INTERSECTION BODIES, POSITIVE DEFINITE DISTRIBUTIONS AND THE BUSEMANN-PETTY PROBLEM.

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Abstract. The 1956 Busemann-Petty problem asks whether symmetric convex bodies with larger central hyperplane sections also have greater volume. In 1988, Lutwak introduced the concept of an intersection body which is closely related to the Busemann-Petty problem. We prove that an origin-symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if $\|x\|_K^{-1}$ is a positive definite distribution on $\mathbb{R}^n$, where $\|x\|_K = \min\{a > 0 : \ x \in aK\}$. We use this result to show that for every dimension n there exist polytopes in $\mathbb{R}^n$ which are intersection bodies (for example, the cross-polytope), the unit ball of every subspace of L_p, $0 < p \le 2$ is an intersection body, the unit ball of the space ℓ_q^n, $2 < q < \infty$ is not an intersection body if $n \ge 5$. Using Lutwak's connection with the Busemann-Petty problem, we present new counterexamples to the problem for $n \ge 5$, and confirm the conjecture of Meyer that the answer to the problem is affirmative if the smaller body is a polar projection body.

1. Introduction

In 1956, Busemann and Petty [3] posed the following problem. Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ so that

$$vol_{n-1}(K \cap \theta^\perp) \le vol_{n-1}(L \cap \theta^\perp)$$

for every θ from the unit sphere S^{n-1} in $\mathbb{R}^n$, where $\theta^\perp = \{x \in \mathbb{R}^n : \ (x,\theta) = 0\}$ is the hyperplane perpendicular to θ, and vol_{n-1} is the $(n-1)$-dimensional volume. Does it follow that $vol_n(K) \le vol_n(L)$?

In 1988, Lutwak [16] introduced the concept of intersection body closely related to the Busemann-Petty problem. Let L be an origin-symmetric star body in $\mathbb{R}^n$. As usual, we denote by $\|x\|_L = \min\{a > 0 : \ x \in aL\}$ the norming functional on $\mathbb{R}^n$ generated by L. Throughout the paper, we assume that the norming functional of a star body is a continuous positive function on S^{n-1}. A body K in R^n is called the intersection body of the star body L if the radial function $\rho_K(x) = \|x\|_K^{-1}$ of K at every point $\theta \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of L by the hyperplane $\theta^\perp$. This can be written as follows: for every $\theta \in S^{n-1}$,

$$\|\theta\|_K^{-1} = vol_{n-1}(L \cap \theta^\perp) = \frac{1}{n-1}\int_{S^{n-1}\cap\theta^\perp} \|\xi\|_L^{-n+1}\, d\xi = R\Big(\frac{1}{n-1}\|\xi\|_L^{-n+1}\Big),$$

1991 *Mathematics Subject Classification.* 46B07, 52A20, 42B10.
Research supported in part by the NSF Grant DMS-9531594

where $\theta \mapsto Rf(\theta) = \int_{S^{n-1}\cap\theta^\perp} f(\xi)\, d\xi,\ \theta \in S^{n-1}$ is the spherical Radon transform defined for every continuous function f on S^{n-1}.

A body K in $\mathbb{R}^n$ is called an intersection body if there exists a finite Borel (non-negative) measure μ on S^{n-1} so that $\|\theta\|_K^{-1} = R\mu$, as functionals on $C(S^{n-1})$, where $R\mu$ is the finite Borel measure on S^{n-1} defined by

$$\langle R\mu, f\rangle = \langle \mu, Rf\rangle = \int_{S^{n-1}} Rf(\theta)\, d\mu(\theta)$$

for every $f \in C(S^{n-1})$. Clearly, every intersection body of a star body is an intersection body for which the corresponding measure μ has continuous density on S^{n-1}.

The results of Lutwak [16], slightly improved by Gardner [4] and Zhang [21,22], imply that if K is an intersection body then the Busemann-Petty problem has positive answer for every body L (we use the same notation as in the formulation of the Busemann-Petty problem). On the other hand, if L is not an intersection body and has C^2 boundary and positive curvature then there exists a body K so that K and L give a counterexample to the Busemann-Petty problem.

A negative answer to the Busemann-Petty problem for $n \geq 5$ was established in a sequence of papers by Larman and Rogers [14] (for $n \geq 12$), Ball [1] ($n \geq 10$), Giannopoulos [9] and Bourgain [2] ($n \geq 7$), Gardner [6] and Papadimitrakis [18] ($n \geq 5$.) Gardner [4] gave a positive answer to the problem for $n = 3$ by showing that every origin-symmetric convex body in $\mathbb{R}^3$ is an intersection body. Zhang [21] claimed that no cube in $\mathbb{R}^n$ with $n \geq 4$ is an intersection body of a star body and argued that this implies a negative answer to the problem for $n \geq 4$. In a later paper Zhang [22] used arguments similar to those from [21] to claim the result that no polytope in $\mathbb{R}^n$, $n \geq 4$ is an intersection body. Besides, [22] includes a counterexample to a conjecture of Meyer [17] that the answer to the Busemann-Petty problem is positive if K is a polar projection body (unit ball of a subspace of L_1).

Our results disprove the statements from [22] mentioned above. Firstly, we show that there exist polytopes in R^n, $n \geq 4$ which are intersection bodies. In Section 3, we present a simple calculation showing that the cross-polytope (unit ball of ℓ_1^n) is an intersection body for every $n \in \mathbb{N}$, and give a precise expression for the corresponding measure μ. In Section 4 we show that the answer to Meyer's question is positive and, moreover, the unit ball of every subspace of L_p, $0 < p \leq 2$ is an intersection body. Note that Meyer [17] has proved that the answer to the Busemann-Petty problem is positive if K is a cross-polytope, and proved the Busemann-Petty inequality up to an absolute constant for polar projection bodies. In Section 5 we prove that the unit ball of the space ℓ_q^n with $2 < q < \infty$ is not an intersection body if $n \geq 5$. In particular, this fact provides new counterexamples to the Busemann-Petty problem for every $n \geq 5$.

All these results are consequences of the following connection between intersection bodies and positive definite distributions: a star body K is an intersection body if and only if the function $\|x\|_K^{-1}$ is a positive definite distribution in $\mathbb{R}^n$. This fact, in conjunction with earlier developed techniques for calculating the Fourier transform of norm dependent functions [10,11], allows us to find precise expressions for generating measures (or signed measures) of certain star bodies.

2. Connection between intersection bodies and positive definite distributions

The main tool of this paper is the Fourier transform of distributions. As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable functions (test functions) in $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform $\hat{f}$ of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle$ for every test function ϕ. A distribution f is called even homogeneous of degree $p \in \mathbb{R}$ if $\langle f(x), \phi(x/t) \rangle = |t|^{n+p} \langle f, \phi \rangle$ for every test function ϕ and every $t \in \mathbb{R},\ t \neq 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$. A distribution f is called positive definite if, for every test function ϕ, $\langle f, \phi * \overline{\phi(-x)} \rangle \geq 0$. L. Schwartz's generalization of Bochner's theorem states that a distribution is positive definite if and only if it is the Fourier transform of a tempered measure in $\mathbb{R}^n$ ([8, p.152]). Recall that a (non-negative, not necessarily finite) measure μ is called tempered if

$$\int_{\mathbb{R}^n} (1 + \|x\|_2)^{-\beta} \, d\mu(x) < \infty$$

for some $\beta > 0$. Every positive distribution (in the sense that $\langle f, \phi \rangle \geq 0$ for every non-negative test function ϕ) is a tempered measure [8, p.147]. Therefore, a distribution is positive definite if and only if its Fourier transform is a positive distribution.

The well-known connection between the Radon transform and the Fourier transform is that, for every $\xi \in S^{n-1}$ and every even integrable function ϕ on $\mathbb{R}^n$, the function $t \to \hat{\phi}(t\xi)$ is the Fourier transform of the function $z \to \int_{(x,\xi)=z} \phi(x)\, dx$. In fact, making a change of variables $(\xi, x) = z$ we get

$$\hat{\phi}(t\xi) = \int_{\mathbb{R}^n} \phi(x) \exp(-it(\xi, x)) \, dx = \int_{\mathbb{R}} \exp(-itz) \, dz \int_{(\xi,x)=z} \phi(x) \, dx.$$

For every even distribution $f \in \mathcal{S}(\mathbb{R}^n)$ we have $(\hat{f})^\wedge = (2\pi)^n f$. Therefore, the Fourier transform of the distribution $t \to \hat{\phi}(t\xi)$ is equal to $2\pi \int_{(\xi,x)=z} \phi(x) \, dx$. If ϕ is an even function from $L_1(\mathbb{R}^n)$ and the function $t \to \hat{\phi}(t\xi)$ belongs to $L_1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \hat{\phi}(t\xi) \exp(-itz) \, dt = 2\pi \int_{(\xi,x)=z} \phi(x) \, dx, \tag{1}$$

We will use this formula mostly with $z = 0$.

Another simple fact, that we often use, is that the norming functional $\|x\|$ of every star body is equivalent to the Euclidean norm in the sense that, for every $x \in \mathbb{R}^n$, $K_1\|x\|_2 \le \|x\| \le K_2\|x\|_2$ for some positive constants K_1, K_2. Hence, $\|x\|^{-p}$ is a locally integrable function on $\mathbb{R}^n$ for every $p \in (0,n)$.

The following Fourier transform formula for sections of star bodies was shown in [13]: for every star body L and every $\theta \in S^{n-1}$

$$vol_{n-1}(L \cap \theta^\perp) = \frac{1}{\pi(n-1)}(\|x\|_L^{-n+1})^\wedge(\theta).$$

This implies that if K is the intersection body of a star body L then $\|u\|_K^{-1} = \frac{1}{\pi(n-1)}(\|x\|_L^{-n+1})^\wedge(u)$ for every $u \in \mathbb{R}^n$. We have $\|x\|_L^{-n+1} = \frac{\pi(n-1)}{(2\pi)^n}(\|u\|_K^{-1})^\wedge(x)$. Therefore, a star body K is an intersection body of a star body if and only if $\|x\|_K^{-1}$ is a distribution whose Fourier transform is a continuous positive function on $\mathbb{R}^n \setminus \{0\}$. In Theorem 1, we will show a more general result.

We need the following fact.

Lemma 1. *Let μ be a tempered measure on R^n, and suppose that μ is also an even homogeneous distribution of degree $-n+1$. Then there exists a finite Borel measure μ_0 on the sphere S^{n-1} so that for every even test function ϕ*

$$(\mu,\phi) = \int_{\mathbb{R}^n} \phi(x)\ d\mu(x) = \int_{S^{n-1}} d\mu_0(\theta) \int_{\mathbb{R}} \phi(t\theta)\ dt. \tag{2}$$

Proof. Let us first show that μ can not have an atom at the origin. In fact, suppose that $\mu = \mu_1 + a\delta$, where $\mu_1(\{0\}) = 0$, and δ is the unit mass at the origin. Since μ is homogeneous of degree $-n+1$, for every non-negative test function ϕ with $\phi(0) > 0$ and every $t > 0$, we have $(\mu, \phi(x/t)) = t(\mu,\phi) \to 0$ as $t \to 0$. On the other hand, $(\mu,\phi(x/t)) = (\mu_1, \phi(x/t)) + a\phi(0)$, so $a = 0$.

For every Borel subset $A \subset S^{n-1}$ and interval $(a,b] \in [0,\infty)$ denote by $A\times(a,b] = \{x \in R^n :\ x = t\theta,\ t \in (a,b],\ \theta \in A\}$, and let $\chi_{A\times(a,b]}$ be the indicator of this set.

By the definition of a homogeneous distribution, we have $(\mu,\phi(x/t)) = t(\mu,\phi)$ for every test function ϕ and $t > 0$. Since μ is a locally finite measure, we can approximate the functions $\chi_{A\times(a,b]}$ by test functions with compact support and use the dominated convergence theorem to show that

$$\mu(A\times[0,k]) = \big(\mu, \chi_{A\times[0,1]}(x/k)\big) = k\mu(A\times[0,1]).$$

Now, for every Borel subset $A \subset S^{n-1}$ and every $0 \le a < b$ we have $\mu(A\times(a,b]) = (b-a)\mu(A\times[0,1])$.

Define a measure μ_0 on S^{n-1} by $\mu_0(A) = \mu(A\times[0,1])$ for every Borel set $A \subset S^{n-1}$. Clearly,

$$\int_{S^{n-1}} d\mu_0(\theta) \int_{\mathbb{R}} \chi_{A\times(a,b]}(t\theta)\ dt = (b-a)\mu_0(A).$$

Therefore, we get the equality (2) with $\phi = \chi_{A\times(a,b]}$ and the result follows since A, a, b are arbitrary. $\square$

Theorem 1. *A star body K is an intersection body if and only if $\|x\|_K^{-1}$ is a positive definite distribution.*

Proof. Using (1) and spherical coordinates in the hyperplane $\theta^\perp$, for every even test function ϕ and every $\theta \in S^{n-1}$ we have

$$\int_{\mathbb{R}} \phi(t\theta)\ dt = \frac{1}{(2\pi)^n} \int_{\mathbb{R}} (\hat{\phi})^\wedge(t\theta)\ dt =$$

$$\frac{1}{(2\pi)^{n-1}} \int_{\theta^\perp} \hat{\phi}(x)\ dx = \frac{1}{(2\pi)^{n-1}} \int_{S^{n-1}\cap\theta^\perp} \Big(\int_0^\infty t^{n-2}\hat{\phi}(t\xi)\ dt \Big)\ d\xi. \tag{3}$$

Suppose that K is an intersection body, and let μ be the measure on S^{n-1} for which $R\mu = \|\theta\|_K^{-1}$. For every non-negative test function ϕ, using (3) and the fact that $\|x\|_K^{-1}$ is a locally integrable function, we get

$$\big((\|x\|_K^{-1})^\wedge, \phi\big) = \big(\|x\|_K^{-1}, \hat{\phi}\big) = \int_{\mathbb{R}^n} \|x\|_K^{-1}\hat{\phi}(x)\ dx =$$

$$\int_{S^{n-1}} \|\theta\|_K^{-1}\ d\theta \int_0^\infty t^{n-2}\hat{\phi}(t\theta)\ dt = \langle \|\theta\|_K^{-1}, \int_0^\infty t^{n-2}\hat{\phi}(t\theta)\ dt\rangle =$$

$$\langle R\mu, \int_0^\infty t^{n-2}\hat{\phi}(t\theta)\ dt\rangle = \langle \mu, R\Big(\int_0^\infty t^{n-2}\hat{\phi}(t\theta)\ dt\Big)\rangle =$$

$$\langle \mu, \int_{S^{n-1}\cap\theta^\perp} \Big(\int_0^\infty t^{n-2}\hat{\phi}(t\xi)\ dt\Big)\ d\xi\rangle = \langle \mu, \int_{\theta^\perp} \hat{\phi}(x)\ dx\rangle =$$

$$(2\pi)^{n-1}\langle \mu, \int_{\mathbb{R}} \phi(t\theta)\ dt\rangle \geq 0.$$

Therefore, $\|x\|_K^{-1}$ is a positive definite distribution.

Conversely, if $\|x\|_K^{-1}$ is a positive definite distribution then, by L. Schwartz's version of Bochner's theorem [8, p. 152], the Fourier transform of $\|x\|_K^{-1}$ is a tempered measure μ on $\mathbb{R}^n$. As the Fourier transform of a homogeneous distribution of degree -1, this measure is a homogeneous distribution of degree $-n+1$. By Lemma 1, there exists a finite Borel measure μ_0 on S^{n-1} so that for every test function ϕ

$$(\mu, \phi) = \int_{\mathbb{R}^n} \phi(x)\ d\mu(x) = \int_{S^{n-1}} d\mu_0(\theta) \int_{\mathbb{R}} \phi(t\theta)\ dt.$$

Now, by (1) and (2), for every test function ϕ

$$(\|x\|_K^{-1}, \phi) = \frac{1}{(2\pi)^n}\big((\|x\|_K^{-1})^\wedge, \hat{\phi}\big) = \frac{1}{(2\pi)^n}(\mu, \hat{\phi}) =$$

$$\frac{1}{(2\pi)^n} \int_{S^{n-1}} d\mu_0(\theta) \int_{\mathbb{R}} \hat{\phi}(t\theta)\ dt = \frac{1}{(2\pi)^{n-1}} \int_{S^{n-1}} d\mu_0(\theta) \int_{\theta^\perp} \phi(x)\ dx =$$

$$\frac{1}{(2\pi)^{n-1}}\langle \mu_0, R\big(\int_0^\infty t^{n-2}\phi(t\theta)\ dt\big)\rangle = \frac{1}{(2\pi)^{n-1}}\langle R\mu_0, \int_0^\infty t^{n-2}\phi(t\theta)\ dt\rangle.$$

On the other hand,

$$(\|x\|_K^{-1}, \phi) = \int_{S^{n-1}} \|\theta\|_K^{-1}\ d\theta \int_0^\infty t^{n-2}\phi(t\theta)\ dt.$$

Every infinitely differentiable function v on the sphere S^{n-1} can be represented in the form $\int_0^\infty t^{n-2}\phi(t\theta)\ dt$ for some $\phi \in \mathcal{S}(\mathbb{R}^n)$ (take $\phi(x) = u(t)v(\theta)$ for every $x = t\theta \in \mathbb{R}^n,\ t > 0,\ \theta \in S^{n-1}$, where $u \in \mathcal{S}(\mathbb{R})$ is a non-negative function supported outside of zero and $\int_{\mathbb{R}} t^{n-2}u(t)dt = 1$). Therefore, the measures $\|\theta\|_K^{-1}$ and $\frac{1}{(2\pi)^{n-1}}R\mu_0$, considered as functionals on $C(S^{n-1})$, coincide on infinitely differentiable functions on S^{n-1}. Hence, those functionals are equal, and K is an intersection body. □

3. The cross-polytope is an intersection body

In this section, we show directly (without using Theorem 1) that, for every $n \geq 2$, the cross-polytope $B_1 = \{x \in \mathbb{R}^n :\ \|x\|_1 = \sum |x_i| \leq 1\}$ is an intersection body.

Let us first note that the function $exp(-\|x\|_1)$ is the Fourier transform of the function $\phi(\xi) = \frac{1}{\pi^n}\prod_{k=1}^n \frac{1}{1+\xi_k^2}$, and both functions ϕ and $\hat{\phi}$ are integrable on $\mathbb{R}^n$. Using (1) and spherical coordinates in the hyperplane $(x,\xi) = 0$, we get

$$\|x\|_1^{-1} = \frac{1}{2}\int_{\mathbb{R}} exp(-|t|\|x\|_1)\ dt = \frac{1}{\pi^{n-1}}\int_{(x,\xi)=0}\prod_{k=1}^n \frac{1}{1+\xi_k^2}\ d\xi = \tag{4}$$

$$\frac{1}{\pi^{n-1}}\int_{S^{n-1}\cap x^\perp}\Big(\int_0^\infty t^{n-2}\prod_{k=1}^n \frac{1}{1+t^2\xi_k^2}\ dt\Big)\ d\xi.$$

This argument suggests that $\|x\|_1^{-1}$ is the spherical Radon transform of the function $h(\xi) = \frac{1}{\pi^{n-1}}\int_0^\infty t^{n-2}\prod_{k=1}^n \frac{1}{1+t^2\xi_k^2}\ dt,\ \xi \in S^{n-1}$. This function belongs to $L_1(S^{n-1})$, but it is not bounded, so B_1 is an intersection body, but not the intersection body of a star body. In this section we give a formal proof of this fact.

The function $\phi(\xi) = \frac{1}{\pi^n}\prod_{k=1}^n \frac{1}{1+\xi_k^2}$ is the density of the standard 1-stable measure γ_1 on $\mathbb{R}^n$ (the Cauchy distribution). A classical result of P. Levy [15] is that the image of this measure under the mapping $\xi \to (x,\xi)/\|x\|_1$ from $\mathbb{R}^n$ to $\mathbb{R}$ is equal to the measure $\nu(t) = \frac{1}{\pi(1+t^2)}\ dt$ on $\mathbb{R}$ independently of the choice of $x \in \mathbb{R}^n,\ x \neq 0$. In fact, fix $x \in \mathbb{R}^n,\ x \neq 0$ and denote by ν_x the measure on $\mathbb{R}$ given by $\nu_x(A) = \gamma_1\{\xi \in \mathbb{R}^n :\ (x,\xi)/\|x\|_1 \in A\}$ for every Borel set A in $\mathbb{R}$. Then, for every $k \in \mathbb{R}$

$$\exp(-|k|\|x\|_1) = \widehat{\gamma_1}(kx) = \int_{\mathbb{R}^n}\exp(-i(kx,\xi))\ d\gamma_1(\xi) =$$

$$\int_{\mathbb{R}^n}\exp(-ik\|x\|_1((x,\xi)/\|x\|_1))\ d\gamma_1(\xi) = \int_{\mathbb{R}}\exp(-ik\|x\|_1 y)\ d\nu_x(y) = \widehat{\nu_x}(k\|x\|_1)$$

We have $\widehat{\nu_x}(k\|x\|_1) = \exp(-|k|\|x\|_1)$ for every $k \in \mathbb{R}$, so $d\nu_x(t) = \frac{1}{\pi(1+t^2)}\ dt$ for every $x \in \mathbb{R}^n \setminus \{0\}$.

We need the following fact.

Lemma 2. *The function*

$$h(\xi) = \frac{1}{\pi^{n-1}} \int_0^\infty t^{n-2} \prod_{k=1}^n \frac{1}{1+t^2\xi_k^2}\, dt$$

is integrable on S^{n-1}).

Proof. The function $g(\xi) = \prod_{k=1}^n \frac{1}{1+\xi_k^2}$ belongs to $L_1(\mathbb{R}^n)$, therefore

$$\int_{\mathbb{R}^n} g(\xi)\, d\xi = \int_{S^{n-1}} d\xi \int_0^\infty t^{n-1} \prod_{k=1}^N \frac{1}{1+t^2\xi_k^2}\, dt < \infty,$$

so the function $\xi \to \int_0^\infty t^{n-1} \prod_{k=1}^n \frac{1}{1+t^2\xi_k^2}\, dt$ belongs to $L_1(S^{n-1})$. We have

$$\pi^{n-1} h(\xi) = \int_0^1 t^{n-2} \prod_{k=1}^n \frac{1}{1+t^2\xi_k^2}\, dt + \int_1^\infty t^{n-2} \prod_{k=1}^n \frac{1}{1+t^2\xi_k^2}\, dt \le$$

$$1 + \int_1^\infty t^{n-1} \prod_{k=1}^n \frac{1}{1+t^2\xi_k^2}\, dt \in L_1(S^{n-1}). \quad \square$$

Theorem 2. *The cross-polytope is an intersection body.*

Proof. Let ϕ be an arbitrary even test function and $\epsilon \in (0,1)$. Consider the integral

$$\epsilon \frac{1}{\pi^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(x,\xi)|^{-1+\epsilon} \prod_{k=1}^n \frac{1}{1+\xi_k^2} \phi(x)\, dx\, d\xi. \tag{5}$$

By the Fubini theorem, the integral (5) is equal to

$$\frac{1}{\pi^n} \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \epsilon |(x,\xi)|^{-1+\epsilon} \prod_{k=1}^n \frac{1}{1+\xi_k^2}\, d\xi \Big) \phi(x)\, dx =$$

$$\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \epsilon \big(\frac{|(x,\xi)|}{\|x\|_1}\big)^{-1+\epsilon} \frac{1}{\pi^n} \prod_{k=1}^n \frac{1}{1+\xi_k^2}\, d\xi \Big)\, \|x\|_1^{-1+\epsilon} \phi(x)\, dx.$$

Now we make a change of variables $t = (x,\xi)/\|x\|_1$ and apply the remark before Lemma 2 to see that the latter integral is equal to

$$(1/\pi) \int_{\mathbb{R}} \epsilon |t|^{-1+\epsilon} \frac{1}{1+t^2}\, dt \int_{\mathbb{R}^n} \|x\|_1^{-1+\epsilon} \phi(x)\, dx.$$

Both integrals converge absolutely, which justifies the use of the Fubini theorem earlier. The integral over $\mathbb{R}$ is equal to

$$2 \int_0^\infty \epsilon t^{-1+\epsilon} \frac{1}{1+t^2}\, dt = 2 \int_0^1 \epsilon t^{-1+\epsilon} \big(\frac{1}{1+t^2} - 1 \big)\, dt +$$

$$2\int_0^1 \epsilon t^{-1+\epsilon}\,dt + 2\epsilon\int_1^\infty |t|^{-1+\epsilon}\frac{1}{1+t^2}\,dt. \tag{6}$$

The second summand in (6) is equal to 2, the first and the third approach zero as $\epsilon \to 0$.

Since the function $\|x\|_1^{-1}$ is locally integrable on $\mathbb{R}^n$, we can apply the dominated convergence theorem to see that

$$\int_{\mathbb{R}^n} \|x\|_1^{-1+\epsilon}\phi(x)\,dx \to \int_{\mathbb{R}^n}\|x\|_1^{-1}\phi(x)\,dx,$$

as $\epsilon \to 0$. In fact, the functions under the integral in the left-hand side are dominated by an integrable function $\max(1,\|x\|_1^{-1})|\phi(x)|$. Therefore, the limit of the integral (5), as $\epsilon \to 0$, is equal to $(2/\pi)\int_{\mathbb{R}^n}\|x\|_1^{-1}\phi(x)\,dx$.

On the other hand, the integral (5) is equal to

$$\frac{1}{\pi^n}\int_{\mathbb{R}^n}\prod_{k=1}^n\frac{1}{1+\xi_k^2}\Big(\int_{\mathbb{R}^n}\epsilon|(x,\xi)|^{-1+\epsilon}\phi(x)\,dx\Big)\,d\xi.$$

Passing to the spherical coordinates in the outer integral and then making a change of variables $(x,\xi)=z$, we get

$$\frac{1}{\pi^n}\int_{S^{n-1}}\Big(\int_0^\infty t^{n-2+\epsilon}\prod_{k=1}^n\frac{1}{1+t^2\xi_k^2}\,dt\Big)\Big(\int_{\mathbb{R}^n}\epsilon|(x,\xi)|^{-1+\epsilon}\phi(x)\,dx\Big)\,d\xi =$$

$$\frac{1}{\pi^n}\int_{S^{n-1}}\Big(\int_0^\infty t^{n-2+\epsilon}\prod_{k=1}^n\frac{1}{1+t^2\xi_k^2}\,dt\Big)\Big(\int_{\mathbb{R}}\epsilon|z|^{-1+\epsilon}\Big(\int_{(x,\xi)=z}\phi(x)\,dx\Big)\,dz\Big)\,d\xi.$$

By the dominated convergence theorem (using estimates similar to those in Lemma 2), for almost every $\xi\in S^{n-1}$, the integral against dt converges to $\pi^{n-1}h(\xi)$, and it is dominated by an integrable (on S^{n-1}) function $1+\int_0^\infty t^{n-1}\prod_{k=1}^n\frac{1}{1+t^2\xi_k^2}\,dt$.

Suppose that the test function ϕ has compact support D. The function $\mathcal{R}(\xi;z) = \int_{(x,\xi)=z}\phi(x)\,dx$ is the Radon transform of the test function ϕ and has continuous derivatives of all orders on $S^{n-1}\times\mathbb{R}$. Similarly to (6), we have

$$\int_{\mathbb{R}}\epsilon|z|^{-1+\epsilon}\mathcal{R}(\xi;z)\,dz = 2\int_0^1\epsilon z^{-1+\epsilon}(\mathcal{R}(\xi;z)-\mathcal{R}(\xi;0))\,dz+ \tag{7}$$

$$2\int_0^1\epsilon z^{-1+\epsilon}\mathcal{R}(\xi;0)\,dz + 2\int_1^\infty\epsilon z^{-1+\epsilon}\mathcal{R}(\xi;z)\,dz.$$

The second summand is equal to $2\mathcal{R}(\xi;0)$. The first summand can be estimated by

$$2\epsilon\max_{\xi\in S^{n-1},\ z\in[0,1]}|\mathcal{R}_z'(\xi;z)|,$$

and the third is less or equal than $2\epsilon\max_{\xi\in S^{n-1}}\int_0^\infty|\mathcal{R}(\xi;z)|\,dz$. It follows that the integral (7) converges to $2\mathcal{R}(\xi;0)$ and is dominated by a constant function, as $\epsilon \to 0$.

Therefore, the integral (5) converges to $(2/\pi)\langle h(\xi), \int_{(x,\xi)=0} \phi(x)\ dx\rangle$, as $\epsilon \to 0$, and, as we showed earlier, the limit of (5) is also equal to $(2/\pi)\int_{\mathbb{R}^n} \|x\|_1^{-1}\phi(x)\ dx$.

Suppose that $\phi(x) = u(z)v(\theta)$ for every $x = z\theta \in \mathbb{R}^n,\ z \in \mathbb{R},\ \theta \in S^{n-1}$, where u is a non-negative test function on $\mathbb{R}$ with compact support outside of zero, and v is any infinitely differentiable function on S^{n-1}. Using spherical coordinates in the hyperplane $(x, \xi) = 0$, we get

$$\int_{(x,\xi)=0} \phi(x)\ dx = \int_0^\infty z^{n-2}u(z)\ dz \int_{S^{n-1}\cap(x,\xi)=0} v(\theta)\ d\theta.$$

On the other hand, using spherical coordinates in $\mathbb{R}^n$, we get

$$\int_{\mathbb{R}^n} \|x\|_1^{-1}\phi(x)\ dx = \int_0^\infty z^{n-2}u(z)\ dz \int_{S^{n-1}} \|\theta\|_1^{-1}v(\theta)\ d\theta.$$

It follows that $\langle h, Rv\rangle = \langle \|\theta\|_1^{-1}, v\rangle$ for every infinitely differentiable function v on S^{n-1}. We conclude that $\|\theta\|_1^{-1} = Rh$. $\square$

4. An affirmative answer to a question of Meyer.

Meyer [17] proved that the answer to the Busemann-Petty problem is positive if K is the cross-polytope and L is any origin-symmetric convex body. Now we can also see this fact as a consequence of Theorem 2 and Zhang's version of Lutwak's connection between intersection bodies and the Busemann-Petty problem mentioned in the Introduction ([22, Theorem 5].) Meyer [17] also proved the Busemann-Petty inequality up to a constant, not depending on the dimension, for unit balls of subspaces of L_p, $0 < p \leq 2$ and asked whether the constant can be removed. Our next result gives an affirmative answer to Meyer's question.

Theorem 3. *The unit ball of any n-dimensional subspace of L_q with $0 < q \leq 2$ is an intersection body.*

Proof. By a well-known result of P.Levy [15], for every n-dimensional subspace $B = (\mathbb{R}^n, \|\cdot\|)$ of L_q with $0 < q \leq 2$, the function $\exp(-\|x\|^q)$ is the Fourier transform of a q-stable symmetric measure μ on $\mathbb{R}^n$.

We have

$$\|x\|^{-1} = \frac{q}{\Gamma(1/q)} \int_0^\infty \exp(-t^q\|x\|^q)\ dt.$$

The function $\|x\|^{-1}$ is locally integrable in $\mathbb{R}^n$ and bounded at infinity, hence, for any test function ϕ the function $\|x\|^{-1}\hat{\phi}(x)$ is integrable on $\mathbb{R}^n$. Therefore, we can use the Fubini theorem in the following calculation:

$$((\|x\|^{-1})^\wedge, \phi) = \int_{\mathbb{R}^n} \|x\|^{-1}\hat{\phi}(x)\ dx =$$

$$\frac{q}{\Gamma(1/q)} \int \hat{\phi}(x)\ dx\Big(\int_0^\infty \exp(-t^q\|x\|^q)\ dt\Big) =$$

$$\frac{q}{\Gamma(1/q)}\int_0^\infty dt\int_{\mathbb{R}^n}\hat{\phi}(x)\exp(-t^q\|x\|^q)\ dx =$$

$$\frac{q}{\Gamma(1/q)}\int_0^\infty dt\int_{\mathbb{R}^n} t^{-n}\hat{\phi}(y/t)\exp(-\|y\|^q)\ dy. \tag{8}$$

The function $y\to t^{-n}\hat{\phi}(y/t)$ is the Fourier transform of the function $\xi\to\phi(t\xi)$. Therefore,

$$\int_{\mathbb{R}^n} t^{-n}\hat{\phi}(y/t)\exp(-\|y\|^q)\ dy = \big(\exp(-\|y\|^q), t^{-n}\hat{\phi}(y/t)\big) =$$

$$(2\pi)^n\big(\mu,\phi(t\xi)\big) = (2\pi)^n\int_{\mathbb{R}^n}\phi(tx)\ d\mu(x).$$

Now (8) can be written as

$$\big((\|x\|^{-1})^\wedge,\phi\big) = \frac{(2\pi)^n q}{\Gamma(1/q)}\int_0^\infty dt\int_{\mathbb{R}^n}\phi(tx)\ d\mu(x).$$

The latter equality shows that $\big((\|x\|^{-1})^\wedge,\phi\big)\geq 0$ if ϕ is any non-negative test function, so $(\|x\|^{-1})^\wedge$ is a positive distribution, and the result follows from Theorem 1. □

It follows from Theorem 3 and [22, Theorem 5] that

Theorem 4. *If K is the unit ball of any finite dimensional subspace of L_q, $0<q\leq 2$ then the answer to the Busemann-Petty problem is positive for any origin-symmetric convex body L.*

5. Unit balls of the spaces ℓ_q^n, $q>2$.

Let $\|x\|_q = (|x_1|^q+\cdots+|x_n|^q)^{1/q}$ be the norm of the space ℓ_q^n, $2<q<\infty$. Denote by γ_q the Fourier transform of the function $z\to\exp(-|z|^q)$, $z\in\mathbb{R}$. The properties of the functions γ_q were studied by Polya [19]. In particular, if q is not an even integer, the function $\gamma_q(t)$ behaves at infinity like $|t|^{-q-1}$. Namely (see [20, Part 3, Problem 154]),

$$\lim_{t\to\infty} t^{1+q}\gamma_q(t) = 2\Gamma(q+1)\sin(\pi q/2).$$

If q is an even integer, the function γ_q decreases exponentially at infinity. The integral

$$S_q(\alpha) = \int_{\mathbb{R}} |t|^\alpha\gamma_q(t)\ dt$$

converges absolutely for every $\alpha\in(-1,q)$. These moments can easily be calculated (see [23] or [10]; α is not an even integer):

$$S_q(\alpha) = 2^{\alpha+2}\pi^{1/2}\Gamma(-\alpha/q)\Gamma((\alpha+1)/2)/(q\Gamma(-\alpha/2)).$$

Clearly, the moment $S_q(\alpha)$ is positive if $\alpha\in(-1,0)\cup(0,2)$, and the moment is negative if $\alpha\in(2,\min(q,4))$.

The Fourier transform of the function $\|x\|^\beta$ was calculated in [10].

Lemma 3. *Let $q > 0$, $n \in \mathbb{N}$, $-n < \beta < qn$, $\beta/q \notin \mathbb{N} \cup \{0\}$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, $\xi_k \neq 0$, $1 \le k \le n$. Then*

$$(\|x\|_q^\beta)^\wedge(\xi) = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_q(t\xi_k)\, dt.$$

Proof. Assume that $-1 < \beta < 0$. By the definition of the Γ–function

$$(|x_1|^q + \cdots + |x_n|^p)^{\beta/q} = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty y^{-1-\beta} \exp(-y^q(|x_1|^q + \cdots + |x_n|^q))\, dy.$$

For every fixed $y > 0$, the Fourier transform of the function $x \to \exp(-y^q(|x_1|^q + \cdots + |x_n|^q))$ at any point $\xi \in \mathbb{R}^n$ is equal to $y^{-n} \prod_{k=1}^n \gamma_q(\xi_k/y)$. Making the change of variables $t = 1/y$ we get

$$((|x_1|^q + \cdots + |x_n|^q)^{\beta/q})^\wedge(\xi) = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty y^{-n-\beta-1} \prod_{k=1}^n \gamma_q(\xi_k/y)\, dy =$$

$$= \frac{q}{\Gamma(-\beta/q)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_q(t\xi_k)\, dt. \tag{9}$$

The latter integral converges if $-n < \beta < qn$ since the function $t \to \prod_{k=1}^n \gamma_q(t\xi_k)$ decreases at infinity like t^{-n-nq} (recall that $\xi_k \neq 0$, $1 \le k \le n$.)

If β is allowed to assume complex values then both sides of (9) are analytic functions of β in the domain $\{-n < Re\beta < nq,\ \beta/q \notin \mathbb{N} \cup \{0\}\}$. These two functions admit unique analytic continuation from the interval $(-1, 0)$. Thus the equality (9) remains valid for all $\beta \in (-n, qn)$, $\beta/q \notin \mathbb{N} \cup \{0\}$ (see [7] for details of analytic continuation in such situations). □

Let us prove that the function $(\|x\|_q^{-p})^\wedge$ is sign-changing for every $0 < p < n-3$. The following argument is similar to that used in the proof of the 1938 Schoenberg's conjecture on positive definite functions in [10].

Lemma 4. *If $q > 2$, $n > 3$, $p \in (0, n-3)$ then the distribution $\|x\|_q^{-p}$ is not positive definite.*

Proof. By Lemma 3 and properties of the moments $S_q(\alpha)$, the integral

$$I(\alpha_1, \dots, \alpha_{n-1}) = \int_{\mathbb{R}} |\xi_1|^{\alpha_1} \dots |\xi_{n-1}|^{\alpha_{n-1}} (\|x\|_q^{-p})^\wedge(\xi_1, ..., \xi_{n-1}, 1)\, d\xi_1 \dots d\xi_{n-1} =$$
$$S_q(\alpha_1) \dots S_q(\alpha_{n-1}) S_q(-\alpha_1 - \cdots - \alpha_{n-1} - p)$$

converges absolutely if the numbers $\alpha_1, \dots, \alpha_{n-1}, -\alpha_1 - \cdots - \alpha_{n-1} - p$ belong to the interval $(-1, q)$. Choosing $\alpha_k \in (-1, 0)$ for every $k = 1, ..., n-1$, we have the moments $S_q(\alpha_k)$, $k = 1, ..., n-1$ positive, and we can make $-\alpha_1 - \cdots - \alpha_{n-1} - p$ equal to any number from $(-p, n-1-p) \cap (-1, q)$. Since $0 < p < n-3$, this interval contains a neighborhood of 2, and, since the moment function S_q changes its sign at 2, we can make the integral $I(\alpha_1, \dots, \alpha_{n-1})$ positive for one choice of α's and negative for another choice. This means that the function $(\|x\|_q^{-p})^\wedge$ is sign-changing. □

Theorem 5. *If the body L is the unit ball of the space ℓ_q^n, $2 < q < \infty$, $n \geq 5$ then there exists K so that the answer to the Busemann-Petty problem in $\mathbb{R}^n$ is negative.*

Proof. For $n \geq 5$ the number 1 belongs to the interval $(0, n-3)$, so, by Lemma 4, the function $\|x\|_q^{-1}$ is not a positive definite distribution. Now the result follows from Theorem 1 and Lutwak's connection between intersection bodies and the Busemann-Petty problem in the form of Zhang [22, Theorem 6]. □

The case of the unit balls of the spaces ℓ_q^4, $2 < q \leq \infty$ requires very complicated calculations (see [11]) and will be considered in a forthcoming paper.

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