arbitrarily close to any point in T^2, in other words, each line is dense in the usual topology of T^2. Each element of ξ is a measurable subset with measure zero of T^2. The problem of finding a basis for the partition ξ is equivalent to the problem of finding a basis for the partition η of T by taking T transversal to the lines of constant η. Two points $x, y \in T$ are in the same element of η iff there exists an integer n such that $y = x + n\alpha \pmod 1$. If we define the map

$$\begin{aligned} f_\alpha : T &\to T \\ x &\mapsto x + \alpha \pmod 1, \end{aligned}$$

then it follows that the elements of the partition η are precisely the orbits of f_α. The η-sets are just the f_α-invariant subsets of T. It is well known that for irrational α the map f_α is ergodic [8], p 64-7. Thus all η-sets have either measure zero or measure unity. This shows that the only measurable partition into η-sets is the trivial partition, with the one set T^2. In other words, the quotient space T^2/ξ of T^2 by the (non-measurable) partition ξ defined by lines of constant irrational slope is not a Lebesgue space, indeed it is not even countably separated. As a topological space, it is not Hausdorff. Using the conventional techniques of integration theory, this quotient appears to be a trivial measure space. For example, all the L^p spaces are one-dimensional, and conventional measure theory is useless. Given two foliations of T^2 by lines of irrational slope

α and α' with $\alpha \neq \alpha'$, conventional measure theory is incapable of distinguishing the two quotient spaces, both appear trivial. Connes [27] calls such spaces which consist of more than one point but are trivial measure theoretically, *e.g.*, quotients of standard spaces by an ergodic equivalence relation, *singular* measure spaces. Connes noncommutative integration theory is developed precisely for the study of these singular spaces. Connes has recently developed techniques of non-commutative geometry for studying singular topological spaces, and singular differentiable manifolds ([61,62], and [63] for an introduction to the ideas) but we shall in the sequel be interested only in the integration theory. We note in concluding the discussion of the toral example, that one can easily construct measurable partitions of T^2 for which the elements are pieces of the lines of slope $\alpha \notin \mathbf{Q}$, but not the entire lines. ∎

The converse of proposition 3.10 is not true, as example 3.14 above illustrates. This fact will be relevant in later sections.

We now introduce ideas associated to increasing partitions, which will be important later. A partition η is said to *refine* the partition ξ if for almost all $x \in M$ $\eta(x) \subset \xi(x)$. This is also written as $\eta > \xi$. Now let $f : M \to M$ be a Borel automorphism.

Definition 3.15. We say a partition η is *increasing for the automorphism* f if $\eta > f\eta$.

The existence of certain types of increasing (measurable) partitions is an important tool for the ergodic theory of hyperbolic differentiable dynamical systems. In particular, increasing partitions subordinate to the unstable manifolds are an essential tool. The prototype was an increasing partition constructed by Sinai from a Markov partition ([53], Lemma 2.2). The elements of this increasing partition are pieces of unstable manifolds. This was generalized by Bowen to Axiom A. Pesin's work used a partition of this type, see [56]. The partition ξ refered to in the discussion of proposition 2.29 is of this type. We shall need in a later chapter some detailed results of Ledrappier and Strelcyn [33] concerning partitions of this type.

3.1 Notes

This chapter collects the results from measure theory which are used in the ergodic theory of differentiable dynamical systems, and results which will be necessary for the understanding and application of the noncommutative integration theory. These are for the most part standard results. I refer to the proofs in all cases where I could find the result in the literature. All proofs which are presented explicitly are original. In this chapter, this comprises two elementary propositions, proposition 3.5 and proposition 3.10.