INTERSECTION BODIES IN $\mathbb{R}^4$

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Abstract. Intersection bodies were introduced in 1988 by Lutwak, who found a close connection between those bodies and the well-known 1956 Busemann-Petty problem asking whether origin symmetric convex bodies with larger central hyperplane sections also have greater volume. The author has recently shown that an origin symmetric star body K is an intersection body if and only if the function $\|x\|_K^{-1}$ is a positive definite distribution, where $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$. We use this result to prove that the unit balls of the spaces ℓ_q^4, $2 < q \leq \infty$, are intersection bodies. For $n \geq 5$ the unit balls of the spaces ℓ_q^n, $2 < q \leq \infty$, are not intersection bodies. The technique of this paper allows to find precise expressions for the generating measures (signed measures) of these bodies. The result that the unit cube in $\mathbb{R}^4$ is an intersection body shows that the negative solution to the Busemann-Petty problem for $n = 4$ in [20] was incorrect.

1. Introduction

The concept of an intersection body was introduced by Lutwak [16] in 1988. Let L be an origin-symmetric star body in $\mathbb{R}^n$. As usual, we denote by $\|x\|_L = \min\{a \geq 0 : \ x \in aL\}$ the norming functional on $\mathbb{R}^n$ generated by L. A body K in R^n is called the intersection body of the star body L if the radial function of K at every point u from the unit sphere Ω in $\mathbb{R}^n$ is equal to the $(n-1)$-dimensional volume of the section of L by the hyperplane $u^\perp = \{x \in \mathbb{R}^n : \ (x, u) = 0\}$. This can be written as follows: for every $u \in \Omega$,

$$\|u\|_K^{-1} = vol_{n-1}(L \cap u^\perp) = \frac{1}{n-1} \int_{\Omega \cap u^\perp} \|x\|_L^{-n+1} \, dx = R\Big(\frac{1}{n-1} \|x\|_L^{-n+1}\Big)(u),$$

where $u \mapsto Rf(u) = \int_{\Omega \cap u^\perp} f(x) \, dx, \ u \in \Omega$, is the spherical Radon transform defined for every continuous function f on Ω.

A body K in $\mathbb{R}^n$ is called an intersection body if there exists a finite Borel (non-negative) measure μ on Ω so that $\|x\|_K^{-1} = R\mu$ (as functionals on $C(\Omega)$), where $R\mu$ is the finite Borel measure on Ω defined by

$$\langle R\mu, f \rangle = \langle \mu, Rf \rangle = \int_\Omega Rf(\theta) \, d\mu(\theta)$$

for every $f \in C(\Omega)$. Clearly, every intersection body of a star body is an intersection body for which the corresponding measure μ has continuous density on Ω.

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Intersection bodies are closely related to the following well-known problem posed by Busemann and Petty [3] in 1956. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, $n \geq 3$, so that

$$vol_{n-1}(K \cap u^{\perp}) \leq vol_{n-1}(L \cap u^{\perp})$$

for every $u \in \Omega$. Does it follow that $vol_n(K) \leq vol_n(L)$?

The results of Lutwak [16], slightly improved by Gardner [4] and Zhang [20,21], imply that if K is an intersection body then the Busemann-Petty problem has a positive answer for every body L. On the other hand, if L is not an intersection body and has C^2 boundary and positive curvature then there exists a body K so that K and L give a counterexample to the Busemann-Petty problem.

The answer to the Busemann-Petty problem is negative if $n \geq 5$, and this was established in a series of papers by Larman and Rogers [15] (for $n \geq 12$), Ball [1] ($n \geq 10$), Giannopoulos [9] and Bourgain [2] ($n \geq 7$), Gardner [5] and Papadimitrakis [17] ($n \geq 5$). Gardner [4] proved that the answer to the problem is positive for $n = 3$.

We show in this article that, contrary to the claims in [20, 21], the cube in $\mathbb{R}^4$ is an intersection body. Thus, the Busemann-Petty problem is still open for $n = 4$. We also prove that the unit balls of the spaces ℓ_q^4, $2 < q < \infty$, are intersection bodies, and that for $n \geq 5$ the unit balls of the spaces ℓ_q^n, $2 < q \leq \infty$, are not intersection bodies.

Our arguments are based on the recent result from [14] that an origin symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|x\|_K^{-1}$ is a positive definite distribution on $\mathbb{R}^n$. It was also shown in [14] that the unit balls of finite dimensional subspaces of L_q with $0 < q \leq 2$ are intersection bodies. In particular, this is true for the unit balls of the spaces ℓ_q^n with $0 < q \leq 2$ and any dimension n. The techniques for calculating the Fourier transform of powers of norms, that we use in this paper, originate in [10,11,12]. The results of this article were announced in [13].

2. The unit cube in $\mathbb{R}^n$ is an intersection body if and only if $n \leq 4$.

We denote by $\|x\|_\infty = \max(|x_1|, \dots, |x_n|)$, $x \in \mathbb{R}^n$, the norm of the space ℓ_∞^n. In view of Theorem 1 from [14], to prove the statement in the title of this section it is sufficient to show that the function $\|x\|_\infty^{-1}$ is a positive definite distribution on $\mathbb{R}^n$ if and only if $n \leq 4$.

Throughout the paper, we use the fact that for every n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ and every $p \in (0, n)$ the function $\|x\|^{-p}$ is locally integrable in $\mathbb{R}^n$.

As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the space of infinitely differentiable rapidly decreasing functions (test functions) on $\mathbb{R}^n$, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. As in [6], the Fourier transform of a distribution f is defined by

$\langle \hat{f}, \hat{\phi}\rangle = (2\pi)^n \langle f, \phi\rangle$ for every test function ϕ. If f is an even distribution (which is the case almost everywhere in this paper), then $(\hat{f})^\wedge = (2\pi)^n f$. A simple consequence of Fubini's theorem is the following well-known connection between the Radon transform and the Fourier transform [8]: for every $\xi \in \Omega$, the function $t \to \hat{\phi}(t\xi)$ is the Fourier transform of the function $z \to \mathcal{R}\phi(\xi; z) = \int_{(x,\xi)=z} \phi(x)\, dx$ ($\mathcal{R}$ stands for the Radon transform). A distribution f is positive definite if and only if its Fourier transform is a positive distribution in the sense that $(\hat{f}, \phi) \geq 0$ for every non-negative test function ϕ (see [7, p.147-152]).

We start with the following simple fact.

Lemma 1. *Let $p_i,\ i \in \mathbb{N}$, be a sequence of numbers from the interval $(0, n)$ so that the limit $p = \lim_{i\to\infty} p_i$ exists and $0 < p < n$. Suppose that $(\mathbb{R}^n, \|\cdot\|)$ is an n-dimensional normed space so that for each $i \in \mathbb{N}$ the function $\|x\|^{-p_i}$ is a positive definite distribution on $\mathbb{R}^n$. Then the distribution $\|x\|^{-p}$ is also positive definite.*

Proof. We can assume that there exists $\epsilon > 0$ so that $0 < p_i < p+\epsilon < n$ for every i. Fix a non-negative test function ϕ. Then for every $i \in \mathbb{N}$ we have $\langle (\|x\|^{-p_i})^\wedge, \phi\rangle \geq 0$. Define a function g on $\mathbb{R}^n$ by $g(x) = \|x\|^{-p-\epsilon}|\hat{\phi}(x)|$ if $\|x\| \leq 1$, and $g(x) = |\hat{\phi}(x)|$ if $\|x\| > 1$. Since $\|x\|^{-p-\epsilon}$ is a locally integrable function, the function g is integrable on $\mathbb{R}^n$ and, for every $i \in \mathbb{N},\ x \in \mathbb{R}^n$, we have $g(x) \geq \|x\|^{-p_i}|\hat{\phi}(x)|$. By the dominated convergence theorem,

$$\langle (\|x\|^{-p})^\wedge, \phi\rangle = \int_{\mathbb{R}^n} \|x\|^{-p}\hat{\phi}(x)\ dx =$$

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} \|x\|^{-p_i}\hat{\phi}(x)\ dx = \lim_{i\to\infty} \langle (\|x\|^{-p_i})^\wedge, \phi\rangle \geq 0,$$

so $(\|x\|^{-p})^\wedge$ is a positive distribution. □

Lemma 2. *If $0 < p < n$ then the function*

$$g(\xi) = \int_0^\infty t^{-p-1} \prod_{k=1}^n \Big|\frac{\sin(t\xi_k)}{\xi_k}\Big|\ dt$$

is locally integrable on $\mathbb{R}^n$.

Proof. Since the function g is homogeneous of degree $-n+p \in (-n, 0)$, it is enough to show that g is integrable on the unit cube Q_n in $\mathbb{R}^n$. We have

$$\int_{Q_n} g(\xi)\ d\xi = \int_0^\infty t^{-p-1}\Big(\prod_{k=1}^n \int_{-1}^1 \Big|\frac{\sin t\xi_k}{\xi_k}\Big|\ d\xi_k\Big)\ dt =$$

$$\int_0^\infty t^{-p-1}\Big(\int_{-t}^t \Big|\frac{\sin u}{u}\Big|\ du\Big)^n\ dt < \infty,$$

because $-n-1 < -p-1 < -1$, and $\int_{-t}^t \left|\frac{\sin u}{u}\right| du$ is bounded by $2t$ at zero, and by $2 + 2\ln t$ at infinity. □

The function g is locally integrable and homogeneous of degree $-n+p$. Therefore, for every test function ϕ on $\mathbb{R}^n$, the function $g(\xi)|\phi(\xi)|$ is integrable on $\mathbb{R}^n$, which justifies the use of Fubini's theorem in the next result, and in Lemma 10 as well.

Lemma 3. *If $p \in (0,n)$ then the Fourier transform (in the sense of distributions) of the function $\|x\|_\infty^{-p}$ is equal to the locally integrable on $\mathbb{R}^n$ function*

$$\xi \mapsto 2^n p \int_0^\infty t^{-p-1} \prod_{k=1}^n \frac{\sin(t\xi_k)}{\xi_k}\, dt. \tag{1}$$

Proof. For every $x \in \mathbb{R}^n,\ x \neq 0$, we have

$$\|x\|_\infty^{-p} = p \int_0^\infty z^{p-1} \chi(z\|x\|_\infty)\, dz,$$

where χ is the indicator of $[-1,1]$, and the integral converges because $p > 0$. Clearly, $\chi(z\|x\|_\infty) = \prod_{k=1}^n \chi(zx_k)$. Therefore, for every $z \neq 0$, $(\chi(z\|x\|_\infty))^\wedge(\xi) = \prod_{k=1}^n \frac{2\sin(\xi_k/z)}{\xi_k}$. Since $0 < p < n$, for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ the integral

$$\langle (\|x\|_\infty^{-p})^\wedge, \phi\rangle = p \int_{\mathbb{R}^n} \hat{\phi}(x)\, dx \int_0^\infty z^{p-1}\chi(z\|x\|_\infty)\, dz \tag{2}$$

converges absolutely, and we can use the Fubini theorem, the definition of the Fourier transform of distributions and the change of variables $t = 1/z$ to show that the expression on the right-hand side of (2) is equal to

$$p \int_0^\infty z^{p-1}\, dz \big(\chi(z\|x\|_\infty), \hat{\phi}(x)\big)\, dz = p \int_0^\infty z^{p-1}\, dz \int_{\mathbb{R}^n} \prod_{k=1}^n \frac{2\sin(\xi_k/z)}{\xi_k}\, \phi(\xi)\, d\xi =$$

$$2^n p \int_{\mathbb{R}^n} \phi(\xi)\, d\xi \int_0^\infty t^{-p-1} \prod_{k=1}^n \frac{\sin(t\xi_k)}{\xi_k}\, dt. \quad \square$$

Since the Fourier transform of the distribution $\|x\|_\infty^{-p}$ is a locally integrable function on $\mathbb{R}^n$, for every $0 < p < n$, to show that $\|x\|_\infty^{-p}$ is a positive definite distribution, it is enough to prove that $(\|x\|_\infty^{-p})^\wedge$ is non-negative almost everywhere with respect to Lebesque measure in $\mathbb{R}^n$. On the other hand, to prove that $\|x\|_\infty^{-p}$ is not positive definite, it is sufficient to find an open set in $\mathbb{R}^n$ on which the Fourier transform is negative.

Our next step is to give a different expression for the Fourier transform of $\|x\|_\infty^{-p}$. The following calculation comes from [12].

Denote by G the set of vectors $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ such that $\xi_k \neq 0$ for $1 \le k \le n$ and $(\delta, \xi) \neq 0$ for every vector $\delta = (\delta_1, \dots, \delta_n)$, with $\delta_k = \pm 1$ for $1 \le k \le n$ (here (δ,ξ) stands for the scalar product in $\mathbb{R}^n$). We say that a function f on $\mathbb{R}$ has power growth at infinity if there exist $A, \rho > 0$ so that $|f(x)| \le A(1+x^2)^\rho$ for every $x \in \mathbb{R}$ with $\|x\|_\infty > 1$

Lemma 4. *Let f be an even continuous function on $\mathbb{R}$ with power growth at infinity so that the distribution $u = (f(t)(sgn(t))^{n-1})^\wedge$ is a continuous function on $\mathbb{R}\setminus\{0\}$. Then, for every $\xi \in G$, we have*

$$\big(f(\|x\|_\infty)\big)^\wedge(\xi) = \frac{(-1)^{n-1} i^{n-1}}{2\xi_1\cdot\ldots\cdot\xi_n}\sum_\delta \delta_1\cdot\ldots\cdot\delta_n\ (\delta,\xi)\ u((\delta,\xi)), \tag{3}$$

where the sum is taken over all changes of signs $\delta = (\delta_1,\ldots,\delta_n)$, $\delta_j = \pm 1$, $j = 1,\ldots,n$.

Proof. Let ϕ be a test function with compact support in G. Then there exists a test function F so that $\partial^n F/\partial x_1\ldots\partial x_n = \widehat{\phi}$. In fact, note that $\psi(x) = \phi(x)/(x_1\ldots x_n)$ belongs to $\mathcal{S}(\mathbb{R}^n)$, and hence $\widehat{\psi}\in\mathcal{S}(\mathbb{R}^n)$. Put $F = i^n\widehat{\psi}$ and now use the connection between the Fourier transform and differentiation:

$$\partial^n F/\partial x_1\ldots\partial x_n = i^n\partial^n\widehat{\psi}/\partial x_1\ldots\partial x_n = (x_1\ldots x_n\psi)^\wedge = \widehat{\phi}.$$

If ϕ is an even function then the function F is even if n is an even integer, and it is an odd function if n is an odd integer.

Denote by B_t the ball $\{x\in\mathbb{R}^n : \|x\|_\infty < t\}$. Now we start the calculation: for every even test function ϕ

$$\begin{aligned}\langle\big(f(\|x\|_\infty)\big)^\wedge,\phi\rangle &= \langle f(\|x\|_\infty),\widehat{\phi}\rangle\\ &= \int_{R^n} f(\|x\|_\infty)\widehat{\phi}(x)\,dx\\ &= \int_0^\infty f(t)\left(\int_{B_t}\widehat{\phi}(x)\,dx\right)_t'\,dt\\ &= \int_0^\infty f(t)\left(\int_{-t}^t\cdots\int_{-t}^t\widehat{\phi}(x)\,dx\right)_t'\,dt\\ &= \int_0^\infty f(t)\sum_\delta \delta_1\cdot\ldots\cdot\delta_n(F(\delta_1 t,\ldots,\delta_n t))_t'\,dt,\end{aligned}$$

which equals

$$\int_0^\infty f(t)\left(\sum_\delta \delta_1\cdot\ldots\cdot\delta_n(\delta_1\frac{\partial F}{\partial x_1}+\cdots+\delta_n\frac{\partial F}{\partial x_n})(\delta_1 t,\ldots,\delta_n t)\right)\,dt. \tag{4}$$

The function in the parentheses is even if n is an odd integer and odd if n is even. Therefore the integral in (4) is equal to

$$\frac{1}{2}\langle f(t)(\mathrm{sgn}(t))^{n-1},\sum\delta_1\cdot\ldots\cdot\delta_n\left(\delta_1\frac{\partial F}{\partial x_1}+\cdots+\delta_n\frac{\partial F}{\partial x_n}\right)(\delta_1 t,\ldots,\delta_n t)\rangle. \tag{5}$$

Since $\partial F/\partial x_j = i^{n-1}(x_j\phi(x)/(x_1\dots x_n))^\wedge$ for each j, we can use the connection between the Radon transform and the Fourier transform, mentioned in the beginning of this section, to rewrite (5) as follows:

$$\frac{i^{n-1}}{2}\langle f(t)(\mathrm{sgn}(t))^{n-1}, \sum_\delta \delta_1\cdot\ldots\cdot\delta_n \left(\sum_{j=1}^n \int_{(\delta,x)=y} \frac{\delta_j x_j \phi(x)}{x_1\dots x_n}\,dx\right)^\wedge \rangle(t),$$

which is equal to (note that for odd distributions h, $\langle\hat{h},\phi\rangle = -\langle h,\hat{\phi}\rangle$)

$$\frac{(-1)^{n-1}i^{n-1}}{2}\langle (f(t)(\mathrm{sgn}(t))^{n-1})^\wedge(y), y\sum_\delta \delta_1\cdot\ldots\cdot\delta_n \int_{(\delta,x)=y}\frac{\phi(x)}{x_1\dots x_n}\,dx\rangle.$$

The distribution $u = (f(t)(\mathrm{sgn}(t))^{n-1})^\wedge$ is a continuous function on $\mathbb{R}\setminus\{0\}$, therefore for every $\phi\in\mathcal{S}(\mathbb{R}^n)$ with compact support in G we have

(6)
$$\langle f(\|x\|_\infty)^\wedge,\phi\rangle = \frac{(-1)^{n-1}i^{n-1}}{2}\int_{\mathbb{R}}\left(\sum_\delta \delta_1\cdot\ldots\cdot\delta_n\int_{(\delta,x)=y}\frac{\phi(x)}{x_1\dots x_n}\,dx\right) y u(y)\,dy.$$

The latter integral converges absolutely because all the functions

$$y \to \int_{(\delta,x)=y}\frac{\phi(x)}{x_1\dots x_n}\,dx \qquad (y\in\mathbb{R})$$

belong to $\mathcal{S}(\mathbb{R})$ and have compact support in $\mathbb{R}\setminus\{0\}$. By Fubini's theorem, the integral on the right-hand side of (6) is equal to

$$\frac{(-1)^{n-1}i^{n-1}}{2}\int_{\mathbb{R}^n}\left(\sum_\delta \delta_1\cdot\ldots\cdot\delta_n\frac{(\delta,x)}{x_1\dots x_n}u((\delta,x))\right)\phi(x)\,dx,$$

which completes the proof. □

The Fourier transform of the functions $t\to|t|^p$ and $t\to|t|^p sgn(t)$, $t\in\mathbb{R}$, can be found in [6 ,p.173]: if p is not an integer then, for every $y\in\mathbb{R}\setminus\{0\}$

$$(|t|^p)^\wedge(y) = \frac{2^{p+1}\sqrt{\pi}\,\Gamma((p+1)/2)}{\Gamma(-p/2)}|y|^{-1-p}$$

and

$$(|t|^p sgn(t))^\wedge(y) = \frac{i2^{p+1}\sqrt{\pi}\,\Gamma((p+2)/2)}{\Gamma((-p+1)/2)}|y|^{-1-p}sgn(y).$$

By applying Lemma 4 to the function $f(t)=|t|^p$, where $p>0$, p is not an integer, we get expressions which can be extended analytically by p to negative values of p that are not integers. Details of analytic continuation can be found in [6], where this method is used in almost every calculation of the Fourier transform.

We get the following

Lemma 5. *Let $p > 0$, p is not an integer, and $\xi \in \mathbb{R}^n$ is a vector from G. Then, if n is odd we have*

$$(\|x\|_\infty^{-p})^\wedge(\xi) =$$

$$\frac{(-1)^{\frac{n-1}{2}} 2^{-p}\sqrt{\pi}\ \Gamma(\frac{-p+1}{2})}{\xi_1 \dots \xi_n\ \Gamma(p/2)} \sum_\delta \delta_1 \dots \delta_n\ |(\delta,\xi)|^p sgn((\delta,\xi)). \tag{7}$$

If n is even

$$(\|x\|_\infty^{-p})^\wedge(\xi) = \frac{(-1)^{\frac{n}{2}+1} 2^{-p}\sqrt{\pi}\ \Gamma((-p+2)/2)}{\xi_1 \dots \xi_n\ \Gamma((p+1)/2)} \sum_\delta \delta_1 \dots \delta_n\ |(\delta,\xi)|^p, \tag{8}$$

where the sum is taken over all choices of signs $\delta = (\delta_1, \dots, \delta_n)$, $\delta_k = \pm 1$, $k = 1, ..., n$.

It is possible to get the result of Lemma 5 by performing the integration in formula (1). First let $-1 < p < 0$ and use the representation of the product $\prod_{k=1}^n \sin(t\xi_k)$ as a sign-changing sum of sins or cosins to calculate the integral (1). Then extend the formula analytically by p. This allows us to double-check the result of Lemma 5.

Let us find the signs of the sums appearing in Lemma 5. For $n \geq 2$ and $p \in \mathbb{R}$ let

$$u_{n,p}(\xi_1, ..., \xi_n) = \sum_\delta \delta_1 \dots \delta_n\ |(\delta,\xi)|^p \mathrm{sgn}((\delta,\xi))$$

if n is an odd integer, and

$$u_{n,p}(\xi_1, ..., \xi_n) = \sum_\delta \delta_1 \dots \delta_n\ |(\delta,\xi)|^p$$

if n is an even integer.

Lemma 6. *Let $n > 3$ and $0 < p < n-3$, or $n = 3$ and $p < 0$, where p is not an integer. Then the function $u_{n,p}$ is sign-changing on $\mathbb{R}_+^n = \{\xi \in \mathbb{R}^n :\ \xi_k > 0,\ k = 1, ..., n\}$.*

Proof. First, let $n = 3$. Then, for every $p < 0$, the number $u_{3,p}(3,1,1)$ is positive, but the number $u_{3,p}(1,3,3)$ is negative.

Now we deduce the result for $n > 3$ and $0 < p < n-3$ from the three-dimensional case. In fact, put $\xi_{n-2} = 3,\ \xi_{n-1} = \xi_n = 1$. Then the limit

$$\lim_{\xi_1 \to +0, \dots, \xi_{n-3} \to +0} \frac{1}{\xi_1 \dots \xi_{n-3}} u_{n,p}(\xi)$$

is equal, up to a non-zero constant, to $u_{3,p-n+3}(3,1,1)$. If $\xi_{n-2} = 1,\ \xi_{n-1} = \xi_n = 3$, the same limit is equal (up to the same non-zero constant) to $u_{3,p-n+3}(1,3,3)$. Since $p - n + 3 < 0$, this shows that, if we make $\xi_1, \dots, \xi_{n-3}$ small enough, the values of the function $u_{n,p}$ at the points $(\xi_1, \dots, \xi_{n-3}, 3, 1, 1)$ and $(\xi_1, \dots, \xi_{n-3}, 1, 3, 3)$ have opposite signs. $\square$

Lemma 7. *Let $n \geq 2$ and $p \in (n-3, n)$, p is not an integer. Then $u_{n,p}$ is a positive function on $\mathbb{R}^n_+$ if $p \in (n-2, n)$, and $u_{n,p}$ is a negative function on $\mathbb{R}^n_+$ if $p \in (n-3, n-2)$.*

Proof. We argue by induction. The case $n = 2$ is trivial. Let $\xi \in \mathbb{R}^n_+$. Without loss of generality we can assume that $\xi_1 \geq \xi_n$. Then

$$u_{n,p}(\xi_1, ..., \xi_n) = p \int_{-\xi_n}^{\xi_n} u_{n-1,p-1}(\xi_1 + x, \xi_2, ..., \xi_{n-1})\, dx.$$

Since $\xi_1 + x \geq 0$, the result follows from the induction hypothesis. □

Theorem 1. *Let $0 < p < n$, $n \geq 3$. The function $\|x\|_\infty^{-p}$ is a positive definite distribution if $p \in [n-3, n)$, and it is not positive definite if $p \in (0, n-3)$. In particular, the unit cube is an intersection body in $\mathbb{R}^n$ if and only if $n \leq 4$.*

Proof. Suppose that $n - 3 < p < n$ and p is not an integer. Then, by Lemmas 5 and 7, the function $(\|x\|_\infty^{-p})^\wedge$ (which is even by each variable, so it is enough to have the result of Lemma 7 on $\mathbb{R}^n_+$ only) is non-negative almost everywhere (with respect to Lebesque measure) on $\mathbb{R}^n$ and is locally integrable by Lemmas 2 and 3. It is easy to check that the coefficients in (7) and (8) agree with the result of Lemma 7, so that the expressions in (7) and (8) are non-negative. This, however, can be proved in a different way. In fact, by Lemma 7, the expressions in (7) and (8) (for the corresponding values of n) are either positive everywhere or negative everywhere in G, depending on the coefficient. However, the Fourier transform of $\|x\|_\infty^{-p}$ with $p \in (0, n)$ cannot be negative almost everywhere in $\mathbb{R}^n$, because its value on the standard Gaussian density γ in $\mathbb{R}^n$ (which is a test function whose Fourier transform is equal to itself, up to a positive constant) is positive. Therefore, for every $p \in (n-3, n)$ which is not an integer, $(\|x\|_\infty^{-p})^\wedge$ is a positive distribution. By Lemma 1, the same is true for $p = n-3, n-2, n-1$.

Let $0 < p < n-3$. By Lemma 6, if p is not an integer then the function $(\|x\|_\infty^{-p})^\wedge$ has opposite signs at two different points, and the function is continuous in neighborhoods of those points, so $(\|x\|_\infty^{-p})^\wedge$ is not a positive distribution. We can show the same thing using a different argument which also applies to the integers p. In fact, if for some $0 < p < n-3$ the function $(\|x\|_\infty^{-p})^\wedge$ is non-negative almost everywhere, then, by Lemma 10 below, so is the function $(\|x\|_q^{-p})^\wedge$ with $q > 2$, which contradicts Lemma 9 below.

Clearly, $1 \in [n-3, n)$ if and only if $n = 3$ or $n = 4$ which, in conjunction with Theorem 1 from [14], shows that the unit cubes in $\mathbb{R}^3$ and $\mathbb{R}^4$ are intersection bodies, but the unit cube in $\mathbb{R}^n$ with $n \geq 5$ is not an intersection body. □

3. Unit balls of the spaces ℓ_q^n, $2 < q < \infty$

Let $\|x\|_q = (|x_1|^q + \cdots + |x_n|^q)^{1/q}$ be the norm of the space ℓ_q^n, $2 < q < \infty$. Denote by γ_q the Fourier transform of the function $z \mapsto \exp(-|z|^q)$, $z \in \mathbb{R}$. The properties of the functions γ_q were studied by Pólya [18]. In particular, if q is not an even integer, the function $\gamma_q(t)$ behaves at infinity like $|t|^{-q-1}$. Namely (see [19, Part 3, Problem 154]),

$$\lim_{t\to\infty} t^{1+q}\gamma_q(t) = 2\Gamma(q+1)\sin(\pi q/2).$$

If q is an even integer, the function γ_q decreases exponentially at infinity. The integral

$$S_q(\alpha) = \int_{\mathbb{R}} |t|^\alpha \gamma_q(t)\ dt$$

converges absolutely for every $\alpha \in (-1,q)$. These moments can easily be calculated (see [22] or [10]; α is not an even integer):

$$S_q(\alpha) = 2^{\alpha+2}\pi^{1/2}\Gamma(-\alpha/q)\Gamma((\alpha+1)/2)/(q\Gamma(-\alpha/2)).$$

Clearly, the moment $S_q(\alpha)$ is positive if $\alpha \in (-1,0)\cup(0,2)$, and the moment is negative if $\alpha \in (2,\min(q,4))$.

The Fourier transform of the function $\|x\|_q^\beta$ was calculated in [10]. We, however, repeat it here to make the paper self-contained.

Lemma 8. *Let $q>0$, $n\in\mathbb{N}$, $-n<\beta<qn$, $\beta/q\notin \mathbb{N}\cup\{0\}$, $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$, $\xi_k\neq 0$, $1\le k\le n$. Then*

$$(\|x\|_q^\beta)^\wedge(\xi) = \frac{q}{\Gamma(-\beta/q)}\int_0^\infty t^{n+\beta-1}\prod_{k=1}^n \gamma_q(t\xi_k)\,dt.$$

Proof. Assume that $-1<\beta<0$. By the definition of the Γ–function

$$(|x_1|^q+\dots+|x_n|^q)^{\beta/q} = \frac{q}{\Gamma(-\beta/q)}\int_0^\infty y^{-1-\beta}\exp(-y^q(|x_1|^q+\dots+|x_n|^q))\,dy.$$

For every fixed $y>0$, the Fourier transform of the function $x\mapsto \exp(-y^q(|x_1|^q+\dots+|x_n|^q))$ at any point $\xi\in\mathbb{R}^n$ is equal to $y^{-n}\prod_{k=1}^n\gamma_q(\xi_k/y)$. Making the change of variables $t=1/y$ we get

$$((|x_1|^q+\dots+|x_n|^q)^{\beta/q})^\wedge(\xi) = \frac{q}{\Gamma(-\beta/q)}\int_0^\infty y^{-n-\beta-1}\prod_{k=1}^n\gamma_q(\xi_k/y)\,dy =$$

$$= \frac{q}{\Gamma(-\beta/q)}\int_0^\infty t^{n+\beta-1}\prod_{k=1}^n\gamma_q(t\xi_k)\,dt. \tag{9}$$

The latter integral converges if $-n<\beta<qn$ since the function $t\to\prod_{k=1}^n\gamma_q(t\xi_k)$ decreases at infinity like t^{-n-nq} (recall that $\xi_k\neq0$, $1\le k\le n$.)

If β is allowed to assume complex values then both sides of (9) are analytic functions of β in the domain $\{-n<Re\beta<nq,\ \beta/q\notin\mathbb{N}\cup\{0\}\}$. These two functions admit unique analytic continuation from the interval $(-1,0)$. Thus the equality (9) remains valid for all $\beta\in(-n,qn)$, $\beta/q\notin\mathbb{N}\cup\{0\}$ (see [6] for details of analytic continuation in such situations). □

It is easy to see that if $p \in (0,n)$ then the function

$$h(\xi) = \int_0^\infty t^{n-p-1} \prod_{k=1}^n \left|\gamma_q(t\xi_k)\right| dt$$

is locally integrable on $\mathbb{R}^n$. In fact, since this function is homogeneous of degree $-n+p$ it is enough to show that it is integrable on the unit cube Q_n in $\mathbb{R}^n$, and this can be done in a way similar to Lemma 2. Recall that the functions γ_q are bounded and behave at infinity like $|t|^{-1-q}$. Also, since the function h is locally integrable and homogeneous of degree $-n+p$, the function $h(\xi)|\phi(\xi)|$ is integrable on $\mathbb{R}^n$ for every test function ϕ.

Let us prove that the function $(\|x\|_q^{-p})^\wedge$ changes its sign if $0 < p < n-3$. The following argument is similar to that used in the proof of Schoenberg's conjecture on positive definite functions in [10].

Lemma 9. *If* $q > 2,\ n > 3,\ p \in (0, n-3)$ *then the distribution* $\|x\|_q^{-p}$ *is not positive definite.*

Proof. By Lemma 8 and properties of the moments $S_q(\alpha)$, the integral

$$I(\alpha_1,\dots,\alpha_{n-1}) = \int_{\mathbb{R}^{n-1}} |\xi_1|^{\alpha_1}\dots|\xi_{n-1}|^{\alpha_{n-1}} (\|x\|_q^{-p})^\wedge(\xi_1,...,\xi_{n-1},1)\ d\xi =$$

$$(q/\Gamma(p/q))\ S_q(\alpha_1)\dots S_q(\alpha_{n-1}) S_q(-\alpha_1-\dots-\alpha_{n-1}-p)$$

converges absolutely if the numbers $\alpha_1,\dots,\alpha_{n-1}, -\alpha_1-\dots-\alpha_{n-1}-p$ belong to the interval $(-1,q)$. Choosing $\alpha_k \in (-1,0)$ for every $k=1,...,n-1$, we have the moments $S_q(\alpha_k),\ k=1,...,n-1$ positive, and we can make $-\alpha_1-\dots-\alpha_{n-1}-p$ equal to any number from $(-p,n-1-p)\cap(-1,q)$. This interval contains a neighborhood of 2, and, since the moment function S_q changes its sign at 2, we can make the integral $I(\alpha_1,\dots,\alpha_{n-1})$ positive for one choice of α's and negative for another choice. This means that the function $(\|x\|_q^{-p})^\wedge$ is sign-changing. $\square$

To show that, for $p \in [n-3, n)$, the function $(\|x\|_q^{-p})^\wedge$ is positive almost everywhere, we first express this function in terms of the function $(\|x\|_\infty^{-p})^\wedge$.

Lemma 10. *Let* $q > 2,\ p \in (0,n)$. *Then, for every* $\xi \in \mathbb{R}^n$ *with non-zero coordinates,*

$$(\|x\|_q^{-p})^\wedge(\xi) = \frac{q^{n+1}}{p\Gamma(p/q)} \int_0^\infty \cdots \int_0^\infty$$

$$(t_1\dots t_n)^q \exp(-\|t\|_q^q)\ (\|x\|_\infty^{-p})^\wedge(t_1\xi_1,...,t_n\xi_n)\ dt_1...dt_n.$$

Proof. For every $x \in \mathbb{R}$, we have

(9) $$\exp(-|x|^q) = q \int_0^\infty \chi(ux)\ u^{-1-q} \exp(-u^{-q})\ du,$$

where, as before, χ is the indicator of $[-1,1]$.

If $u \in BbbR,\ u \neq 0$ then the Fourier transform of the function $x \mapsto \chi(ux)$ is equal to $(\chi(ux))^\wedge(\xi) = 2\sin(\xi/u)/\xi$. Calculating the Fourier transforms of both sides of (9) and making the change of variables $t = 1/u$, we get an integral representation for the function γ_q : for every $\xi \in \mathbb{R}$,

$$\gamma_q(\xi) = 2q\int_0^\infty \frac{\sin(t\xi)}{t\xi} t^q \exp(-t^q)\ dt.$$

By Lemma 8 and the remark after Lemma 2,

$$(\|x\|_q^{-p})^\wedge(\xi) = \frac{q}{\Gamma(p/q)}\int_0^\infty z^{n-p-1}\prod_{k=1}^n \gamma_q(z\xi_k)\,dz =$$

$$\frac{2^n q^{n+1}}{\Gamma(p/q)}\int_0^\infty \cdots \int_0^\infty (t_1\dots t_n)^q \exp(-\|t\|_q^q)\Big(\int_0^\infty z^{n-p-1}\prod_{k=1}^n \frac{\sin(t_k\xi_k z)}{t_k\xi_k z}\ dz\Big)\ dt,$$

and the result follows from Lemma 3. □

Theorem 2. *Let $0 < p < n,\ n \geq 3$. If $2 < q < \infty$ then $\|x\|_q^{-p}$ is a positive definite distribution if $p \in [n-3,n)$, and it is not positive definite if $p \in (0,n-3)$. Therefore the unit ball of the space ℓ_q^n is an intersection body if and only if $n \leq 4$.*

Proof. If $p \in (n-3,n)$ and p is not an integer, then the Fourier transform of $\|x\|_\infty^{-p}$ is non-negative, and, by Lemma 10, the function $(\|x\|_q^{-p})^\wedge$ is non-negative almost everywhere. It is also locally integrable by the remark after Lemma 8. Therefore, $(\|x\|_q^{-p})^\wedge$ is a positive distribution. One can use Lemma 1 to add the integers $n-3, n-2$ and $n-1$.

In the case where $0 < p < n-3$, the result follows from Lemma 9. Since $1 \in [n-3,n)$ if and only if $n \leq 4$, by Theorem 1 from [14], the unit ball of ℓ_q^n is an intersection body if and only if $n \leq 4$. □

By Lutwak's connection between intersection bodies and the Busemann-Petty problem in the form of Zhang [21, Theorems 5 and 6], we get the following:

Corollary. *The answer to the Busemann-Petty problem is positive if the body K is the unit ball of one of the spaces $\ell_q^n,\ 2 < q \leq \infty,\ n \leq 4$. If the body L is the unit ball of the space $\ell_q^n,\ 2 < q < \infty,\ n \geq 5$, then there exists K so that the answer to the Busemann-Petty problem is negative.*

The latter result on the unit balls of the spaces $\ell_q^n,\ n \geq 5,\ 2 < q < \infty$, can be significantly generalized. In a forthcoming paper, we will present a simple test for intersection bodies in terms of the second derivative of the norm and exhibit a rather large class of bodies in $\mathbb{R}^n,\ n \geq 5$, that are not intersection bodies. This class includes, for example, the q-sum of any normed spaces X and Y with $q > 2$ and $dim(X) \geq 4,\ dim(Y) \geq 1.$

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