ASPECTS OF THE ISOMETRIC THEORY OF BANACH SPACES

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1. Introduction.

The isometric theory of Banach spaces was born and developed in inseparable connection with other areas of the Banach space theory. Many articles of this Handbook contain isometric and almost isometric results and problems. A remarkable result of Ball [Ba2] on the maximal volume of hyperplane sections of the unit cube in $\mathbb{R}^n$, Burkholder's martingale inequalities with exact constants [Bu], exact probabilistic estimates are just a few examples. Isometric and almost isometric operators appear in numerous situations. In spite of this close connection with the rest of the Banach space theory, the isometric theory has its specific features and methods. The goal of this article is to show how these methods work in different settings and to emphasize connections between the isometric theory and other areas of mathematics, such as Fourier analysis, geometry, probability and combinatorics.

The classical direction, initiated by the work of Banach in the early 30's, is the characterization of the isometries of Banach spaces. Numerous results in this direction can be found in the survey [FJ]. Surjective isometries of Banach spaces are now well understood in a long sequence of results culminating in complete characterization of such isometries for all rearrangement invariant spaces by Zaidenberg [Za] (complex case) and Kalton, Randrianantoanina [KR]. The characterization of injective isometries is far from being complete, though recent results by Carothers, Haydon, Lin [CHL] (for Lorentz spaces) and Randrianantoanina [Ran2] (Orlicz spaces) can be considered as a successful start. However, the ultimate goal of this direction of the isometric theory, to our opinion, is to generalize to other classes of spaces the extension method for L_p-isometries discovered in the 70's by Plotkin [P1-P5] and, independently, Rudin [Ru]. In Section 2, we describe this method and its connections with Fourier analysis, potential theory and probability.

In Section 3, we present some other results that belong to the intersection of the isometric theory, harmonic analysis, probability and combinatorics. The connections between positive definite functions, stable measures and isometric embeddings of Banach spaces into L_p were discovered in the 30's by P.Lévy. Since then,

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The first author was supported in part by the NSF grant DMS-9531594 these connections have been under intensive investigation. We discuss recent developments, in particular, a solution to Schoenberg's problem on positive definite functions and its generalizations. This section also includes recent results explaining the difference in the isometric structure of the spaces L_p and ℓ_p, and showing the special place of the spaces L_p with p being an even integer. We explain recent results of Delbaen, Jarchow and Pelczynski [DJP] when subspaces X of L_p imbed isometrically into ℓ_p or ℓ_p^N; the results differ for $p \in 2\mathbb{N}$ and $p \notin 2\mathbb{N}$. In the case of $X = \ell_2^n$ and $p \in 2\mathbb{N}$, we study concrete imbeddings into ℓ_p^N, with small $N = N(n,p)$.

Recently discovered connections between convexity and the Fourier transform are subject of Section 4 and are explained there through a complete analytic solution to the Busemann-Petty problem on section of convex bodies, which was considered one of the most important isometric questions in convexity.

An almost isometric counterpart of the problem of imbedding of subspaces of L_p into ℓ_p studied in Section 3 is considered in Section 5 for $p = 1$: Given $\epsilon > 0$, when can finite dimensional spaces $X \subseteq L_1$ be $(1+\epsilon)$-isomorphically imbedded into ℓ_1^N, $N = N(X,\epsilon)$? This is the dual setting of the problem of approximating zonoids by zonotopes which was studied by various authors, e.g. Bourgain, Lindenstrauss, Milman, Matousek, and Talagrand. The best known estimates for $N(X,\epsilon)$ as a function of ϵ are given in Section 5.

In Section 6 we derive precise upper bounds for the projection constants of general finite dimensional Banach spaces which improve the square root of dimension estimate of Kadec-Snobar. In the case of symmetric spaces better bounds hold. The examples, which show that these estimates are almost exact, are related to the techniques used to imbed ℓ_2^n into ℓ_4^N.

2. EXTENSION OF ISOMETRIES IN L_p-SPACES.

The extension method for L_p-isometries provides an effective tool for characterization of the isometries of subspaces of L_p, and it also has different applications to other areas of functional analysis, probability and harmonic analysis. This method is based on three principles: the Uniqueness Theorem for measures, the Equimeasurability Theorem and the Extension Theorem. The first versions of these principles appeared in early 70's in the papers of Plotkin [P1-P4], and in 1976 all three results were published in their final form in [P5]. The Uniqueness and Equimeasurability Theorems were discovered independently by Rudin [Ru] in 1976. The Extension Theorem was proved independently by Hardin [Ha1] in 1981.

We start with a simple proof of the Uniqueness Theorem.

Uniqueness Theorem. *Let $p > 0$, $p \notin 2\mathbb{N}$, $C \in \mathbb{R}$, μ and ν are finite Borel measures on $\mathbb{R}$ so that, for every $a \in \mathbb{R}$,*

$$\int_{\mathbb{R}} |x-a|^p \ d\mu(x) + C = \int_{\mathbb{R}} |x-a|^p \ d\nu(x) \ < \infty \tag{2.1}$$

Then $\mu = \nu$ and $C = 0$.

Proof. Let $\phi \in \mathcal{S}(\mathbb{R})$ be a test function supported outside of the origin. Then $\int_{\mathbb{R}} \hat{\phi}(t)\, dt = 2\pi\phi(0) = 0$. Since p is not an even integer, for every fixed $x \in \mathbb{R}$ the Fourier transform (in the sense of distributions) of the function $a \mapsto |x-a|^p$ is equal to $(|x-a|^p)^\wedge(t) = c_p|t|^{-1-p}\exp(-itx)$, where $c_p = (2^{p+1}\pi^{1/2}\Gamma((p+1)/2))/\Gamma(-p/2)$ (see [GS,p. 173]). Using this fact and Fubuni's theorem, we get

$$\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |x-a|^p \, d\mu(x) \Big) \, \hat{\phi}(a) \, da = c_p \int_{\mathbb{R}} \langle |t|^{-1-p} \exp(-itx), \phi(t) \rangle \, d\mu(x) =$$

$$c_p \int_{\mathbb{R}} |t|^{-1-p} \phi(t) \Big(\int_{\mathbb{R}} \exp(-itx) \, d\mu(x) \Big) \, dt = c_p \int_{\mathbb{R}} |t|^{-1-p} \hat{\mu}(t) \phi(t) \, dt. \tag{2.2}$$

Integrating both sides of (2.1) with respect to $\hat{\phi}(a)\, da$ and using (2.2), we get

$$\int_{\mathbb{R}} |t|^{-1-p} \hat{\mu}(t) \phi(t) \, dt = \int_{\mathbb{R}} |t|^{-1-p} \hat{\nu}(t) \phi(t) \, dt$$

for every test function ϕ supported outside of the origin. Therefore, continuous functions $\hat{\mu}$ and $\hat{\nu}$ are equal everywhere on $\mathbb{R}$, which implies that $\mu = \nu$. $\square$

If p is an even integer, the statement of the Uniqueness Theorem is no longer true, because in this case the equalities (2.1) imply only that a finite number of moments of the measures μ and ν are equal.

The proofs of the Equimesurability and Extension Theorems below are essentially the same as in [P5]. Recall that the distribution of a function $f \in L_p(\Omega, \sigma)$ is the measure μ on $\mathbb{R}$ defined by $\mu(A) = \sigma\{\omega \in \Omega : f(\omega) \in A\}$ for every Borel set $A \subset \mathbb{R}$.

Equimeasurability Theorem. *Let $p > 0$, $p \notin 2\mathbb{N}$, (Ω_1, σ_1) and (Ω_2, σ_2) are probability spaces, H is a subspace of $L_p(\Omega_1, \sigma_1)$ containing the constant function $1(\omega) = 1$, and $T : H \mapsto L_p(\Omega_2, \sigma_2)$ is a linear isometry such that $T1 = 1$. Then every function $f \in H$ and its image Tf have equal distributions with respect to the measures σ_1 and σ_2, correspondingly.*

Proof. Let μ and ν be the distributions of f and Tf. Since $T1 = 1$, for every $a \in \mathbb{R}$

$$\int_{\Omega_1} |f(\omega) - a|^p \, d\sigma_1(\omega) = \int_{\Omega_2} |Tf(\omega) - a|^p \, d\sigma_2(\omega).$$

Making the changes of variables $x = f(\omega)$ and $x = Tf(\omega)$ in both integrals, we get (2.1). By the Uniqueness Theorem, $\mu = \nu$. $\square$

In order to prove the Extension Theorem, we need the following

Lemma 2.1. *We remain under the conditions of the Equimeasurability Theorem, except for $T1 = 1$ which is no longer required. Then for every function $f \in H$, the support of Tf is contained in the support of $T1$. Besides, for any $k \in \mathbb{N}$ the* *equal to the joint distribution of the functions* $Tf_1/T1, ..., Tf_k/T1$ *with respect to the measure* $|T1|^p \, d\sigma_2$.

Proof. Let $f \in H$. Then, for every $a \in \mathbb{R}$

$$\int_{\Omega_1} |f(\omega) - a|^p \, d\sigma_1(\omega) = \int_{\Omega_2} |Tf(\omega) - aT1(\omega)|^p \, d\sigma_1(\omega) =$$

$$\int_{\{T1(\omega)\neq 0\}} |\frac{Tf}{T1}(\omega) - a|^p |T1(\omega)|^p \, d\sigma_2(\omega) + \int_{\{T1(\omega)=0\}} |Tf(\omega)|^p \, d\sigma_2(\omega).$$

If we denote the second summand in the right-hand side by C, and pass to the distributions of the functions f and $Tf/T1$ with respect to the measures $d\sigma_1$ and $|T1|^p d\sigma_2$, correspondingly, we find ourselves in the situation of the Uniqueness Theorem. Hence $C = 0$, and the function Tf is equal to zero σ_2-almost everywhere on $\{T1(\omega) = 0\}$, which means that the support of Tf is contained in the support of $T1$.

Now the operator $\tilde{T} : H \mapsto L_p(\Omega_2, |T1|^p \, d\sigma_2)$ defined by $\tilde{T}f = Tf/T1$ is a linear isometry satisfying the condition $\tilde{T}1 = 1$. Denote by μ and ν the measures on $\mathbb{R}^k$, which are the joint distributions from the statement of lemma. By the Equimeasurability Theorem, for every $b = (b_1, ..., b_k) \in \mathbb{R}^k$, the functions $\sum b_i f_i$ and $\sum b_i Tf_i/T1$ have equal distributions with respect to σ_1 and $|T1|^p \, d\sigma_2$, respectively. Therefore, the Fourier transforms of the measures μ and ν are equal at the point b. Since b is arbitrary, the result follows. $\square$

Extension Theorem. *Under the conditions of the Equimeasurability Theorem, except for* $T1 = 1$ *which is no longer required, there exists a linear isometry* $T' : L_p(\Omega_1, \mathcal{A}, \sigma_1) \mapsto L_p(\Omega_2, \sigma_2)$ *so that* $T'|_H = T$, *where* $\mathcal{A}$ *is the smallest* σ*-algebra of subsets of* Ω_1 *making all functions from* H *measurable.*

Proof. It is easily seen that the space $L_p(\Omega_1, \mathcal{A}, \sigma_1)$ is the closure in $L_p(\Omega_1, \sigma_1)$ of the set of functions that can be represented in the form $B(f_1, ..., f_k)$, where $k \in \mathbb{N}$, B is a Borel function on $\mathbb{R}^k$ and $f_1, ..., f_k$ belong to H. Define an operator $T' : L_p(\Omega_1, \mathcal{A}, \sigma_1) \mapsto L_p(\Omega_2, \sigma_2)$ by

$$T'(B(f_1, ..., f_k)) = T1 \cdot B(Tf_1/T1, ..., Tf_k/T1)$$

for every choice of $k, B, f_1, ..., f_k$. The operator T' is well-defined. In fact, suppose that $B_1(f_1, ..., f_k) = B_2(g_1, ..., g_m)$ for $k, m \in \mathbb{N}$, Borel functions B_1, B_2 on $\mathbb{R}^k$ and $\mathbb{R}^m$, and functions $f_1, ..., f_k, g_1, ..., g_m \in H$. By Lemma 2.1 applied to the functions $f_1, ..., f_k, g_1, ..., g_m$,

$$\int_{\Omega_1} |B_1(f_1, ..., f_k) - B_2(g_1, ..., g_m)|^p \, d\sigma_1 =$$

$$\int |B_1(Tf_1/T1, ..., Tf_k/T1) - B_2(Tg_1/T1, ..., Tg_m/T1)|^p |T1|^p \, d\sigma_2.$$

Therefore, $T1\cdot B_1(Tf_1/T1,...,Tf_k/T1)=T1\cdot B_2(Tg_1/T1,...,Tg_m/T1)$ as elements of $L_p(\Omega_2,\sigma_2)$. The same argument (with $B_2=0$) shows that T' is a linear isometry, and its restriction to H is equal to T. $\square$

The Extension Theorem applies only to subspaces of L_p containing constant functions (unital subspaces). However, the general case can be reduced to the unital case by changing the density, since every separable subspace of L_p contains a function with maximal support (see, for example, [An]).

Let us show a typical application of the Extension Theorem. Let E be a domain in $\mathbb{R}^n$. Denote by $L^h_p(E)$ the subspace of $L_p(E)$ (with Lebesque measure) which consists of harmonic functions. The following result was proved in [P4].

Theorem 2.2. *Let $p>0$, $p\notin 2\mathbb{N}$, E_1 and E_2 be two domains in $\mathbb{R}^n$. The spaces $L^h_p(E_1)$ and $L^h_p(E_2)$ can be isometric only in one of the following situations: (i) the domains E_1 and E_2 are similar; (ii) $p=2n/(n-1)$ and the domains E_1 and E_2 coincide up to the composition of an inversion and a similarity.*

Sketch of Proof. The smallest σ-algebra $\mathcal{A}$, making all functions from $L^h_p(E_1)$ measurable, is the σ-algebra of all Borel subsets of E_1. By the Extension Theorem, every linear isometry T from $L^h_p(E_1)$ to $L^h_p(E_2)$ can be extended to an isometry T' from the whole space $L_p(E_1)$ to $L_p(E_2)$. Using the classical characterization of the isometries between L_p-spaces (see [La]), one can show that T' is a composition operator, i.e. $T'f(\omega)=T1(\omega)f(\tau(\omega))$, where τ is a measurable mapping from E_2 to the closure of E_1. Now our problem is reduced to characterization of all composition operators mapping harmonic functions to harmonic functions. Direct differentiation shows that this happens only if τ is a conformal mapping. If the dimension $n\geq 3$, one can apply Liouville's theorem that every conformal mapping is the composition of a similarity and an inversion. It can be shown that an inversion can be included only when $p=2n/(n-1)$. The case $n=2$ must be considered separately. $\square$

The Uniqueness Theorem was generalized in [GK], [Lin3] to the case of measures on certain finite and infinite dimensional normed spaces. Suppose that $p>0$, E is a separable Banach space, and μ,ν are finite Borel measures on E so that, for every $a\in E$,

$$g(a)=\int_E \|x-a\|^p d\mu(x)=\int_E \|x-a\|^p d\nu(x)<\infty. \tag{2.3}$$

We say that p is an exceptional exponent for E if (2.3) does not necessarily imply that $\mu=\nu$. The exceptional exponents for the space L_q, $q>0$ are those for which p/q is an integer, and, in the case of the n-dimensional space ℓ^n_q, besides the condition $\frac{p}{q}\in\mathbb{N}$, one of the following must be satisfied: i) $p/q<n$, ii) q is an even integer, iii) q and $\frac{p}{q}-n$ are both odd integers. If $E=C(K)$, where K is a compact without isolated points, then there are no exceptional exponents. For the

complex space ℓ_∞^n exceptional exponents are even integers, and in the real case p is exceptional if and only if $n+p$ is an odd integer. Other results in this direction include formulae for calculating the measure μ out of the potential g (see [Ko3]), uniqueness theorems for Gaussian measures (see [Lew], [Lin4], [Lin5]).

Applications and generalizations of the extension method also include [Al], [DJP] (see Section 2), [GW], [Ha2], [HP], [Ko2], [Ko5], [Lin1], [Lin2], [Lop], [LP1], [LP2], [Lus], [Mat], [Ran1], [Ro], [St], [Va], [ZKK].

3. POSITIVE DEFINITE FUNCTIONS AND ISOMETRIC EMBEDDING OF NORMED SPACES IN L_p-SPACES.

The problem of how to check whether a given Banach space is isometric to a subspace of L_p was posed by P. Lévy [Le] in 1937. A well-known fact is that a Banach space embeds isometrically in a Hilbert space if and only if its norm satisfies the parallelogram law [Fr], [JN]. Neyman [Ne] (see also [We] and [Bu]) proved that subspaces of L_p with $p \neq 2$ can not be characterized by a finite number of equations or inequalities.

There is a close connection between isometric embeddings in L_p and positive definite functions. Recall that a complex valued function f defined on $\mathbb{R}^n$ is called positive definite if, for every finite sequence $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ and every choice of complex numbers $\{c_i\}_{i=1}^m$,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_i \bar{c}_j f(x_i - x_j) \geq 0.$$

By Bochner's theorem, every continuous positive definite function on $\mathbb{R}^n$ is the Fourier transform of a positive finite measure. It was known already to P.Lévy [Le] that if $B = (\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p, $0 < p \leq 2$ then $\exp(-\|x\|^p)$ is a positive definite function, and, hence, is the Fourier transform of a symmetric p-stable random vector. The actual equivalence of the two notions was discovered by Bretagnolle, Dacunha-Castelle and Krivine [BDK] who proved that, for $0 < p \leq 2$, a Banach space B is isometric to a subspace of L_p if and only if the function $\exp(-\|x\|^p)$ is positive definite. For different proofs and generalizations see [BDK], [Kr], [AMM], [MR], [JMST], [WW] and [BL, Chapter 11].

Bretagnolle, Dacunha-Castelle and Krivine used their result to prove that the space L_q embeds isometrically in L_p when $0 < p < q \leq 2$. However, it is more difficult to check whether $\exp(-\|x\|^p)$ is positive definite for other norms. For example, the following problem posed by Schoenberg in 1938 (see [Sch]) was open for more than fifty years: for which $p \in (0,2)$ is the function $\exp(-\|x\|_q^p)$ positive definite, where $\|x\|_q$ is the norm of the space ℓ_q^n, $2 < q \leq \infty$? An equivalent formulation asks whether the spaces ℓ_q^n embed in L_p with $0 < p \leq 2$. After Dor [Do] answered the question for $p \in [1,2)$, the complete solution (including $p \in (0,1)$)

$\ell_q^n,\ 2 < q \le \infty$ do not embed isometrically in $L_p,\ 0 < p < 2$, and for $n = 2$ the embedding exists if and only if $0 < p \le 1$. The fact that every two-dimensional normed space embeds in L_p for every $p \in (0,1]$ was established much earlier (see [Fe], [He], [Li]).

We now present the solution to the case $2 < q < \infty$ of Schoenberg's problem from [Ko1]. Parts of this solution will also be used in Section 4. The solution is based on the following Fourier transform characterization of subspaces of L_p (see [Ko4], [Ko6] for details). In [DK] this criterion was also applied to Lorentz spaces.

Theorem 3.1. *Let $p > 0,\ p \notin 2\mathbb{N}$ and let $(\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional subspace of L_p. Then the Fourier transform of the function $\Gamma(-p/2)\|x\|^p$ is a positive distribution on $\mathbb{R}^n \setminus \{0\}$.*

Proof. We have to prove that, for every non-negative even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ supported in $\mathbb{R}^n \setminus \{0\}$, $\langle \Gamma(-p/2)(\|x\|^p)^\wedge, \phi\rangle \ge 0$. A simple fact going back to P.Lévy [Le] is that an n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_p,\ p > 0$ if and only if there exists a finite Borel measure μ on the unit sphere S^{n-1} in $\mathbb{R}^n$ so that, for every $x \in \mathbb{R}^n$, $\|x\|^p = \int_{S^{n-1}} |(x,\xi)|^p\, d\mu(\xi)$. Therefore,

$$\langle (\|x\|^p)^\wedge, \phi\rangle = \langle \|x\|^p, \hat{\phi}\rangle = \int_{S^{n-1}} \int_{\mathbb{R}^n} |(x,\xi)|^p \hat{\phi}(x)\, dx\, d\mu(\xi) =$$

$$\int_{S^{n-1}} \langle |t|^p, \int_{(x,\xi)=t} \hat{\phi}(x)\, dx\rangle\, d\mu(\xi) = (2\pi)^{n-1} c_p \int_{S^{n-1}} \int_{\mathbb{R}} |t|^{-1-p} \phi(t\xi)\, dt\, d\mu(\xi) \ge 0,$$

where c_p is the constant from the proof of the Uniqueness Theorem in Section 2, and we use the fact that $(2\pi)^n \phi(t\xi)$ is the Fourier transform of the function $t \mapsto \int_{(x,\xi)=t} \hat{\phi}(x)\, dx$. Since $\Gamma(-p/2)c_p \ge 0$, the result follows. $\square$

Let $\|x\|_q = (|x_1|^q + \cdots + |x_n|^q)^{1/q}$ be the norm of the space $\ell_q^n,\ 2 < q < \infty$. Denote by γ_q the Fourier transform of the function $z \to \exp(-|z|^q),\ z \in \mathbb{R}$. The properties of the functions γ_q were studied by Polya [Po]. In particular, if q is not an even integer, the function $\gamma_q(t)$ behaves at infinity like $|t|^{-q-1}$. Namely (see [PS, Part 3, Problem 154]),

$$\lim_{t\to\infty} t^{1+q}\gamma_q(t) = 2\Gamma(q+1)\sin(\pi q/2).$$

If q is an even integer, the function γ_q decreases exponentially at infinity. The integral

$$S_q(\alpha) = \int_{\mathbb{R}} |t|^\alpha \gamma_q(t)\, dt$$

converges absolutely for every $\alpha \in (-1, q)$. These moments can easily be calculated (see [Zo]; α is not an even integer):

$$S_q(\alpha) = 2^{\alpha+2}\pi^{1/2}\Gamma(-\alpha/q)\Gamma((\alpha+1)/2)/(q\Gamma(-\alpha/2)).$$

Clearly, the moment $S_q(\alpha)$ is positive if $\alpha \in (-1,0) \cup (0,2)$, and the moment is negative if $\alpha \in (2, \min(q,4))$.

Lemma 3.2. *Let* $q > 0$, $n \in \mathbb{N}$, $-n < \beta < qn$, $\beta/q \notin \mathbb{N} \cup \{0\}$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, $\xi_k \neq 0$, $1 \leq k \leq n$. *Then*

$$(\|x\|_q^\beta)^\wedge(\xi) = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_q(t\xi_k)\, dt.$$

Proof. Assume that $-1 < \beta < 0$. By the definition of the Γ–function

$$(|x_1|^q + \cdots + |x_n|^p)^{\beta/q} = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty y^{-1-\beta} \exp(-y^q(|x_1|^q + \cdots + |x_n|^q))\, dy.$$

For every fixed $y > 0$, the Fourier transform of the function $x \to \exp(-y^q(|x_1|^q + \cdots + |x_n|^q))$ at any point $\xi \in \mathbb{R}^n$ is equal to $y^{-n} \prod_{k=1}^n \gamma_q(\xi_k/y)$. Making the change of variables $t = 1/y$ we get

$$((|x_1|^q + \cdots + |x_n|^q)^{\beta/q})^\wedge(\xi) = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty y^{-n-\beta-1} \prod_{k=1}^n \gamma_q(\xi_k/y)\, dy =$$

$$= \frac{q}{\Gamma(-\beta/q)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_q(t\xi_k)\, dt. \tag{3.1}$$

The latter integral converges if $-n < \beta < qn$ since the function $t \to \prod_{k=1}^n \gamma_q(t\xi_k)$ decreases at infinity like t^{-n-nq} (recall that $\xi_k \neq 0$, $1 \leq k \leq n$.)

If β is allowed to assume complex values then both sides of (3.1) are analytic distributions of β in the domain $\{-n < Re\beta < nq,\ \beta/q \notin \mathbb{N} \cup \{0\}\}$. These two functions admit unique analytic continuation from the interval $(-1, 0)$. Thus the equality (3.1) remains valid for all $\beta \in (-n, qn)$, $\beta/q \notin \mathbb{N} \cup \{0\}$ (see [GS] for details of analytic continuation in such situations). $\square$

Lemma 3.3. *Let* $q > 2$. *If* $n > 3$, $p \in (-n+3, 0)$, *or* $n = 3$, $p \in (0, 2)$, *or* $n = 2$, $p \in (1, 2)$, *then the Fourier transform of the distribution* $\|x\|_q^p$ *is a sign-changing function on* S^{n-1}.

Proof. By Lemma 3.2 and properties of the moments $S_q(\alpha)$, the integral

$$I(\alpha_1, \dots, \alpha_{n-1}) = \int_{\mathbb{R}} |\xi_1|^{\alpha_1} \dots |\xi_{n-1}|^{\alpha_{n-1}} (\|x\|_q^p)^\wedge(\xi_1, ..., \xi_{n-1}, 1)\, d\xi_1 \dots d\xi_{n-1} =$$

$$S_q(\alpha_1) \dots S_q(\alpha_{n-1}) S_q(-\alpha_1 - \cdots - \alpha_{n-1} - p)$$

converges absolutely if the numbers $\alpha_1, \dots, \alpha_{n-1}, -\alpha_1 - \cdots - \alpha_{n-1} + p$ belong to the interval $(-1, q)$. Choosing $\alpha_k \in (-1, 0)$ for every $k = 1, ..., n-1$, we have the moments $S_q(\alpha_k)$, $k = 1, ..., n-1$ positive, and we can make $-\alpha_1 - \cdots - \alpha_{n-1} + p$ equal to any number from $(p, n-1+p) \cap (-1, q)$. Because of our conditions on p, this interval contains a neighborhood of 2, and, since the moment function S_q changes its sign at 2, we can make the integral $I(\alpha_1, \dots, \alpha_{n-1})$ positive for one choice of α's and negative for another choice. This means that the (homogeneous of degree $-n-p$) function $(\|x\|_q^p)^\wedge$ is sign-changing. $\square$

Theorem 3.1 and the cases $n=3$ and $n=2$ of Lemma 3.3 imply that the spaces ℓ_q^n with $n \geq 3,\ 2<q<\infty$ can not embed in $L_p,\ 0<p\leq 2$, and that the spaces ℓ_q^2 with $2<q<\infty$ can not embed in L_p with $1<p<2$, which answers Schoenberg's question.

Soon after the paper [Ko1] appeared, Zastavnyi [Za1],[Za2] and Lisitsky [Lis] independently proved a more general result that there are no non-trivial positive definite functions of the form $f(\|\cdot\|_q)$, where $q>2,\ n\geq 3$. Note that for $q=\infty$ a similar result was established earlier in [Mi2]. Zastavnyi's result is even stronger:

Theorem 3.4. *Let E be a three dimensional normed space with a basis e_1, e_2, e_3 such that the function $x \mapsto \|xe_1+ye_2+ze_3\|$ is differentiable at the point $x=1$ for almost all $(y,z)\in\mathbb{R}^2$, and suppose that the function*

$$(y,z)\mapsto \|xe_1+ye_2+ze_3\|_x^{'}(1,y,z)/\|e_1+ye_2+ze_3\|,\ \ y,z\in\mathbb{R}$$

belongs to the space $L_1(\mathbb{R}^2)$. Assume that, for some function $f:\mathbb{R}\mapsto\mathbb{R}$, the function $(x,y,z)\mapsto f(\|xe_1+ye_2+ze_3\|)$ is positive definite on $\mathbb{R}^3$. Then f is constant.

Sketch of Proof. First, one can assume without loss of generality that f is differentiable on $(0,\infty)$, $\lim_{t\to 0} tf'(t)=0$, and there exists a constant $C>0$ so that $|tf'(t)|<C$ for every $t>0$. In fact, if this is not the case, one can replace f by a function $F(t)=\int_0^\infty f(ts)g(s)\,ds$, where g is an infinitely differentiable non-negative function with compact support in $\mathbb{R}$.

It is easy to check that the function

$$h(y,z)=\begin{cases}(1-|y|)(1-|z|), & \max\{|y|,|z|\}\leq 1\\ 0, & \text{otherwise}\end{cases}$$

is positive definite on $\mathbb{R}^2$. Let us prove that

$$\psi(t)=\int_{\mathbb{R}^2} f(\|te_1+ye_2+ze_3\|)h(y,z)\ dy\ dz.$$

is positive definite on $\mathbb{R}$. In fact, for every $\epsilon>0$

$$\psi_\epsilon(t,y,z)=f(\|te_1+ye_2+ze_3\|)h(y,z)\exp(-\epsilon|t|)$$

is integrable and positive definite on $\mathbb{R}^3$ (as the product of positive definite functions). Therefore, for every $s\in\mathbb{R}$

$$\hat{\psi}_\epsilon(s,0,0)=\int_{\mathbb{R}}\exp(-ist)\exp(-\epsilon|t|)\psi(t)\ dt\geq 0,$$

which means that the function $\exp(-\epsilon|t|)\psi(t)$ is positive definite on $\mathbb{R}$. Since ϵ is arbitrary, ψ is positive definite on $\mathbb{R}$.

Using the fact that the derivative of the norm by the first coordinate is a homogeneous function of degree 0 and making the change of variables $y = tu,\ z = tv$, we compute the derivative of the function ψ at a point $t \neq 0:$

$$\psi'(t) = \int_{\mathbb{R}^2} f'(\|te_1 + ye_2 + ze_3\|)\|te_1 + ye_2 + ze_3\|'_t(t, y, z)h(y, z)\ dy\ dz =$$

$$t^2 sign(t) \int_{\mathbb{R}^2} f'(|t|\|e_1 + ue_2 + ve_3\|)\|te_1 + ue_2 + ve_3\|'_t(1, u, v)h(tu, tv)\ du\ dv.$$

Multiplying and dividing the expression under the latter integral by $\|e_1 + ue_2 + ve_3\|$, we can use the condition of the theorem and the properties of the function f mentioned in the beginning of the proof to show that $\lim_{t\to 0} \psi'(t)/t = 0$. (Note that this is possible because we have t^2 in front of the integral, which is due to the fact that the dimension of the space is three. This is the point of the proof that fails in dimension 2.) This condition on the derivative of ψ implies that $\psi'(0) = \psi''(0) = 0$. It follows now from [Luk, Th 4.1.1] that ψ is a constant function.

The same is true if we replace the function $h(y, z)$ by the function $n^2 h(ny, nz)$, where $n \in \mathbb{N}$. As $n \to \infty$, the fact that the corresponding functions ψ are constant immediately implies that f is constant. $\square$

Clearly, every normed space of dimension ≥ 3, which is the q-sum with $q > 2$ of two non-trivial normed spaces, satisfies the conditions of Theorem 3.4.

Another necessary condition for isometric embedding in $L_p,\ 0 < p \leq 2$ was given in [KL]. The proof in [KL] is a modification of Zastavnyi's argument.

Theorem 3.5. *Let X be a three dimensional normed space with a normalized basis $\{e_1, e_2, e_3\}$ so that*

1. *For every fixed $(x_2, x_3) \in \mathbb{R}^2 \backslash \{0\}$, the function $x_1 \to \|x_1 e_1 + x_2 e_2 + x_3 e_3\|$ has a continuous second derivative everywhere on $\mathbb{R}$, and*

   $$\|x\|'_{x_1}(0, x_2, x_3) = \|x\|''_{x_1^2}(0, x_2, x_3) = 0,$$

   where $\|x\|'_{x_1}$ and $\|x\|''_{x_1^2}$ stand for the first and second derivatives by x_1, respectively.
2. *There exists a constant K so that for every $x_1 \in \mathbb{R}$ and every $(x_2, x_3) \in \mathbb{R}^2$ with $\|x_2 e_2 + x_3 e_3\| = 1$, one has $\|x\|''_{x_1^2}(x_1, x_2, x_3) \leq K$.*

Then, for every $0 < p \leq 2$, the space X does not embed isometrically in L_p.

Theorem 3.5 applies to some spaces not covered by Zastavnyi's result, for example, to Orlicz spaces ℓ_M^n whose Orlicz function M has continuous second derivative and $M'(0) = M''(0) = 0$.

In 1981, Eaton [Ea] introduced a concept of an isotropic random vector generalizing stable vectors in the following way: a random vector $X = (X_1, ..., X_n)$ is called isotropic if there exists a quasi-norm $\|\cdot\|$ on $\mathbb{R}^n$ so that, for every $a \in \mathbb{R}^n$, the random variables $\sum a_i X_i$ and $\|a\| X_1$ have equal distributions. A simple fact is that a random vector X is isotropic if and only if its characteristic functional (Fourier transform) has the form $f(\|\cdot\|)$, where $\|\cdot\|$ is a quasi-norm on $\mathbb{R}$, and $f : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function. In view of Bochner's theorem, the problem of characterization of isotropic vectors reduces to the study of positive definite functions of the form $f(\|\cdot\|)$. On the other hand, if $f(\|\cdot\|)$ is a positive definite function on $\mathbb{R}^n$, then f is positive definite on $\mathbb{R}$, and is the characteristic function of a finite measure on $\mathbb{R}$. It is easy to show that if this measure has finite moment of order p for some $p > 0$ then the space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p. These results represent another connection between probability, positive definite functions and isometric embeddings in L_p. The classes of positive definite functions of the form $f(\|\cdot\|)$ have been studied by several authors. We refer the reader to surveys [Ko6], [Mi3] for proofs of the results mentioned above and for numerous references. The following question is probably the most important open problem in this direction: is it true that a non-constant positive definite function of the form $f(\|\cdot\|)$ exists only if the corresponding quasi-normed space embeds isometrically in one of the spaces $L_p,\ 0 < p < 2$?

Isometric embeddings of finite dimensional normed spaces in L_1 play important role in convex geometry, due to the fact that the unit ball of every such space is a polar to a zonoid (see Section 5 for more on zonoids). Surveys [SW], [GoW] include different characterizations of zonoids and related results. We would like to mention the following problem posed by Lindenstrauss. Let $d_2(X)$ denote the Banach-Mazur distance between X and the Hilbert space of the same dimension. For each n, consider the number $a_n = \sup\{d_2(X) : X$ and X^* embed isometrically in $L_1, dim X = n\}$. Is it true that $\limsup_{n\to\infty} a_n = 1$?

Note that Schneider [S1] gave the first example of an n-dimensional normed space X such that both X and X^* embed isometrically in L_1 but X is not isometric to ℓ_2^n. This result of Schneider solved a problem going back to Grothendieck [G] (see also [Bol]). Answering a question of Bourgain, Lonke [Lo] showed that in dimensions 3 and 4 there exist non-smooth spaces X with the same property. Schneider's construction was modified in [Ko11] to show that there exist non-Hilbertian normed spaces of any finite dimension, which are isometric to subspaces of L_p with $p < 2$ and of L_q with $q > 2$, simultaneously.

In the following, L_p denotes the space $L_p(0,1)$ with respect to Lebesgue measure. If X is a subspace of L_p, a change of density argument shows that there is a subspace Y of L_p isometric to X and containing constant functions (*unital subspace*). For $X \subset L_p$ it is a natural question whether X embeds isometrically into ℓ_p or not. This is of interest even for $X = \ell_2^n$. A very satisfactory answer to this problem, which depends on whether p is an even integer ($p \in 2\mathbb{N}$) or not, was recently given by Delbaen, Jarchow and Pelczynski [DJP]:

Theorem 3.6. *Let $0 < p < \infty$ and X be a closed subspace of L*

(i) If $p \notin 2\mathbb{N}$, X is isometric to a subspace of ℓ_p if and only if every (equivalently, some) unital subspace $Y \subset L_p$ isometric to X consists of functions with discrete distribution.

(ii) If $p \in 2\mathbb{N}$ and X is finite dimensional, X embeds isometrically in ℓ_p. For $n \in \mathbb{N}$ and $p = 2s \in 2\mathbb{N}$, let

$$N_{\mathbb{R}}(n,p) := \binom{n+p-1}{p} - 1, \; N_{\mathbb{C}}(n,p) := \binom{n+s-1}{s}^2 - 1.$$

Then, more precisely, every n-dimensional subspace of L_p embeds isometrically into ℓ_p^N where $N \le N_{\mathbb{K}}(n,p)$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Here the distribution λ_f of a function f is *discrete* if $\lambda_f = \sum_{j \in \mathbb{N}} a_j \delta_{x_j}$, where $x_j, a_j \in \mathbb{K}$, $a_j \ge 0$, $\sum_{j \in \mathbb{N}} a_j = 1$. Following [DJP], we present an

Idea of the proof of (i). Let $p \notin 2\mathbb{N}$.

(a) Assume $Y \subset L_p$ is a unital subspace isometric to a subspace of ℓ_p, via an isometry $w : Y \mapsto \ell_p$. Let $\gamma = w(1)$, $S := supp(\gamma)$. Then $w(Y)$ can be regarded as a subspace of $\ell_p^S = \{\xi \in \ell_p : \xi_s = 0 \text{ for all } s \notin S\}$. The formula $\nu(B) := \sum_{s \in B} |\gamma(s)|^p$ for $B \subset S$ defines a probability measure on S since $\nu(S) = \|w(1)\|_p^p = 1$. Moreover, define

$$u : \ell_p^S \mapsto L_p(\nu), \; \xi \mapsto \big(\xi(s) sgn(\gamma(s))/\gamma(s)\big)_{s \in S}.$$

This is an isometric isomorphism with $u(\gamma) = 1$. Thus $uw : Y \mapsto L_p(\nu)$ is an isometric isomorphism onto the unital subspace $uw(Y)$ of $L_p(\nu)$ with $uw(1) = 1$. Take $f \in Y$. If $f = z1$ for $z \in \mathbb{K}$, f has discrete distribution. If not, uw maps the 2-dimensional subspace $[1,f] \subset L_p$ onto $[1, uw(f)] \subset L_p(\nu)$ isometrically. By the Equimeasurability Theorem (see Section 2), the distribution λ_f of the function f with respect to Lebesgue measure is equal to the distribution of $uw(f)$ with respect to ν, which is discrete. Therefore, λ_f is discrete.

(b) If $Y \subset L_p$ has finite dimension and $f_1, ..., f_k$ is a basis of functions having discrete distribution, we may write for $n = 1, ..., k$

$$f_n = \sum_{j \in J(n)} z_{j,n} \chi_{C(j,n)},$$

$\big(C(j,n)\big)_{j \in J(n)}$ a partition of $[0,1]$ into Borel sets. These sets generate a purely atomic σ-algebra generating a space isometric to ℓ_p^m for some $m \in \mathbb{N} \cup \{\infty\}$ containing Y. For $dim Y = \infty$, more measure theory is needed, see [DJP] for details. $\square$

Proof of (ii) for $\mathbb{K} = \mathbb{R}$. Let $p \in 2\mathbb{N}$, $p = 2s$. For a Banach space Z, let $BM_n(Z)$ be the set of isometry classes of n-dimensional Banach spaces $[E]$ generated by the n-dimensional subspaces $E \subseteq Z$ with respect to the Banach-Mazur metric.

Let $n \in \mathbb{N}$ and $N = N_{\mathbb{R}}(n,p)$ as in the claim (ii). It suffices to show that there that $A_n \subset BM_n(\ell_p^N)$. Continuity of the embedding and compactness of $BM_n(\ell_p^N)$ then yield $BM_n(L_p) \subset BM_n(\ell_p^N)$, i.e. the statement we want to prove.

For $r \in \mathbb{N}$, let $I_{k,r} := [\frac{k-1}{r}, \frac{k}{r}]$. The averaging operator

$$u_r : L_p \mapsto L_p, \ f \mapsto \sum_{k=1}^{r} r\Big(\int_{I_{k,r}} f(t)\ dt\Big)\chi_{I_{k,r}}$$

has norm 1 and $u_r(1) = 1$. Let $A_{n,r}$ be the family generated by the n-dimensional subspaces of $u_r(L_p) \subset L_p$, and put $A_n := \cup_{r\in\mathbb{N}} A_{n,r}$. Then A_n is dense in $BM_n(L_p)$. We will show that $A_n \subset BM_n(\ell_p^N)$. The important fact is that N is independent of r.

Since $u_r(1) = 1$, if $Y \subset L_p$ is unital, so is $u_r(Y)$. We thus start with a unital subspace $Y \subset u_r(L_p)$ of dimension n. Since $u_r(L_p) \subset L_\infty$, Y has a basis $\{f_0, f_1, ..., f_{n-1}\}$ with $f_0 = 1$ and $\|f_j\|_\infty = 1$ for $j = 1, ..., n-1$. Moreover, the functions in Y are constant on small intervals by our reduction to $u_r(L_p)$. Let

$$S := \{\alpha \in \mathbb{Z}_+^{n-1} : 1 \le \sum_{j=1}^{n-1} \alpha_j \le p = 2s, \ \alpha = (\alpha_j)_{j=1}^{n-1}\}.$$

Then $|S| = N_{\mathbb{R}}(n,p) = N$. We use standard multiindex notation: for $\alpha \in Z_+^{n-1}$ and $x \in \mathbb{R}^{n-1}$, $x^\alpha := \prod_{j=1}^{n-1} x_j^{\alpha_j}$. Put $F := (f_j)_{j=1}^{n-1}$. Then for all $(b_j)_{j=0}^{n-1} \subset \mathbb{R}^n$

$$\|\sum_{j=0}^{n-1} b_j f_j\|_{L_p}^p = \int_0^1 |\sum_{j=0}^{n-1} b_j f_j|^{2s}\ d\lambda = c_0 + \sum_{\alpha\in S} c_\alpha m_\alpha,$$

where the c_α are polynomials in the coefficients (b_j) and $m_\alpha := \int_0^1 F^\alpha d\lambda$. Let $m = (m_\alpha)_{\alpha\in S} \in \mathbb{R}^N$. We will show that m can be also obtained by integration of similar functions v^α on a discrete set of size N. Let $I = [-1, 1]$ and

$$L := \{(x^\alpha)_{\alpha\in S} : x = (x_j)_{j=1}^{n-1} \in I^{n-1}\} \subset \mathbb{R}^N.$$

Put $x^{(k)} := \big(f_j((k-1/2)/r)\big)_{j=1}^{n-1}$ for $k = 1, ..., r$. Then $(x^{(k)})^\alpha \in L$ since $\|f_j\|_\infty = 1$. Since the f_j's are constant on the small intervals $I_{k,r}$,

$$m = \frac{1}{r}\sum_{k=1}^{r} (x^{(k)})^\alpha \in C := conv(L).$$

By (an improvement of) Caratheodory's theorem, m is a convex combination of at most N elements of $L \subset \mathbb{R}^N$ (see [DJP]). Thus there are $\xi^{(d)} \in L$ and $\nu_d \ge 0$ (for $d = 1, ..., N$) with $\sum_{d=1}^N \nu_d = 1$ such that $m = \sum_{d=1}^N \nu_d \xi^{(d)}$. Let $x^{(d)} = (x_j^{(d)})_{j=1}^{n-1}$ be the (unique) point of I^{n-1} such that $\big((x^{(d)})^\alpha\big)_{\alpha\in S} = \xi^{(d)} \in L$. Let $\mathcal{N} = \{1, ..., N\}$ and

$$v : \mathcal{N} \mapsto \mathbb{R}, \ d \in \mathcal{N} \mapsto x^{(d)} \ (j = 0, ..., n-1)$$

and $v_0 \equiv 1$. Also, $\nu = (\nu_d)_{d=1}^N$ defines a probability measure on $\mathcal{N}$. Let $Z = span[v_0, v_1, ..., v_{n-1}] \subset L_p(\nu) \equiv \ell_p^N$. Then

$$Y \mapsto Z,\ f_j \mapsto v_j\ (j = 0, ..., n-1)$$

extends to an isometric isomorphism: note that with $V = (v_j)_{j=1}^{n-1}$

$$\Big(\int_{\mathcal{N}} V^\alpha d\nu\Big)_{\alpha\in S} = \Big(\sum_{d=1}^N \nu_d (x^{(d)})^\alpha\Big)_{\alpha\in S} = \sum_{d=1}^N \nu_d \xi^{(d)} = m = (m_\alpha)_{\alpha\in S},$$

and hence

$$\|\sum_{j=0}^{n-1} b_j v_j\|_{L_p(\nu)}^p = \int_{\mathcal{N}} \big|\sum_{j=0}^{n-1} b_j v_j\big|^p \, d\nu =$$

$$c_0 + \sum_{\alpha\in S} c_\alpha \int_{\mathcal{N}} V^\alpha d\nu = c_0 + \sum_{\alpha\in S} c_\alpha m_\alpha =$$

$$c_0 + \sum_{\alpha\in S} c_\alpha \int_0^1 F^\alpha d\lambda = \|\sum_{j=0}^{n-1} b_j f_j\|_{L_p}^p.$$

Hence Y is isometric to a subspace of ℓ_p^N. $\square$

In the case of embedding ℓ_2^n into ℓ_{2s}^N, the idea of using Caratheodory's theorem was employed by Gromov-Milman [M].

Corollary 3.7. *Let $0 < p < \infty$, $p \notin 2\mathbb{N}$. Then X is isometric to a subspace of ℓ_p if and only if every 2-dimensional subspace of X is isometric to a subspace of ℓ_p.*

This is immediate from (i). Since no infinite dimensional subspace of ℓ_p is isomorphic to ℓ_2 for $p \neq 2$, we get

Corollary 3.8. *Let $0 < p < \infty$, $p \notin 2\mathbb{N}$. Then the 2-dimensional Hilbert space ℓ_2^2 is not isometric to a subspace of ℓ_p.*

This was a question of Pietsch motivating a part of [DJP]. It was known before that ℓ_2^2 does not embed isometrically into ℓ_p^N if $p \notin 2\mathbb{N}$, cf. [L].

For $p = 2s \in 2\mathbb{N}$, let $M_{\mathbb{K}}(n,p)$ denote the minimal value of N such that (ii) of the theorem holds; thus $M_{\mathbb{K}}(n,p) \leq N_{\mathbb{K}}(n,p)$. For $n = 2,\ p = 4$, (ii) gives $M_{\mathbb{R}}(2,4) \leq 4$, $M_{\mathbb{C}}(2,4) \leq 8$. The following beautiful result in [DJP] is very surprising:

Proposition 3.9. *Every two-dimensional subspace Y of L_4 is isometric to a subspace of ℓ_4^3 if $\mathbb{K} = \mathbb{R}$ and of ℓ_4^4 if $\mathbb{K} = \mathbb{C}$. In the real case, such a subspace admits a 1-symmetric basis.*

In the case of the two-dimensional Hilbert space $Y = \ell_2^2 \subset L_4$ and $\mathbb{K} = \mathbb{R}$ it is easy to give an explicit embedding into ℓ_4^3 since

$$B_2 = \{x = (x_j)_{j=1}^3 : \|x\|_4 = 1,\ \sum^3 x_j = 0\}$$

is a circle (of radius $2^{1/4}$). Of course, for all $n \in \mathbb{N}$ and $s \in \mathbb{N}$, ℓ_2^n embeds isometrically into L_p for $p = 2s$. By $H_{\mathbb{K}}(n,p)$ we denote the minimal dimension N such that ℓ_2^n embeds isometrically into ℓ_p^N, $p = 2s$. Hence by Theorem 3.6 (ii), $H_{\mathbb{K}}(n,p) \leq M_{\mathbb{K}}(n,p) \leq N_{\mathbb{K}}(n,p)$.

Let $\mathcal{P}_{p,n}^{hom}$ denote the space of polynomials in n variables homogeneous of degree $p = 2s$, restricted to the sphere S^{n-1}, in the real case. In the complex case, we mean all polynomials $q(z_1, ..., z_n, \overline{z_1}, ..., \overline{z_n})$ which are homogeneous of degree s in each of the sets of variables $(z_1, ..., z_n)$ and $(\overline{z_1}, ..., \overline{z_n})$, restricted to $S^{n-1}(\mathbb{C})$. Let σ denote the normalized surface measure on S^{n-1}. Hilbert's formula for $x \in \mathbb{K}^n$,

$$\int_{S^{n-1}} |(x,y)|^{2s} \, d\sigma(y) = c_{n,s} \|x\|_2^{2s}$$

shows that $c_{n,s}\|x\|_2^{2s}$ is in the (closed) convex hull of the set $\{|(\cdot, y)|^{2s} : \ y \in S^{n-1}\} \in \mathcal{P}_{2s,n}^{hom}$. Using again (the improvement of) Carathéodory's theorem, one gets again the upper bound

$$H_{\mathbb{K}}(n,p) \leq dim \mathcal{P}_{p,n}^{hom} - 1 = N_{\mathbb{K}}(n,p), \ p = 2s,$$

cf. Milman [M] for this argument. A lower bound is known too [K1]. We summarize these bounds in the

Proposition 3.10. *For $n, s \in \mathbb{N}$ and $p = 2s$, let $L_{\mathbb{R}}(n,p) := \binom{n+s-1}{s}$, $L_{\mathbb{R}}(n,p) := \binom{n+[(s+1)/2]-1}{[(s+1)/2]}\binom{n+[s/2]-1}{[s/2]}$. Then*

$$L_{\mathbb{K}}(n,p) \leq H_{\mathbb{K}}(n,p) \leq N_{\mathbb{K}}(n,p).$$

The upper bounds for $H_{\mathbb{K}}(n,p)$ and $M_{\mathbb{K}}(n,p)$ being identical, the following conjecture is made in [DJP].

Conjecture. *$H_{\mathbb{K}}(n,p) = M_{\mathbb{K}}(n,p)$ for $n, s \in \mathbb{N}$, $p = 2s$.*

At least it is true for $n = 2$, $p = 4$. For fixed $p = 2s$ and $n \to \infty$, $L_{\mathbb{K}}(n,p) \sim n^{p/2}$, $N_{\mathbb{K}}(n,p) \sim n^p$, i.e. both bounds differ substantially. By Dvoretsky's theorem for ℓ_p, cf. [FLM], ℓ_2^n embeds $(1+\epsilon)$-isomorphically into ℓ_p^N, where $N = c_\epsilon n^{p/2}$ if $p > 2$, $\epsilon > 0$. If for $p \in 2\mathbb{N}$, c_ϵ could be chosen to be bounded as $\epsilon \to 0$, one would achieve the order of the lower bound, $H_{\mathbb{K}}(n,p) \sim n^{p/2}$. For $p = 4$, this is true, see Proposition 3.12 below. By results of Reznick [R], Goethals-Seidel [GoS] and Lyubich-Vaserstein [LV], the existence of isometric embeddings of ℓ_2^n into ℓ_{2s}^N is equivalent to the existence of certain cubature formulas on the sphere S^{n-1} with N points; this would seem to suggest that the upper bound is reasonable:

Proposition 3.11. *Let $n, s, N \in \mathbb{N}$. Then the following are equivalent:*

(1) There exists an isometric embedding of ℓ_2^n into ℓ_{2s}^N;

(2) There are N points $(x_k)_{k=1}^N \subset S^{n-1}$ and a probability sequence $(\mu_k)_{k=1}^N \subset \mathbb{R}_+$, $\sum_{k=1}^N \mu_k = 1$, such that for all polynomials $p \in \mathcal{P}_{2s,n}^{hom}$

$$\int_{S^{n-1}} p(y) \, dy = \sum^{N} \mu_k p(x_k); \tag{3.2}$$

(3) There exist (x_k) *and* (μ_k) *as in (2) such that*

$$c_{n,s} := \int_{S^{n-1}} \int_{S^{n-1}} |(x,y)|^{2s} \, d\sigma(x) d\sigma(y) = \sum_{k,\ell=1}^{N} \mu_k \mu_\ell |(x_k, x_\ell)|^{2s}. \tag{3.3}$$

Sketch of the proof ([K1]). **(1)⇒ (3).** Any embedding of ℓ_2^n into ℓ_{2s}^N has the form $x \mapsto \big((x, z_k)\big)_{k=1}^N$ with $z_k \in \mathbb{K}^n$. Let $x_k := z_k / \|z_k\|_2$ and $\mu_k := \|z_k\|_2^{2s} / \sum_{l=1}^N \|z_l\|_2^{2s}$. Then (3) holds.

(3)⇒ (2). Let $\mathbb{K} = \mathbb{R}$. For $x \in \mathbb{R}^n$, we denote by $x^{\otimes 2s} \in \mathbb{R}^{n^{2s}}$ the $(2s)$-fold tensor product of copies of x. Then $(x^{\otimes 2s}, y^{\otimes 2s}) = (x,y)^{2s}$. Consider

$$\xi := \sum_{k=1}^N \mu_k x_k^{\otimes 2s} - \int_{S^{n-1}} x^{\otimes 2s} \, d\sigma(x) \in \mathbb{R}^{n^{2s}}.$$

Then Sidelnikov's inequality holds,

$$0 \le (\xi, \xi) = \sum_{k,l=1}^N \mu_k \mu_\ell |(x_k, x_\ell)|^{2s} - c_{n,s} = 0,$$

which implies that $\xi = 0$: all monomials of degree $2s$ and hence all $p \in \mathcal{P}_{2s,n}^{hom}$ are integrated exactly as indicated in (3.2).

(2) ⇒ (1). Let $x \in \mathbb{K}^n$. Apply (2) to $p(y) = |(y,x)|^{2s}$. Then

$$\sum_{k=1}^N \mu_k |(x, x_s)|^{2s} = \int_{S^{n-1}} |(x,y)|^{2s} \, d\sigma(y) = c_{n,s} \|x\|_2^{2s},$$

i.e. $x \mapsto \big((\mu_k / c_{n,s})^{1/2s} (x, x_s)\big)_{k=1}^N$ defines an isometry of ℓ_2^n into ℓ_{2s}^N. □

Condition (3) is useful to prove the existence of certain explicit embeddings since only one equality - (3.2) - has to be checked. The lower bound is exact, $L_{\mathbb{K}}(n,p) = H_{\mathbb{K}}(n,p) =: N$, if and only if the spherical design $(x_s)_{s=1}^N$ is *tight*, i.e. the set of scalar products $C = \{|(x_k, x_\ell)| : k \neq \ell\}$ is a very small set, and all μ_k's are equal to $1/N$. For $p = 4$, $|C| = 1$, and the vectors $(x_k)_{k=1}^N$ span *equiangular* lines. Known cases of equality are

$$H_{\mathbb{R}}(n,4) = \frac{n(n+1)}{2}, \; n = 2,3,7,23, \quad H_{\mathbb{C}}(n,4) = n^2, \; n = 2,3,8.$$

For $p = 4$ it is not known whether there are more values of n such that this holds. For $p \geq 6$, there are only very few cases of equality, the most spectacular being $H_{\mathbb{R}}(24,10) = \binom{28}{5}$, cf. [R], [LV], [K1].

To show that $H_{\mathbb{K}}(n,4)$ is close to $L_{\mathbb{K}}(n,4)$, one may study systems (x_k) which

Proposition 3.12 ([K1], [Ra]). *The space ℓ_2^n embeds isometrically into ℓ_4^N in the following cases*

(a) $\mathbb{K} = \mathbb{C},\ n = q + 1,\ q$ *prime power,* $N = n^2 + 1$
(b) $\mathbb{K} = \mathbb{C},\ n = q,\ q$ *odd prime power,* $N = n^2 + n$
(c) $\mathbb{K} = \mathbb{R},\ n = 2^m,\ m$ *integer,* $N = n^2 + n$
(d) $\mathbb{K} = \mathbb{R},\ n = 4^m,\ m$ *integer,* $N = n(n+1)/2.$

Hence $H_{\mathbb{K}}(n, 4)$ is of the order of the lower bound. To show e.g. (b), define for $k = (k_1, k_2) \in \{1, ..., n\}^2$ the vectors

$$x_k := \frac{1}{\sqrt{n}}\big(\exp\{\frac{2\pi i}{n}(k_1 j + k_2 j^2)\}\big)_{j=1}^{n} \in S^{n-1}(\mathbb{C}).$$

Then for $\ell = (\ell_1, \ell_2) \neq k = (k_1, k_2)$,

$$|(x_k, x_\ell)| = \begin{cases} 1/\sqrt{n} & \text{if } \ k_2 \neq \ell_2 \\ 0 & \text{if } \ k_2 = \ell_2 \end{cases}$$

In addition to these n^2 vectors one takes the n unit vectors $e_1, ..., e_n$. These $n^2 + n$ vectors satisfy (3.2) provided all $\mu_k = 1/N$.

Case (a) is done by using B_2-sequences and exponential vectors defined by them, case (d) follows by using vectors from the Kerdock code, cf. [K1]. Case (c) uses a construction with Walsh functions, as observed by Schechtman and explained in Rabinovich [Ra]. For $p = 2s \geq 6$, the problem is completely open.

Problem. *Let* $s \in \mathbb{N},\ s \geq 3,\ p = 2s$. *Is it true that for large* n

$$H_{\mathbb{K}}(n, p) \sim n^{p/2}?$$

4. THE BUSEMANN-PETTY PROBLEM ON SECTIONS OF CONVEX BODIES

The Busemann-Petty problem (see [BP]) asks the following question. Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$vol_{n-1}(K \cap \xi^{\perp}) \leq vol_{n-1}(L \cap \xi^{\perp})$$

for every ξ from the unit sphere S^{n-1} in $\mathbb{R}^n$, where $\xi^{\perp} = \{x \in \mathbb{R}^n :\ (x, \xi) = 0\}$ is the central hyperplane perpendicular to ξ. Does it follow that

$$vol_n(K) \leq vol_n(L)?$$

The answer is negative if $n \geq 5$ and affirmative for $n \leq 4$. The solution to the problem was based on the concept of an intersection body introduced by Lutwak [Lut] in 1988. A star body A in $\mathbb{R}^n$ is called an intersection body of a star body B

if for every $\xi \in S^{n-1}$ the radius of A in the direction of ξ is equal to the $(n-1)$-dimensional volume of the section of B by the central hyperplane perpendicular to ξ :

$$\rho_A(\xi) = \|\xi\|_A^{-1} = vol_{n-1}(B \cap \xi^{\perp}), \tag{4.1}$$

where $\|\xi\|_A = \min\{t \geq 0 : \xi \in tA\}$ is the Minkowski functional of A, ρ_A is the radial function, and $\xi^{\perp} = \{x \in \mathbb{R}^n : (x, \xi) = 0.\}$ A star body A is called an intersection body of a star body if there exists a star body B satisfying the equalities (4.1). The class of intersection bodies can be defined as the closure of the class of intersection bodies of star bodies in the radial metric $d(A, B) = max_{\xi \in S^{n-1}} |\rho_A(\xi) - \rho_B(\xi)|$. Lutwak [Lut] found the following connection between intersection bodies and the Busemann-Petty problem (the original results of Lutwak were slightly improved by Gardner [Ga1] and Zhang [Zh1]): (i) If K is an intersection body then the answer to the Busemann-Petty problem is affirmative for every star body L; (ii) If L_0 is an origin symmetric convex body that is not an intersection body, then one can perturb L_0 twice so that the resulting bodies K and L are convex and give a counterexample to the Busemann-Petty problem. Therefore,

Theorem 4.1. *The Busemann-Petty problem has an affirmative answer in $\mathbb{R}^n$ if and only if every origin symmetric convex body in $\mathbb{R}^n$ is an intersection body.*

The Busemann-Petty problem has a long and dramatic history. The negative answer for $n \geq 5$ was established in a series of papers by Larman and Rogers [LR] (for $n \geq 12$), Ball [Ba] ($n \geq 10$), Giannopoulos [Gi] and Bourgain [Bo] ($n \geq 7$), Gardner [Ga1] and Papadimitrakis [Pa] ($n \geq 5$). Gardner [Ga2] proved that every symmetric convex body in $\mathbb{R}^3$ is an intersection body, and, hence, the problem has an affirmative answer in the case $n = 3$. For several years the answer in the dimension 4 was believed to be negative, because Zhang [Zh2] claimed that the unit cube in $\mathbb{R}^4$ is not an intersection body, which would imply the negative answer in view of Lutwak's connection. Zhang's claim was disproved in [Ko8], where it was shown that the unit ball of the space ℓ_q^n, $2 < q \leq \infty$ is an intersection body if and only if $n \leq 4$. This result is a consequence of the following connection between intersection bodies and the Fourier transform established in [Ko7]:

Theorem 4.2. *A star body K in $\mathbb{R}^n$ is an intersection body if and only if $\|\cdot\|_K^{-1}$ is a positive definite distribution.*

Recall that a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is positive definite if and only if its Fourier transform $\hat{f}$ is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ (see [GV] for definitions and details).

Another consequence of Theorem 4.2 (see [Ko7]) is that the unit ball of every finite dimensional subspace of L_p with $0 < p \leq 2$ is an intersection body. Theorem 4.2 was also applied in [Ko9] to present a variety of counterexamples to the

Busemann-Petty problem in dimensions ≥ 5. For example, if $q > 2$ the unit ball of the q-sum of any normed spaces X, $dim(X) \geq 4$ and Y is not an intersection body.

These results prompted a reexamination of the four dimensional case, and finally Zhang [Zh3] proved that the answer in the four dimensional case is affirmative, and, two months later, a unified solution to the Busemann-Petty problem was given in [GKS].

The proof in [GKS] is based on

Theorem 4.3. *Let K be an origin-symmetric star body in $\mathbb{R}^n$ with C^∞ boundary, and let $k \in \mathbb{N} \cup \{0\}$, $k \neq n-1$. Suppose that $\xi \in S^{n-1}$, and let*

$$A_{K,\xi}(t) = vol_{n-1}(K \cap \{x \in \mathbb{R}^n : (x,\xi) = t\}),\ t \in \mathbb{R}$$

be the parallel section function of K in the direction of ξ.

(a) *If k is even, then*

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_{K,\xi}^{(k)}(0);$$

(b) *if k is odd, then*

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{(k+1)/2}\, 2(n-1-k)k!$$

$$\int_0^\infty \frac{A_{K,\xi}(z) - A_{K,\xi}(0) - A''_{K,\xi}(0)\frac{z^2}{2!} - \cdots - A_{K,\xi}^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\, dz,$$

where $A_{K,\xi}^{(k)}(0)$ is the derivative of the order k of the parallel section function at zero, and $(\|x\|_K^{-n+k+1})^\wedge$ is the Fourier transform in the sense of distributions.

Theorem 4.3 is a direct consequence of

Theorem 4.4. *Let K be an origin-symmetric infinitely smooth star body in $\mathbb{R}^n$. Suppose that $\xi \in S^{n-1}$, and let $A_{K,\xi}$ be the corresponding parallel section function of K. For $q \in \mathbb{C}$ with $\Re q > -1$, $q \neq n-1$ we have*

$$A_{K,\xi}^{(q)}(0) = \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)}(\|x\|_K^{-n+q+1})^\wedge(\xi).$$

Here $A_{K,\xi}^{(q)}(0)$ is the fractional derivative of order q at zero, defined by

$$A_{K,\xi}^{(q)}(0) = \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}A_{K,\xi}(t)\ dt$$

if $-1 < \Re q < 0$, and by

$$A_{K,\xi}^{(q)}(0) = \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}\left(A_{K,\xi}(t) - \sum^{k-1} \frac{t^{2j}}{(2j)!}A_{K,\xi}^{(2j)}(0)\right) dt,$$

whenever $2k-2 < \Re q < 2k,\ k \in \mathbb{N}$. The function $q \mapsto A^{(q)}_{K,\xi}(0),\ q \in \mathbb{C}$, can then be extended to an analytic function on all of $\mathbb{C}$. Note that the function $A_{K,\xi}$ is even and that if q is an even integer the fractional derivative of order q coincides with the ordinary derivative of the same order. We refer the reader to [GS, pp. 48–56] for details.

To prove Theorem 4.4, suppose first that $-1 < q < 0$. The function $A_{K,\xi}(z) = \int_{\langle x,\xi\rangle = z} \chi(\|x\|)\, dx$ is even. Applying Fubini's theorem and passing to spherical coordinates, we get

$$\begin{aligned} A^{(q)}_{K,\xi}(0) &= \frac{1}{2\Gamma(-q)} \int_{-\infty}^{\infty} |z|^{-q-1} A_{K,\xi}(z)\, dz \\ &= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}^n} |\langle x,\xi\rangle|^{-q-1} \chi(\|x\|_K)\, dx \\ &= \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} |\langle \theta,\xi\rangle|^{-q-1} \int_0^{\infty} r^{n-q-2} \chi(r\|\theta\|_K)\, dr\, d\theta \\ &= \frac{1}{2(n-q-1)\Gamma(-q)} \int_{S^{n-1}} |\langle \theta,\xi\rangle|^{-q-1} \|\theta\|_K^{-n+q+1}\, d\theta. \end{aligned}$$

We now consider $A^{(q)}_{K,\xi}(0)$ as a function of $\xi \in \mathbb{R}^n \setminus \{0\}$. Using the same argument as in the end of the proof of Theorem 3.1, we see that, for every even test function $\varphi \in \mathcal{S}$

(4.2)

$$\begin{aligned} \langle A^{(q)}_{K,\xi}(0), \varphi(\xi)\rangle &= \frac{1}{2(n-q-1)\Gamma(-q)} \int_{S^{n-1}} \|\theta\|_K^{-n+q+1}\, d\theta \int_{\mathbb{R}^n} |\langle \theta,\xi\rangle|^{-q-1} \varphi(\xi)\, d\xi \\ &= \frac{-1}{4(n-q-1)\Gamma(-q)\Gamma(q+1)\sin\frac{q\pi}{2}} \int_{S^{n-1}} \|\theta\|_K^{-n+q+1} \int_{-\infty}^{\infty} |t|^q \widehat{\varphi}(t\theta)\, dt\, d\theta \\ &= \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)} \left\langle (\|x\|_K^{-n+q+1})^\wedge(\xi), \varphi(\xi) \right\rangle, \end{aligned}$$

where the last equation follows from the property $\Gamma(-q)\Gamma(q+1) = -\pi/\sin(q\pi)$ of the Γ-function and the simple calculation

$$\langle (\|x\|_K^{-n+q+1})^\wedge(\xi), \varphi(\xi)\rangle = \int_{\mathbb{R}^n} \|x\|_K^{-n+q+1} \widehat{\varphi}(x)\, dx =$$

$$\int_{S^{n-1}} \|\theta\|_K^{-n+q+1} \int_0^{\infty} t^q \widehat{\varphi}(t\theta)\, dt\, d\theta$$

(note that the function $\|x\|^{-n+q+1}$ is locally integrable on $\mathbb{R}^n$ because $-1 < q < 0$).

Since (4.2) holds for every even test function φ, Theorem 4.4 is proved when $-1 < q < 0$.

To prove the theorem for other values of q, we first observe that $(\|x\|_K^{-n+q+1})^\wedge$ is an analytic distribution (with respect to q) on $\{q \in \mathbb{C} :\ \Re q > -1\}$. It follows that for every even test function $\varphi \in \mathcal{S}$, the functions $q \mapsto \langle A^{(q)}_{K,\xi}(0), \varphi\rangle$ and

$$q \mapsto \left\langle \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)} (\|x\|_K^{-n+q+1})^\wedge(\xi), \varphi \right\rangle$$

are analytic on the connected region $\{q \in \mathbb{C} : \Re q > -1,\ q \neq n-1\}$ (for details of analytic continuation in such situations, see [8]). These functions coincide on the interval $-1 < q < 0$, so they coincide on $\{q \in \mathbb{C} : \Re q > -1,\ q \neq n-1\}$. Since φ is an arbitrary even test function, we have proved Theorem 4.4.

Part (a) of Theorem 4.3 immediately follows from Theorem 4.4 and the fact that fractional derivatives coincide with ordinary derivatives. To prove part (b) of Theorem 4.3, divide both sides of the formula in Theorem 4.4 by $\cos(q\pi/2)$, and compute the limit as $q \to k$, where k is an odd integer.

Another short proof of Theorem 4.3 (with the Fourier transform replaced by the spherical Radon transform) was recently given by Barthe, Fradelizi and Maurey [BFM].

Let us show how the solution to the Busemann-Petty problem follows from Theorems 4.1, 4.2 and 4.3. Let $n = 4$ and K be any symmetric infinitely smooth convex body in $\mathbb{R}^4$. Put $k = 2$ in part (a) of Theorem 4.3. By the Brunn-Minkowski theorem (see [S2]), the volume of the central section is maximal among volumes of sections perpendicular to a given direction, therefore, $A''_{K,\xi}(0) \leq 0$ for every ξ. By part (a) of Theorem 4.3, we conclude that $\|x\|_K^{-1}$ is positive definite, and, by Theorem 4.2, K is an intersection body. The positive answer to the Busemann-Petty problem in dimension 4 follows now from Theorem 4.1. If $n = 5$, we have to put $k = 3$ and positive definiteness of the function $\|x\|^{-1}$ depends on the properties of the third derivative of parallel section functions. Since convexity does not control the third derivative, it is easy to construct a counterexample using part (b) of Theorem 4.3.

We now present a proof of a result from [Ko10] that generalizes the solution to the Busemann-Petty problem. This proof no longer uses intersection bodies and is based on a version of Parseval's formula. The result is as follows:

Theorem 4.5. *Let K and L be $(k-1)$-smooth origin symmetric convex bodies in $\mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$,*

$$(-1)^{(k-1)/2} A_{K,\xi}^{(k-1)}(0) \leq (-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0), \tag{4.3}$$

where k is an odd integer and $1 \leq k \leq n-1$. Then

(i) if $k \geq n-3$ then $vol_n(K) \leq vol_n(L)$;

(ii) if $k < n-3$ then it is still possible that $vol_n(K) > vol_n(L)$.

Clearly, the case $k = 1$ of Theorem 4.5 is the answer to the Busemann-Petty problem. Also it is enough to prove Theorem 4.5 in the case where K and L are infinitely smooth. The crucial point of the proof is a version of Parseval's formula on the sphere that can be proved by extending functions to $\mathbb{R}^n$ and using the classical Parseval's formula.

Lemma 4.6. *Let K and D be origin symmetric star bodies with C^∞-boundaries in $\mathbb{R}^n$ and $k \in \mathbb{N},\ k < n$. Then*

$$\int_{S^{n-1}} (\|x\|_K^{-k})^\wedge(\theta)(\|x\|_D^{-n+k})^\wedge(\theta)\, d\theta = (2\pi)^n \int_{S^{n-1}} \|\theta\|_K^{-k} \|\theta\|_D^{-n+k}\, d\theta.$$

Theorem 4.7. *Let k be an odd integer, $1 \le k \le n-1$, and let K and L be origin symmetric $(k-1)$-smooth star bodies in $\mathbb{R}^n$. Suppose that the distributions $\|x\|_K^{-k}$ and $\|x\|_L^{-n+k} - \|x\|_K^{-n+k}$ are positive definite. Then $vol_n(K) \le vol_n(L)$.*

Proof. We have

$$\int_{S^{n-1}} (\|x\|_K^{-k})^\wedge(\theta)(\|x\|_L^{-n+k})^\wedge(\theta)\ d\theta \ge \int_{S^{n-1}} (\|x\|_K^{-k})^\wedge(\theta)(\|x\|_K^{-n+k})^\wedge(\theta)\ d\theta.$$

By Lemma 4.6,

$$\int_{S^{n-1}} \|\theta\|_K^{-k}\|\theta\|_L^{-n+k}\ d\theta \ge \int_{S^{n-1}} \|\theta\|_K^{-n}\ d\theta.$$

By Holder's inequality and since $(1/n)\int_{S^{n-1}} \|\theta\|_K^{-n}\ d\theta = vol_n(K)$, we have

$$(vol_n(K))^{k/n}(vol_n(L))^{(n-k)/n} \ge vol_n(K). \quad \square$$

Theorem 4.8. *Let $0 < k < n$ and let L be an origin symmetric convex body in $\mathbb{R}^n$ with C^∞-boundary and positive curvature so that the distribution $\|x\|_L^{-k}$ is not positive definite. Then there exists an origin symmetric convex body K in $\mathbb{R}^n$ with C^∞-boundary such that the distribution $\|x\|_L^{-n+k} - \|x\|_K^{-n+k}$ is positive definite but $vol_n(K) > vol_n(L)$.*

Proof. Since $(\|x\|_L^{-k})^\wedge$ is a continuous sign-changing function on S^{n-1}, there exists an open subset Ω in S^{n-1} on which $(\|x\|_L^{-k})^\wedge$ is negative. Let $f \in C^\infty(S^{n-1})$ be a non-negative (and not identically zero) function supported in Ω. One can prove (see [Ko10, Lemma 5]) that the function $f(\theta)r^{-k}$ is the Fourier transform of a function $g(\theta)r^{-n+k}$, where $g \in C^\infty(S^{n-1})$.

Define a body K by

$$\|x\|_K^{-n+k} = \|x\|_L^{-n+k} - \frac{\epsilon}{(2\pi)^n} g(x),$$

where $\epsilon > 0$ is small enough so that the body K is convex (a standard perturbation argument is that, given an infinitely differentiable function on S^{n-1}, one can choose a small enough ϵ so that the differential properties of the norm $\|\cdot\|_L^{-n+p}$ equivalent to convexity of L are preserved after adding an ϵ-multiple of the $(-n+k)$-homogeneous extension of this function.) We have

$$(\|x\|_L^{-n+p})^\wedge - (\|x\|_K^{-n+p})^\wedge = \epsilon f(\theta) r^{-p}, \tag{4.4}$$

so the distribution $\|x\|_L^{-n+p} - \|x\|_K^{-n+p}$ is positive definite.

On the other hand, by (4.4) and Lemma 4.6

$$\int (\|x\|_L^{-k})^\wedge(\theta) f(\theta)\ d\theta = \frac{(2\pi)^n}{\ }\Big(vol_n(L) - \int \|\theta\|_L^{-p}\|\theta\|_K^{-n+p}\Big).$$

Since the quantity in the left-hand side of the latter formula is negative, we use Holder's inequality (as in Theorem 1) to see that $vol_n(K) > vol_n(L)$. $\square$

Proof of Theorem 4.5. Putting $k = 2$ in part (a) of Theorem 4.3 and using the fact that the central section of a convex symmetric body is maximal among sections perpendicular to a given direction, we conclude that the function $\|x\|_K^{-n+3}$ is a positive definite distribution for every symmetric convex body K. Similarly, $\|x\|_K^{-n+2}$ and $\|x\|_K^{-n+1}$ are positive definite (put $k = 1$ and $k = 0$ in Theorem 4.3). Now part (i) of Theorem 4.5 immediately follows from Theorems 4.7 and 4.3. To show (ii), let L be the unit ball of the space with the norm $\|x\|_L = \|x\|_4 + \epsilon\|x\|_2$, where $\epsilon > 0$ and $\|\cdot\|_q$ stands for the norm of the space ℓ_q^n. By Lemma 3.3, the distribution $\|x\|_4^{-k}$ is not positive definite, therefore $\|x\|_L^{-k}$ is not positive definite for small enough ϵ. Using this value of ϵ in the definition of L (the perturbation of the ℓ_4^n-norm was made to ensure that L has positive curvature) and using Theorem 4.8 we get a body K giving the desired example (again use Theorem 4.3 to connect the Fourier transform with the derivatives of parallel section functions). $\square$

The condition that $\|x\|_K^{-k}$ is positive definite, that we use in the proofs, has a clear geometric interpretation. For $1 \leq k < n$, let us say that a star body K in $\mathbb{R}^n$ is a k-intersection body of a star body if there exists a star body L in $\mathbb{R}^n$ so that, for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$vol_k(K \cap H^\perp) = vol_{n-k}(L \cap H).$$

It was proved in [Ko10] that an infinitely smooth symmetric star body K is a k-intersection body of a star body if and only if $\|x\|_K^{-k}$ is a positive definite distribution.

Finally, we would like to mention that the isomorphic version of the Busemann-Petty problem is open and equivalent to the famous hyperplane (or slicing) problem (see [MP] for details).

5. Approximation of zonoids by zonotopes

A zonotope in $\mathbb{R}^n$ is a special convex polytope, namely the Minkowski sum of finitely many segments $I_j,\ j = 1, ..., N$ in $\mathbb{R}^n$,

$$Z_N = \sum_{j=1}^{N} I_j = \{\sum_{j=1}^{N} y_j :\ y_j \in I_j,\ j = 1, ..., N\}.$$

By a segment we mean a compact one-dimensional convex set. For simplicity, we will assume that 0 is the center of all segments; then 0 is the center of symmetry of Z_N. A zonoid B is a convex body which can be approximated arbitrarily well by zonotopes in the Hausdorff metric. For $n = 2$, all centrally symmetric convex bodies are zonoids, for $n \geq 3$, the unit balls B_p^n of ℓ_p^n are zonoids if and only if $2 \leq p \leq \infty$. Several authors studied the problem of approximating zonoids by zonotopes: what is the minimal number $N = N(B, \epsilon)$ of segments I_j needed to approximate a zonoid $B \subset \mathbb{R}^n$ up to $\epsilon > 0$ by a zonotope $Z_N = \sum_{j=1}^N I_j$ given by N segments, i.e. $Z_N \subset B \subset (1+\epsilon)Z_N$? This is of particular interest for the Euclidean ball $B = B_2^n$.

Before stating the estimates for $N(B, \epsilon)$ known for general B and $B = B_2^n$, we explain the dual functional analytic formulation. By definition, a zonotope Z_N is a linear image of the unit cube B_∞^N. Thus, if the interior of Z_N is non-empty, the norm $\| \cdot \|$ induced by Z_N in $\mathbb{R}^n$ is a quotient norm of ℓ_∞^N. Consequently, the polar of the zonotope, $P = Z_N^\circ$, is the unit ball of an n-dimensional subspace of ℓ_1^N with norm

$$\|x\|_* = \sum_{j=1}^N \lambda_j |(x, x_j)|, \ x \in \mathbb{R}^n.$$

Choosing $x_j \in S^{n-1}$, the value λ_j is just 1/2 of the length of I_j. Similarly, the polar B° of a zonoid B is the unit ball of a norm

$$\|x\|_* = \int_{S^{n-1}} |(x, y)| \, d\mu(y)$$

where μ is a (positive) measure on S^{n-1}. In other words, B is a zonoid if and only if B° is the unit ball of an n-dimensional subspace of $L_1(S^{n-1}, \mu)$; one may also take $L_1([0,1], \mu)$, cf. [Bol]. For $B = B_2^n$, μ is multiple of the usual surface measure. Thus, using the Banach-Mazur distance d, the above problem can be restated as follows: Given an n-dimensional subspace X of $L_1 = L_1(S^{n-1}, \mu)$ and $\epsilon > 0$, what is the minimal number $N = N(X, \epsilon)$ such that there is an n-dimensional subspace $Y$of ℓ_1^N with $d(X, Y) \leq 1 + \epsilon$. Here X and Y have as their unit balls the polars of a zonoid and the approximating zonotope. The results of Section 3 show that e.g. $X = \ell_2^2 \subset L_1(S^{n-1})$ but that ℓ_2^2 does not embed isometrically into ℓ_1, so one may not take $\epsilon = 0$. In this sense, the results presented here form an almost isometric counterpart to those in Section 3.

Figiel, Lindenstrauss, Milman [FLM] showed that $N(\ell_2^n, \epsilon) \leq c\epsilon^{-2}(\ln \epsilon^{-1})n$, Gordon [Go] improved it to $\leq c\epsilon^{-2}n$. Johnson, Schechtman [JS] proved that $N(\ell_p^n, \epsilon) \leq c_p \epsilon^{-p'} n$ for $1 \leq p < 2$. Bourgain, Lindenstrauss, Milman [BLM] then established a general estimate for all Banach spaces X_B with zonoid unit ball B, $N(X_B, \epsilon) \leq c\epsilon^{-2}(\ln \epsilon^{-1})n(\ln n)^3$. Talagrand [Ta] simplified their proof and improved the estimate at the same time, obtaining $N(X_B, \epsilon) \leq c\epsilon^{-2}n(\ln n)$. The constants c, c_p here do not depend on n and ϵ. Up to the $(\ln n)$-terms, the estimates are clearly optimal as far as the dependence on n is concerned. However, fixing n, the dependence on ϵ is not the best possible. Better results on this setting were obtained in a series of papers by Bourgain, Lindenstrauss, Milman, Matousek and Wagner, for the Euclidean ball B_2^n as well as for the general zonoids B. In terms of the previous notation

$$N(B, \epsilon) = \min\{N \in \mathbb{N} : \ \exists Z_N = \sum^N I_j, \ Z_N \subset B \subset (1+\epsilon)Z_N\}$$

the best known result is

Theorem 5.1. *(1) Let* $n \geq 2$*. Then there is a constant* $c(n) > 0$ *such that for all* $\epsilon > 0$

$$N(B_2^n, \epsilon) \geq c(n)\epsilon^{-2(n-1)/(n+2)}. \tag{5.1}$$

(2) Let $n \geq 2$ *and* $B \subset \mathbb{R}^n$ *be a zonoid. Then there is* $d(n) > 0$ *such that for all* $1 \geq \epsilon > 0$, *if* $n \geq 5$ *or* $n = 2$

$$N(B, \epsilon) \leq d(n)\epsilon^{-2(n-1)/(n+2)}, \tag{5.2}$$

and if $n = 3, 4$

$$N(B, \epsilon) \leq d(n)(\epsilon^{-2}\ln \epsilon^{-1})^{(n-1)/(n+2)}. \tag{5.3}$$

The segments of the approximating zonotope may be taken of equal length if $n \geq 5$ *or* ($B = B_2^n$ *and* $n \geq 2$).

The lower estimate (5.1) was proved in [BLM] using spherical harmonics. Bourgain, Lindenstrauss [BL1] proved the upper estimate (5.3) for all zonoids (with a slightly worse logarithmic term if $n = 4$) and for $B = B_2^n$ in all dimensions $n \geq 2$. Wagner [W] had proved that the approximating zonotope for B_2^n may be chosen to have segments of equal length if $n \leq 6$. Bourgain, Lindenstrauss [BL2] in a second paper removed this restriction $n \leq 6$ for $B = B_2^n$ (with logarithmic terms). Matousek [Ma] improved these upper estimates to the form stated in (2) of Theorem 5.1.

We sketch some basic ideas of the proof of Theorem 5.1.

(i) To approximate a zonoid B by a zonotope Z_N generated by N segments up to $\epsilon > 0$ means by the dual formulation that, for a given probability measure μ on S^{n-1}, one has to find a discrete probability measure $\lambda = (\lambda_j)$ on S^{n-1} supported in N points $(x_j) \subset S^{n-1}$ such that for all $x \in S^{n-1}$

$$\Big| \int_{S^{n-1}} |(x, y)| \, d\mu(y) - \sum_{j=1}^{N} \lambda_j |(x, x_j)| \Big| < \epsilon. \tag{5.4}$$

For $B = B_2^n$ the measure μ is the normalized surface measure. In this case $\int_{S^{n-1}} |(x, y)| \, d\mu(y) = \beta_n$ is a constant depending only on n and (5.4) reads

$$\Big|\beta_n - \sum_{j=1}^{N} \lambda_j |(x, x_j)| \Big| < \epsilon, \; x \in S^{n-1}. \tag{5.5}$$

That the segments of Z_N have equal length then means that λ is uniform, i.e. all λ_j are equal, $\lambda_j = 1/N$, $j = 1, \ldots, N$.

(ii) For the lower estimate of $N(B_2^n, \epsilon)$ in (5.1), assume (5.5) to hold and put $h(x) := \sum_{j=1}^N \lambda_j |(x, x_j)|$, $x \in S^{n-1}$. Then $\epsilon > \|\beta_n - h\|_{L_2(m)}$, m=normalized surface measure on S^{n-1}. For each $k \geq 0$, let $(Y_{kj})_{j=1}^{M(n,k)}$ be an orthonormal basis of spherical harmonics of degree k on S^{n-1}. Expand h into spherical harmonics

$$h = \beta_n + \sum_{k=1}^{\infty} \sum_{j=1}^{M(n,k)} \langle h, Y_{kj} \rangle \tag{5.6}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(m)$. By the Funk-Henke formula [Mü]

$$\langle h, Y_{kj} \rangle = \sum_{j=1}^N \lambda_j \int_{S^{n-1}} |(x, x_j)| Y_{kj}(x) \, dm(x) = \alpha_k \sum_{j=1}^N \lambda_j Y_{kj}(x_j)$$

where calculations show that $\alpha_k \sim k^{-(1+n/2)}$ for even k, $\alpha_0 = 1$ and $\alpha_k = 0$ for odd $k \in \mathbb{N}$. The generating function of the generalized Legendre polynomials and the addition formula for spherical harmonics imply that

$$\frac{1-r^2}{(1+r^2-2r(x,y))^{n/2}} = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{M(n,k)} Y_{kj}(x) Y_{kj}(y), \ 0 \leq r < 1, \ x, y \in S^{n-1}. \tag{5.7}$$

Hence

$$\sum_{j=1}^N \lambda_j \frac{1-r^2}{(1+r^2-2r(x,x_j))^{n/2}} = \sum_{k=0}^{\infty} r^k / \alpha_k \sum_{j=1}^{M(n,k)} \langle h, Y_{kj} \rangle Y_{kj}(x),$$

and using (5.6) in the form

$$\epsilon^2 > \|\beta_n - h\|_{L_2(m)}^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{M(n,k)} \langle h, Y_{kj} \rangle^2,$$

we find using $\alpha_0 = 1$ and (5.7)

$$\begin{aligned}
\epsilon^2 \max_{k \geq 1} (r^k / \alpha_k) &\geq \|1 - \sum_{j=1}^N \lambda_j \frac{1-r^2}{(1+r^2-2r(\cdot, x_j))^{n/2}} \|_{L_2(m)}^2 \\
&= \sum_{j,i=1}^N \lambda_i \lambda_j \frac{1-r^4}{(1+r^4-2r^2(x_i,x_j))^{n/2}} - 1 \\
&\geq \Big(\sum_{i=1}^N \lambda_i^2 \Big) \frac{1-r^4}{(1-r^2)^n} - 1 \geq \frac{1}{N} (1-r^2)^{1-n} - 1. \qquad (5.8)
\end{aligned}$$

Choosing r such that $(1-r^2)^{1-n} = 2N$, the $\max_k (r^k/\alpha_k)$ is attained for $k \sim (1-r^2)^{-1}$ and (5.8) implies the desired estimate

$$c(n) \epsilon^2 N^{(n+2)/(n-1)} \geq 1.$$

For more details, see [BLM], section 6.

(iii) To prove the upper estimate for $N(B,\epsilon)$ in (5.2) and (5.3), given a measure μ, one has to construct points $(x_j) \subseteq S^{n-1}$ and a probability measure $\lambda = (\lambda_j)$ on them so that (5.4) holds. The proofs in [BL1], [BL2] and [M] are technically involved. In the first deterministic procedure, the sphere S^{n-1} is divided into N small pieces $(Q_j)_{j=1}^N$ having diameters of order of magnitude $N^{-1/(n-1)}$ and (approximately) the same μ measure, $\mu(Q_j) = 1/N$. In the second step, in each of the Q_j's, $(n+2)$ or $(n+1)$ points $(y_{j\ell})_\ell \subseteq Q_j$ and weights $(\lambda_{j\ell})_\ell$ are chosen, in a probabilistic way or by employing methods of the geometric discrepancy theory ("irregularities of distribution"); the union of the points $y_{j\ell}$ constitutes the sequence $(x_j) \subseteq S^{n-1}$ and $(\frac{1}{N}\lambda_{j\ell})$ the measure on them. In [BL1] the set Σ_j of probability measures σ_j on Q_j is considered which have the form $\sigma_j = \sum_{\ell=1}^{n+2} \lambda_{j\ell}\delta_{y_{j\ell}}$ with $y_{j\ell} \in Q_j$ such that linear functions h are integrated exactly on Q_j

$$N\int_{Q_j} h\,d\mu = \int_{Q_j} h\,d\sigma_j \;= \Big(\sum_{\ell=1}^{n+2} \lambda_{j\ell} h(y_{j\ell})\Big),$$

i.e. the barycenter of σ_j coincides with the one of the probability measures $N\mu|_{Q_j}$. Using Caratheodory's theorem and the separation theorem, one shows that $N\mu|_{Q_j}$ belongs to the weak* closure of the convex hull of Σ_j. Since Σ_j is compact, the Krein-Milman theorem implies that there is a probability measure ν_j on Σ_j such that

$$N\mu|_{Q_j} = \int_{\Sigma_j} \sigma_j\, d\nu_j(\sigma_j),\ 1 \le j \le N.$$

Then $\nu := \prod_{j=1}^N \nu_j$ is a probability measure on $\Sigma := \prod_{j=1}^N \Sigma_j$ and a probabilistic deviation inequality shows that the probability of the set of those $\sigma = (\sigma_1, ..., \sigma_N) \in \Sigma$, σ_j of the above form, so that

$$(5.9) \qquad \Big|\int_{S^{n-1}} |(x,y)|\, d\mu(y) - \frac{1}{N}\sum_{j=1}^{N}\sum_{\ell=1}^{n+2} \lambda_{j\ell}|(x, y_{j\ell})|\Big| > \epsilon/2$$

is exponentially small for a fixed $x \in S^{n-1}$, namely at most

$$2\exp(-d_1(n)N^{(n+2)/(n-1)}\epsilon^2),$$

and still less than 1 for all points in a suitable $(\epsilon/4)$-net in S^{n-1} provided that

$$d_2(n)N^{(n+2)/(n-1)}\epsilon^2/\ln\epsilon^{-1} < 1$$

which is the type of the condition in (5.3). Hence there is $\sigma = (\sigma_1, ..., \sigma_N) \in \Sigma$ so that (5.9) is false for any x in the $(\epsilon/4)$-net on S^{n-1}. For this σ, the left-hand side in (5.9) is at most ϵ for <u>all</u> $x \in S^{n-1}$, giving a measure $\lambda = (\lambda_{j\ell})$ on the $N(n+2)$ points $(y_{j\ell}) \subseteq S^{n-1}$ with (5.4). It does not matter that N is replaced by $N(n+2)$ since we disregard constants depending on n. The measure λ is not uniform in general, however, and so the approximating zonotope will not (yet) be generated by segments of equal length. For the use of methods of the geometric discrepancy theory we refer to [Ma]

6. Exact estimates for projection constants

By Lindenstrauss-Tzafriri's complemented subspaces theorem [LT], a Banach space Z is isomorphic to a Hilbert space if and only if every closed subspace X of Z is complemented. This means that the *relative projection constant* of X in Z,

$$\lambda(X, Z) := \inf\{\|P\| : \ P^2 = P \in \mathcal{L}(Z) \text{ is a projection onto } X\}$$

is finite. The proof shows that Z is isomorphic to H if and only if there is a constant $c > 0$ such that $\lambda(X, Z) \leq c$ for every finite dimensional subspace X of Z. Conversely, if Z is not Hilbertian, there will be a sequence of subspaces X_n of Z such that $\lambda(X_n, Z)$ tends to infinity together with $dim X_n$. If there is no projection in $\mathcal{L}(Z)$ from Z to X, like in the case $X = c_0, \ Z = \ell_\infty$, put $\lambda(X, Z) = \infty$. The *absolute projection constant* is defined as

$$\lambda(X) := \sup\{\lambda(X, Z) : \ Z \text{ is a Banach space containing } X \text{ as a subspace}\}.$$

Let $n := dim X_n < \infty$. Then $\lambda(X_n) < \infty$. In fact, by Kadec-Snobar [KS], $\lambda(X_n) \leq \sqrt{n}$ in this n-dimensional case. In this section, we give some general (almost) exact estimates for absolute and relative projection constants in terms of n, improving this estimate slightly. Estimates for $\lambda(X)$ are useful to construct extensions of operators into X (or from X) since

$$\lambda(X) = \inf\{\lambda > 0 : \forall Y \subseteq Z, \ T \in \mathcal{L}(Y, X) \ \exists \tilde{T} \in \mathcal{L}(Z, X), \ \tilde{T}|_Y = T, \ \|\tilde{T}\| \leq \lambda \|T\|\}.$$

This follows by embedding X isometrically into $\ell_\infty(I)$ and using the extension property of $\ell_\infty(I)$, i.e. coordinatewise application of the Hahn-Banach theorem. One gets for any such embedding $\lambda(X) = \lambda(X, \ell_\infty(I))$. Further,

$$\lambda(X) = \gamma_\infty(X) := \inf\{\|R\|\|S\| : \ R \in \mathcal{L}(X, \ell_\infty(I)), \ S \in \mathcal{L}(\ell_\infty(I), X), \ SR = Id_X.\}$$

As an example of $\lambda(X) > 1$, consider $X_2 = \{x \in \ell_\infty^3 \ : \ \sum_{i=1}^3 x_i = 0\} \subseteq \ell_\infty^3$. This 2-dimensional space has the regular hexagon as its unit ball. The orthogonal projection P from $\mathbb{R}^3$ onto X_2 is the minimal norm projection with $\lambda(X_2) = \|P\| = 4/3$.

To estimate projection constants, we use trace-duality. By ν and π_p we denote the nuclear and p-summing norms, respectively. For their basic properties, see [DiJP].

Lemma 6.1. *Let X and Y be finite dimensional with $X \subseteq Y$. Then*

$$\lambda(X, Y) = \sup\{|tr(T : X \mapsto X) : \ T \in \mathcal{L}(Y), \ \nu(T) = 1, \ T(X) \subseteq X\}.$$

Proof. Since for projections $P : Y \mapsto X$ onto X,

$$|tr(T : X \mapsto X)| = |tr(TP : Y \mapsto Y)| \leq \nu(TP) \leq \nu(T)\|P\| = \|P\|$$

only the inequality $\leq$ is non-trivial. There is a projection $P_0 : Y \mapsto X \subseteq Y$ onto X of minimal norm, $\|P_0\| = \lambda(X, Y)$. The convex sets

$$\mathcal{B} := \{S \in \mathcal{L}(Y) : \ \|S\| \leq \|P_0\|\}$$

and

$$\mathcal{P} := \{P \in \mathcal{L}(Y) : \ P = P_0 + \sum_{i=1}^{n} x_i^* \otimes x_i, \ n \in \mathbb{N}, \ x_i^* \in X^\perp \subseteq Y^*, \ x_i \in X\}$$

are disjoint since $\mathcal{P}$ consists of projections. Hence $\mathcal{B}$ and $\mathcal{P}$ can be separated by a linear functional on $\mathcal{L}(Y)$. By trace-duality, there is $T \in \mathcal{L}(Y)$ such that

$$Re(tr(TS)) \leq \|P_0\| \leq Re(tr(TP)), \ S \in \mathcal{B}, \ P \in \mathcal{P}.$$

This implies that $\|P_0\| = tr(TP_0)$ and

$$\nu(T) = \sup\{|tr(TS)|/\|S\| : \ 0 \neq S \in \mathcal{L}(Y)\} = 1.$$

Considering $P = P_0 + x^* \otimes x$ for $x^* \in X^\perp, \ x \in X$ yields that $T(X) \subseteq X$. $\square$

Let $X_n \subseteq Y_N$ with $dim X_n = n$ and $dim Y_N = N < \infty$. Using Lemma 6.1, it was shown in [KLL] that

$$\lambda(X_n, Y_N) \leq f(n, N) := \sqrt{n}(\sqrt{n}/N + \sqrt{(N-1)(N-n)}/N). \tag{6.1}$$

In particular, $\lambda(X_2, Y_3) \leq 4/3$ (equality in the above example). Cases of equality in (6.1) are discussed in [KLL]. Spaces with polytopes as unit balls embed isometrically into ℓ_∞^N, N finite. For operators T on ℓ_∞^N, $\nu(T) = \pi_1(T)$. Then Lemma 6.1 and an approximation argument imply

Corollary 6.2. *Let X be finite dimensional. Then*

$$\lambda(X) = \sup\{|tr(T : X \mapsto X)|; \ X \subseteq \ell_\infty, \ T \in \mathcal{L}(\ell_\infty), \ \pi_1(T) = 1, \ T(X) \subseteq X\}.$$

Since by [K2, prop.5] the ℓ_2-norm of the eigenvalues of T (on ℓ_∞) is bounded by $\pi_2(T) \leq \pi_1(T) = \nu(T) = 1$, and since the trace $tr(T : X \mapsto X)$ is the sum of n eigenvalues where $n = dim X$, Hölder's inequality yields $\lambda(X) \leq \sqrt{n}$, the result of Kadec-Snobar.

In the case of $X = \ell_p^n$, the map $T|_X$ is in fact the identity map and

$$\lambda(\ell_p^n)\pi_1(Id_{\ell_p^n}) = tr(I|_{\ell_p^n}) = n.$$

For $p = 2$, using the Grothendieck-Pietsch factorization theorem one can easily calculate the π_1-norm of ℓ_2^n, yielding $\lambda(\ell_2^n) \simeq \sqrt{\frac{2}{\pi}}\sqrt{n}$ for $\mathbb{K} = \mathbb{R}$. A similar fact holds for ℓ_1^n; both estimates were proved first by Grünbaum [Gr]. Thus ℓ_2^n is very badly complemented as a subspace of ℓ_∞; of course it is norm 1 complemented in any larger Hilbert space.

The Kadec-Snobar estimate was improved in [KT2]:

Theorem 6.3. *Let X_n be n-dimensional. Then*

$$\lambda(X_n) \leq g(n) := \begin{cases} (2+(n-1)\sqrt{n+2})/(n+1) & \text{if } \mathbb{K}=\mathbb{R} \\ (1+(n-1)\sqrt{n+1})/n & \text{if } \mathbb{K}=\mathbb{C}. \end{cases} \tag{6.2}$$

We remark that $g(n) = f(n, N(n))$ where $N(n) := n(n+1)/2$ if $\mathbb{K} = \mathbb{R}$, and $N(n) := n^2$ if $\mathbb{K} = \mathbb{C}$. There exist spaces X_n with equality $\lambda(X_n) = g(n)$ if and only if there exist $N(n)$ equiangular vectors $(x_s) \subseteq \mathbb{K}^n$, i.e. $\|x\|_2 = 1$ and $|(x_s, x_t)| = \alpha$ for α independent of $s, t = 1, ..., N(n)$ with $s \neq t$. If $\mathbb{K} = \mathbb{R}$, this is true for $n = 2, 3, 7, 23$, if $\mathbb{K} = \mathbb{C}$, for $n = 2, 3, 8$. For $\mathbb{K} = \mathbb{R}$, $n = 2, 3$, the unit balls of these X_n are the regular hexagon and the regular dodecahedron, respectively.

Note that asymptotically $g(n) = \sqrt{n} - \frac{1}{\sqrt{n}} + O(\frac{1}{n})$ if $\mathbb{K} = \mathbb{R}$ and $g(n) = \sqrt{n} - \frac{1}{2\sqrt{n}} + O(\frac{1}{n})$ if $\mathbb{K} = \mathbb{C}$. The estimate (6.2) is almost precise also if n is such that $N(n)$ equiangular vectors do not exist, as shown by examples in [K3], [KT1]:

Proposition 6.4. *(a) Let $n = q + 1$, q a prime power and $N = n^2 - n + 1$. Then there exist complex n-dimensional spaces $X_n \subseteq \ell_\infty^N$ with $\lambda(X_n) = f(n, N)$. Thus*

$$0 \leq g(n) - \lambda(X_n) \leq 1/(2n^{3/2}).$$

(b) Let $n = 4^m + 2^m + 1$ for $m \in \mathbb{N}$. Then there exist real n-dimensional subspaces X_n of $\ell_\infty^n(\ell_1^n)$ with a 1-unconditional basis and (relative) projection constant

$$\lambda(X_n) \geq \sqrt{n} - 1.$$

We now indicate the main steps of the

Proof of Theorem 6.3. (i) By approximation, we may assume that X is embedded isometrically into ℓ_∞^N for some finite $N \in \mathbb{N}$, $i : X_n \mapsto \ell_\infty^N$. By Corollary 6.2, there is $T \in \mathcal{L}(\ell_\infty^N)$, $\pi_1(T) = 1$ with $T(X_n) \subseteq X_n$ and $tr(T : X_n \mapsto X_n) = \lambda(X_n)$. The norm $\pi_1 = \nu$ on $\mathcal{L}(\ell_\infty^N)$ is the trace-dual of the operator norm, i.e. $\pi_1(T) = \sum_{s=1}^N \|Te_s\|_\infty$, where $e_s = (0, ..., 1, ...0)$ are the standard unit vectors. In fact, $\mu = (\mu_s)_{s=1}^N$, $\mu_s := \|Te_s\|_\infty$, is the (discrete) Pietsch probability measure on the extreme points $(e_s)_{s=1}^N$ of the unit ball in the dual $(\ell_\infty^N)^* = \ell_1^N$ with

$$\|Tx\|_\infty \leq \sum_{t=1}^N \mu_t |(x, e_t)|, \ x \in \ell_\infty^N. \tag{6.3}$$

Since $\|\mu\|_1 = 1$, the formal identity map $j : \ell_\infty^N \mapsto \ell_2^N(\mu)$ has norm 1. Choose an orthonormal basis $f_1, ..., f_n$ in $ji(X_n) \subseteq \ell_2^N(\mu)$. Then using (6.3)

$$\lambda(X_n) = tr(T|_{X_n}) = \sum_{u=1}^n \langle Tf_u, f_u \rangle_{\ell_2^N(\mu)} = \sum_{s=1}^N \mu_s \sum_{u=1}^n \overline{f_u(s)} (Tf_u)(s)$$

$$\leq \sum^N \mu_s \| \sum^n \overline{f_u(s)} T f_u \|_\infty = \sum^N \mu_s \| T(\sum^n \overline{f_u(s)} f_u) \|_\infty$$

$$\leq \sum_{s=1}^{N} \sum_{t=1}^{N} \mu_s \mu_t \Big| \sum_{u=1}^{n} \overline{f_u(s)} f_u(t) \Big| = n \sum_{s,t=1}^{N} \mu_s \mu_t |(x_s, x_t)| \tag{6.4}$$

where $x_s = \frac{1}{\sqrt{n}} (f_u(s))_{u=1}^{n} \in \mathbb{K}^n$. The ℓ_2-norm of the x_s is (up to $\sqrt{n}$) the square function of the f_u's, $\sqrt{n} \|x_s\|_2 = (\sum_{u=1}^{n} (f_u(s)^2)^{1/2}$. It is a non-trivial fact, proved by using the Lagrange multipliers in [KT2], that in the extremal case of spaces with maximal (relative) projection constant in ℓ_∞^N, the square function is constant, i.e. $\|x_s\|_2$ is independent of s. Since $f_1, ..., f_n$ were chosen of norm 1 in $\ell_2^N(\mu)$, we get that $\|x_s\|_2 = 1$. Hence $|(x_s, x_t)| \leq 1$.

(ii) Let $0 < \alpha < 1$. Then there is a unique polynomial p of the fourth order such that

$$|u| \leq p(u) = \gamma_0 + \gamma_2 u^2 - \gamma_4 u^4, \; u \in [-1, 1] \tag{6.5}$$

which touches u at α ($p(\alpha) = \alpha$, $p^{'}(\alpha) = 1$) and satisfies $p(1) = 1$. Here $\gamma_0, \gamma_2, \gamma_4$ depend on α but are positive. Estimating the right-hand side in (6.4), we find

$$\lambda(X_n) \leq n \sum_{s,t=1}^{N} \mu_s \mu_t (\gamma_0 + \gamma_2 |(x_s, x_t)|^2 - \gamma_4 |(x_s, x_t)|^4)$$

$$= n\gamma_0 + \gamma_2 - \gamma_4 \sum_{s,t=1}^{N} \mu_s \mu_t |(x_s, x_t)|^4.$$

Here $n \sum_{s,t=1}^{N} \mu_s \mu_t |(x_s, x_t)|^2 = 1$ in view of the orthogonality of the f_j's was used. By Sidelnikov's inequality used already in Section 3,

$$\sum_{s,t=1}^{N} \mu_s \mu_t |(x_s, x_t)|^4 \geq \int_{S^{n-1}} \int_{S^{n-1}} |(x, y)|^4 \, dm(x) \, dm(y) = \beta$$

where $\beta = 3/(n+2)$ if $\mathbb{K} = \mathbb{R}$ and $\beta = 2/(n+1)$ if $\mathbb{K} = \mathbb{C}$. Thus

$$\lambda(X_n) \leq n\gamma_0 + \gamma_2 - \gamma_4 \beta.$$

Since $\gamma_0, \gamma_2, \gamma_4$ depend on α, the right-hand side depends on α. It is minimal for $\alpha = 1/\sqrt{n+2}$ if $\mathbb{K} = \mathbb{R}$ and $\alpha = 1/\sqrt{n+1}$ if $\mathbb{K} = \mathbb{C}$ and gives the value $g(n)$ stated in Theorem 6.3. The proof shows that equality in the estimates requires that $|(x_s, x_t)| = \alpha$ or 1 since otherwise a strict inequality holds in (6.5): the vectors should be equiangular (the case of $|(x_s, x_t)| = 1$ for $s \neq t$ is a degenerate one); also one needs "many" such (x_s) to get a large double sum in (6.4). In $\mathbb{R}^2$, the 3 points $e_1, (-1/2e_1 \pm \sqrt{3}/2e_2)$ provide such equiangular vectors giving as X_2 the hexagonal space, in $\mathbb{R}^3$ the 6 diagonals of the regular icosahedron provide such points; the dual of the icosahedron, i.e. the dodecahedron, giving the unit ball of the extremal example X □

Idea for the examples in Proposition 6.4. In a converse to the previous proof, $N(n)$ equiangular vectors in $\mathbb{K}^n$ allow us to construct an n-dimensional space X_n with $\lambda(X_n) = g(n)$ by taking as its dual unit ball the absolutely convex hull of the points (x_s). In $\mathbb{C}^n$, $N(n) = n^2$. For the values of n in (a), $n^2 - n + 1$ (almost as many as n^2) equiangular vectors can be constructed by looking at certain exponentials using B_2-sequences. An easier construction of "almost" equiangular vectors (yielding slightly smaller projection constants) is given by the n^2 vectors in $\mathbb{C}^n$,

$$x_{(s_1,s_2)} = \frac{1}{\sqrt{n}}\big(\exp(\frac{2\pi i}{n}(s_1 j + s_2 j^2))\big)_{j=1}^n \in \mathbb{C}^n,$$

where $s_1, s_2 \in \{1, ..., n\}$ and n is a prime number. Gaussian sums yield

$$|(x_{(s_1,s_2)}, x_{(t_1,t_2)})| = \begin{cases} \frac{1}{\sqrt{n}}, & s_2 \neq t_2 \\ 0 & s_2 = t_2,\ s_1 \neq t_1 \end{cases}$$

cf. Proposition 3.12 and [KT1]. The real construction in (b) uses incidence matrix combinations for projective planes of order 2^m, cf. [K3]. □

Thus there are spaces X_n with a 1-unconditional basis with $\lambda(X_n)/\sqrt{n} \to 1$ as $n \to \infty$. For spaces with a 1-symmetric basis this cannot happen, see [KST]:

Theorem 6.5. *There is $0 < c < 1$ such that the projection constant of any n-dimensional space X_n with a 1-symmetric basis is bounded by*

$$\lambda(X_n) \le c\sqrt{n}.$$

Recall that a basis - which we may and will identify with the standard unit vector basis (e_j) in $\mathbb{K}^n$ - is 1-symmetric if the norm of a vector x does not change if the coordinates of x with respect to (e_j) are permuted or multiplied coordinatewise by scalars of modulus 1. For example, the spaces ℓ_p^n have a 1-symmetric basis.

Let G denote the symmetry group generated by the permutations and multiplication by scalars of modulus 1; in the real case G is discrete with $|G| = 2^n n!$. Let m denote the normalized Haar measure on G.

Idea of proof of Theorem 6.5. For symmetric spaces X_n, the estimate proceeds as in (6.4) except that we may average over G leading to

$$\lambda(X_n) \le n \sup\{\sum_{s,t=1}^N \mu_s \mu_t \int_G |(g(x_s), x_t)|\, dm(g)\} \tag{6.6}$$

where the sup is extended over finite sequences $(x_s)_1^N \subseteq S^{n-1}$ and $\sum_{s=1}^N \mu_s = 1$. Khintchine's inequality states that

$$\frac{1}{\sqrt{2}}\|x\|_2 \le \mathbb{E}\|\sum^n x_i r_i\|_{L_1} \le \|x\|_2,\ x = (x_i)_{i=1}^n \in \mathbb{R}^n.$$

For "most" x, the expectation in the center behaves like $\sqrt{2/\pi}\|x\|_2$; the following variant of Khintchine's inequality is proved in [KST]:

$$\big|\mathbb{E}\|\sum_{i=1}^{n} x_i r_i\|_{L_1} - \sqrt{\frac{2}{\pi}}\|x\|_2\big| \leq (1-\sqrt{\frac{2}{\pi}})\|x\|_\infty. \tag{6.7}$$

With a large constant c instead of $(1-\sqrt{\frac{2}{\pi}})$, this also follows from a corollary of the Barry-Esséen theorem. Averaging over the subgroup G_0 of sign changes of G, (6.7) yields for arbitrary $x, y \in S^{n-1}(\mathbb{R})$

(6.8)

$$\int_{G_0} |(gx,y)|\; dm_0(g) = \mathbb{E}\|\sum_{i=1}^{n} x_i y_i r_i\|_{L_1} \leq \sqrt{\frac{2}{\pi}}\|(x_i y_i)\|_2 + (1-\sqrt{\frac{2}{\pi}})\|(x_i y_i)\|_\infty.$$

Further averaging over permutations σ is required in (6.6), i.e. x_i should be replaced by $x_{\sigma(i)}$; however, as easily seen,

$$Average_\sigma \|x_{\sigma(i)} y_i\|_2 \leq \frac{1}{\sqrt{n}}.$$

If for the smaller ℓ_∞-norm, the corresponding average would be bounded by $\frac{d}{\sqrt{n}}$ with $d < 1$, (6.6) and (6.8) would imply Theorem 6.5 with $c = \sqrt{\frac{2}{\pi}} + (1-\sqrt{\frac{2}{\pi}})d < 1$. This is false, however, if x and y are of very different form. In the sum (6.6), however, "diagonal" terms involve twice the same vector $x = x_s$ $(s = t)$, and for these $x \in S^{n-1}$

$$Average_\sigma \|x_{\sigma(i)} x_i\|_2 \leq \frac{d}{\sqrt{n}}, \; d = 49/50, \tag{6.9}$$

cf. [KST]; the methods to prove (6.9) are similar to those employed by Kwapien-Schütt [KSch] to average norms over permutations.

If $x = x_s$ and $y = y_t$ have a rather similar form, (6.9) still holds for $(x_{\sigma(i)} y_i)$. Combining this estimate "close" to the diagonal with a (worse) general estimate "off" the diagonal enables us to prove Theorem 6.5 by using (6.6) and (6.8). It is important here that the constant $(1-\sqrt{\frac{2}{\pi}})$ is optimal in (6.7). In the complex case, the Rademacher variables r_i in (6.7) have to be replaced by Steinhaus variables and $\sqrt{2/\pi}$ by $\sqrt{\pi}/2$. $\square$

Letting $c = \overline{\lim}_n \{\lambda(X_n)/\sqrt{n} : \; X_n \text{ 1-symmetric }\}$, examples in [K2] show $c \geq (2-\sqrt{2/\pi})^{-1}$ if $\mathbb{K} = \mathbb{R}$ and $c \geq (2-\sqrt{\pi}/2)^{-1}$ if $\mathbb{K} = \mathbb{C}$. Problem: Are the latter inequalities in fact equalities?

Inequality (6.7) is used in [KST] also to prove that $\lambda(\ell_p^n)/\sqrt{n} \to d$ for $1 \leq p \leq 2$ and $n \to \infty$ where $d = \sqrt{2/\pi}$ if $\mathbb{K} = \mathbb{R}$ and $d = \sqrt{\pi}/2$ for $\mathbb{K} = \mathbb{C}$, independently of $1 \leq p \leq 2$.

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