AVERAGE VOLUME OF SECTIONS OF STAR BODIES

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Abstract. We study the asymptotic behavior, as the dimension goes to infinity, of the volume of sections of the unit balls of the spaces $\ell_q^n,\ 0 < q \le \infty$. We compute the precise asymptotics of the average volume of central sections and then prove a concentration inequality of exponential type. For the case of non-central hyperplane sections of the cube, we prove a local limit theorem confirming the conjecture on the asymptotically Gaussian dependence of the volume of sections on the distance from the hyperplane to the origin. Note that a weak limit theorem was established very recently in [ABP] for a larger class of bodies. Our calculations are based on connections between volume and the Fourier transform.

1. Introduction

For a star body K in $\mathbb{R}^n$ and an integer $0 < p < n$, consider the average volume of p-dimensional sections of K :

$$AV_p(K) = \int_{Gr(n,p)} vol_p(K \cap H)\ dH,$$

where $Gr(n,p)$ is the Grassman manifold of p-dimensional subspaces of $\mathbb{R}^n$ equipped with the probability Haar measure. Our study of the quantities $AV_p(K)$ and, in particular, of their behavior as the dimension goes to infinity is motivated by connections with the local theory of Banach spaces.

In this paper we consider the case where $K = B_q^n$ is the unit ball of the space $\ell_q^n,\ 0 < q \le \infty$. In Sections 4-5 we compute precise asymptotics of the averages $AV_p(B_q^n)$, as $n \to \infty$ and $p = p(n)$, using the Fourier transform formula introduced in Section 3. The results are different for the cases where $p = n - d$ with d fixed, where $p \sim \alpha n,\ 0 < \alpha < 1$, and where p is fixed and $n \to \infty$.

A problem that logically follows is to estimate the concentration properties of $vol_p(K \cap H)$ as a function of H on the Grassman manifold. In Section 6, we prove

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an exponential concentration inequality for the volume of hyperplane sections of the bodies B_q^n, $0 < q \le \infty$: there exist n_0 and c depending on q only so that for all $n > n_0$ and all $\epsilon \in (0,1)$,

$$mes\{\xi \in S^{n-1}: \ |V(\xi) - m_V| > \epsilon\} \le 4\exp\{-c\,\epsilon^2(n-1)\}, \tag{1.1}$$

where $V(\xi)$ is the volume of the central hyperplane section of B_q^n orthogonal to ξ, and $mes(\cdot)$ is the probability uniform measure on S^{n-1}. Our calculations are based on the connections between volumes and the Fourier transform and on Lévy's isoperimetric inequality.

In Section 7, we consider another problem from the local theory that was communicated to us by V. Milman. The problem is to show that the volume of sections of origin-symmetric convex bodies by hyperplanes located at distance r from the origin converges (in some sense) to the Gaussian density function of r, as the dimension goes to infinity. Note that both Laplace [La] and Polya [P] proved this for the sections of the n-cubes perpendicular to the main diagonal, and that a simple calculation gives an affirmative answer for the Euclidean balls. We confirm this conjecture for the hyperplane sections of the cubes by proving first that

$$\lim_{n\to\infty} \frac{1}{vol_n(B_\infty^n)|S^{n-1}|} \int_{S^{n-1}} vol_{n-1}(B_\infty^n \cap (r\xi + \xi^\perp))d\xi = \sqrt{3/2\pi}\exp\{-3r^2/2\},$$

and then noting that exactly the same concentration argument, as in Section 6, works for non-central hyperplane sections and leads to a local limit theorem similar to (1.1). After this work was completed, we learned about an excellent earlier paper [ABP], where an exponential concentration inequality (for the distribution function instead of the density) was established for a class of bodies including B_q^n, $1 \le q \le \infty$. The result of our Theorem 6.2 can be proved by methods from [ABP] (except for the case $0 < p < 1$, where one can not use Busemann's theorem). However, in the case of non-central sections, we do not immediately see how can one deduce the local limit result (for the density) of our Section 7 from the weak limit theorem (for the distribution function) of [ABP]. In both Sections 6 and 7, our methods are completely different from those of [ABP]. Our argument is based on the approximation of spheric averages by Gaussian ones and on a Fourier transform representation for the volumes, while the proofs in [ABP] use the tools of convexity. We have also learned after this work was completed that important earlier papers [BV] and [V] contain limit theorems for the density (in individual directions) with convergence in L_1 and L_∞ norms. These results, however, do not imply the exponential concentration. The methods there are also different from ours. For several related probabilistic results, see [DF], [R], [S], [W].

In the sequel, $f(n) \sim g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$.

2. An upper bound for the average volume of sections

Let K be a body that is starshaped with respect to the origin. We call K a *star body* if the origin is an interior point of K and the Minkowski functional of K (defined by $\|x\|_K = \min\{a > 0 : \ x \in aK\}$) is continuous on $\mathbb{R}^n$.

Let $Gr(n,p)$ be the Grassman manifold of p-dimensional subspaces of R^n. In the sequel, we consider the Grassman manifolds equipped with their normalized Haar measures, while the Haar measures on the sphere S^{n-1} and its sections are not normalized. For every continuous function f on S^{n-1},

$$\int_{S^{n-1}} f(x)\ dx = \frac{|S^{n-1}|}{|S^{p-1}|} \int_{Gr(n,p)} \Big(\int_{S^{n-1}\cap H} f(x)\ dx \Big)\ dH, \tag{2.1}$$

where $|S^{p-1}| = 2\pi^{p/2}/\Gamma(p/2)$ is the surface area of the unit sphere S^{p-1} in $\mathbb{R}^p$.

We use an elementary formula for the p-dimensional volume of the section of K by a subspace $H \in Gr(n,p)$:

$$vol_p(K\cap H) = \int_{S^{n-1}\cap H} \int_0^{1/\|\xi\|} r^{p-1} dr\ d\xi = p^{-1} \int_{S^{n-1}\cap H} \|\xi\|^{-p}\ d\xi. \tag{2.2}$$

We need the following simple fact.

Lemma 2.1. *For every pair of integers* $0 \le d < n$,

$$1 \le \frac{n^{(n-d)/n} |S^{n-d-1}|}{(n-d)|S^{n-1}|^{(n-d)/n}} = \frac{\big(\Gamma(n/2+1)\big)^{(n-d)/n}}{\Gamma(\frac{n-d}{2}+1)} \le e^{d/2}. \tag{2.3}$$

Proof. To prove the lower bound, use the well-known fact that the function $\log(\Gamma(x))$ is convex. We have

$$\frac{\log(\Gamma(\frac{n}{2}+1)) - \log(\Gamma(1))}{n/2} \ge \frac{\log(\Gamma((n-d)/2+1)) - \log(\Gamma(1))}{(n-d)/2},$$

which implies the result.

To prove the upper bound, let us write the inequality (2.3) in the form

$$\frac{\Gamma(n/2+1)}{\Gamma((n-d)/2+1)} \frac{1}{(\Gamma(n/2+1))^{d/n} e^{d/2}} \le 1. \tag{2.4}$$

Using again the log-convexity of the Γ-function, we get

$$\Gamma^2(n/2+1) \le \Gamma(n/2+3/2)\Gamma(n/2+1/2) = (n/2+1/2)\Gamma^2(n/2+1/2).$$

Therefore,

$$\frac{\Gamma(n/2+1)}{\Gamma(n/2+1/2)} \le (n/2+1)^{1/2}. \tag{2.5}$$

It immediately follows that for bigger d

$$\frac{\Gamma(n/2+1)}{\Gamma((n-d)/2+1)} \le (n/2)^{d/2}, \quad d>1. \tag{2.6}$$

For the second fraction in (2.4), we use Stirling's formula in the form of [A, p.24]: for every $z \ge 1/2$,

$$\Gamma(z+1) \ge \sqrt{2\pi}(z+1)^{z+1/2}e^{-z-1} = \frac{\sqrt{2\pi}}{e}(z/e)^z(z+1/z)^z\sqrt{z+1} \ge (z/e)^z\sqrt{z+1}.$$

Letting in (2.7) $z=n/2$ we obtain for the second fraction in (2.4) the upper bound

$$(n/2)^{-d/2}(n/2+1)^{-d/2n}. \tag{2.7}$$

Combining this bound with (2.6) immediately proves (2.4) for $d>1$. For $d=1$ after combining (2.7) and (2.5) we still have to check

$$(1+2/n)^{1/2}(n/2+1)^{-1/2n} < 1.$$

This reduces to

$$(1+2/n)^{n/2} < e < (n/2+1)^{1/2}$$

and is true for $n \ge 16$. In the case $d=1, n<16$ the inequality can be checked directly. $\square$

The following inequality follows from a more general result of Lutwak [Lu]. We give here a simple proof.

Proposition 2.2. *Let K be a star body in R^n. Then for every positive integer $p<n$*

$$AV_p(K) \le R_{n,p} = \frac{|S^{p-1}|}{p\,|S^{n-1}|^{p/n}}\left(n\ vol_n(K)\right)^{p/n}. \tag{2.8}$$

In particular, for $p=n-d$ we have

$$AV_{n-d}(K) \le e^{d/2}(vol_n(K))^{(n-d)/n}, \tag{2.9}$$

and if $p = n/\alpha$ *and* $K = K_n \subset \mathbb{R}^n$ *so that* $vol_n(K_n) = c$ *then*

$$R_{n,p} \sim (\pi n)^{1/2\alpha - 1/2} \alpha^{n/2\alpha + 1/2} vol_n(K)^{1/\alpha}. \tag{2.10}$$

Proof. Denote by $c = |S^{p-1}|/(p|S^{n-1}|)$. By (2.1), (2.2) and Hölder's inequality,

$$\int_{Gr(n,p)} vol_p(K \cap H)\, dH = p^{-1} \int_{Gr(n,p)} \int_{S^{n-1}\cap H} \|\xi\|^{-p}\, d\xi = c \int_{S^{n-1}} \|\xi\|^{-p}\, d\xi \leq$$

$$c \left(\int_{S^{n-1}} \|\xi\|^{-n}\, d\xi \right)^{p/n} \left(\int_{S^{n-1}} 1\, d\xi \right)^{1-p/n} = c\, (n\; vol_n(K))^{p/n}\, |S^{n-1}|^{1-p/n},$$

which gives (2.8). The estimates (2.9) and (2.10) follow from Lemma 2.1 and

$$|S^{p-1}| \sim \frac{\pi^{p/2-1/2} 2^{p/2} e^{p/2}}{p^{p/2-1/2}}, \qquad |S^{n-1}|^{p/n} \sim \frac{\pi^{p/2-p/2n} 2^{p/2} e^{p/2}}{n^{p/2-p/2n}}. \quad \square$$

The estimate (2.8) turns into equality if $K = B_2^n$ is the Euclidean ball. Also the estimates (2.9) and (2.10) are asymptotically sharp in this case. The results of Section 5.1 of this paper, together with the formula $vol_n(B_q^n) = (2\Gamma(1+1/q))^n/\Gamma(1+n/q)$, imply that

$$\lim_{n\to\infty} \frac{AV_{n-d}(B_q^n)}{(vol_n(B_q^n))^{(n-d)/n}} = \left(\frac{4\Gamma(1/q)(\Gamma(1+1/q))^2 e^{2/q}}{2\pi\Gamma(3/q)} \right)^{d/2}.$$

In particular, if $q = \infty$ the limit is equal to $(6/\pi)^{d/2}$. Note that the minimal volume of a d-codimensional section of the n-cube is 1 (see [Va]), and the maximal volume is $2^{d/2}$ (see [Ba3]).

The situation is different for proportional sections. For example, if $K_n = B_\infty^n$ is the cube with side 2 and $p = n/2$, then the main term of the asymptotic upper bound (2.10) is

$$\alpha^{n/2\alpha} vol_n(K_n)^{1/\alpha} = 2^{n/4+n/2} = (2.828\ldots)^{n/2},$$

while, as it will be shown in Section 4.3.2, the leading term of the true asymptotics of $AV_{n/2}(K_n)$ is $(2.516\ldots)^{n/2}$.

3. Volumes of sections and the Fourier transform

The Fourier transform formulas for the volume of sections have already been applied to several geometric problems. The first formula of this kind was known to Laplace [La], who proved that the volume of the section of $\frac{1}{2}B_\infty^n$ perpendicular to the main diagonal is equal to

$$\frac{2\sqrt{n}}{\pi}\int_0^\infty \left(\frac{\sin(t)}{t}\right)^n dt,$$

and applied the law of large numbers to show that the limit of this expression, as $n \to \infty$, is equal to $\sqrt{6/\pi}$. This formula was generalized in [P], [H], [Va], [Ba1,Ba3], [MeP], [K1] to the case of arbitrary central sections of the bodies B_q^n, $0 < q \le \infty$ and applied to find the maximal and minimal sections of some of these bodies. It was noticed in [K1] that, in the case of hyperplane central sections, these formulas represent particular cases of the following relation: for every origin-symmetric star body K in $\mathbb{R}^n$ and every $\xi \in S^{n-1}$,

$$vol_{n-1}(K\cap\xi^\perp) = \frac{1}{\pi(n-1)}(\|x\|_K^{-n+1})^\wedge(\xi), \tag{3.1}$$

where $\hat{f}$ stands for the Fourier transform of f in the sense of distributions.

The latter formula was generalized in [K3, Lemma 7] to central sections of arbitrary dimension. We say that a star body K in $\mathbb{R}^n$ is *k-smooth* if the restriction of the Minkowski functional of K to the sphere S^{n-1} belongs to the space $C^{(k)}(S^{n-1})$ of continuously differentiable up to order k functions. If K is a $(k-1)$-smooth symmetric star body in $\mathbb{R}^n$, $1 \le k < n$, then for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$vol_{n-k}(K\cap H) = \frac{1}{n-k}\int_{S^{n-1}\cap H}\|x\|_K^{-n+k}\,dx =$$

$$\frac{1}{n-k}\frac{1}{(2\pi)^k}\int_{S^{n-1}\cap H^\perp}(\|x\|_K^{-n+k})^\wedge(\theta)\,d\theta. \tag{3.2}$$

The formulas (3.1) and (3.2), in conjunction with Propositions 3.2 and 3.3 below, reproduce the formulas for central sections of the bodies B_q^n that appeared in [H],[Ba1,3], [MeP], [K1].

The formulas (3.1) and (3.2) can be used to get an expression for the average volume of sections in terms of the Fourier transform. However, applying (3.2) requires a certain approximation argument, because of the smoothness condition. Instead, we present here a simple direct proof. If $0 < p < n$ then the function $\|x\|_K^{-p}$ is locally integrable on $\mathbb{R}^n$. The Fourier transform $(\|\cdot\|_K^{-p})^\wedge$ is defined as the distribution satisfying $\langle(\|x\|_K^{-p})^\wedge, \phi\rangle = \langle\|x\|_K^{-p}, \hat{\phi}\rangle$ for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$. Note that the distribution $(\|\cdot\|_K^{-p})^\wedge$ is homogeneous of degree $-n+p$.

Lemma 3.1. *Let K be a star body in R^n so that $(\|\cdot\|_K^{-p})^\wedge$ is a locally integrable function on R^n, and p is an integer, $0<p<n$. Then*

$$AV_p(K) = \frac{|S^{n-p-1}|}{p(2\pi)^{n-p}|S^{n-1}|}\int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\theta)\ d\theta.$$

Proof. We have

$$\langle(\|x\|_K^{-p})^\wedge, \exp(-\|x\|_2^2/2)\rangle = (2\pi)^{n/2}\langle\|x\|_K^{-p}, \exp(-\|x\|_2^2/2)\rangle =$$

$$(2\pi)^{n/2}\int_{R^n}\|x\|_K^{-p}\exp(-\|x\|_2^2/2)\ dx =$$

$$(2\pi)^{n/2}\int_{S^{n-1}}\|\theta\|_K^{-p}\ d\theta\int_0^\infty t^{n-p-1}\exp(-t^2/2)\ dt =$$

$$2^{n-p/2-1}\pi^{n/2}\Gamma((n-p)/2)\int_{S^{n-1}}\|\theta\|_K^{-p}\ d\theta. \tag{3.3}$$

On the other hand, since $(\|x\|_K^{-p})^\wedge$ is a locally integrable homogeneous function of degree $-n+p$ on R^n, we have

$$\langle(\|x\|_K^{-p})^\wedge, \exp(-\|x\|_2^2/2)\rangle = \int_{S^{n-1}}(\|x\|_K^{-p})^\wedge(\theta)\ d\theta\int_0^\infty t^{p-1}\exp(-t^2/2)\ dt =$$

$$2^{p/2-1}\Gamma(p/2)\int_{S^{n-1}}(\|x\|_K^{-p})^\wedge(\theta)\ d\theta. \tag{3.4}$$

Finally, using (2.1) and (2.2), we get

$$\int_{Gr(n,p)} vol_p(K\cap H)\ dH = \frac{1}{p}\int_{Gr(n,p)}\int_{S^{n-1}\cap H}\|\theta\|_K^{-p}\ d\theta =$$

$$\frac{|S^{p-1}|}{p|S^{n-1}|}\int_{S^{n-1}}\|\theta\|_K^{-p}\ d\theta. \tag{3.5}$$

The desired result now follows from (3.3), (3.4) and (3.5). □

It is known (see [K3]) that $(\|\cdot\|_K^{-p})^\wedge$ is a continuous function on $\mathbb{R}^n\setminus\{0\}$ if K is a $(n-p-1)$-smooth star body. This immediately implies that $(\|\cdot\|_K^{-p})^\wedge$ is locally integrable because this function is also homogeneous of degree $-n+p>-n$. However, Lemma 3.1 can also be applied to the bodies B_q^n that are not necessarily smooth. In order to apply Lemma 3.1 to the bodies B_q^n we use simple direct computations of the Fourier transform from [K2, Lemma 3 and Lemma 8]:

Proposition 3.2. *If $p \in (0, n)$ then the Fourier transform of the function $\|x\|_\infty^{-p}$ is equal to a locally integrable on $\mathbb{R}^n$ function*

$$\xi \mapsto 2^n p \int_0^\infty t^{-p-1} \prod_{k=1}^n \frac{\sin(t\xi_k)}{\xi_k}\, dt.$$

Denote by γ_q the Fourier transform of the function $z \mapsto \exp(-|z|^q)$, $z \in \mathbb{R}$.

Proposition 3.3. *Let $q > 0$, $n \in \mathbb{N}$, $0 < p < n$. Then the Fourier transform of the function $\|x\|_q^{-p}$ is equal to a locally integrable on $\mathbb{R}^n$ function*

$$\xi \mapsto \frac{q}{\Gamma(p/q)} \int_0^\infty t^{n-p-1} \prod_{k=1}^n \gamma_q(t\xi_k)\, dt.$$

4. Sections of the cube

Combining Lemma 3.1 with Proposition 3.2 we get

$$AV_p(B_\infty^n) = 2^p \pi^{p-n} |S^{n-p-1}|\ I_n, \tag{4.1}$$

where

$$I_n = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \int_0^\infty t^{n-p-1} \prod_{k=1}^n \sin(t\xi_k)/(t\xi_k) dt d\xi.$$

In this section we study the asymptotics of these integrals and the corresponding average volumes as $n \to \infty$. We consider three typical cases. First, we consider "hypersections", where $p = n - d$ and d is fixed, while $n \to \infty$. The second is the case of proportional sections, where $p \sim \alpha n$, $0 < \alpha < 1$. Finally, in the case of low-dimensional sections we have p fixed and $n \to \infty$. In fact, the methods exposed below are applicable to any reasonable dependence $p = p(n)$.

The Fourier transform of the function $\|x\|_\infty^{-p}$ is a sign-changing function if $p < n - 3$ which shows that the cube in $\mathbb{R}^n$ is a p-intersection body only if $p \geq n - 3$ (see [K2, Th.1], [K4] for details). Therefore it would be interesting to see how large is the set of those points $\xi \in S^{n-1}$ where $(\|x\|_\infty^{-p})^\wedge(\xi) < 0$. For this reason, along with the integrals I_n we study the asymptotics of the integrals

$$A_n = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \int_0^\infty t^{n-p-1} \prod_{k=1}^n |\sin(t\xi_k)/(t\xi_k)| dt d\xi.$$

The results of Ball [Ba1, Th.4], [Ba3, Th.6 and Proposition 4] suggest that the behaviour of the integrals A_n must be similar to that of I_n if $d = n - p$ is fixed. We show below that this is, indeed, the case. Moreover, these integrals still have the same main asymptotic term $n^{(1-\alpha)n/2}$ but differ at most by c^n if $p \sim \alpha n,\ 0 < \alpha < 1$. However, if p is fixed, the integrals A_n grow much faster than I_n.

To compute the asymptotics of our integrals, we first link the integral over the sphere to the integral over the Gaussian distribution G with zero mean and covariance $\frac{1}{n}U$, where U is the unit matrix. Then the correspondent Gaussian integrals take the form

$$I_n^G = \int_0^\infty t^{n-p-1}[E_G \sin(t\xi_1)/(t\xi_1)]^n dt = \int_0^\infty t^{n-p-1} g\left(\frac{t}{\sqrt{n}}\right)^n dt;$$

$$A_n^G = \int_0^\infty t^{n-p-1}[E_G |\sin(t\xi_1)/(t\xi_1)|]^n dt = \int_0^\infty t^{n-p-1} h\left(\frac{t}{\sqrt{n}}\right)^n dt$$

where

$$g(\tau) = E\sin(\tau X)/(\tau X), \qquad h(\tau) = E|\sin(\tau X)/(\tau X)|, \tag{4.2}$$

and X follows the standard normal distribution.

Note that there is an explicit relation between I_n and I_n^G, A_n and A_n^G, which follows from the fact that the functions under the integrals are homogeneous.

Lemma 4.1. *Let $f : R^n \to R$ be a homogeneous function of degree β. Then*

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(\xi)d\xi = \frac{\Gamma(n/2)}{\Gamma\left(\frac{n+\beta}{2}\right)}(n/2)^{\beta/2}\int_{\mathbb{R}^n} f(x)G(dx).$$

Proof. Writing the integrals in polar coordinates we get

$$\int_{\mathbb{R}^n} f(x)G(dx) = \int_0^\infty \int_{S^{n-1}} f(r\xi)r^{n-1}p_G(r\xi)d\xi dr =$$

$$\int_0^\infty \int_{S^{n-1}} f(\xi)r^{n+\beta-1}(n/2\pi)^{n/2}\exp\{-r^2n/2\}d\xi dr =$$

$$(n/2\pi)^{n/2}\int_0^\infty r^{n+\beta-1}\exp\{-r^2n/2\}dr\int_{S^{n-1}} f(\xi)d\xi =$$

$$(n/2\pi)^{n/2}\frac{\Gamma\left(\frac{n+\beta}{2}\right)}{n}(2/n)^{(n+\beta)/2-1}\frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(\xi)d\xi =$$

$$\frac{\Gamma\left(\frac{n+\beta}{2}\right)}{\Gamma(n/2)}(2/n)^{\beta/2}\frac{1}{|S^{n-1}|}\int_{S^{n-1}} f(\xi)d\xi. \quad \square$$

The function

$$f(\xi) = \int_0^\infty t^{n-p-1} \prod_{k=1}^n \sin(t\xi_k)/(t\xi_k) dt$$

is homogenious of degree $\beta = p - n$, and the same is true for the integrals with the absolute values. Now Lemma 4.1 yields

$$I_n = \frac{\Gamma(n/2)}{\Gamma\left(\frac{p}{2}\right)} (n/2)^{(p-n)/2} \; I_n^G, \qquad A_n = \frac{\Gamma(n/2)}{\Gamma\left(\frac{p}{2}\right)} (n/2)^{(p-n)/2} \; A_n^G.$$

4.1. Asymptotic behavior of the Gaussian integrals I_n^G.

The function g appearing in the expression for the integrals I_n^G can easily be calculated. In fact,

$$(g(\tau)\tau)' = E\cos(\tau X) = e^{-\tau^2/2}.$$

It follows that

$$g(\tau) = \sqrt{\pi/2}(2\Phi(\tau) - 1)\tau^{-1}, \tag{4.3}$$

where $\Phi(\tau) = P\{X < \tau\}$ denotes the standard normal distribution function. Therefore,

$$g(0) = 1, \;\; g'(0) = 0, \;\; g''(0) = -1/3$$

and, as $t \to \infty$,

$$g(\tau) \sim \sqrt{\pi/2} \; \tau^{-1}. \tag{4.4}$$

In order to study the integrals I_n^G we need to find the point τ_* which maximizes the integrand:

$$t^{n-p-1} g(\frac{t}{\sqrt{n}})^n \to \max.$$

Letting the derivative vanish and using (4.3), we get the equation

$$\frac{\Phi'(\tau_*)\tau_*}{\Phi(\tau_*) - \Phi(0)} = \frac{p+1}{n}. \tag{4.5}$$

Different behavior of τ_* at $n \to \infty$ in three following cases explains the difference of the asymptotics for $AV_p(B_n^\infty)$.

4.1.1. Hypersections, $p = n - d$ with constant d.

In this case the solution of the equation (4.5) tends to zero and the behavior of I_n^G is determined by the asymptotic of the integrand at zero. Split the domain of integration into the union of $[0, r\sqrt{n}]$ and $(r\sqrt{n}, \infty)$. For the latter interval

$$\int_{r\sqrt{n}}^{\infty} t^{d-1} g(\frac{t}{\sqrt{n}})^n dt \le (\max_{\tau > r} g(\tau))^{n-d-1} \int_{r\sqrt{n}}^{\infty} t^{d-1} g(\frac{t}{\sqrt{n}})^{d+1} dt \le$$

$$(\max_{\tau > r} g(\tau))^{n-d-1} \int_{r\sqrt{n}}^{\infty} t^{d-1} (\frac{c\sqrt{n}}{t})^{d+1} dt = (\max_{\tau > r} g(\tau))^{n-d-1} c^{d+1} n^{d/2} r^{-1} \to 0,$$

since the maximum is strictly less than one. On the other hand,

$$\int_0^{r\sqrt{n}} t^{d-1} g(\frac{t}{\sqrt{n}})^n dt = \int_0^{r\sqrt{n}} t^{d-1} \left(1 - \frac{t^2(1+o(r))}{6n}\right)^n dt \sim$$

$$\int_0^{\infty} \exp\{-\frac{t^2(1+o(r))}{6}\} t^{d-1} dt.$$

Sending $r \to 0$, we finally obtain for $p = n - d$

$$\lim_n I_n^G = \int_0^{\infty} \exp\{-\frac{t^2}{6}\} t^{d-1} dt = \frac{\Gamma(d/2) 6^{d/2}}{2}. \tag{4.6}$$

4.1.2. Proportional sections, $p = \alpha n - 1, 0 < \alpha < 1$.

In this case the equation (4.5) turns into

$$\frac{\Phi'(\tau_*)\tau_*}{\Phi(\tau_*) - \Phi(0)} = \alpha.$$

Hence, the solution τ_* does not depend on n. Make a linear scaling $t = \sqrt{n}\tau_* + u$ and denote by

$$g_\alpha(\tau) = g(\tau)\tau^{1-\alpha}.$$

Then

$$I_n^G = \int_0^{\infty} t^{(1-\alpha)n} g(\frac{t}{\sqrt{n}})^n \, dt = \int_0^{\infty} \left[t^{1-\alpha} g(\frac{t}{\sqrt{n}})\right]^n \, dt =$$

$$n^{(1-\alpha)n/2} \int_0^{\infty} g_\alpha(\frac{t}{\sqrt{n}})^n dt = n^{(1-\alpha)n/2} \int_{-\sqrt{n}\tau_*}^{\infty} g_\alpha(\tau_* + \frac{u}{\sqrt{n}})^n \, du =$$

$$n^{(1-\alpha)n/2} g_\alpha(\tau_*)^n \int_{-\sqrt{n}\tau_*}^{\infty} \left[\frac{g_\alpha(\tau_* + \frac{u}{\sqrt{n}})}{g_\alpha(\tau_*)}\right]^n \, du \sim$$

$$n^{(1-\alpha)n/2} g_\alpha(\tau_*)^n \int_{-\sqrt{n}\tau_*}^{\infty} \left[1 + \frac{g_\alpha''(\tau_*)u^2}{2n g_\alpha(\tau_*)}\right]^n du \sim$$

$$n^{(1-\alpha)n/2} g_\alpha(\tau_*)^n \int_{-\sqrt{n}\tau_*}^{\infty} \exp\left\{\frac{g_\alpha''(\tau_*)u^2}{2 g_\alpha(\tau_*)}\right\} du \sim$$

$$n^{(1-\alpha)n/2} g_\alpha(\tau_*)^n \sqrt{\frac{2\pi g_\alpha(\tau_*)}{|g_\alpha''(\tau_*)|}} =$$

$$n^{(1-\alpha)n/2} g_\alpha(\tau_*)^{n+1/2} \sqrt{\frac{2\pi}{|g_\alpha''(\tau_*)|}}. \tag{4.7}$$

4.1.3. Low-dimensional sections, $p = const$**.**

In this case we make a linear scaling $t = \sqrt{n}\tau$,

$$I_n^G = \int_0^\infty t^{n-p-1} g(\frac{t}{\sqrt{n}})^n dt = n^{(n-p)/2} \int_0^\infty \tau^{n-p-1} g(\tau)^n d\tau =$$

$$n^{(n-p)/2} \int_0^\infty \tau^{-p-1} [\tau g(\tau)]^n dt = n^{(n-p)/2} \int_0^\infty \tau^{-p-1} [\sqrt{\pi/2}(2\Phi(\tau) - 1)]^n d\tau.$$

The solution τ_* of the equation (4.5) now tends to infinity. We have

$$\tau_* \sim \sqrt{2 \ln n}, \qquad \exp\{-{\tau_*}^2/2\} \sim \frac{(p+1)\sqrt{2\pi}}{2\tau_* n} << \frac{1}{n}.$$

Therefore, $(2\Phi(\tau_*) - 1)^n \sim 1$ and the main contribution emerges from

$$\int_{\tau_*}^\infty \tau^{-p-1} (2\Phi(\tau) - 1)^n d\tau \sim {\tau_*}^{-p}/p \sim (2 \ln n)^{-p/2}/p.$$

Note that the integral over $[0, \alpha\tau_*]$ does not give any significant contribution for every $\alpha < 1$, since on this interval

$$(2\Phi(\tau) - 1)^n \le (2\Phi(\alpha\tau_*) - 1)^n \le \exp\left\{-c(\alpha)(\ln n)^{-(1+\alpha^2)/2} n^{1-\alpha^2}\right\} << (\ln n)^{-p/2}.$$

The resulting asymptotics turns out to be

$$I_n^G \sim n^{(n-p)/2} (\pi/2)^{n/2} (2 \ln n)^{-p/2}/p. \tag{4.8}$$

4.2. Asymptotic behavior of A_n^G.

In order to study the integral A_n^G we need to know the properties of the function h (see (4.2)). Note that the asymptotic behavior of g and h at zero is the same. In fact,

$$|h(\tau) - g(\tau)| \le P\{|X| > \frac{\pi}{\tau}\} << \exp\{-\pi^2/2\tau^2\}.$$

In particular, $h(0) = 1,\ h'(0) = 0,\ h''(0) = -1/3$. But the behavior of h at infinity is different. As $\tau \to \infty$, we have

$$\tau h(\tau) \sim \sqrt{2/\pi} \int_0^\tau \frac{|\sin y| dy}{y} \sim \sqrt{2/\pi} \frac{2}{\pi} \ln \tau = (2/\pi)^{3/2} \ln \tau. \tag{4.9}$$

4.2.1. Hypersections, $p = n - d$ with constant d.

Since in this case only a neighbourhood of zero is important, the result is the same as in (4.6):

$$\lim_n A_n^G = \lim_n I_n^G = \int_0^\infty \exp\{-\frac{t^2}{6}\} t^{d-1} dt = \frac{\Gamma(d/2) 6^{d/2}}{2}.$$

4.2.2. Proportional sections, $p = \alpha n - 1,\ 0 < \alpha < 1$.

Denote by

$$h_\alpha(\tau) = h(\tau)\tau^{1-\alpha}$$

and let τ_* be the maximal point of this function. Then, as in (4.7),

$$A_n^G \sim n^{(1-\alpha)n/2} h_\alpha(\tau_*)^{n+1/2} \sqrt{\frac{2\pi}{|h_\alpha{}''(\tau_*)|}}. \tag{4.10}$$

We see that the main term is the same for A_n^G and I_n^G, and they essentially differ by a constant to the power n.

4.2.3. Low-dimensional sections, $p = const$.

In this case we again make the scaling $t = \sqrt{n}\tau$,

$$A_n^G = \int_0^\infty t^{n-p-1} h(\frac{t}{\sqrt{n}})^n dt = n^{(n-p)/2} \int_0^\infty \tau^{n-p-1} h(\tau)^n d\tau =$$

$$n^{(n-p)/2} \int_0^\infty \tau^{-p-1} [\tau h(\tau)]^n d\tau.$$

Taking into account the asymptotics (2.9), we get

$$\int_R^\infty \tau^{-p-1} [\tau h(\tau)]^n d\tau = (1 + o(R))(2/\pi)^{3n/2} \int_R^\infty \tau^{-p-1} [\ln \tau]^n d\tau =$$

$$= (1+o(R))(2/\pi)^{3n/2} \int_{p \ln R}^{\infty} e^{-u} [u/p]^n \frac{du}{p} \sim$$

$$(2/\pi)^{3n/2} n! p^{-n-1} \sim (2/\pi)^{3n/2} (n/e)^n \sqrt{2\pi n} p^{-n-1}.$$

Therefore, the final answer is

$$A_n^G \sim (2n/\pi)^{3n/2} (pe)^{-n} n^{(1-p)/2} \frac{\sqrt{2\pi}}{p}, \tag{4.11}$$

which is significantly bigger than (4.8).

4.3. Asymptotics for the average volume of sections of the cube.

We consider the sections of the unit balls B_∞^n of the spaces ℓ_∞^n, which are cubes with side 2.

4.3.1. Hypersections, $p = n - d$ with constant d.

In this case, we have

$$I_n = \frac{\Gamma(n/2)}{\Gamma\left(\frac{n-d}{2}\right)} (n/2)^{-d/2}\ I_n^G = (1+o(1)) I_n^G.$$

Therefore, I_n converges to the same constant (4.6) as the Gaussian integral. The same is true for A_n. Now we see from (4.1) and (4.6) that

$$AV_{n-d}(B_\infty^n) \sim 2^{n-d} \pi^{-d} |S^{d-1}| \frac{\Gamma(d/2) 6^{d/2}}{2} =$$

$$2^{n-d} \pi^{-d} \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{\Gamma(d/2) 6^{d/2}}{2} =$$

$$2^{n-d} (6/\pi)^{d/2} = (6/\pi)^{d/2} vol_n(B_\infty^n)^{(n-d)/n}.$$

One can see that this result is fairly close to the general estimate (2.9) and to Ball's bound for the maximal section of the cube.

4.3.2. Proportional sections, $p = \alpha n - 1,\ 0 < \alpha < 1$.

We have

$$I_n = \frac{\Gamma(n/2)}{\Gamma\left(\frac{\alpha n - 1}{2}\right)} (n/2)^{((\alpha-1)n-1)/2}\ I_n^G.$$

Since

$$\Gamma(n/2) \sim \frac{n^{n/2-1/2} \pi^{1/2}}{(2e)^{n/2-1} e},$$

$$\Gamma\left(\frac{\alpha n - 1}{2}\right) \sim (\pi \alpha n)^{1/2} e^{-3/2} \left(\frac{\alpha n}{2e}\right)^{\alpha n/2 - 3/2},$$

we have

$$\frac{\Gamma(n/2)}{\Gamma\left(\frac{\alpha n-1}{2}\right)}(n/2)^{((\alpha-1)n-1)/2} \sim \alpha^{1-\alpha n/2} e^{(\alpha-1)n/2}. \tag{4.12}$$

It follows from (4.7) that

$$I_n \sim n^{(1-\alpha)n/2} g_\alpha(\tau_*)^{n+1/2} \sqrt{\frac{2\pi}{|g_\alpha''(\tau_*)|}} \alpha^{1-\alpha n/2} e^{(\alpha-1)n/2}. \tag{4.13}$$

Combining this expression with (4.1) and using the asymptotics

$$|S^{n-p-1}| \sim 2^{1/2} \left(\frac{2\pi e}{n(1-\alpha)}\right)^{n/2-\alpha n/2} \tag{4.14}$$

we get

$$AV_{\alpha n-1}(B_\infty^n) \sim 2^{n/2+\alpha n/2} \pi^{-1/2-n/2+\alpha n/2} \alpha^{1-\alpha n/2} (1-\alpha)^{\alpha n/2-n/2} \frac{g_\alpha(\tau_*)^{n+1/2}}{\sqrt{|g_\alpha''(\tau_*)|}}.$$

It is worthwhile to note that the main (exponential) term of this expression is

$$\left(\frac{2^{1+\alpha} g_\alpha(\tau_*)^2}{\pi^{1-\alpha}\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)^{n/2}.$$

For example, if $\alpha = 1/2$, solving the equation (4.5) numerically, one can find $\tau_* \approx 1.4$, $\Phi(\tau_*) \approx 0.91924$. Hence, by (4.3)

$$g_\alpha(\tau_*) = \sqrt{\pi/2}(2\Phi(\tau_*) - 1)\tau_*^{-1/2} \approx 0.888,$$

and the main term of the asymptotics of the average volume is

$$\left(\frac{2^{5/2} g_\alpha(\tau_*)^2}{\pi^{1/2}}\right)^{n/2} = (2.516\ldots)^{n/2}.$$

Similarly to (4.12), it follows from (4.10) that

$$A_n \sim n^{(1-\alpha)n/2} h_\alpha(\tau_*)^{n+1/2} \sqrt{\frac{2\pi}{|h_\alpha''(\tau_*)|}} \alpha^{1-\alpha n/2} e^{(\alpha-1)n/2}.$$

4.3.3. Low-dimensional sections, $p = const$.

We have

$$I_n = \frac{\Gamma(n/2)}{\Gamma\left(\frac{p}{2}\right)}(n/2)^{(p-n)/2}\ I_n^G;$$

Since

$$\Gamma(n/2)(n/2)^{(p-n)/2} \sim \frac{n^{n/2-1/2}\pi^{1/2}}{(2e)^{n/2-1}e}(n/2)^{(p-n)/2} = \frac{n^{p/2-1/2}\pi^{1/2}}{2^{p/2-1}e^{n/2}}, \tag{4.15}$$

we get from (4.8) that

$$I_n \sim \frac{n^{p/2-1/2}\pi^{1/2}}{2^{p/2-1}\Gamma(p/2)e^{n/2}} n^{(n-p)/2}(\pi/2)^{n/2}(2\ln n)^{-p/2}/p \sim$$
$$\frac{n^{(n-1)/2}\pi^{(n+1)/2}}{2^{n/2+p-1}\Gamma(p/2)e^{n/2}(\ln n)^{p/2}\ p}.$$

Now using (4.1) and

$$|S^{n-p-1}| \sim \left(\frac{2\pi}{n}\right)^{n/2-p/2}\sqrt{\frac{n}{\pi}}e^{n/2}, \tag{4.16}$$

we see that

$$AV_p(B_\infty^n) \sim 2^p\pi^{p-n}|S^{n-p-1}|\frac{n^{(n-1)/2}\pi^{(n+1)/2}}{2^{n/2+p-1}\Gamma(p/2)e^{n/2}(\ln n)^{p/2}\ p} \sim$$
$$\frac{2}{p\Gamma(p/2)}\left(\frac{\pi n}{2\ln n}\right)^{p/2}. \tag{4.17}$$

It is interesting to compare this formula with the well-known behaviour of the 1-dimensional sections of the cube. The length of such sections is given by

$$\frac{2\left(\sum_{j=1}^n |\xi_j|^2\right)^{1/2}}{\max_{1\le j\le n}|\xi_j|},$$

where ξ is a random vector with spherically symmetric distribution. Taking the standard normal vector as ξ, we see that for big dimensions the numerator is equivalent to $2\sqrt{n}$ (by the law of large numbers) and the denominator is equivalent to $\sqrt{2\ln n}$ (by the well-known behavior of the maximal value of a Gaussian i.i.d. sequence). Therefore, the average section length is precisely $(2n/\ln n)^{1/2}$, as suggested by (4.17) with $p = 1$. Moreover, this example suggests that the volume of sections is highly concentrated near the average, at least for the low-dimensional sections. It also shows that the average differs from the minimal and maximal section lengths which are of the orders 2 and $2\sqrt{n}$, respectively.

Similarly to (4.17), we get from (4.11) that

$$A_n \sim \frac{n^{p/2-1/2}\pi^{1/2}}{\Gamma(p/2)2^{p/2-1}e^{n/2}}(2n/\pi)^{3n/2}(pe)^{-n}n^{(1-p)/2}\frac{\sqrt{2\pi}}{p} \sim \frac{n^{3n/2}2^{(3n-p+3)/2}}{\Gamma(p/2)\pi^{3n/2}p^{n+1}e^{3n/2}}.$$

5. Sections of the balls B_q^n, $0 < q < \infty$.

Recall that, by Proposition 3.3,

$$(\|\cdot\|_q^{-p})^\wedge(\xi) = \frac{q}{\Gamma(p/q)} \int_0^\infty t^{n-p-1} \prod_{k=1}^n \gamma_q(\xi_k t)dt,$$

where

$$\gamma_q(t) = \int_{-\infty}^\infty e^{itx} \exp(-|x|^q)\, dx, \qquad t \in R^1.$$

5.1. Hypersections, $p = n - d$.

In this case, our calculations are based on the behavior of the function γ_q at zero. In particular, we need

$$\gamma_q(0) = \int_{-\infty}^\infty \exp\{-|x|^q\}dx = \frac{2\Gamma(1/q)}{q} \tag{5.1}$$

and

$$\gamma_q''(0) = -\int_{-\infty}^\infty x^2 \exp\{-|x|^q\}dx = \frac{-2\Gamma(3/q)}{q}.$$

We normalize the function γ_q at zero by introducing $\bar\gamma_q(t) = \gamma_q(t)/\gamma_q(0)$. Then we have

$$(\|\cdot\|_q^{-p})^\wedge(\xi) = \frac{q\gamma_q(0)^n}{\Gamma(p/q)} \int_0^\infty t^{n-p-1} \prod_{k=1}^n \bar\gamma_q(\xi_k t)dt.$$

Now we proceed as in Sections 4.1.1 and 4.3.1. The analog of (4.6) reads as

$$\lim_n \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \int_0^\infty t^{n-p-1} \prod_{k=1}^n \bar\gamma_q(\xi_k t)dtd\xi = \int_0^\infty \exp\{-\frac{t^2|\bar\gamma_q''(0)|}{2}\}t^{d-1}dt =$$

$$\frac{\Gamma(d/2)}{2(|\bar\gamma_q''(0)|/2)^{d/2}} = \frac{\Gamma(d/2)2^{d/2-1}\Gamma(1/q)^{d/2}}{\Gamma(3/q)^{d/2}}.$$

We end up with

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} (\|\cdot\|_q^{-p})^\wedge(\xi)\ d\xi \sim \frac{(2\Gamma(1/q)/q)^n}{\Gamma(p/q)} \quad \frac{q\ \Gamma(d/2)2^{d/2-1}\Gamma(1/q)^{d/2}}{\Gamma(3/q)^{d/2}}$$

and since

$$\Gamma(p/q) = \Gamma\left(\frac{n-d}{q}\right) \sim \sqrt{2\pi}(n/q)^{\frac{n-d}{q}-1/2}e^{-n/q},$$

we get

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} (\|\cdot\|_q^{-p})^\wedge(\xi)\ d\xi \sim \frac{(2\Gamma(1/q)/q)^n e^{n/q}}{(n/q)^{\frac{n-d}{q}-1/2}} \quad \frac{q\ \ \Gamma(d/2)2^{d/2-1}\Gamma(1/q)^{d/2}}{\sqrt{2\pi}\Gamma(3/q)^{d/2}}.$$

The main term $n^{-n/q}$ shows a significant difference with the ℓ_∞-case when $q \to \infty$.

Now applying Lemma 3.1 we get

$$AV_{n-d}(B_q^n) \sim n^{-1}(2\pi)^{-d}|S^{d-1}| \; \frac{(2\Gamma(1/q)/q)^n e^{n/q}}{(n/q)^{\frac{n-d}{q}-1/2}} \quad \frac{q \;\; \Gamma(d/2)2^{d/2-1}\,\Gamma(1/q)^{d/2}}{\sqrt{2\pi}\Gamma(3/q)^{d/2}} =$$

$$(2\pi)^{-(d+1)/2} \left(\frac{\Gamma(1/q)}{\Gamma(3/q)}\right)^{d/2} \frac{(2\Gamma(1/q)/q)^n e^{n/q}}{(n/q)^{\frac{n-d}{q}+1/2}}.$$

Note that if $q = 2$ we deal with the Euclidean balls of dimension p. Our asymptotics gives in this case

$$2^{-d/2}\pi^{-d/2-1/2}n^{d/2-1/2}\left(\frac{2\pi e}{n}\right)^{n/2},$$

which is equivalent to the volume of the Euclidean ball, i.e. to the expression $vol_{n-d}(B_2^{n-d}) = (n-d)^{-1}|S^{n-d-1}|$.

5.2. Proportional sections, $p = \alpha n - 1,\ 0 < \alpha < 1$.

Using Lemma 3.1, Proposition 3.3, Lemma 4.1, formulae (4.12), (4.14), (5.1) and repeating the calculation from subsection 4.1.2 we obtain

$$AV_{\alpha n-1}(B_n^q) \sim \frac{2^{n/2+\alpha n/2}\Gamma(1/q)^n g_{\alpha,q}(\tau_*)^{n+1/2}}{\sqrt{\pi|g''_{\alpha,q}(\tau_*)|}\, n\pi^{n/2-\alpha/2}(1-\alpha)^{n/2-\alpha n/2}\alpha^{\alpha n/2}q^{n-1}\Gamma(p/q)},$$

where $g_{\alpha,q}(\tau) = \tau^{1-\alpha}E\bar{\gamma}(\tau X)$ and τ_* is the argmax of this function. Therefore, using the asymtotics

$$\Gamma(p/q) \sim \sqrt{2\pi}(\alpha n/q)^{\alpha n/q-1/2-1/q}e^{-\alpha n/q},$$

we get

$$AV_{\alpha n-1}(B_n^q) \sim$$

$$\frac{2^{n/2+\alpha n/2-1/2}\Gamma(1/q)^n g_{\alpha,q}(\tau_*)^{n+1/2}e^{\alpha n/q}}{\sqrt{|g''_{\alpha,q}(\tau_*)|}\, n\pi^{(1-\alpha)n/2+1}(1-\alpha)^{(1-\alpha)n/2}\alpha^{(1/2+1/q)(\alpha n-1)}(n/q)^{(\alpha n-1)/q+1/2}}.$$

The main term of this asymptotics is $n^{-\alpha n/q}$.

5.3. Low-dimensional sections, $p = const$.

Beginning again with general Lemma 3.1 and Proposition 3.3, then using specific asymptotic expressions (4.15) and (4.16), we arrive at

$$AV_p(B_q^n) \sim \frac{q\gamma_q(0)^n}{n^{n/2-p}p2^{n/2-1}\pi^{n/2-p/2}\Gamma(p/q)\Gamma(p/2)}\; I_n^G \tag{5.2}$$

with the Gaussian integral

$$I_n^G = \int_0^\infty t^{n-p-1}[E\bar{\gamma}_q(tX/\sqrt{n})]^n dt =$$

$$n^{n/2-p/2}\gamma_q(0)^{-n}\int_0^\infty \tau^{-p-1}[\tau\, E\gamma_q(\tau X)]^n d\tau. \tag{5.3}$$

Note that we have

$$\tau E\gamma_q(\tau X) = \tau \int_{-\infty}^\infty E e^{i\tau X x} e^{-|x|^q} dx =$$

$$\tau \int_{-\infty}^\infty e^{-\tau^2 x^2/2-|x|^q} dx = \int_{-\infty}^\infty e^{-u^2/2-|u/\tau|^q} dx.$$

Replacing for large τ the expression $e^{-|u/\tau|^q}$ by $1-|u/\tau|^q$ we get

$$\tau E\gamma_q(\tau X) = \sqrt{2\pi}(1-E_q(\tau^{-q}+o(\tau^{-q}))) = \sqrt{2\pi}\exp(-E_q(\tau^{-q}+o(\tau^{-q}))),\ \tau\to\infty,$$

where

$$E_q = E|X|^q = 2^{q/2}\pi^{-1/2}\Gamma\left(\frac{q+1}{2}\right).$$

Therefore,

$$\int_0^\infty \tau^{-p-1}[\tau E\gamma(\tau X)]^n d\tau \sim$$

$$(2\pi)^{n/2}\int_0^\infty \tau^{-p-1}\exp\{-E_q\tau^{-q}\}d\tau = (2\pi)^{n/2}q^{-1}\Gamma(p/q)(E_q n)^{-p/q}.$$

Chaining this estimate with (5.2) and (5.3), we get the final answer,

$$AV_p(B_q^n) \sim \frac{\pi^{p/2} n^{p/2-p/q}}{\Gamma(p/2+1)E_q^{p/q}}\ . \tag{5.4}$$

In the simplest case where $p=1$, this behavior corresponds to the intuitive picture. Namely, the length of the one dimensional central section in direction ξ is

$$\frac{\left(\sum_{k=1}^n \xi_k^2\right)^{1/2}}{\left(\sum_{k=1}^n |\xi_k|^q\right)^{1/q}}\ .$$

Considering ξ as a standard Gaussian vector in R^n and using twice the law of large numbers, we see that the typical value of the above fraction is $2n^{1/2}/(nE_q)^{1/q} = 2n^{1/2-1/q}E_q^{-1/q}$, just as stated in (5.4).

6. Exponential concentration of hyperplane section volumes

Let $\xi \in S^{n-1}$. Combining (3.1) with Proposition 3.2 or Proposition 3.3, respectively, we obtain the following formula for the volume of the central section of the ℓ_q^n-ball by the hyperplane orthogonal to ξ,

$$vol_{n-1}(B_q^n \cap \xi^\perp) = a_n V(\xi), \tag{6.1}$$

where

$$V(\xi) = \int_0^\infty \prod_{k=1}^n \bar{\gamma}(\xi_k t)dt$$

and

$$\bar{\gamma}(\tau) = \sin\tau/\tau,\ a_n = \frac{2^n}{\pi}, \qquad q = \infty,$$

or

$$\bar{\gamma}(\tau) = \frac{q}{2\Gamma(1/q)} \int_{-\infty}^{\infty} e^{i\tau x - |x|^q} dx,\ a_n = \frac{2^n\Gamma(1/q)^n}{\pi(n-1)\Gamma(\frac{n-1}{q})q^{n-1}}, \qquad q < \infty.$$

In this section we show that the "typical" volume of a hyperplane section is, in fact, very close to the average volume calculated in Sections 4.3.1 and 5.1, respectively. To quantify this statement we prove the exponential concentration property of $V(\cdot)$ considering it as a random variable defined on the unit sphere S^{n-1} equipped with the unit Haar measure $mes(\cdot)$. Our study of the concentration properties of V was inspired by the following classical result (functional form of Lévy's isoperimetric inequality). Denote by $Lip(C)$ the class of functions on the sphere S^{n-1} satisfying the Lipschitz condition

$$|f(\xi) - f(\xi')| \le Cd(\xi, \xi')$$

with respect to the geodesic distance $d(\cdot,\cdot)$.

Theorem 6.1. *(cf [Le], Section 2) Let $f \in Lip(C)$ and let m_f be the median of f. Then for all $r \ge 0$*

$$mes\{\xi :\ |f(\xi) - m_f| \ge r\} \le 2\exp\{-r^2(n-1)/2\,C^2\}. \tag{CI}$$

With this main ingredient we are able to prove the following concentration inequality for V.

Theorem 6.2. *There exist n_0 and c depending on the function $\bar\gamma$ such that for all $n > n_0$ and all $r \in (0,1)$,*

$$mes\{|V - m_V| > r\} \le 4\exp\{-cr^2(n-1)\}.$$

Proof. Define some parameters, depending on the function $\bar\gamma$, namely $\tau_0, \sigma, \alpha, I$, as follows. We choose $\tau_0, \sigma > 0$ so that for all τ satisfying $|\tau| \le \tau_0$

$$|\bar\gamma(\tau)| \le \exp\{-\sigma^2\tau^2\}.$$

Next, fix

$$\alpha = \sup_{|\tau|\ge\tau_0} \bar\gamma(\tau) < 1$$

and let

$$I = \int_0^\infty \tau^2|\bar\gamma(\tau)|^4 d\tau < \infty.$$

The reason for considering I comes from the following Hölder-type estimate: for all real $x_1 \dots, x_4$

$$\int_0^\infty t^2 \prod_{k=1}^4 |\bar\gamma(x_k t)| dt \le \prod_{k=1}^4 \left(\int_0^\infty t^2 |\bar\gamma(x_k t)|^4 dt\right)^{1/4} = \prod_{k=1}^4 \left(|x_k|^{-3} I\right)^{1/4} =$$

$$I \prod_{k=1}^4 |x_k|^{-3/4} \le I\ (\min_{1\le k\le 4} |x_k|)^{-3}. \tag{6.2}$$

Furthermore, for every $\xi = (\xi_k) \in S^{n-1}$ introduce the sum of four maximal squares,

$$\Lambda(\xi) = \max_{k_1<k_2<k_3<k_4} |\xi_{k_1}|^2 + \dots + |\xi_{k_4}|^2.$$

We try to show that the functional V is Lipschitz on the sphere in order to apply the concentration inequality (CI). Unfortunately, our estimate for the derivative V' works only for ξ with $\Lambda(\xi)$ not approaching 1.

The partial derivative of V by ξ_j is equal to

$$V_j'(\xi) = \int_0^\infty t\ \bar\gamma'(\xi_j t) \prod_{k\le n, k\ne j} \bar\gamma(\xi_k t) dt.$$

Using an elementary estimate

$$|\bar\gamma'(\tau)| \le D_q|\tau|, \tag{6.3}$$

where $D_\infty = 1/3$ and $D_q = \Gamma(3/q)/\Gamma(1/q)$ for $0 < q < \infty$, we get

$$|V_j'(\xi)| \le D_q|\xi_j| \int_0^\infty t^2 \prod_{k\le n, k\ne j} |\bar\gamma(\xi_k t)| dt. \tag{6.4}$$

Here is the key estimate for the integral in the right-hand side of (6.4).

Lemma 6.3. *There exist $n_0 = n_0(\bar\gamma), C = C(\bar\gamma)$ so that for all $n > n_0$, all $j \le n$, and all $\xi \in S^{n-1}$ satisfying $\Lambda(\xi) \le 1/3$ one has $|V(\xi)| \le C$ and*

$$\int_0^\infty t^2 \prod_{k\le n, k\ne j} |\bar\gamma(\xi_k t)| dt \le C. \tag{6.5}$$

Proof. Before proceeding with rather technical proof of the general case, let us indicate a short proof for $q \in (0,2)$. In this case $\bar\gamma(\cdot)$ is positive and the function $\log \bar\gamma(\sqrt{\cdot})$ is logarithmically convex, as stated in [K1, Lemma 3]. It follows immediately from the log-convexity that for every $t > 0$ and all positive $\eta_1 \le \xi_1 \le \xi_2 \le \eta_2$ with $\eta_1^2 + \eta_2^2 = \xi_1^2 + \xi_2^2$ one has

$$\bar\gamma(t\xi_1)\bar\gamma(t\xi_2) \le \bar\gamma(t\eta_1)\bar\gamma(t\eta_2).$$

Considering now the products from (6.5) and modifying step by step the pairs of coordinates one can easily show that

$$\prod_{k\le n, k\ne j} \bar\gamma(\xi_k t) \le \prod_{k\le n} \bar\gamma(\eta_k t)$$

where $\eta_1 \ge \cdots \ge \eta_4 \ge 1/6$, $\eta_5 = \cdots = 0$. It follows by (6.1) that

$$\int_0^\infty t^2 \prod_{k\le n, k\ne j} |\bar\gamma(\xi_k t)| dt \le 6^3 I.$$

Now we give a proof of the general case, which does not rely on log-convexity. Without lost of generality, assume that $n > 5$, $j = n$ and $|\xi_1| \le \cdots \le |\xi_{n-1}|$. Let $l = [\frac{3\ln|\xi_{n-4}|}{\ln\alpha}] + 1$, $\kappa = n - 4 - l$, $\tau_* = \frac{\tau_0}{|\xi_\kappa|}$. All these parameters depend on ξ. To ensure that κ is well defined note that

$$|\xi_{n-4}| \ge \frac{1}{n-4} \sum_{k=1}^{n-4} |\xi_k|^2 \ge \frac{1-\Lambda(\xi)}{n-4} \ge \frac{2}{3n}.$$

Hence,

$$l \le \frac{3\ln(3n/2)}{|\ln\alpha|} + 2 \ << \ n$$

and, therefore, κ is positive, say for $n > n_0(\bar\gamma)$. The dependence of n_0 on $\bar\gamma$ comes via the parameter α.

Finally, let $\beta = \beta(\bar\gamma) < 1$ be so small that

$$\sup_{0\le x\le\beta} \frac{3x\ln x}{\ln\alpha} \ \le \ \frac{1}{6}. \tag{6.6}$$

Now consider two cases.

1) If $|\xi_{n-4}| \geq \beta$, we may use the estimate (6.2) and write

$$\int_0^\infty t^2 \prod_{k \leq n-1} |\bar{\gamma}(\xi_k t)| dt \leq \int_0^\infty t^2 \prod_{k=n-4}^{n-1} |\bar{\gamma}(\xi_k t)| dt \leq I |\xi_{n-4}|^{-3} \leq I\beta^{-3}, \tag{6.7}$$

thus proving (6.5).

2) If $|\xi_{n-4}| \leq \beta$, write separate estimates for two integration domains. For the domain $0 \leq t \leq \tau_*$ first note that by the estimate (6.6) and the assumption on $\Lambda(\xi)$

$$\sum_{k=1}^{\kappa} \xi_k^2 = 1 - \sum_{k=\kappa+1}^{n-4} \xi_k^2 - \sum_{k=n-3}^{n} \xi_k^2 \geq 1 - l|\xi_{n-4}| - \Lambda(\xi) \geq 1/2,$$

and then write (applying the definitions of τ_0, σ, τ_*)

$$\int_0^{\tau_*} t^2 \prod_{k \leq n-1} |\bar{\gamma}(\xi_k t)| dt \leq \int_0^{\tau_*} t^2 \prod_{k \leq \kappa} |\bar{\gamma}(\xi_k t)| dt \leq$$

$$\int_0^\infty t^2 \prod_{k \leq \kappa} \exp\{-\sigma^2 \xi_k^2 t^2\} dt \leq \int_0^\infty t^2 \exp\{-\sigma^2 t^2/2\} dt = \sqrt{\pi/2}\sigma^{-3}.$$

For the domain $\tau_* \leq t < \infty$, apply the definition of α, τ_* and the second inequality in (6.7) to see that

$$\int_{\tau_*}^\infty t^2 \prod_{k \leq n-1} |\bar{\gamma}(\xi_k t)| dt \leq \max_{t \geq \tau_*} \prod_{k=\kappa}^{n-5} |\bar{\gamma}(\xi_k t)| \int_0^\infty t^2 \prod_{k=n-4}^{n-1} |\bar{\gamma}(\xi_k t)| dt \leq \alpha^l I |\xi_{n-4}|^{-3}.$$

Moreover, by the definition of l

$$\alpha^l |\xi_{n-4}|^{-3} \leq 1.$$

By summing up two estimates, we get

$$\int_0^\infty t^2 \prod_{k \leq n-1} |\bar{\gamma}(\xi_k t)| dt \leq \sqrt{\pi/2}\sigma^{-3} + I,$$

and we have proved (6.5) in this case either. We omit the proof of the inequality $|V(\xi)| \leq C$ since it follows the same lines but is even easier. □

Note that, in the case of the cube, the inequality $|V(\xi)| \leq C$ was proved in [H] and [B1].

Corollary 6.4. *Under the assumptions of Lemma 6.3, we have the Lipschitz property*

$$|V_j'(\xi)| \le D_q\ C\ . \tag{6.8}$$

Remark 6.5. We could obtain the same result under a weaker assumption $\Lambda(\xi) \le 1-\delta$, but then the constant C depends on δ and tends to infinity when δ tends to zero.

We now finish the **proof of Theorem 6.2.** Note that every sum of squares of coordinates belongs to $Lip(2)$. Hence, the function $\Lambda(\cdot)$ being the maximum of such sums also belongs to $Lip(2)$. Let $\theta(\cdot)$ be a smooth decreasing function on the real line such that $\theta(r) = 1$ for $r \le 1/5$ and $\theta(r) = 0$ for $r \ge 1/4$. Then define a smoothened functional

$$W(\xi) = \theta(\Lambda(\xi))V(\xi).$$

The point is that the first term kills V in the domain where the Lipschitz properties of V are unknown and, moreover, $V = W$ on a big set $\{\xi : \Lambda(\xi) \le 1/5\}$ that contains the set $\{\xi : \max|\xi_k|^2 \le 1/20\}$. It follows from the definition that W belongs locally to a Lipschitz class $Lip(C')$, hence it belongs to this class globally on the sphere. Applying (CI) to W and to the maximum of coordinate functionals $M(\xi) = \max_k |\xi_k| \in Lip(1)$, we observe that for some c, all $n > n_0$ and all r

$$mes\{|V - m_W| > r\} \le mes\{V \ne W\} + mes\{|W - m_W| > r\} \le$$

$$mes\{\xi : M(\xi)^2 > 1/20\} + 2\exp\{-c\,r^2(n-1)\} \le$$

$$2\exp\{-(n-1)(\sqrt{1/20} - m_M)^2/2\} + 2\exp\{-cr^2(n-1)\}. \tag{6.9}$$

It follows from this estimate that

$$\lim_{n\to\infty} |m_V - m_W| = 0.$$

Hence, slightly changing n_0 and c, we can replace in (6.9) m_W by m_V. Since for $r \le 1$ the second term dominates the first one, we get, as claimed,

$$mes\{|V - m_V| > r\} \le 4\exp\{-c\,r^2(n-1)\}. \quad \square$$

Remark 6.6. One can see from explicit formulae in Sections 4 and 5 that the volumes of one-dimensional sections also exhibit the concentration behavior. Therefore, one could conjecture that the same is true for all intermediate dimensions of sections. To address this question, one has to consider the isoperimetry on more complicated Grassman manifolds than the unit sphere, which could be a subject of another research.

Remark 6.7. Another general question arizes naturally from our result. Which systems of bodies $K_n \subset R^n$ exhibit the same strong concentration effect for hyperplane section volumes ?

7. Average volume of non-central sections of the cube.

We investigate here the behavior of the average volume of non-central sections of the cube B^n_∞ and encounter the Gaussian dependence of this volume on the displacement of the slicing hyperplane.

7.1. Approximation of spheric averages by Gaussian averages.

As in Section 6, denote $\bar\gamma(v) = \frac{\sin v}{v}$. Obviously, $|\bar\gamma(v)| \le 1$ and, by (6.3), $|\bar\gamma'(\tau)| \le \frac{|\tau|}{3}$.

Consider the function $\Pi(t,\xi) = \prod_{k=1}^n \bar\gamma(t\xi_k),\ t > 0,\ \xi \in R^n$. Then

$$|\Pi'_t(t,\xi)| \le \sum_{k=1}^n |\bar\gamma'(t\xi)|\ |\xi_k| \prod_{j\neq k} |\bar\gamma(t\xi_j)| \le \frac{t}{3}\sum_{k=1}^n \xi_k^2 = \frac{t\ |\xi|^2}{3}\ . \tag{7.1}$$

In what follows we make use of the "Gaussian" functions $g(\cdot)$ and $h(\cdot)$ introduced in (4.2). Consider the following spheric and Gaussian averages:

$$I_n(t) = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \Pi(t,\xi)d\xi \qquad \text{and} \qquad I_n^G(t) = E_G\Pi(t,\xi) = g(t/\sqrt{n})^n.$$

Note that by (7.1) we have $|I'_n(t)| \le |t|/3$. We show now that $I_n(t)$ and $I_n^G(t)$ are close.

Theorem 7.1. *For every bounded function f on $[0,\infty)$, we have*

$$\lim_{n\to\infty}\left(\int_0^\infty f(t)I_n(t)dt - \int_0^\infty f(t)I_n^G(t)dt\right) = 0.$$

Proof. First, we show that I_n^G is an integral transform of I_n. Using the equality $\Pi(t,r\xi) = \Pi(rt,\xi)$ and representing the Gaussian measure in polar coodinates, we get

$$I_n^G(t) = \int_0^\infty I_n(rt)p_n(r)dr, \tag{7.2}$$

where

$$p_n(r) = \left(\frac{n}{2\pi}\right)^{n/2} |S^{n-1}|\ r^{n-1} e^{-r^2 n/2} \tag{7.3}$$

is the probability density of the random variable $\sqrt{\sum_{k=1}^n X_k^2/n}$ with X_k i.i.d. standard normal. Take any $\delta \in (0,1)$ and recall that $1-\delta < \sqrt{1-\delta} < \sqrt{1+\delta} < 1+\delta$. By our probabilistic interpretation and Chebyshev's inequality,

$$\int_{|r-1|>\delta} p_n(r)dr \le$$

$$P\left\{\sum_{k=1}^n X_k^2/n \le 1-\delta\right\} + P\left\{\sum_{k=1}^n X_k^2/n \ge 1+\delta\right\} \le \frac{Var X^2}{\delta^2 n} = \frac{2}{\delta^2 n}\ .$$

Therefore, by (7.2),

$$|I_n(t) - I_n^G(t)| \le \int_0^\infty |I_n(t) - I_n(rt)| p_n(r)dr \le$$

$$2\int_{|r-1|>\delta} p_n(r)dr + \sup_{(1-\delta)t \le u \le (1+\delta)t} |I_n(t) - I_n(u)| \le$$

$$\frac{4}{\delta^2 n} + 2\delta \sup_{u \le (1+\delta)t} |I_n'(u)| \le \frac{4}{\delta^2 n} + \frac{2\delta(1+\delta)\ t}{3}\ .$$

Letting $\delta = n^{-1/3}$ we have

$$|I_n(t) - I_n^G(t)| \le (4 + \frac{4\ t}{3}) n^{-1/3}. \tag{7.4}$$

(More precise probabilistic tools would yield approximation of the order $n^{-1/2}$).

We see from (7.4) that $I_n(t)$ and $I_n^G(t)$ are asymptotically uniformly close on compacts. We also need some estimates to handle big values of t. Let

$$A_n(t) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} |\Pi(t,\xi)| d\xi \qquad \text{and} \qquad A_n^G(t) = E_G|\Pi(t,\xi)| = h(t/\sqrt{n})^n.$$

Then, as in (7.2),

$$A_n^G(t) = \int_0^\infty A_n(rt) p_n(r)dr.$$

We focus our attention on an integration domain near $r=1$. Let $a_n = 1 - 1/\sqrt{2n}$, $b_n = 1 + 1/\sqrt{2n}$. Then the local central limit theorem suggests (and it is possible to check directly from the definition) that there exists a constant $c > 0$ such that for all n and all $u \in [a_n, b_n]$ we have $p_n(u) \geq c\sqrt{n}$. Then

$$A_n^G(t) \geq c\sqrt{n} \int_{a_n}^{b_n} A_n(rt)dr = \frac{c\sqrt{n}}{t} \int_{a_n t}^{b_n t} A_n(u)du.$$

For every $T > 0$,

$$\int_T^\infty A_n^G(t)dt \geq c\sqrt{n} \int_T^\infty \int_{a_n t}^{b_n t} A_n(u)du \frac{dt}{t} \geq c\sqrt{n} \int_{b_n T}^\infty A_n(u) \int_{u/b_n}^{u/a_n} \frac{dt}{t} du =$$

$$c\sqrt{n}(\ln b_n - \ln a_n) \int_{b_n T}^\infty A_n(u)du \geq c \int_{2T}^\infty A_n(u)du. \tag{7.5}$$

We get

$$\int_T^\infty |I_n(t)|dt \leq \int_T^\infty A_n(t)dt \leq$$

$$c^{-1} \int_{T/2}^\infty A_n^G(t)dt = c^{-1} \int_{T/2}^\infty h(t/\sqrt{n})^n dt.$$

Since we know the behavior of the function h, we can evaluate the latter integral. Let $\delta > 0$ and choose $\sigma = \sigma(h(\cdot), \delta)$ that satisfies

$$|h(\tau)| \leq 1 - \sigma\tau^2, \qquad 0 < \tau < \delta.$$

Introduce also a constant $\beta = \beta(h(\cdot), \delta)$ by

$$\beta = \sup_{\tau \geq \delta} h(\tau) < 1.$$

Then

$$\int_{T/2}^\infty h(t/\sqrt{n})^n dt \leq \int_{T/2}^{\delta\sqrt{n}} (1 - \sigma t^2/n)^n dt + \beta^{n-2} \int_{\delta\sqrt{n}}^\infty h(t/\sqrt{n})^2 dt \leq$$

$$\int_{T/2}^\infty \exp\{-\sigma t^2\}dt + \beta^{n-2}\sqrt{n} \int_0^\infty h(\tau)^2 d\tau. \tag{7.6}$$

Since $\beta < 1$, we get from (7.5) and (7.6) that

$$\lim_{T\to\infty} \limsup_{n\to\infty} \int_T^\infty |I_n(t)|dt = 0. \tag{7.7}$$

The same property of I_n^G is now obvious.

As a consequence of (7.4) and (7.7), we get the statement of the Theorem 7.1. $\square$

7.2. An application to non-central sections

For every $\xi \in S^{n-1}$, denote by

$$A_\xi(r) = vol_{n-1}(B_\infty^n \cap \{r\xi + \xi^\perp\}),\ r \in [0, \infty)$$

the parallel section function of B_∞^n in the direction of ξ.

Let χ be the indicator function of the interval $[-1, 1]$. The Fourier transform of the function $A_\xi(\cdot)$ is equal to

$$\hat{A}_\xi(t) = \int_{-\infty}^{\infty} A_\xi(z) e^{-izt}\ dt =$$

$$\int_{\mathbb{R}^n} \chi(\|x\|_\infty) e^{-i(x,\xi)t}\ dx = \prod_{k=1}^{n} \int_{-\infty}^{\infty} \chi(x_k) e^{-ix_k \xi_k t}\ dt = 2^n \Pi(t, \xi).$$

Inverting the Fourier transform we get

$$A_\xi(r) = \frac{2^n}{2\pi} \int_{-\infty}^{\infty} e^{irt} \Pi(t, \xi) dt = \frac{2^n}{\pi} \int_0^{\infty} \cos(rt) \Pi(t, \xi) dt. \tag{7.8}$$

Respectively, the average volume of sections at distance r from the origin is

$$\frac{1}{|S^{n-1}|} \int_{S^{n-1}} A_\xi(r) d\xi = \frac{2^n}{\pi} \int_0^{\infty} \cos(rt) I_n(t) dt.$$

Using Theorem 7.1 and arguing as in Section 4.1.1, we get

$$\lim_{n\to\infty} \int_0^{\infty} \cos(rt)\ I_n(t) dt = \lim_{n\to\infty} \int_0^{\infty} \cos(rt) I_n^G(t) dt =$$

$$\int_0^{\infty} \cos(rt)\ \exp\{-t^2/6\} dt = \sqrt{3\pi/2} \exp\{-3r^2/2\}.$$

This shows that the average volume of non-central hyperplane sections of B_∞^n at distance r from the origin is asymptotically Gaussian with respect to r. Namely,

$$\lim_{n\to\infty} \frac{1}{vol_n(B_\infty^n)|S^{n-1}|} \int_{S^{n-1}} vol_{n-1}(B_\infty^n \cap (r\xi + \xi^\perp)) d\xi = \sqrt{3/2\pi} \exp\{-3r^2/2\}.$$

Remark 7.2. Having at hand the representation (7.8) for the average volume it is natural to ask for the concentration properties of the volume of non-central section as a function of direction ξ (with fixed displacement r). Without any changes, all the reasonings of Section 6 hold in this case, starting from (7.8) instead of (6.1). We get the same exponential inequality as in Theorem 6.2. Perhaps, one might wish to obtain an inequality which becomes stronger when the displacement parameter r increases.

Remark 7.3. We conjecture that the results of this section also hold for non-central sections of B_q^n with finite q. Unfortunately, in this case we don't see such a convenient starting point, as (7.8).

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