BANACH SPACES EMBEDDING ISOMETRICALLY INTO L_p WHEN $0 < p < 1$

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Abstract. For $0 < p < 1$ we give examples of Banach spaces isometrically embedding into L_p but not into any L_r with $p < r \leq 1$.

1. Introduction

It is a consequence of the Maurey-Nikishin factorization theory that every Banach space which embeds isomorphically into $L_p(0,1)$ for some $0 < p < 1$ embeds into every $L_p(0,1)$ for $0 < p < 1$ (see [6], [7] and [11] pp. 257ff.). It is, however, an open problem whether every Banach space which embeds isomorphically into L_p for some $0 < p < 1$ must also embed isomorphically into L_1. This problem was formulated by Kwapien [5] in 1969; see [2] where it is shown that X embeds into L_1 if and only if $\ell_1(X)$ embeds into L_p for some $p < 1$. The isometric version of the problem asks whether if X isometrically embeds into L_p for some $p < 1$ it follows that X isometrically embeds into L_1. This problem was solved negatively by the second author in 1996 [3] who showed that there is a Banach space embedding into $L_{1/2}$ but not into L_1. The construction also yielded an example of a Banach space embedding into $L_{1/4}$ but not $L_{1/2}$. Later J. Borwein and the Center for Computational Mathematics at Simon Fraser University (unpublished) showed by computer methods that this algorithm yields examples of Banach spaces embedding into $L_{a/64}$ but not into $L_{(a+1)/64}$ for $a = 1, 2 \cdots, 63$.

The aim of this note is to show that for every $0 < p < 1$ we can find a Banach space X embedding isometrically into L_p but not into any L_r for $p < r \leq 1$. The example constructed in [3] is finite-dimensional and is obtained by a perturbation method. By contrast, our spaces are infinite-dimensional and we use probabilistic ideas to construct them. It is, of course, true that an infinite-dimensional space X embeds isometrically into L_p if and only if every finite-dimensional subspace does, and so our methods also imply the existence of finite-dimensional examples.

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We start in Section 2 by discussing the Plotkin-Rudin Equimeasurability and Uniqueness Theorems which we need for our applications. In Section 3 we construct a very basic example, which we denote by E_p. This is the subspace of $L_p(0,1)$ spanned by a constant function and a sequence of symmetric 1-stable random variables. It turns out that this space is a Banach space which is an absolute direct sum of a one-dimensional space and an isometric copy of ℓ_1. The spaces E_p provide our first family of examples. We show this by establishing that they have a certain extremal property (see Proposition 3.5).

In Section 4 we provide a second family of examples which are renormings of Hilbert spaces. For each $0 < p < 1$ we construct an example of such a space X_p which embeds isometrically into L_p but not into any L_r for $r > p$. These spaces are absolute direct sums of two infinite-dimensional Hilbert spaces. We observe that these examples have the additional property that no subspace of finite codimension can be embedded into any L_r where $r > p$.

2. REMARKS ON THE PLOTKIN-RUDIN THEOREM

In this section we discuss some essentially known results based on the Plotkin-Rudin theorems on isometric embeddings ([8], [9], [10]). See [4] for a discussion of these results.

We will always work in the setting of a Polish space Ω equipped with a nonatomic Borel probability measure μ; we then say (Ω, μ) is a standard probability space. All functions will be assumed Borel; if $f_1, \cdots, f_n$ are real Borel functions their joint distribution is the Borel measure on $\mathbb{R}^n$ given by $\mu \circ (f_1, \cdots, f_n)^{-1}$ and this will be denoted by $\rho_{f_1,\cdots,f_n}$.

Let us say that if (Ω_1, μ_1) and (Ω_2, μ_2) are two standard probability spaces then a Borel map $\sigma : \Omega_1 \to \Omega_2$ is a measure isomorphism if there is a Borel map $\tau : \Omega_2 \to \Omega_1$ (an essential inverse) such that

- $\tau\sigma(\omega_1) = \omega_1$, μ_1-a.e.
- $\sigma\tau(\omega_2) = \omega_2$, μ_2-a.e.
- $\mu_2 \circ \tau^{-1} = \mu_1$ and $\mu_1 = \mu_2 \circ \sigma^{-1}$.

If σ is a measure isomorphism then it may be modified on a set of μ_1-measure zero to become a Borel isomorphism (i.e. an invertible Borel map). If (Ω, μ) is a standard probability space then there is always a Borel isomorphism $\sigma : \Omega \to [0,1]$ such that $\lambda = \mu \circ \sigma^{-1}$ where λ is Lebesgue measure.

We shall need the following fact.

Proposition 2.1. *Let (Ω, μ) be a standard probability space and suppose K is a Polish space. Suppose $\sigma : \Omega \to K$ is a Borel map and $\nu = \mu \circ \sigma^{-1}$. Suppose there exists a Borel function f on Ω such that $\rho_f = \mu \circ f^{-1}$ is nonatomic and f is independent of σ (i.e. f is independent of the $\sigma-$algebra of sets of the form $\sigma^{-1}B$ for B a Borel subset of K.). Then there is a Borel map $\tau : \Omega \to [0,1]$ so that $\sigma \times \tau$ is a measure isomorphism of Ω onto $(K \times [0,1], \nu \times \lambda)$.*

Proof. This is surely well-known, but we do not know an explicit reference. It follows for example from Proposition 2.2 of [1] once one observes that σ is anti-injective (i.e. if B is a Borel set such that σ is injective on B then $\mu(B) = 0$).) It suffices by Lusin's theorem to consider the case when B is compact and σ is continuous on B; then σ is a Borel isomorphism of B onto $\sigma(B)$. To see this suppose $C_1, \cdots, C_N$ form a partition of $\mathbb{R}$ so that $\rho_f(C_k) = N^{-1}$. Let $B_k = B \cap f^{-1}(C_k)$. Then $\sigma(B_k)$ is Borel and $\mu(f^{-1}(C_k) \cap \sigma^{-1}\sigma(B_k)) = N^{-1}\nu(\sigma(B_k))$. Hence $\mu(B) \le N^{-1}\sum_{k=1}^{N} \nu(\sigma(B_k)) \le N^{-1}$. □

Let X be a separable normed space, and $T : X \to L_p(\Omega, \mu)$ be an isometric embedding. We say that T is *in canonical position* if it satisfies the following two conditions:

- There exists $x \in X$ so that Tx has full support i.e. $\mu(Tx \neq 0) = 1$
- There exists a function f with ρ_f nonatomic such that f is independent of the smallest $\sigma-$algebra Σ such that each Tx is $\Sigma-$measurable.

It is well-known that if X embeds into L_p then there is also an embedding in canonical position.

Let us say that two embeddings $S : X \to L_p(\Omega_1, \mu_1)$ and $T : X \to L_p(\Omega_2, \mu_2)$ are *equivalent* if

$$\rho_{Sx_1,\cdots,Sx_n} = \rho_{Tx_1,\cdots,Tx_n} \qquad x_1, \cdots, x_n \in X.$$

Theorem 2.2. [8] [9][10]*(1) Suppose p is not an even integer and (Ω, μ_1) and (Ω_2, μ_2) are two standard probability spaces. If $S : X \to L_p(\Omega, \mu_1)$ and $T : X \to L_p(\Omega, \mu_2)$ are isometric embeddings such that for some x_0 we have $Sx_0 = \chi_{\Omega_1}$ and $Tx_0 = \chi_{\Omega_2}$ then S and T are equivalent.*
(2) If, in addition, S and T are in canonical position then there exists a measure isomorphism $\sigma : \Omega_1 \to \Omega_2$ such that $\mu_2 = \mu_1 \circ \sigma^{-1}$ and $Tx \circ \sigma = Sx$ for $x \in X$.

Proof. (1) is the usual Plotkin-Rudin equimeasurability theorem [8],[9],[10], [4]. (2)is surely well-known, and follows directly from Proposition 2.1. Let us indicate one proof. Let (x_n) be any dense sequence in X and define, for $j = 1, 2$, $\tau_j : \Omega_j \to \mathbb{R}^{\mathbb{N}}$ by $\tau_1(\omega_1) = (Sx_n(\omega_1))$ and $\tau_2(\omega_2) = ((Tx_n)(\omega_2))$. Then by (1) $\mu_1 \circ \tau_1^{-1} = \mu_2 \circ \tau_2^{-1} = \nu$, say. By Proposition 2.1 we can define Borel maps $\kappa_j : \Omega_j \to [0, 1]$ so that $\tau_j \times \kappa_j$ is a measure isomorphism of (Ω_j, μ_j) onto $(\mathbb{R}^{\mathbb{N}} \times [0, 1], \nu \times \lambda)$. The map σ is then the composition $\alpha(\tau_1 \times \kappa_1)$ where α is the essential inverse of $\tau_2 \times \kappa_2$. □

If $T : X \to L_p(\Omega, \mu)$ is an isometric embedding then we can always construct a new embedding by a change of density. If φ is a nonvanishing Borel function, and $\int |\varphi|^p d\mu = 1$ we define $d\nu = |\varphi|^p d\mu$ and $T'x = \varphi^{-1}Tx$; then $T' : X \to L_p(\Omega, \nu)$ is a new isometric embedding. We then say that T' is obtained from T by a change of density.

Theorem 2.3. *Suppose p is not an even integer and $S : X \to L_p(\Omega, \mu)$ is an isometric embedding of canonical type. Then if $T : X \to L_p(\Omega_1, \mu_1)$ is any other isometric embedding then there exists a nonvanishing Borel function φ so that T' is equivalent to T where $T' : X \to L_p(\Omega, |\varphi|^p d\mu)$ is given by $T'x = \varphi^{-1}Sx$. (Thus T is obtained from S by a change of density.)*

Proof. We assume S is also of canonical type. Pick any x_0 with $\|x_0\| = 1$ so that $Sx_0 = f$ and $Tx_0 = g$ have full support. Consider $V_1x = f^{-1}Sx$ and $V_2x = g^{-1}Tx$. Then $V_1 : X \to L_p(\Omega, |f|^p d\mu)$ and $V_2 : X \to L_p(\Omega_1, |g|^p d\mu_1)$ are isometric embeddings with $V_1x_0 = \chi_\Omega$ and $V_2x_0 = \chi_{\Omega_1}$. It follows that there is a measure isomorphism $\sigma : \Omega \to \Omega_1$ so that $|g|^p\mu_1 = |f|^p\mu \circ \sigma^{-1}$ and $V_1x = V_2x \circ \sigma$. Now $Tx \circ \sigma = g \circ \sigma V_2x \circ \sigma = g \circ \sigma f^{-1}Sx$. and if B is a Borel subset of $\mathbb{R}^n$ and $x_1, \cdots, x_n \in X$ then

$$\mu_1((Tx_1, \cdots, Tx_n) \in B) = \int |g \circ \sigma|^{-p}|f|^p \chi_{(Tx_1\circ\sigma,\cdots,Tx_n\circ\sigma)\in B)} d\mu$$

and the conclusion follows with $\varphi = f(g \circ \sigma)^{-1}$. □

Corollary 2.4. *Let X be a (separable) Banach space which embeds into L_p where $p < 1$. Let E be a subspace of X and suppose $T : E \to L_p(\Omega, \mu)$ is a given isometric embedding. Then there is an isometric embedding $S : X \to L_p(\Omega_1, \mu_1)$ such that the restriction of S to E is equivalent to T.*

Proof. Let $R : X \to L_p(\Omega, \mu)$ be any isometric embedding of canonical type. We note R is also of canonical type when restricted to E. In fact it is only necessary to note that for *every* $x \in X$ Rx has full support in Ω. Indeed if Rx_0 has full support then

$$\int |Rx + tRx_0|^p d\mu \geq \|x\|^p + |t|^p \int_{Rx=0} |Rx_0|^p d\mu$$

which contradicts convexity of the norm unless Rx has full support. It follows that we make a change of density so that the new embedding S restricted to E is equivalent to T. □

A random variable f is called symmetric p-stable $0 < p < 2$ if the Fourier transform of ρ_f is of the form $e^{-c|t|^p}$ for some $c > 0$. We recall that there is an isometric embedding T of $L_r(0, 1)$ into $L_p(0, 1)$ when $0 < p < r < 2$ so that each Tf has a symmetric r-stable distribution. We will call this the r-stable embedding. A particular case is that ℓ_1 can be embedded into L_p for $p < 1$ by mapping the basic vectors to a sequence of independent 1-stable random variables.

3. THE SPACES E_p FOR $0 < p < 1$.

Lemma 3.1. *Suppose* $0 < p < 1$. *Then for* $-\pi/2 < \theta \leq \pi/2$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|x\cos\theta+\sin\theta|^p}{1+x^2}dx = \frac{\cos p\theta}{\cos p\pi/2} \tag{3.1}$$

.

Proof. We consider the case $\theta \neq 0$ of (3.1); the other cases are similar. We define $f(z)$ to be the branch of $(z\cos\theta+\sin\theta)^p$ defined in $\mathbb{C}\setminus\{-\tan\theta - it : t \geq 0\}$ such that $f(x)$ is real and positive if $x \geq -\tan\theta$. Now by a routine contour integration we have:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{1+x^2}dx = e^{ip(\frac{\pi}{2}-\theta)}.$$

Taking imaginary parts gives

$$\frac{1}{\pi}\int_{-\infty}^{-\tan\theta}\frac{|x\cos\theta+\sin\theta|^p}{1+x^2}dx = \frac{\sin p(\frac{\pi}{2}-\theta)}{\sin p\pi}.$$

Taking real parts and substituting in,

$$\frac{1}{\pi}\int_{-\tan\theta}^{\infty}\frac{|x\cos\theta+\sin\theta|^p}{1+x^2}dx = \cos p(\frac{\pi}{2}-\theta) - \cot p\pi\sin p(\frac{\pi}{2}-\theta) = \frac{\sin p(\frac{\pi}{2}+\theta)}{\sin p\pi}.$$

Combining gives (3.1). □

Lemma 3.2. *Let* $M : \mathbb{C} \to [0,\infty)$ *be a continuous non-negative function. Suppose* M *is subharmonic and positively homogeneous (i.e.* $M(az) = aM(z)$ *for* $a \geq 0$.*) Then* M *is convex.*

Proof. First assume M is C^2 on $\mathbb{C}\setminus\{0\}$. Then for any $z = x+iy \neq 0$ the second derivative of M is given by a symmetric 2×2 matrix which has rank at most one. To see this note that the equation $M(az) = aM(z)$ implies on differentiation by a and then setting $a = 1$ that

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = M.$$

Differentiating again with respect to x and y gives

$$x\frac{\partial^2 M}{\partial x^2} + y\frac{\partial^2 M}{\partial x\partial y} = 0$$
$$x\frac{\partial^2 M}{\partial x\partial y} + y\frac{\partial^2 M}{\partial y^2} = 0$$

and hence the second derivative has determinant zero. Thus if $\nabla^2 M \geq 0$ the second derivative of M is non-negative at z. This shows that M is convex.

If M is not C^2 then we may approximate it by functions of the form

$$\tilde{M}(z) = \int_0^{2\pi} \varphi(\theta) M(ze^{i\theta}) d\theta$$

where φ is smooth and non-negative. Each such function $\tilde{M}$ is convex and so M is convex. $\square$

Now let us define a function $N_p(x,y)$ on $\mathbb{R}^2$ by setting

$$N_p(x,y) = r\left(\frac{\cos p\theta}{\cos\frac{p\pi}{2}}\right)^{\frac{1}{p}},$$

whenever $x \geq 0$ and $x = r\cos\theta,\ y = r\sin\theta$ with $r \geq 0,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then extend N_p to be an even function i.e. so that $N_p(x,y) = N_p(-x,-y)$ whenever $x \leq 0$. Note also that $N_p(0,1) = 1$ but $N_p(1,0) = (\sec\frac{p\pi}{2})^{\frac{1}{p}}$.

Lemma 3.3. *N_p is an absolute norm on $\mathbb{R}^2$, i.e. N_p is a norm so that $N_p(x,y) = N_p(|x|,|y|)$.*

Proof. Let $u(z) = r^p \cos p\theta$ when $z = re^{i\theta}$ with $-\pi < \theta \leq \pi$. Then u is subharmonic and $N_p(x,y) = (\sec\frac{p\pi}{2})^{\frac{1}{p}}(\max(u(z),u(-z)))^{\frac{1}{p}}$ where $z = x+iy$. Hence N_p is a norm by Lemma 3.2. The fact that N_p is absolute is trivial. $\square$

We now define a Banach space E_p for $0 < p < 1$. We define this to be the space $\ell_1 \oplus \mathbb{R}$ with the norm $\|(x,y)\|_{E_p} = N_p(\|x\|,|y|)$.

Let (f_n) be a sequence of independent 1-stable random variables on some probability space (Ω,μ) so that $\int e^{itf_n}d\mu = e^{-|t|}$. Then for any finitely non-zero sequence $(\xi_n)_{n=1}^\infty$ and any η we have

$$\|\sum_{n=1}^\infty \xi_n f_n + \eta\|_p = N_p(\sum_{n=1}^\infty |\xi_n|, |\eta|).$$

It follows that:

Proposition 3.4. *E_p is isometric to a closed subspace of L_p for $0 < p < 1$.*

We next show that E_p cannot be embedded into L_r for any $p < r < 1$. To do this we introduce the quantity

$$a_p = \lim_{t\to 0}\frac{N((\cos\frac{p\pi}{2})^{\frac{1}{p}}t,1) - 1}{t} = (\cos\frac{p\pi}{2})^{\frac{1}{p}-1}\sin\frac{p\pi}{2}.$$

Proposition 3.5. *Suppose $0 < p < 1$ and that (g_n) is a sequence in $L_p(\Omega,\mu)$ which is 1-equivalent to the standard unit vector basis of ℓ_1. Suppose $h \in L_p$ and $\|h\|_p = 1$. Then*

$$\lim_{n\to\infty}\|h + tg_n\|_p \geq N_p((\cos\frac{p\pi}{2})^{\frac{1}{p}}t,1) \geq 1 + a_p|t|.$$

Proof. It follows from the Theorem 2.3 and Corollary 2.4, that it suffices to consider the case when $g_n = (\cos\frac{p\pi}{2})^{\frac{1}{p}} f_n$ where (f_n) is a sequence of independent 1-stable random variables with $\int e^{itf_n} d\mu = e^{-|t|}$. We now apply Lemma 4.2:

$$\begin{aligned}\lim_{n\to\infty} \int |h + \tau f_n|^p d\mu &= \frac{1}{\pi} \int_\Omega \int_{-\infty}^{\infty} \frac{|h(\omega) + \tau x|^p}{1 + x^2} dx\, d\mu(\omega) \\ &= \int N_p(\tau, h(\omega))^p d\mu(\omega).\end{aligned}$$

Now since N is an absolute norm

$$\begin{aligned}\int N_p(\tau,1)^{1-p} N_p(\tau, h(\omega))^p d\mu &\geq \int N_p(\tau, |h(\omega)|^p) d\mu \\ &\geq N_p(\tau, 1)\end{aligned}$$

and hence

$$\int N_p(\tau, h(\omega))^p d\mu(\omega) \geq N_p(\tau, 1)^p.$$

This gives us the first inequality.

For second part observe that

$$\lim_{t\to 0+} \frac{N_p((\cos\frac{p\pi}{2})^{\frac{1}{p}} t, 1) - 1}{t} = a_p$$

and use the fact that N_p is a norm. ☐

Theorem 3.6. *For $0 < p < 1$ the space E_p is a Banach space isometric to a subspace of L_p, which is not isometric to a subspace of any L_r for $r > p$.*

Proof. This is immediate from Proposition 3.5 once we show that the function $p \to a_p$ is strictly increasing on $(0, 1)$. Since L_r embeds into L_p when $p < r$ and E_r embeds into L_r it is clear from Proposition 3.5 that $p \to a_p$ is increasing. This function is non-constant since $\lim_{p\to 1} a_p = 1$ and $a_{\frac{1}{2}} = \frac{1}{2}$. Since it is a real-analytic function it must therefore be strictly increasing. ☐

4. Perturbed Hilbert spaces

We will need the following Lemma:

Lemma 4.1. *Suppose X is a Banach space and $T : X \to L_p(\Omega, \mu)$ is an isometric embedding where $0 < p < 1$. Then $\{|Tx|^p : \|x\| \leq 1\}$ is equi-integrable.*

Proof. This follows by contradiction: if $\{|Tx|^p : \|x\| \leq 1\}$ is not equi-integrable then (see [11] p.137) there exists $\delta > 0$, a disjoint sequence of Borel sets (A_k)

and x_k with $\|x_k\| \leq 1$ so that $\int_{A_k} |Tx_k|^p d\mu > \delta^p$. Then by an application of Khintchine's inequality we have for suitable $c > 0$,

$$\begin{aligned} N^p &\geq \operatorname*{Ave}_{\epsilon_k=\pm 1} \| \sum_{k=1}^N \epsilon_k x_k \|^p \\ &\geq c^p \int (\sum_{k=1}^N |Tx_k|^2)^{\frac{p}{2}} d\mu \\ &\geq c^p N \delta^p, \end{aligned}$$

and for large enough N this gives a contradiction. □

Lemma 4.2. *Let $F : \mathbb{R}^{m+1} \to \mathbb{R}$ be a continuous function. Suppose $g_1, \cdots, g_m$ and h are measurable functions on (Ω, μ) and that $(f_n)_{n=1}^\infty$ is any sequence of identically distributed independent random variables with common distribution $\rho = \rho_{f_n}$. Then if the functions $F(g_1, \cdots, g_m, f_n)$ are equi-integrable for $n = 1, 2, \cdots$ then $F(g_1, \cdots, g_m, f_0)$ is integrable and*

$$\lim_{n\to\infty} \int F(g_1, \cdots, g_m, f_n) d\mu = \int_\Omega \int_{\mathbb{R}} F(g_1, \cdots, g_m, t) d\rho(t) d\mu. \tag{4.1}$$

Proof. Suppose first that $F, g_1, \cdots, g_m, f_n$ are all bounded functions. Note that for $a_1, \cdots, a_m, b = 0, 1, 2 \cdots,$ we have

$$\lim_{n\to\infty} \int g_1^{a_1} g_2^{a_2} \cdots g_m^{a_m} f_n^b d\mu = \left(\int g_1^{a_1} \cdots g_m^{a_m} d\mu \right) \left(\int t^b d\rho(t) \right)$$

since f_n^b converges weakly in L_2 to the constant $\int f_n^b d\mu$. Hence for any polynomial P

$$\lim_{n\to\infty} \int P(g_1, \cdots, g_m, f_n) d\mu = \int_\Omega \int_{\mathbb{R}} P(g_1, \cdots, g_m, t) d\rho(t) d\mu.$$

If $|f_n|, |g_1|, \cdots |g_m| \leq M$ and $\epsilon > 0$ we approximate F on the cube $[-M, M]^{m+1}$ by a polynomial P so that range of

$$|P(x_1, \cdots, x_m, y) - F(x_1, \cdots, x_m, y)| \leq \epsilon \qquad |x_j| \leq M, \ 1 \leq j \leq m, \ |y| \leq M.$$

Then it follows that we have

$$\left| \lim_{n\to\infty} \int F(g_1, \cdots, g_m, f_n) d\mu - \int_\Omega \int_{\mathbb{R}} F(g_1, \cdots, g_m, t) d\rho(t) d\mu \right| \leq \epsilon.$$

Letting $\epsilon \to 0$ we obtain (4.1) under the assumption that $f, g_1 \cdots, g_m$ are bounded.

Next assume $|F|$ is bounded by M, but allow f and g_j to be unbounded. For any $m \in \mathbb{N}$ let $f_{k,n} = f_n\chi_{|f_n|\le k}$, and $g_{k,j} = g\chi_{|g|\le k}$. Then for $n \ge 0$,

$$\begin{aligned}&\left|\int F(g_1,\cdots,g_m,f_n)d\mu - \int F(g_{k,1},\cdots,g_{k,m},f_{k,n})d\mu\right| \\ &\le 2M\left(\mu(|f_0|>k) + \sum_{j=1}^m \mu(|g_j|>k)\right).\end{aligned}$$

Since we have (4.1) for bounded $f_n, g_1, \cdots, g_m$ we obtain the result in general for F bounded.

Now assume $F(g_1,\cdots,g_m,f_n)$ equi-integrable and let $F_k = \min(F,k)$ if $F \ge 0$ and $F_k = \max(F,-k)$ if $F \le 0$. Then

$$\lim_{n\to\infty}\int_\Omega |F_k(g_1,\cdots,g_m,f_n)|d\mu = \int_\Omega\int_{\mathbb{R}} |F_k(g_1,\cdots,g_m,t)|d\rho(t)d\mu$$

and it follows that $F(g_1,\cdots,g_m,t)$ is integrable with respect to $\mu\times\rho$. We also have

$$\lim_{k\to\infty}\int F_k(g_1\cdots,g_m,f_n)d\mu = \int F(g_1,\cdots,g_m,f_n)d\mu$$

uniformly in k so the general result follows by uniform convergence. ☐

Lemma 4.3. *Suppose* $0 < p < 1$. *Then there exists* $\epsilon(p) > 0$ *so that if* $0 < a < \epsilon(p)$ *the following equation defines an absolute norm on* $\mathbb{R}^2$.

$$N(x,y)^p = \frac{1}{2}(x^2 + (1+a)^{\frac{2}{p}}y^2)^{\frac{p}{2}} + (x^2 + (1-a)^{\frac{2}{p}}y^2)^{\frac{p}{2}} \tag{4.2}$$

Proof. This follows easily from Lemma 3.2 since if a is small enough $(x^2+(1+a)^{\frac{2}{p}}y^2)^{\frac{p}{2}}$ and $(x^2+(1-a)^{\frac{2}{p}}y^2)^{\frac{p}{2}}$ are both subharmonic. ☐

Theorem 4.4. *Suppose* $0 < p < 1$ *and* N *is given by (4.2). Then the space* $X = \ell_2 \oplus_N \ell_2$ *embeds into* L_p *but does not embed into any space* L_r *where* $r > p$.

Proof. We first establish an embedding of X into $L_p(\Omega,\mu)$. Let (e_n) and (e'_n) be the canonical orthonormal bases of the two factors of X. Let $(f_n), (g_n)$ be two mutually independent sequences of independent normalized Gaussians; we denote by γ their common distribution so that $d\gamma(t) = (2\pi)^{-\frac{1}{2}}\exp(-\frac{t^2}{2})dt$. Let E be a Borel set independent of (f_n, g_n) with $\mu E = \frac{1}{2}$. Let $h = (1+a)^{\frac{1}{p}}\chi_E + (1-a)^{\frac{1}{p}}\chi_{\tilde{E}}$. We define our embedding by

$$\begin{aligned}Te_n &= b_1 f_n \\ Te'_n &= b_1 h g_n\end{aligned}$$

where $b_1^{-p} = \|f_n\|_p^p = \int |t|^p d\gamma(t)$. We can and do assume that T is of canonical type. Suppose $(\xi_n), (\eta_n)$ are two finitely non-zero sequences of reals. Then

$$\begin{aligned}\int_\Omega |\sum_{n=1}^\infty \xi_n Te_n + \sum_{n=1}^\infty \eta_n Te'_n|^p d\mu &= b_1^p \int_\Omega |\sum_{n=1}^\infty \xi_n f_n + h \sum_{n=1}^\infty \eta_n g_n|^p d\mu \\ &= \int_\Omega (\sum_{n=1}^\infty |\xi|^2 + h^2 \sum_{n=1}^\infty \eta_n^2)^{\frac{p}{2}} d\mu \\ &= N((\sum_{n=1}^\infty \xi_n^2)^{\frac12}, (\sum_{n=1}^\infty \eta_n^2)^{\frac12})^p.\end{aligned}$$

Now assume X also embeds isometrically into L_r for some $p < r < 2$. Then X can also be embedded into L_p by an r-stable embedding S. In view of Theorem 2.3 it may be supposed that S is obtained from T by a change of density, i.e. there exists a nonvanishing Borel function φ with $\|\varphi\|_p = 1$ such that $S : X \to L_p(\Omega, |\varphi|^p d\mu)$ is given by $Sx = \varphi^{-1}Tx$. Fix any $0 < q < p$. It follows for an appropriate choice of b_2 the map $S'x = b_2 Sx$ embeds X into $L_q(\Omega, |\varphi|^p d\mu)$. Now we make a further change of density. Let $b_3^q = \int_\Omega |\varphi|^{p-q} d\mu$ and define $\psi = b_3^{-1}\varphi^{-1}$. Let $R : X \to L_q(\Omega, |\psi|^q|\varphi|^p d\mu)$ by $Rx = \psi^{-1}S'x$. Then $Rx = b_3 b_2 Tx$. Let $b_0 = b_3 b_2 b_1$.

We now use Lemma 4.1 and Theorem 4.2. Suppose $x, y \in \mathbb{R}$.

$$\begin{aligned}N(x,y)^q &= b_0^q \lim_{m\to\infty} \lim_{n\to\infty} \int_\Omega |xf_m + yhg_n|^q |\varphi|^p |\psi|^q d\mu \\ &= b_0^q \lim_{m\to\infty} \int_\Omega \int_{\mathbb{R}} |xf_m + yth|^q d\gamma(t) |\varphi|^p |\psi|^q d\mu \\ &= b_0^q \int_\Omega \int_{\mathbb{R}} \int_{\mathbb{R}} |xs + yth|^q d\gamma(s) d\gamma(t) ||\varphi|^p |\psi|^q d\mu \\ &= b_0^q \int_{\mathbb{R}} |t|^q d\gamma(t) \int_\Omega (x^2 + t^2h^2)^{\frac{q}{2}} |\varphi|^p |\psi|^q d\mu.\end{aligned}$$

Since h takes only the values $(1 \pm a)^{\frac1p}$ this implies we can find positive constants c_1, c_2 so that for all x, y,

$$N(x,y)^q = c_1(x^2 + (1-a)^{\frac2p} y^2)^{\frac{q}{2}} + c_2(x^2 + (1+a)^{\frac2p} y^2)^{\frac{q}{2}}.$$

Since $N(1,0) = N(0,1) = 1$ this requires

$$\begin{aligned}c_1 + c_2 &= 1 \\ c_1(1-a)^{\frac{q}{p}} + c_2(1+a)^{\frac{q}{p}} &= 1.\end{aligned}$$

Note also that

$$\begin{aligned}\lim_{t\to 0}\frac{N(1,t)^2-1}{t^2} &= \frac{1}{2}((1+a)^{\frac{2}{p}}+(1-a)^{\frac{2}{p}})\\ &= c_1(1-a)^{\frac{2}{p}}+c_2(1+a)^{\frac{2}{p}}.\end{aligned}$$

It is clearly impossible to satisfy these three conditions. This contradiction shows that we cannot embed X into L_r for any $r > p$. □

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