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A correlation inequality for stable random vectors

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Abstract. Let $X_1, ..., X_n$ and $Y_1, ..., Y_n$ be jointly q-stable symmetric random variables, $0 < q \le 2$, so that, for some $k \in \mathbb{N},\ 1 \le k < n$, the vectors $(X_1, ..., X_k)$ and $(X_{k+1}, ..., X_n)$ have the same distributions as $(Y_1, ..., Y_k)$ and $(Y_{k+1}, ..., Y_n)$, respectively, but Y_i and Y_j are independent for every choice of $1 \le i \le k,\ k+1 \le j \le n$. Let $(\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional normed space such that $\|(u,v)\| = \|(u,-v)\|$ for every $u \in \mathbb{R}^k,\ v \in \mathbb{R}^{n-k}$. We prove that, for every $p \in [n-3, n)$, $\mathbb{E}(\|X\|^{-p}) \ge \mathbb{E}(\|Y\|^{-p})$.

1. Introduction

Let $X_1, ..., X_n$ and $Y_1, ..., Y_n$ be jointly q-stable symmetric random variables, $0 < q \le 2$, so that, for some $k \in \mathbb{N},\ 1 \le k < n$, the vectors $(X_1, ..., X_k)$ and $(X_{k+1}, ..., X_n)$ have the same distributions as $(Y_1, ..., Y_k)$ and $(Y_{k+1}, ..., Y_n)$, respectively, but Y_i and Y_j are independent for every choice of $1 \le i \le k,\ k+1 \le j \le n$. Let $B = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional normed space.

The following result was established in [K, Th.4] and later proved by Houdré [H, Remark 2.4] by different methods:

Theorem A. *If $0 < p < n$, $\|x\|^{-p}$ is a positive definite distribution and the norm satisfies a symmetry condition $\|(u,v)\| = \|(u,-v)\|$ for every $u \in \mathbb{R}^k,\ v \in \mathbb{R}^{n-k}$, then $\mathbb{E}(\|X\|^{-p}) \ge \mathbb{E}(\|Y\|^{-p})$.*

Here we consider $\|x\|^{-p}$ as a tempered distribution. Recall that by L.Schwartz's generalization of Bochner's theorem (see [GV, p. 152]), a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is positive definite if and only if its Fourier transform $\hat{f}$ is a positive distribution. The latter means that $\langle \hat{f}, \phi \rangle \ge 0$ for every non-negative test function $\phi \in \mathcal{S}(\mathbb{R}^n)$.

It was shown in [K, Corollary 2(ii)] that if B is a subspace of L_r with $0 < r \le 2$ then $\|x\|^{-p}$ is positive definite for every $0 < p < n$. However, if $B = \ell_r^n,\ 2 < r \le \infty,\ n \ge 3$ then $\|x\|^{-p}$ is positive definite if and only if $p \in [n-3, n)$. In particular, for every $p \in [n-3, n)$, $n \ge 3$

$$\mathbb{E}(\max_{i=1,\dots,n} |X_i|^{-p}) \ge \mathbb{E}(\max_{i=1,\dots,n} |Y_i|^{-p}).$$

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In this article we show that the latter inequality is a part of a more general result:

THEOREM 1. *For every $p \in [n-3, n)$ and every n-dimensional normed space $B = (\mathbb{R}^n, \|\cdot\|)$, $n \geq 3$, whose norm satisfies the symmetry condition $\|(u,v)\| = \|(u,-v)\|$ for each $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$, we have $\mathbb{E}(\|X\|^{-p}) \geq \mathbb{E}(\|Y\|^{-p})$.*

The proof is based on the fact that, for $p \in [n-3, n)$, the distribution $\|x\|^{-p}$ is positive definite for every n-dimensional normed space $B = (\mathbb{R}^n, \|\cdot\|)$. Theorem 1 follows immediately from this result and Theorem A.

2. Proof of Theorem 1

We use methods of convex geometry to prove positive definiteness of powers of the norm. Let $K = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the unit ball of the space B. For every unit vector $\xi \in S^{n-1}$ the parallel section function A_ξ in the direction of ξ is defined as a function on $\mathbb{R}$ so that for each $t \in \mathbb{R}$, $A_\xi(t)$ is the $(n-1)$-dimensional volume of the section of K by the hyperplane perpendicular to ξ and located at the distance t from the origin. We say that the space B is infinitely smooth if the restriction of the norm of B to the unit sphere S^{n-1} belongs to the space $C^\infty(S^{n-1})$ of infinitely differentiable functions on the sphere. If B is infinitely smooth then, for every $\xi \in S^{n-1}$, A_ξ is an infinitely differentiable function in a neighborhood of zero. For $\beta \in (-1, 0)$, the fractional derivative of order β of the function A_ξ at zero is defined by

$$A_\xi^{(\beta)}(0) = \frac{1}{\Gamma(-\beta)} \int_0^\infty t^{-1-\beta} A_\xi(t)\, dt. \tag{1}$$

If $\beta \in (0,2)$, $\beta \neq 1$ then

$$A_\xi^{(\beta)}(0) = \frac{1}{\Gamma(-\beta)} \int_0^\infty t^{-1-\beta} (A_\xi(t) - A_\xi(0))\, dt \tag{2}$$

(note that A_ξ is an even function so its first derivative at zero is equal to zero; for more on fractional derivatives see, for example, [GKS, Section 3]).

Our main tool is the following theorem, which was proved in [GKS, Th.2] in a more general form (for every $\beta \in \mathbb{C}$, $\Re(\beta) > -1$, $\beta \neq n-1$).

THEOREM B. *Let B be an infinitely smooth n-dimensional normed space, K is the unit ball of B, $\beta \in (-1, 2)$, β is not an integer. Then for every $\xi \in S^{n-1}$*

$$A_\xi^{(\beta)}(0) = \frac{\cos\frac{\beta\pi}{2}}{\pi(n-\beta-1)} (\|x\|^{-n+\beta+1})^\wedge(\xi). \tag{3}$$

Note that this result in its general form was used in [GKS] as one of the major ingredients of the solution to the Busemann-Petty problem on sections of convex bodies.

THEOREM 2. *Let $B = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional normed space. Then for every $p \in [n-3, n)$ the distribution $\|x\|^{-p}$ is positive definite.*

PROOF. First assume that B is infinitely smooth and p is not an integer. Put $\beta = n-p-1 \in (-1,2)$. We are going to show that $(\|x\|^{-n+\beta+1})^\wedge$ is a non-negative continuous function on S^{n-1}. Since this function is also homogeneous of degree $-\beta-1$ on $\mathbb{R}^n$, $n \geq 3$, we deduce that it is non-negative and locally integrable on $\mathbb{R}^n$. This would mean, in particular, that $(\|x\|^{-n+\beta+1})^\wedge = (\|x\|^{-p})^\wedge$ is a positive distribution, and $\|x\|^{-p}$ is positive definite.

Since the restriction of the norm to S^{n-1} is infinitely smooth and the volume of every section can be expressed in terms of the norm, it is easily seen that $A_\xi^{(\beta)}(0)$ is a continuous function of $\xi \in S^{n-1}$. By Theorem B, the restriction of the function $(\|x\|^{-n+\beta+1})^\wedge$ to the sphere is continuous on S^{n-1}.

Let us show that $(\|x\|^{-n+\beta+1})^\wedge$ is a non-negative function. First let $p \in (n-1,n)$. Then $\beta \in (-1,0)$, so $\Gamma(-\beta) > 0$ and, by (1), $A_\xi^{(\beta)}(0) > 0$ for every $\xi \in S^{n-1}$. Also $\cos\frac{\beta\pi}{2} > 0$, so (3) implies non-negativity.

If $p \in (n-2,n-1)$ then $\beta \in (0,1)$, so $\Gamma(-\beta) < 0$. But, since the unit ball K of the space B is a convex body, the function A_ξ has maximum at zero (the central section has maximal volume among all sections perpendicular to ξ; this follows for example from the Brunn-Minkowski theorem, see [S, Th. 6.1.1].) Therefore, the integral in (2) is less or equal to zero, and again $A_\xi^{(\beta)}(0) \geq 0$ for every $\xi \in S^{n-1}$. Also $\cos\frac{\beta\pi}{2} > 0$, so the result follows from (3).

If $p \in (n-3,n-2)$ then $\beta \in (1,2)$, so $\Gamma(-\beta) > 0$. The integral in (2) is less or equal to zero for the same reason as in the case $\beta \in (0,1)$, so $A_\xi^{(\beta)}(0) \leq 0$ for every $\xi \in S^{n-1}$. But now $\cos\frac{\beta\pi}{2} < 0$.

Now we have to free ourselves from the restrictions imposed in the beginning of the proof.

Suppose that B is not infinitely smooth. We can approximate the unit ball K of B in the Hausdorff metric by infinitely smooth convex bodies K_m, $m \in \mathbb{N}$ so that $K_m \subset K$ for every m. Let $\|\cdot\|_m$ be the norm on $\mathbb{R}^n$ with the unit ball K_m. Since $p < n$, the functions $\|x\|_m^{-p}$ are locally integrable on $\mathbb{R}^n$. Hence, for every test function ϕ, the functions $\|x\|_m^{-p}|\hat{\phi}(x)|$ are integrable on $\mathbb{R}^n$. Also these functions are majorated by an integrable function $\|x\|^{-p}|\hat{\phi}(x)|$. By definition of the Fourier transform of distributions and the dominated convergence theorem, for every non-negative test function ϕ and every $p \in [n-3,n)$,

$$\langle(\|x\|^{-p})^\wedge, \phi\rangle = \int_{\mathbb{R}^n} \|x\|^{-p}\hat{\phi}(x)\ dx =$$

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} \|x\|_m^{-p}\hat{\phi}(x)\ dx = \lim_{m\to\infty} \langle(\|x\|_m^{-p})^\wedge, \phi\rangle \geq 0$$

because we have already proved that the distributions $\|x\|_m^{-p}$ are positive definite.

Finally, let us show that the statement of Theorem 2 is true for $p = n-3, n-2, n-1$. Suppose that $0 < p < n$ and p_i is a sequence of numbers that are not integers, belong to $[n-3,n)$ and $\lim_{i\to\infty} p_i = p$. We can assume that there exists $\epsilon > 0$ so that $0 < p_i < p+\epsilon < n$ for every i. Fix a non-negative test function ϕ. Then for every $i \in \mathbb{N}$ we have $\langle(\|x\|^{-p_i})^\wedge, \phi\rangle \geq 0$. Define a function g on $\mathbb{R}^n$ by $g(x) = \|x\|^{-p-\epsilon}|\hat{\phi}(x)|$ if $\|x\| \leq 1$, and $g(x) = |\hat{\phi}(x)|$ if $\|x\| > 1$. Since $\|x\|^{-p-\epsilon}$ is a locally integrable function, the function g is integrable on $\mathbb{R}^n$ and, for every $i \in \mathbb{N}$, $x \in \mathbb{R}^n$, we have $g(x) \geq \|x\|^{-p_i}|\hat{\phi}(x)|$. By the dominated convergence theorem,

$$\langle (\|x\|^{-p})^\wedge, \phi \rangle = \int_{\mathbb{R}^n} \|x\|^{-p} \hat{\phi}(x)\ dx =$$

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} \|x\|^{-p_i} \hat{\phi}(x)\ dx = \lim_{i\to\infty} \langle (\|x\|^{-p_i})^\wedge, \phi \rangle \geq 0,$$

so $(\|x\|^{-p})^\wedge$ is a positive distribution, since we have already proved that $(\|x\|^{-p_i})^\wedge$ is positive for every i. □

Theorem 1 immediately follows from Theorems A and 2. If $n = 2$ the statement of Theorem 1 remains valid for $p \in (0,2)$, and the inequality for the expectations reverses if $p \in (-\min(1,q), 0)$. To see that, note that every two-dimensional normed space embeds isometrically in L_1, and use [K, Corollary 2(ii)] and [K, Proposition 1]. Also, note that if $p \geq n$ in Theorem 1, then the function $\|x\|^{-p}$ is not locally integrable, and the expectations do not exist.

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