SECTIONS OF STAR BODIES
AND THE FOURIER TRANSFORM

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Abstract. A new approach to the study of sections of star bodies, based on methods of Fourier analysis, has recently been developed. The idea is to express certain geometric properties of bodies in terms of the Fourier transform and then apply methods of harmonic analysis to solve geometric problems. This approach has already led to several results including an analytic solution to the Busemann-Petty problem on sections of convex bodies. In this article we bring these results together and present short proofs of major connections.

1. Introduction and notation.

The study of geometric properties of bodies based on information about sections of these bodies has important applications to many areas of mathematics and science. A new approach, based on methods of Fourier analysis, has been developed recently. The idea is to express certain geometric properties of bodies in terms of the Fourier transform and then use methods of harmonic analysis to solve geometric problems. This approach has already led to several results including an analytic solution to the Busemann-Petty problem. In this article, we bring these results together and present short proofs of major connections.

The Fourier transform formula for the volume of hyperplane sections of the unit cube Q_n in $\mathbb{R}^n$ has been known for a long time:

$$vol_{n-1}(Q_n \cap \xi^\perp) = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{k=1}^{n} \frac{\sin(r\xi_k)}{r\xi_k}\, dr,$$

where vol_{n-1} is the $(n-1)$-dimensional volume, and $\xi^\perp = \{x \in \mathbb{R}^n : (x,\xi) = 0\}$ is the central hyperplane perpendicular to $\xi \in S^{n-1}$. This formula has many applications, the most remarkable of them being the result of Ball that the maximal volume of hyperplane sections of the cube (in every dimension) is $\sqrt{2}$. We outline

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the proof of this result in Section 2. A similar formula can be proved for the volume of central hyperplane sections of l_p-balls, and it also leads to results on critical sections of these balls.

In Section 3, we show that the above mentioned formulas are particular cases of a general formula that applies to any origin symmetric star body K in $\mathbb{R}^n$:

$$vol_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)} (\|\cdot\|_K^{-n+1})^\wedge(\xi), \tag{1.1}$$

where $\|\cdot\|_K$ is the Minkowski functional of K, and the Fourier transform is considered in the sense of distributions. The latter formula has several immediate applications. For example, a well-known result of Minkowski is that a symmetric star body is uniquely determined by volumes of its cental hyperplane sections. By (1.1), Minkowski's theorem follows from the uniqueness theorem for the Fourier transform.

Another application of (1.1) is a characterization of intersection bodies in terms of positive definite distributions. Let K and L be origin symmetric star bodies in $\mathbb{R}^n$. Following Lutwak [Lu], we say that K is the *intersection body of* L if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\|\xi\|_K^{-1} = vol_{n-1}(L \cap \xi^\perp).$$

A more general class of *intersection bodies* can be defined as the closure in the radial metric of the class of intersection bodies of star bodies. We apply formula (1.1) to prove that an origin symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$.

Intersection bodies play an important role in the solution to the following problem posed by Busemann and Petty in 1956: suppose that K and L are origin symmetric convex bodies in $\mathbb{R}^n$ so that, for every $\xi \in S^{n-1}$,

$$vol_{n-1}(K \cap \xi^\perp) \le vol_{n-1}(L \cap \xi^\perp).$$

Does it follow that $vol_n(K) \le vol_n(L)$?

The answer is affirmative if $n \le 4$ and negative if $n \ge 5$. The solution has appeared as a result of work of many mathematicians. In Section 4, we present a complete analytic solution of the problem. The solution makes use of the connection between intersection bodies and the Busemann-Petty problem established by Lutwak [Lu]: a solution to the problem in $\mathbb{R}^n$ is affirmative if and only if every origin symmetric convex body in $\mathbb{R}^n$ is an intersection body. Then we use the characterization of intersection bodies, mentioned above, and a formula expressing the derivatives of parallel section functions in terms of the Fourier transform. In the same section we present a generalization of the Busemann-Petty problem, whose proof is completely Fourier analytic.

In Section 5, we consider the classes of k-intersection bodies. These bodies are related to the generalization of the Busemann-Petty problem from Section 4 in the same way as intersection bodies are related to the original problem. We prove a characterization of these bodies in terms of positive definite distributions. The proof is based on yet another Fourier transform formula, this time for the volume of lower dimensional sections. Then we show that k-intersection bodies represent in some sense an extension of the concept of L_p-spaces to negative values of p. The article ends with a number of examples of k-intersection bodies.

Historical remarks and references can be found at the end of each section. We refer the reader to [MiP], [S], [Ga3] for more information on sections of convex bodies.

We start with definitions and notation. We say that a set K in $\mathbb{R}^n$ is a *star body* if it is starshaped with respect to the origin, has the origin as its interior point and has a continuous boundary. For $\xi \in S^{n-1}$, we define the *parallel section function* of K in the direction of ξ as a function on $\mathbb{R}$ given by

$$A_{K,\xi}(t) = vol_{n-1}(K \cap \{\xi^\perp + t\xi\}),$$

where $\{\xi^\perp + t\xi\}$ is the hyperplane perpendicular to ξ at distance t from the origin.

The *Minkowski functional* of K is defined by

$$\|x\|_K = \min\{a \geq 0 : \ x \in aK\}.$$

A star body is called k-smooth if the restriction of the Minkowski functional to the sphere S^{n-1} belongs to the space $C^{(k)}(S^{n-1})$ of k times continuously differentiable functions.

If χ is the indicator function of the interval $[-1,1]$ then

$$A_{K,\xi}(t) = \int_{(x,\xi)=t} \chi(\|x\|_K)\, dx. \tag{1.2}$$

For $t = 0$, writing the integral in the right-hand side in the polar coordinates of the hyperplane $(x,\xi) = 0$, we get the *polar formula for the volume of sections*:

$$A_{K,\xi}(0) = vol_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1} \int_{S^{n-1}\cap\xi^\perp} \|\theta\|_K^{-n+1}\, d\theta. \tag{1.3}$$

Similarly, if H is an m-dimensional subspace of $\mathbb{R}^n$ then

$$vol_m(K\cap H)=\frac{1}{m}\int_{S^{n-1}\cap H}\|\theta\|_K^{-m}\,d\theta.$$

Our main tool is the Fourier transform of distributions. We use the notation from [GS]. As usual, we denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$ with values in $\mathbb{C}$. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. Every locally integrable real valued function f on $\mathbb{R}^n$ with power growth at infinity represents a distribution acting by integration: for every $\phi\in\mathcal{S}$,

$$\langle f,\phi\rangle=\int_{\mathbb{R}^n} f(x)\phi(x)\,dx.$$

The *Fourier transform* of a distribution f is defined by $\langle\hat{f},\hat{\varphi}\rangle=(2\pi)^n\langle f,\varphi\rangle$ for every test function φ. If a test function φ is even, we have

$$(\hat{\varphi})^{\wedge}=(2\pi)^n\varphi \qquad \text{and} \qquad \langle\hat{f},\varphi\rangle=\langle f,\hat{\varphi}\rangle$$

for every $f\in\mathcal{S}'$. If q is not an integer, then the Fourier transform of the function $|z|^q,\ z\in\mathbb{R}$ is equal to (see [GS, p. 173])

$$(|z|^q)^{\wedge}(t)=-2\Gamma(1+q)\sin\frac{q\pi}{2}\,|t|^{-q-1}=c_q|t|^{-1-q},\quad t\in\mathbb{R}. \tag{1.4}$$

A distribution f is called *positive definite* if, for every test function φ,

$$\left\langle f,\varphi*\overline{\varphi(-x)}\right\rangle\geq 0.$$

A distribution is positive definite if and only if its Fourier transform is a positive distribution (in the sense that $\langle f,\varphi\rangle\geq 0$ for every non-negative test function φ). L. Schwartz's generalization of Bochner's theorem (see, for example, [GV, p. 152]) states that a distribution is positive definite if and only if it is the Fourier transform of a tempered measure on $\mathbb{R}^n$. Recall that a (non-negative, not necessarily finite) measure μ is called tempered if

$$\int_{\mathbb{R}^n}(1+\|x\|_2)^{-\beta}\,d\mu(x)<\infty$$

for some $\beta>0$.

The Radon transform of a function ϕ is defined as a function of $(\xi;t),\ \xi\in S^{n-1},\ t\in\mathbb{R}$ in the following way:

$$\mathcal{R}(\xi;t)=\int_{(x,\xi)=t}\phi(x)\,dx.$$

A well-known connection between the Radon and Fourier transforms is that, for a fixed ξ, the Fourier transform of the function $t\mapsto\mathcal{R}(\xi;t)$ is equal to the function $z\mapsto\hat{\phi}(z\xi)$.

2. Critical sections of l_q-balls.

We begin this section with the Fourier transform formulas for the volume of hyperplane sections of l_q-balls. Denote by

$$B_\infty^n = \{x \in \mathbb{R}^n : \ \|x\|_\infty = \max_{k=1,\dots,n} |x_k| \le 1\},$$

$$B_q^n = \{x \in \mathbb{R}^n : \ \|x\|_q = \big(\sum_{k=1}^n |x_k|^q\big)^{1/q} \le 1\}.$$

the unit balls of the spaces $l_q^n,\ 0 < q \le \infty$.

Theorem 2.1 ([Po]). For $\xi \in S^{n-1},\ t \in \mathbb{R}$,

$$A_{B_\infty^n,\xi}(t) = vol_{n-1}(B_\infty^n \cap \{\xi^\perp + t\xi\}) = \frac{2^n}{\pi}\int_0^\infty \cos(tr)\prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k}\ dr.$$

Proof. Let χ be the indicator function of the interval $[-1,1]$. For every $x \in \mathbb{R}^n$,

$$\chi(\|x\|_\infty) = \prod_{k=1}^n \chi(|x_k|). \tag{2.1}$$

For any $r \in \mathbb{R}$, using first (1.2) and then the Fubini theorem and (2.1), we get

$$\hat{A}_\xi(r) = \int_{\mathbb{R}} A_\xi(t)e^{-itr}\ dt = \int_{\mathbb{R}} e^{-itr}\Big(\int_{(x,\xi)=t} \chi(\|x\|_\infty)\ dx\Big)\ dt =$$

$$\int_{\mathbb{R}^n} \chi(\|x\|_\infty)e^{-ir(x,\xi)}\ dx = \prod_{k=1}^n \int_{-\infty}^\infty e^{-irx_k\xi_k}\ dx_k = 2^n\prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k}.$$

Since the function A_ξ is even and $\hat{A}_\xi$ is integrable on $\mathbb{R}$, we can invert the Fourier transform:

$$2\pi A_\xi(t) = (\hat{A}_\xi)^\wedge(t) = 2^n\int_{\mathbb{R}} e^{-itr}\prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k}\ dr. \quad \square$$

The advantage of this formula is that the variables ξ_k are separated under the integral in the right-hand side. This allows to reduce different problems to one-dimensional questions. Let us show how this formula helps to find the maximal hyperplane section of the cube. Note that the unit cube $Q_n = \frac{1}{2}B_\infty^n$.

Theorem 2.2 ([Ba1]). For every $\xi \in S^{n-1}$, $vol_{n-1}(Q_n \cap \xi^\perp) \le \sqrt{2}$, with equality for $\xi = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0)$.

Sketch of proof. Suppose first that $|\xi_k| > \frac{1}{\sqrt{2}}$ for some k. Project the set $Q_n \cap \xi^\perp$ to the coordinate hyperplane $x_k = 0$ in $\mathbb{R}^n$. The $(n-1)$-dimensional volume of the projection is equal to $|\xi_k|$ times the $(n-1)$-dimensional volume of $Q_n \cap \xi^\perp$. On the other hand, the projection is inside the face of the cube, so the $(n-1)$-dimensional volume of the projection is less or equal to 1. This implies that the $(n-1)$-dimensional volume of $Q_n \cap \xi^\perp$ is less than $\sqrt{2}$.

Now let us consider the main case, where $|\xi_k| \le \frac{1}{\sqrt{2}}$ for all $k = 1, ..., n$. We use the formula of Theorem 2.1 with $t = 0$ and then Hölder's inequality (note that $\xi_1^2 + ... + \xi_n^2 = 1$):

$$vol_{n-1}(Q_n \cap \xi^\perp) \le \int_{-\infty}^{\infty} \prod_{k=1}^n \Big| \frac{\sin(\pi r \xi_k)}{\pi r \xi_k} \Big| \, dt \le$$

$$\prod_{k=1}^n \Big(\int_{-\infty}^{\infty} \Big| \frac{\sin(\pi r \xi_k)}{\pi r \xi_k} \Big|^{1/\xi_k^2} dt \Big)^{\xi_k^2} = \prod_{k=1}^n (\frac{1}{\xi_k})^{\xi_k^2} \Big(\int_{-\infty}^{\infty} \Big| \frac{\sin(\pi x)}{\pi x} \Big|^{1/\xi_k^2} dx \Big)^{\xi_k^2}. \tag{2.2}$$

The most difficult part of the proof is the following one-dimensional inequality (called Ball's integral inequality; we do not include the proof here): for $s \ge 2$,

$$\int_{-\infty}^{\infty} \Big| \frac{\sin(\pi x)}{\pi x} \Big|^s \, dx \le \sqrt{\frac{2}{s}}.$$

Applying this inequality with $s = 1/\xi_k^2 \ge 2$, we see that the quantity in (2.2) is less or equal than

$$\prod_{k=1}^n \big(\sqrt{2\xi_k^2}\big)^{\xi_k^2} (1/\xi_k)^{\xi_k^2} = \prod_{k=1}^n (\sqrt{2})^{\xi_k^2} = \sqrt{2}. \quad \square$$

An analog of the formula of Theorem 2.1 can be proved for central hyperplane sections of l_q-balls, $0 < q < \infty$.

Theorem 2.3 ([MeP],[K2]). For every $\xi \in S^{n-1}$

$$vol_{n-1}(B_q^n \cap \xi^\perp) = \frac{q}{\pi(n-1)\Gamma((n-1)/q)} \int_0^\infty \prod_{k=1}^n \gamma_q(t\xi_k) \, dt,$$

where γ_q is the Fourier transform of the function $\exp(-|\cdot|^q)$ on $\mathbb{R}$.

This formula will later be proved as a part of a more general result (see Theorem 3.2). Let us show an application to the problem of finding the minimal hyperplane section of l_q^n-balls with $0 < q \le 2$.

Theorem 2.4 ([K2]). For $0 < q \leq 2$, the minimal central hyperplane section of B_q^n is the one perpendicular to the vector $(1, 1, .., 1)$.

Proof. A well-known fact is that, for $0 < q \leq 2$, the function γ_q is positive everywhere on $\mathbb{R}$. As noted in [MeP], to prove the theorem, it is enough to show that the function $\log(\gamma_q(\sqrt{\cdot}))$ is convex on $[0, \infty)$. In fact, the convexity of this function implies that, for any $0 < \xi_1 < \eta_1 < \eta_2 < \xi_2$ with $\xi_1^2 + \xi_2^2 = \eta_1^2 + \eta_2^2 = 1$, one has

$$\gamma_q(t\xi_1)\gamma_q(t\xi_2) \geq \gamma_q(t\eta_1)\gamma_q(t\eta_2)$$

for all $t > 0$. This means, in particular, that the integral of Theorem 2.3 is minimal when all the coordinates of the vector ξ are equal (also this integral is maximal when one of the coordinates of ξ equals 1 and the rest are equal to zero).

To prove that the function $\log(\gamma_q(\sqrt{\cdot}))$ is convex on $[0, \infty)$, we use another well-known fact from the theory of stable processes (see [Zo]) that for every $\alpha \in (0, 1)$ there exists a probability measure μ_α on $[0, \infty)$ whose Laplace transform is equal to the function $\exp(-|\cdot|^\alpha)$ (this fact also immediately follows from Bernstein's theorem on completely monotone functions, see for example [F].) Since $0 < q \leq 2$, there exists a measure μ_q on $[0, \infty)$ so that, for every $z > 0$,

$$\exp(-|z|^{q/2}) = \int_0^\infty exp(-uz)\ d\mu_q(u).$$

The latter formula can be written in the form

$$\exp(-|z|^q) = \int_0^\infty \exp(-uz^2)\ d\mu_q(u)$$

for every $z \in \mathbb{R}$. Computing the Fourier transform of both sides of the latter equality as functions of $z \in \mathbb{R}$ (use the Fubini theorem in the right-hand side), we get that for every $t \in \mathbb{R}$,

$$\gamma_q(t) = \sqrt{2\pi} \int_0^\infty u^{-1/2} \exp(-\frac{t^2}{4u})\ d\mu_q(u).$$

By the Cauchy-Schwartz inequality, for any $t_1, t_2 > 0$,

$$\gamma_q^2(\sqrt{\frac{t_1 + t_2}{2}}) = 2\pi\Big(\int_0^\infty u^{-1/2} \exp(-\frac{t_1}{8u}) \exp(-\frac{t_2}{8u})\ d\mu_q(u)\Big)^2 \leq$$

$$2\pi \int_0^\infty u^{-1/2} \exp(-\frac{t_1}{4u})\ d\mu_q(u) \int_0^\infty u^{-1/2} \exp(-\frac{t_2}{4u})\ d\mu_q(u) = \gamma_q(\sqrt{t_1})\gamma_q(\sqrt{t_2}),$$

which proves our claim. $\square$

Remarks. (i) To the best of author's knowledge, the formula of Theorem 2.1 first appeared in the literature in the paper [Po] by Polya. Since then, this formula has been independently derived and applied by several authors. Hensley [He] used the formula to prove that hyperplane sections of the cube are bounded from above by a constant not depending on the dimension. Theorem 2.2 giving the exact value $\sqrt{2}$ for the maximal volume of hyperplane sections of the cube was proved by Ball [Ba1]. Different proofs of Ball's integral inequality were later found by Nazarov and Podkorytov [NP] and by Oleszkiewicz and Pelczynski [OP]. The minimal volume of hyperplane sections of the cube is equal to 1 and corresponds to sections parallel to the faces. This follows from a result of Hadwiger [Ha1] and was later proved for sections of arbitrary codimension by Vaaler [Va]. Ball [Ba3] used the Brascamp-Lieb inequality to find the maximal lower-dimensional sections of the cube.

(ii) The formula of Theorem 2.3 for sections of the l_q^n-balls was first established by Meyer and Pajor [MeP] for $1 \leq q \leq 2$. The fact that this formula is valid for all $q > 0$ was proved later in [K2]. Meyer and Pajor used this formula to prove that the minimal hyperplane section of the l_1^n-ball is the one perpendicular to the vector $(1, 1, ..., 1)$ and conjectured that the same is true for every $q \in [1, 2]$. This conjecture was proved for $0 < q \leq 2$ in [K2]. Meyer and Pajor proved in [MeP] that the maximal section of l_q^n-balls with $1 \leq q \leq 2$ is the one perpendicular to the vector $(1, 0, ...0)$ (note that the proof of Theorem 2.4 confirms this result and shows that this is also true for $0 < q < 1$). As also proved in [MeP], the direction $(1, 0, ..., 0)$ gives the minimal section of l_q^n-balls with $q > 2$. It is still an open question which sections are maximal for l_q^n-balls with $2 < q < \infty$. Oleszkiewicz [O] has recently shown that the answer must depend on q and n.

(iii) Oleszkiewicz and Pelczynski [OP] have proved a complex analog of Theorem 2.2. Barthe and Naor [BN] found extremal hyperplane projections of l_q^n-balls using techniques surprisingly similar (in a non-trivial way) to those from [MeP] and [K2].

3. Intersection bodies and positive definite distributions.

We start this section with a calculation showing, in particular, that the formulas of Theorems 2.1 and 2.3 have something in common.

Lemma 3.1 ([K3]). Let $0 < p < n$, $\xi \in \mathbb{R}^n$ with non-zero coordinates. For $0 < q < \infty$, the Fourier transform of the function $\|\cdot\|_q^{-p}$ in the sense of distributions is given by

$$(\|\cdot\|_q^{-p})^\wedge(\xi) = \frac{q}{\Gamma(p/q)} \int_0^\infty t^{n-p-1} \prod_{k=1}^n \gamma_q(t\xi_k)\ dt.$$

Also

$$(\|\cdot\|_\infty^{-p})^\wedge(\xi) = 2^n p \int_0^\infty t^{n-p-1} \prod_{k=1}^n \frac{\sin(t\xi_k)}{t\xi_k}\ dt.$$

Both formulas represent locally integrable functions of the variable $\xi \in \mathbb{R}^n$.

Proof of the first formula. By the definition of the Γ-function, for every $x \in \mathbb{R}^n$,

$$\|x\|_q^{-p} = \frac{q}{\Gamma(p/q)} \int_0^\infty y^{p-1} \exp(-y^q(|x_1|^q + ... + |x_n|^q))\ dy.$$

The function $\|x\|_q^{-p}$ is locally integrable on $\mathbb{R}^n$, since $0 < p < n$. Using the latter formula, the Fubuni theorem and Parseval's formula (note that ξ has non-zero coordinates and $\gamma_q(t) = O(|t|^{-1-q})$, as $t \to \infty$), we get that for every even test function ϕ,

$$\langle (\|\cdot\|_q^{-p})^\wedge, \phi\rangle = \langle \|x\|_q^{-p}, \hat{\phi}\rangle = \int_{\mathbb{R}^n} \|x\|_q^{-p} \hat{\phi}(x)\ dx =$$

$$\frac{q}{\Gamma(p/q)} \int_0^\infty y^{p-1}\ dy \int_{\mathbb{R}^n} \prod_{k=1}^n \exp(-|y|^q |x_k|^q) \hat{\phi}(x)\ dx =$$

$$\frac{q}{\Gamma(p/q)} \int_{\mathbb{R}^n} \Big(\int_0^\infty y^{-n+p-1} \prod_{k=1}^n \gamma_q(\xi_k/y) dy \Big) \phi(\xi)\ d\xi.$$

Making a change of variables $t = 1/y$ in the inner integral, we get the desired formula. $\Box$

Comparing the formulas of Lemma 3.1 with those from Theorems 2.1 and 2.3, we see that in both cases the volume of the section perpendicular to a vector ξ is equal (up to the same constant) to the Fourier transform of $\|\cdot\|_q^{-n+1}$ at the point ξ. This fact appears to be true for any star body.

Theorem 3.2 ([K2]). Let K be an origin symmetric star body in $\mathbb{R}^n$. For every $\xi \in S^{n-1}$

$$A_{K,\xi}(0) = vol_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)} (\|\cdot\|_K^{-n+1})^\wedge(\xi).$$

Proof. For every even test function ϕ,

$$\langle (\|\cdot\|_K^{-n+1})^\wedge, \phi\rangle = \langle \|\cdot\|_K^{-n+1}, \hat{\phi}\rangle = \int_{\mathbb{R}^n} \|x\|_K^{-n+1} \hat{\phi}(x)\ dx =$$

$$\int_{S^{n-1}} \|\theta\|_K^{-n+1} d\theta \int_0^\infty \hat{\phi}(t\theta)\ dt. \tag{3.1}$$

Using the connection between the Fourier and Radon transforms (see the end of Section 1), we get

$$\int_{-\infty}^{\infty} \hat{\phi}(t\theta)dt = (\hat{\phi}(t\theta))^{\wedge}_t(0) = 2\pi \int_{(x,\theta)=0} \phi(x)dx.$$

Substituting the latter integral in (3.1) and writing it in polar coordinates of the hyperplane $(x,\theta)=0$, we get that the expression in (3.1) is equal to

$$\pi \int_{S^{n-1}} \|\theta\|_K^{-n+1} d\theta \int_{S^{n-1}\cap\theta^\perp} d\xi \int_0^\infty r^{n-2}\phi(r\xi)dr. \tag{3.2}$$

We now use self-duality of the spherical Radon transform, see for example [Gr] or [SW, p.150]: for any continuous functions f, g on S^{n-1}

$$\int_{S^{n-1}} f(\theta)d\theta \int_{S^{n-1}\cap\theta^\perp} g(\xi)d\xi = \int_{S^{n-1}} g(\xi)d\xi \int_{S^{n-1}\cap\xi^\perp} f(\theta)d\theta.$$

Applying this formula to (3.2) with $f(\theta) = \|\theta\|_K^{-n+1}$ and $g(\xi) = \int_0^\infty r^{n-2}\phi(r\xi)dr$, we get

$$(3.2) = \pi \int_{S^{n-1}} \Big(\int_0^\infty r^{n-2}\phi(r\xi)dr\Big)d\xi \int_{S^{n-1}\cap\xi^\perp} \|\theta\|_K^{-n+1}d\theta =$$

$$\pi \int_{\mathbb{R}^n} \|x\|_2^{-1}\Big(\int_{S^{n-1}\cap(x/\|x\|_2)} \|\theta\|_K^{-n+1}d\theta\Big)\phi(x)\ dx,$$

where $\|\cdot\|_2$ stands for the Euclidean norm in $\mathbb{R}^n$, and the last equality can be checked by passing to the polar coordinates $x = r\xi$.

We see now that the distribution $(\|\cdot\|_K^{-n+1})^\wedge$ is equal to the locally integrable function $\pi\|x\|_2^{-1}\int_{S^{n-1}\cap(x/\|x\|_2)} \|\theta\|_K^{-n+1}d\theta$. In particular, for every unit vector $x \in \mathbb{R}^n$, using the polar formula (1.3) for the volume of sections we get

$$(\|\cdot\|_K^{-n+1})^\wedge(x) = \pi \int_{S^{n-1}\cap x^\perp} \|\theta\|_K^{-n+1}d\theta = \pi(n-1)vol_{n-1}(K\cap x^\perp). \qquad \square$$

Theorem 3.2 implies the following characterization of intersection bodies (the definition is given in Section 1) in terms of the Fourier transform.

Theorem 3.3 ([K5]). An origin symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$.

Idea of proof. Suppose K is the intersection body of some symmetric star body L. Then, using the definition and Theorem 3.2, we get

$$\|x\|_K^{-1} = vol_{n-1}(L \cap x^\perp) = \frac{1}{\pi(n-1)}(\|\cdot\|_L^{-n+1})^\wedge(x).$$

Since K is symmetric, the Fourier transform is self-invertible (up to a constant), so

$$(\|\cdot\|_K^{-1})^\wedge(\xi) = \frac{(2\pi)^n}{\pi(n-1)}\|\xi\|_L^{-n+1} > 0,$$

which means that the locally integrable function $\|\cdot\|_K^{-1}$ is a positive definite distribution.

Conversely, suppose that $\|\cdot\|_K^{-1}$ is positive definite and its Fourier transform is a continuous positive function on S^{n-1}. Then define a symmetric star body L by

$$\|\xi\|_L = \Big(\frac{\pi(n-1)}{(2\pi)^n}(\|\cdot\|_K^{-1})^\wedge(\xi)\Big)^{-1/(n-1)}.$$

It is easy to see now that K is the intersection body of L. To finish the proof one needs an approximation argument that we do not include here. □

Let us show how Theorem 3.3 works for l_q-balls with $q > 2$. We need a few more facts about the function γ_q (see Theorem 2.3 for the definition of this function). If q is not an even integer then the function $\gamma_q(t)$ decreases at infinity like $|t|^{-1-q}$ (see [PS, Ch.4, Problem 154]). If q is an even integer the rate of decrease is exponential. Therefore, for $-1 < \alpha < q$, the integrals

$$s_q(\alpha) = \int_{\mathbb{R}} |t|^\alpha \gamma_q(t)\ dt$$

converge absolutely. These moments can easily be computed. First assume that $-1 < \alpha < 0$ and apply Parseval's formula and (1.4), then use the fact that $s_q(\alpha)$ is an analytic function of α in the domain $\{\alpha \in \mathbb{C},\ -1 < Re(\alpha) < q\}$. The result is that

$$s_q(\alpha) = \frac{2^{\alpha+2}\sqrt{\pi}\Gamma(-\alpha/q)\Gamma((\alpha+1)/2)}{q\Gamma(-\alpha/2)}.$$

It is easily seen now that the moments $s_q(\alpha)$ are positive for $\alpha \in [0,2]$ and negative for $\alpha \in (2, \min(q,4))$.

Theorem 3.4 ([K3]). Let $2 < q \le \infty$ and $0 < p < n$. The function $\|\cdot\|_q^{-p}$ represents a positive definite distribution if and only if $p \in [n-3, n)$.

Proof. We only prove here that $\|\cdot\|_q^{-p}$ is not a positive definite distribution if $2<q<\infty$ and $0<p<n-3$ (see Remark (iv) at the end of this section).

Suppose that $0<p<n-3$. For $\alpha_1,...,\alpha_{n-1}\in\mathbb{R}$, consider the integral

$$I(\alpha_1,...,\alpha_{n-1})=\int_{\mathbb{R}^{n-1}}|\xi_1|^{\alpha_1}...|\xi_{n-1}|^{\alpha_{n-1}}(\|\cdot\|_q^{-p})^\wedge(\xi_1,...,\xi_{n-1},1)\ d\xi_1...d\xi_{n-1}.$$

Using the first formula of Lemma 3.1, we see that this integral converges absolutely and is equal to

$$I(\alpha_1,...,\alpha_{n-1})=\frac{q}{\Gamma(p/q)}s_q(\alpha_1)...s_q(\alpha_{n-1})s_q(-\alpha_1-...-\alpha_{n-1}-p)$$

if each of the numbers $\alpha_1,...,\alpha_{n-1},-\alpha_1-...-\alpha_{n-1}-p$ belongs to the interval $(-1,q)$. Choose $\alpha_1,...,\alpha_{n-1}\in(-1,0)$. The corresponding moments $s_q(\alpha_k),\ k=1,...,n-1$ are all positive. But then we can make the number $-\alpha_1-...-\alpha_{n-1}-p$ equal to any number from the interval $(-p,n-p-1)$. Since $q>2$ and $p<n-3$, the intersection of this interval with the interval $(2,\min(q,4))$ is non-empty. Choosing $\alpha_1,...,\alpha_{n-1}\in(-1,0)$ in such a way that

$$-\alpha_1-...-\alpha_{n-1}-p\in(-p,n-p-1)\cap(2,\min(q,4)$$

and using the remark before the theorem, we see that the integral $I(\alpha_1,...,\alpha_{n-1})$ is negative. This means that the function $(\|\cdot\|_q^{-p})^\wedge$ is sign-changing, and $\|\cdot\|_q^{-p}$ is not a positive definite distribution. $\Box$

It follows from Theorems 3.3 and 3.4 that

Corollary 3.5 ([K3]). For $2<q\le\infty$, the unit ball of the space l_q^n is an intersection body if and only if $n\le 4$.

Remarks. (i) The concept of an intersection body of a star body was introduced by Lutwak [Lu] in 1988, as a part of his dual Brunn-Minkowski theory. The more general concept of an intersection body first appeared in [GLW]. However, several results related to intersection bodies were proved earlier (without naming this special class of bodies). For instance, Busemann [Bu1] proved that if L is convex then the intersection body of L is also convex. Another proof and a generalization of this result were found by Milman and Pajor [MiP]. Lutwak [Lu] established a connection between intersection bodies and the Busemann-Petty problem that played an important role in the solution of the problem.

(ii) The proof of Theorem 3.3 in [K5] is written in such a way that the underlying argument (presented in the proof here) can not be explicitly seen.

(iii) The formula for the moments of the functions γ_q can be found in [Zo] or [K1].

(iv) An argument similar to that in the proof of Theorem 3.4 was previously used by the author in the solution to Schoenberg's problem on positive definite functions, see [K1]. The case $p \in [n-3, n)$ of Theorem 3.4 is not proved in this section, but the result holds for any norm in $\mathbb{R}^n$: for any n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ and any $p \in [n-3, n)$, the function $\|\cdot\|^{-p}$ represents a positive definite distribution. In Section 4, we prove the case $p = n-3$ of this result. For the proof for all $p \in [n-3, n)$, see [K6]. One can, however, prove this result for l_q-norms directly, see [K3].

4. An analytic solution to the Busemann-Petty problem.

Our next formula generalizes the result of Theorem 3.2 and serves as one of the main ingredients in the unified solution to the Busemann-Petty problem.

Theorem 4.1 ([GKS]). Let D be an infinitely smooth origin symmetric star body in $\mathbb{R}^n$, $k \in \mathbb{N} \cup \{0\}$, $k \neq n-1$, $\xi \in S^{n-1}$. We denote by A_ξ the parallel section function of D in the direction of ξ. Then:

(i) If k is even

$$(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_\xi^{(k)}(0);$$

(ii) If k is odd

$$(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) =$$

$$(-1)^{(k+1)/2}2(n-k-1)k! \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - ... - A_\xi^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\, dz,$$

where $A_\xi^{(k)}$ stands for the derivative of the order k of the function A_ξ.

Before starting the proof let us recall the definition and some basic properties of *fractional derivatives*. Let $m \in \mathbb{N} \cup \{0\}$. For $q \in \mathbb{C}$, $-1 < Re(q) < m$, $q \neq 0, 1, ..., m-1$, the fractional derivative of the order q of the function A_ξ at zero is defined by

$$A_\xi^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q}\big(A_\xi(t) - A_\xi(0) - ... - A_\xi^{(m-1)}(0)\frac{t^{m-1}}{(m-1)!}\big)\, dt +$$

$$\frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} A_\xi(t) dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{A_\xi^{(k)}(0)}{k!(k-q)}.$$

It is easy to see that for a fixed q the definition does not depend on the choice of $m > Re(q)$, so the fractional derivatives $A_\xi^{(q)}(0)$ are correctly defined for all non-integer $q \in \mathbb{C},\ Re(q) > -1$. Note that without dividing by $\Gamma(-q)$ the expression for the fractional derivative represents an analytic function in the domain $\{q \in \mathbb{C} : Re(q) > -1\}$ not including integers, and has simple poles at integers. The function $\Gamma(-q)$ is analytic in the same domain and also has simple poles at non-negative integers, so after the division we get an analytic function in the whole domain $\{q \in \mathbb{C} :\ Re(q) > -1\}$, which also defines fractional derivatives of integer orders. Moreover, computing the limit as $q \to k$, where k is a non-negative integer we see that the fractional derivatives of integer orders coincide with usual derivatives up to a sign:

$$A_\xi^{(k)}(0) = (-1)^k \frac{\partial^k}{\partial t^k} A_\xi(t)|_{t=0}.$$

Proof of Theorem 4.1. We are going to prove a more general fact: for every $q \in \mathbb{C},\ Re(q) > -1,\ q \neq n-1,$

$$A_\xi^{(q)}(0) = \frac{\cos(\pi q/2)}{\pi(n-q-1)} (\|\cdot\|_D^{-n+q+1})^\wedge(\xi). \tag{4.1}$$

The result of the theorem follows from (4.1). In fact, if $q = k$ is an even integer we immediately get the first formula of the theorem. If q is an odd integer, both sides of (4.1) are equal to zero (A_ξ is an even function). To get the second formula, we assume that $q \to k$, divide both sides of (4.1) by $\cos(\pi q/2)$ and compute the limit using the definition of the fractional derivative.

To prove (4.1), we start with the case where $-1 < q < 0$ and then use analytic continuation. In the case $-1 < q < 0$ we have

$$\begin{aligned} A_\xi^{(q)}(0) &= \frac{1}{2\Gamma(-q)} \int_{-\infty}^{\infty} |z|^{-q-1} A_\xi(z)\, dz \\ &= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}^n} |\langle x, \xi\rangle|^{-q-1} \chi(\|x\|_D)\, dx \\ &= \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} |\langle \theta, \xi\rangle|^{-q-1} \int_0^\infty r^{n-q-2} \chi(r\|\theta\|_D)\, dr\, d\theta \\ &= \frac{1}{2(n-q-1)\Gamma(-q)} \int_{S^{n-1}} |\langle \theta, \xi\rangle|^{-q-1} \|\theta\|_D^{-n+q+1}\, d\theta. \end{aligned}$$

We now consider $A_\xi^{(q)}(0)$ as a function of $\xi \in \mathbb{R}^n \setminus \{0\}$. For every even test function $\varphi \in \mathcal{S}$, we have

$$\langle A_\xi^{(q)}(0), \varphi(\xi)\rangle = \frac{1}{2(n-q-1)\Gamma(-q)} \int_{S^{n-1}} \|\theta\|_D^{-n+q+1} \int_{\mathbb{R}^n} |\langle \theta, \xi\rangle|^{-q-1} \varphi(\xi)\, d\xi =$$

$$\frac{1}{2(n-q-1)\Gamma(-q)}\int_{S^{n-1}}\|\theta\|_D^{-n+q+1}\int_{\mathbb{R}}|z|^{-1-q}\Big(\int_{(\theta,\xi)=z}\phi(\xi)\ d\xi\Big)\ dz.$$

The inner integral as the action of the distribution $|z|^{-1-q}$ on the test function $\int_{(\theta,\xi)=z}\phi(\xi)\ d\xi$, which is the Radon transform of ϕ in the direction of θ. Using formula (1.4), the connection between the Fourier and Radon transforms and well-known properties of the Γ-function, we get

$$\langle A_\xi^{(q)}(0),\varphi(\xi)\rangle=\frac{\cos(\pi q/2)}{2\pi(n-q-1)}\int_{S^{n-1}}\|\theta\|_D^{-n+q+1}d\theta\int_{-\infty}^{\infty}|t|^q\widehat{\varphi}(t\theta)\,dt.$$

On the other hand,

$$\langle(\|\cdot\|_D^{-n+q+1})^\wedge(\xi),\varphi(\xi)\rangle=\int_{\mathbb{R}^n}\|x\|^{-n+q+1}\widehat{\varphi}(x)\,dx=$$

$$\int_{S^{n-1}}\|\theta\|_D^{-n+q+1}d\theta\int_0^{\infty}t^q\widehat{\varphi}(t\theta)\,dt.$$

Therefore, for every even test function ϕ,

$$\langle A_\xi^{(q)}(0),\varphi(\xi)\rangle=\frac{\cos(\pi q/2)}{\pi(n-q-1-)}\langle(\|\cdot\|_D^{-n+q+1})^\wedge(\xi),\varphi(\xi)\rangle. \tag{4.2}$$

This proves (4.1) for $-1<q<0$. To finish the proof, observe that both sides of (4.2) are analytic functions of q in the domain $Re(q)>-1,\ q\neq n-1$. By analytic continuation, the equality (4.2) holds for any even test function ϕ and for any q from this domain. □

Corollary 4.2. For every n-dimensional normed space $E=(\mathbb{R}^n,\|\cdot\|)$ and every $p\in[n-3,n)$, the function $\|\cdot\|^{-p}$ represents a positive definite distribution.

Proof. We shall prove only the case $p=n-3$ and refer the reader to [K6] for the general case. First assume that the function $\|\cdot\|$ is infinitely smooth on S^{n-1}. Since E is a normed space, the unit ball D of the space E is convex and symmetric. By Brunn's theorem (see for example [S]), the central section has maximal volume among all hyperplane sections perpendicular to a fixed direction. Therefore, for every ξ the function A_ξ has maximum at zero and $A_\xi^{''}(0)\leq 0$. By Theorem 4.1 with $k=2$,

$$(\|\cdot\|_D^{-n+3})^\wedge(\xi)=-\pi(n-3)A_\xi^{''}(0)\geq 0,$$

which implies that the locally integrable function $\|\cdot\|_D^{-n+3}=\|\cdot\|^{-n+3}$ represents a positive definite distribution on $\mathbb{R}^n$.

Let us now consider the case where D is not necessarily infinitely smooth. In this case we use an approximation argument. Though we omit similar arguments in other proofs in this paper, at this time we present the argument in detail so that the reader can reproduce it in other places.

Let $D_m,\ m \in \mathbb{N}$ be a sequence of infinitely smooth bodies so that $D_m \subset D$ and $\|\cdot\|_{D_m}$ converges to $\|\cdot\|_D$ pointwise on the sphere. Let ϕ be any non-negative test function. Then for every m and every $x \in S^{n-1}$,

$$\|x\|_{D_m}^{-n+3}|\hat{\phi}(x)| \leq \|x\|_D^{-n+3}|\hat{\phi}(x)|.$$

The function in the right-hand side is integrable on $\mathbb{R}^n$ as the product of a locally integrable on $\mathbb{R}^n$, bounded outside of the origin function $\|\cdot\|_D^{-n+3}$ and a bounded integrable function $|\hat{\phi}|$. By the dominated convergence theorem,

$$\langle (\|\cdot\|_D^{-n+3})^\wedge, \phi \rangle = \int_{\mathbb{R}^n} \|x\|_D^{-n+3}\hat{\phi}(x)\ dx =$$

$$\lim_{m\to\infty}\int_{\mathbb{R}^n} \|x\|_{D_m}^{-n+3}\hat{\phi}(x)\ dx = \lim_{m\to\infty}\langle (\|\cdot\|_{D_m}^{-n+3})^\wedge, \phi \rangle.$$

But the latter limit is non-negative, since we have already proved the result for infinitely smooth bodies. Therefore, $\langle (\|\cdot\|_D^{-n+3})^\wedge, \phi \rangle \geq 0$ for any non-negative test function ϕ, so $\|\cdot\|_D^{-n+3}$ represents a positive definite distribution. $\square$

We are now ready to present a unified solution to the Busemann-Petty problem on sections of convex bodies (the BP problem) formulated in Section 1. We use the connections between the BP problem and intersection bodies found by Lutwak [Lu]: (i) If K is an intersection body then the answer to the BP problem is affirmative for K and any star body L, whose central section have greater volume than the corresponding sections of K; (ii) If L is not an intersection body, is infinitely smooth and has positive curvature, then one can perturb L to get a body K giving together with L a counterexample to the BP problem. We shall prove Lutwak's connections later in a more general form.

Theorem 4.3 The solution to the BP problem is affirmative if $n \leq 4$ and it is negative if $n \geq 5$.

Proof. Let $n = 4$. By Corollary 4.2, for any origin symmetric convex body K in $\mathbb{R}^4$, the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution. By Theorem 3.3, every such K is an intersection body, and the affirmative answer to the BP problem in dimension 4 follows now from the first Lutwak's connection.

Let $n \geq 5$. By Corollary 3.5, the unit ball of the space l_4^n is not an intersection body. This unit ball is infinitely smooth, but does not have positive curvature everywhere, so we can not use the second Lutwak's connection directly. We can, however, slightly perturb the l_4^n-ball to get a non-intersection body with positive curvature. For every $\epsilon > 0$ define a body D_ϵ by

$$\|\cdot\|_{D_\epsilon} = \|\cdot\|_4 + \epsilon\|\cdot\|_2.$$

The bodies D_ϵ are convex, have positive curvature everywhere and are infinitely smooth. Also for small enough ϵ, the body D_ϵ is not an intersection body. In fact, if D_ϵ is an intersection body for every $\epsilon > 0$ then $\|\cdot\|_{D_\epsilon}^{-1}$ is a positive definite distribution. But then $\|\cdot\|_4^{-1}$ is also positive definite by a simple approximation argument (similar to the one in the proof of Corollary 4.2), which contradicts to the result of Theorem 3.4. The negative answer to the BP problem in dimensions 5 and higher follows now from the second Lutwak's connection. □

Next, we prove a generalization of the solution to the Busemann-Petty problem. The solution to the original problem shows that for $n \geq 5$ it is not enough to know that all central sections are smaller to conclude that the volume is smaller. The question is what does one have to know about the behavior of parallel section functions at zero to be able to make conclusions about the n-dimensional volume. An answer to this question is given by

Theorem 4.4 ([K7]). Let D and L be origin symmetric, $(k-1)$-smooth convex bodies in $\mathbb{R}^n$, k is an odd integer, $0 < k < n$. Suppose that for every $\xi \in S^{n-1}$,

$$(-1)^{(k-1)/2} A_{D,\xi}^{(k-1)}(0) \leq (-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0). \tag{4.3}$$

Then:

(i) If $k \geq n-3$ we have $vol_n(D) \leq vol_n(L)$;

(ii) If $k < n-3$ it is still possible that $vol_n(D) > vol_n(L)$.

This means that one has to know the inequalities (4.3) for the derivatives of parallel section functions of order $n-4$ and higher to be able to conclude that the volumes are related in the same way. Note that the case $k = 1$ gives the answer to the original BP problem. If k is an even integer one has to replace the derivatives in (4.3) by the expressions from the second formula of Theorem 4.1.

Our main tool here is the following version of Parseval's formula: if D and L are infinitely smooth symmetric star bodies in $\mathbb{R}^n$, and $0 < k < n$ then

$$\int_{S^{n-1}} (\|\cdot\|_D^{-k})^\wedge(\xi)(\|\cdot\|_L^{-n+k})^\wedge(\xi)\, d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_D^{-k}\|x\|_L^{-n+k}\, dx. \tag{4.4}$$

A straightforward proof of this formula can be found in [K7]. There also exists a simple proof using spherical harmonics. The formula (4.4) has different applications. For example putting $k = 1$, $D = L$ and using Theorem 3.2 and the polar formula for the volume, one gets an expression for the volume of a body in terms of volumes of its central hyperplane sections:

$$vol_n(D) = \frac{(2\pi)^n \pi(n-1)}{n} \int_{S^{n-1}} vol_{n-1}(D \cap \xi^\perp)(\|\cdot\|_D^{-1})^\wedge(\xi)\ d\xi.$$

The next two theorems generalize Lutwak's connections between intersection bodies and the BP problem.

Theorem 4.5 ([K7]). Let D and L be infinitely smooth, origin symmetric star bodies in $\mathbb{R}^n$, $0 < k < n$. If $\|\cdot\|_D^{-k}$ is a positive definite distribution and $\|\cdot\|_L^{-n+k} - \|\cdot\|_D^{-n+k}$ is a positive definite distribution, then $vol_n(D) \leq vol_n(L)$.

Proof. By Theorem 4.1, $(\|\cdot\|_D^{-k})^\wedge$ and $(\|\cdot\|_L^{-n+k})^\wedge - (\|\cdot\|_D^{-n+k})^\wedge$ are continuous functions on S^{n-1}. These functions are also non-negative by the assumptions of this theorem. Therefore,

$$\int_{S^{n-1}} (\|\cdot\|_D^{-k})^\wedge(\xi)(\|\cdot\|_L^{-n+k})^\wedge(\xi)\ d\xi \geq \int_{S^{n-1}} (\|\cdot\|_D^{-k})^\wedge(\xi)(\|\cdot\|_D^{-n+k})^\wedge(\xi)\ d\xi.$$

Applying Parseval's formula (4.4) to both sides we get

$$\int_{S^{n-1}} \|x\|_D^{-k}\|x\|_L^{-n+k}\ dx \geq \int_{S^{n-1}} \|x\|_D^{-k}\|x\|_D^{-n+k}\ dx = n\ vol_n(D).$$

By Hölder's inequality, the left-hand side is less or equal to

$$\Big(\int_{S^{n-1}} \|x\|_D^{-n}\ dx\Big)^{k/n}\Big(\int_{S^{n-1}} \|x\|_L^{-n}\ dx\Big)^{(n-k)/n}.$$

Finally, we get

$$n\ vol_n(D) \leq (n\ vol_n(D))^{k/n}(n\ vol_n(L))^{(n-k)/n}. \quad \Box$$

To get the second connection we need

Theorem 4.6 ([K7]). If L is an infinitely smooth, origin symmetric convex body in $\mathbb{R}^n$ with positive curvature, and $\|\cdot\|_L^{-k}$ is not a positive definite distribution, then there exists a convex body D in $\mathbb{R}^n$ so that the distribution $\|\cdot\|_L^{-n+k} - \|\cdot\|_D^{-n+k}$ is positive definite, but $vol_n(D) > vol_n(L)$.

Proof. Since L is C^∞, $(\|\cdot\|_L^{-k})^\wedge$ is a continuous function on S^{n-1}. This function is negative on some open symmetric subset Ω of S^{n-1}. Let $f \in C^\infty(S^{n-1})$ be a non-negative even function supported in Ω. Extend f to a homogeneous function $f(\theta)r^{-k}$ of degree $-k$ on $\mathbb{R}^n$. By the same argument as in Theorem 4.1, the Fourier transform of $f(\theta)r^{-k}$ is a homogeneous function of degree $-n+k$ on $\mathbb{R}^n$ whose restriction to the sphere is infinitely smooth: $(f(\theta)r^{-k})^\wedge = g(\theta)r^{-n+k}$, where $g \in C^\infty(S^{n-1})$. Choosing a small $\epsilon > 0$, we now define the body D:

$$\|x\|_D^{-n+k} = \|x\|_L^{-n+k} - \epsilon g(\theta)r^{-n+k}.$$

Since L is convex and has positive curvature, one can choose a small enough ϵ so that the body D is convex. (This follows from a simple two-dimensional argument: if f is a strictly concave function on a bounded interval and g is twice continuously differentiable on this interval, then for some small ϵ the second derivative of $f + \epsilon g$ is still negative on the whole interval.) By the definition of g and since f is non-negative, we have

$$(\|\cdot\|_D^{-n+k})^\wedge = (\|\cdot\|_L^{-n+k})^\wedge - (2\pi)^n \epsilon f(\theta)r^{-k} \le (\|\cdot\|_L^{-n+k})^\wedge,$$

so the distribution $\|\cdot\|_L^{-n+k} - \|\cdot\|_D^{-n+k}$ is positive definite. On the other hand, the function f is positive only where $(\|\cdot\|_L^{-k})^\wedge$ is negative, so

$$\int_{S^{n-1}} (\|\cdot\|_D^{-n+k})^\wedge(\xi)(\|\cdot\|_L^{-k})^\wedge(\xi)\ d\xi = \int_{S^{n-1}} (\|\cdot\|_L^{-n+k})^\wedge(\xi)(\|\cdot\|_L^{-k})^\wedge(\xi)\ d\xi -$$

$$\epsilon \int_{S^{n-1}} (\|\cdot\|_L^{-k})^\wedge(\xi) f(\xi)\ d\xi \ge \int_{S^{n-1}} (\|\cdot\|_L^{-n+k})^\wedge(\xi)(\|\cdot\|_L^{-k})^\wedge(\xi)\ d\xi.$$

Now apply Parseval's formula and Hölder's inequality in the same way as in Theorem 4.5 to see that $vol_n(D) > vol_n(L)$. $\square$

We are now ready to prove Theorem 4.4.

Proof of Theorem 4.4. By a simple approxmation argument, it is enough to prove the theorem for infinitely smooth D and L. By Theorem 4.1, the assumption (4.3) is equivalent to positive definiteness of the distribution $\|\cdot\|_L^{-n+k} - \|\cdot\|_D^{-n+k}$.

If $k \ge n-3$ then, by Corollary 4.2, $\|\cdot\|_D^{-k}$ is a positive definite distribution, so the result follows from Theorem 4.5.

If $k < n-3$ then $\|\cdot\|_4^{-k}$ is not positive definite. As in the solution to the original BP problem, we define a body L by

$$\|\cdot\|_L = \|\cdot\|_4 + \epsilon\|\cdot\|_2$$

where ϵ is chosen small enough so that $\|\cdot\|_L^{-k}$ is still not positive definite. Then L satisfies the conditions of Theorem 4.6, and there exists D giving together with L the desired counterexample. $\square$

Remarks. (i) A different proof of Theorem 4.1, not using distributions, was given in [BFM].

(ii) It is easy to prove the negative answer to the BP problem in dimension 5 using the second formula of Theorem 4.1, instead of the l_4-balls. A simple example is given in [GKS].

It is also possible to give an informal explanation of the fact that the transition occurs between the dimensions 4 and 5. This explanation is based on the first formula of Theorem 4.1. If $n = 4$ then to make a conclusion about the behavior of $\|\cdot\|_D^{-1}$ we put $k = 2$, and positive definiteness follows from the negativity of the second derivative of the parallel section function. If $n = 5$ one has to put $k = 3$ and positive definiteness of $\|\cdot\|_D^{-1}$ depends on the behavior of the third derivative. But the third derivative is not controlled by convexity and can be made sign-changing.

(iii) The second Lutwak's connection is presented here in the form of [Ga1] and [Zh1,Zh2], which slightly improve the original result of Lutwak [Lu].

(iv) The BP problem was posed in [BP] in 1956. Clearly, the solution is affirmative if $n = 2$, because in this case the inequalities for one-dimensional sections imply $K \subset L$. During the first twenty years the results related to the problem all went in the positive direction. Busemann [Bu2] proved that the answer is affirmative if K is an ellipsoid. Hadwiger [Ha2] proved the affirmative answer in the case where K and L are solids of revolution in $\mathbb{R}^3$. However, in 1975, Larman and Rogers [LR] used probabilistic methods to prove that the answer is negative if the dimension $n \geq 12$. Ball [Ba2] used his $\sqrt{2}$ bound (see Theorem 2.2) to construct counterexamples for $n \geq 10$. In Ball's examples the body with smaller sections is the unit cube and the body with larger sections is the Euclidean ball whose hyperplane sections have volume $\sqrt{2}$. Giannopoulos [Gi] and Bourgain [Bo] constructed counterexamples for $n \geq 7$. In 1992, Papadimitrakis [Pa] proved that the answer is negative for $n \geq 5$, and Gardner [Ga1] and Zhang [Z2] proved the same by constructing the first examples of non-intersection bodies in $\mathbb{R}^5$. In 1994, Gardner [Ga2] proved that every origin symmetric convex body in $\mathbb{R}^3$ is an intersection body, and, hence, the answer to the BP problem in dimension 3 is affirmative. For several years the problem was considered as completely solved, since the paper [Zh2] published in 1994 claimed that the unit cube in $\mathbb{R}^4$ was not an intersection body, which would imply a negative answer in dimension 4. However, in 1997, it was proved (Corollary 3.5 from this paper) in [K3] that the cube in $\mathbb{R}^4$ is an intersection body which reopened the four-dimensional case and suggested that the answer for $n = 4$ must be affirmative instead of negative. After this, Zhang [Zh3] proved that the answer to the BP problem in dimension 4 is indeed affirmative, and two months later a unified solution to the problem in all dimensions was given in [GKS]. The solution from [GKS] is presented in this section.

(v) Many more examples of non-intersection bodies can be found in [K4]. These include the unit balls of q-sums of normed spaces with $q > 2$ (one of the components must have dimension ≥ 4), the unit balls of Orlicz spaces satisfying the condition $M^{'}(0) = M^{''}(0) = 0$. For more results on intersection bodies, see [Zh4], [Ca1], [Ca2].

(vi) Other generalizations of the positive part of the solution to the Busemann-Petty problem can be found in [K8], [K9]. A very short proof of the affirmative part and its generalization based on the Funk-Hecke formula for spherical harmonics was given in [K11].

(vii) One can ask an isomorphic version of the Busemann-Petty problem: does there exist a constant C not depending on n, K, L so that if K and L are origin symmetric convex bodies in $\mathbb{R}^n$ satisfying $vol_{n-1}(K \cap \xi^{\perp}) \leq vol_{n-1}(L \cap \xi^{\perp})$ for all $\xi \in S^{n-1}$, then $vol_n(K) \leq C vol_n(L)$. This problem is open and is equivalent to the slicing (or hyperplane) problem: does there exist an absolute constant c so that, for every symmetric convex body K with volume 1, $\max_{\xi \in S^{n-1}} vol_{n-1}(K \cap \xi^{\perp}) \geq c$. For the status of these problems and for other equivalent formulations, see [MiP], [Bo2], [Da], [Ju], [KMP], [Pao].

5. k-intersection bodies.

Let H be an $(n-k)$-dimensional subspace of $\mathbb{R}^n$, $0 < k < n$, and let $\xi_1, ..., \xi_k$ be an orthonormal basis in $H^{\perp}$. We define the parallel section function of a star body D in $\mathbb{R}^n$ corresponding to the subspace H as a function on $\mathbb{R}^k$ given by

$$A_{D,H}(u) = vol_{n-k}(D \cap \{H + u_1\xi_1 + ... + u_k\xi_k\}),\ u \in \mathbb{R}^k.$$

Theorem 5.1 ([K8]). For every $m \in \mathbb{N} \cup \{0\}$, $m \neq (n-k)/2$, and any infinitely smooth symmetric star body D in $\mathbb{R}^n$,

$$\Delta^m A_{D,H}(0) = \frac{(-1)^m}{2^k \pi^k (n-2m-k)} \int_{S^{n-1} \cap H^{\perp}} (\| \cdot \|_D^{-n+2m+k})^{\wedge}(\xi)\ d\xi,$$

where Δ is the Laplace operator on $\mathbb{R}^k$.

Putting $m = 0$ in Theorem 5.1, we get the Fourier transform formula for lower dimensional sections:

$$vol_{n-k}(D \cap H) = \frac{1}{2^k \pi^k (n-k)} \int_{S^{n-1} \cap H^{\perp}} (\| \cdot \|_D^{-n+k})^{\wedge}(\xi)\ d\xi. \tag{5.1}$$

Applying the formula of Theorem 4.1 to the right-hand side of (5.1), we get an expression for the volume of lower dimensional sections in terms of the derivatives of hyperplane parallel section functions: for odd k,

$$vol_{n-k}(D \cap H) = \frac{1}{2^k \pi^{k-1}} \int_{S^{n-1}\cap H^\perp} A_\xi^{(k-1)}(0)\ d\xi.$$

Another geometric application of the formula (5.1) is the Fourier transform characterization of k-intersection bodies. Let D and L be origin symmetric star bodies in $\mathbb{R}^n$. We follow [K7] in saying that D is a *k-intersection body of L* if for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$

$$vol_k(D \cap H^\perp) = vol_{n-k}(L \cap H). \tag{5.2}$$

Unlike the case of intersection bodies (corresponding to $k = 1$ in the latter definition, up to a constant 2), it is not immediately clear when does a k-intersection body exist for a given L. We define the class of *k-intersection bodies* as the closure of k-intersection bodies of star bodies in the radial metric.

Theorem 5.2 ([K7,K8]). An origin symmetric star body D in $\mathbb{R}^n$ is a k-intersection body if and only if $\|\cdot\|_D^{-k}$ is a positive definite distribution. A k-intersection body exists for a given origin symmetric star body L if and only if $(\|\cdot\|_L^{-n+k})^\wedge$ is a positive continuous function on S^{n-1}.

Proof. We consider only the case where D is an infinitely smooth k-intersection body of some star body L. The general case can be proved after that by an approximation argument. Using polar formulas for the volume of sections we can write (5.2) as follows: for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$\frac{1}{k}\int_{S^{n-1}\cap H^\perp} \|x\|_D^{-k}\ dx = \frac{1}{n-k}\int_{S^{n-1}\cap H} \|\xi\|_L^{-n+k}\ d\xi.$$

Applying (5.1) to the left-hand side, we get that for every H

$$\frac{1}{(2\pi)^{n-k}k}\int_{S^{n-1}\cap H} (\|x\|_D^{-k})^\wedge(\xi)\ d\xi = \frac{1}{n-k}\int_{S^{n-1}\cap H} \|\xi\|_L^{-n+k}\ d\xi.$$

The standard uniqueness theorem for the spherical Radon transform (see for example [Ga3, p.248]) implies that for every $\xi \in S^{n-1}$

$$(\|\cdot\|_D^{-k})^\wedge(\xi) = \frac{(2\pi)^{n-k}k}{n-k}\|\xi\|_L^{-n+k} > 0.$$

One can immediately prove the second statement of the theorem by reversing the Fourier transform in the latter formula and using (5.1) again. □

We can see now why Theorems 4.5 and 4.6 generalize Lutwak's connection. Let us translate these theorems into the language of geometry using Theorems 4.1 and 5.2. The result of Theorem 4.5 reads as follows (one again needs a simple approximation argument to pass from infinitely smooth to $(k-1)$-smooth bodies):

Corollary 5.3. If D and L are origin-symmetric, $(k-1)$-smooth star bodies in $\mathbb{R}^n$ so that D is a k-intersection body and, for every $\xi \in S^{n-1}$,

$$(-1)^{(k-1)/2} A_{D,\xi}^{(k-1)}(0) \le (-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0).$$

Then $vol_n(D) \le vol_n(L)$.

Next we translate Theorem 4.6:

Corollary 5.4. Let L be an origin-symmetric, infinitely smooth convex body in $\mathbb{R}^n$ with positive curvature. Suppose that $0 < k < n$ and L is not a k-intersection body. Then there exists an origin symmetric convex body D in $\mathbb{R}^n$ so that, for every $\xi \in S^{n-1}$,

$$(-1)^{(k-1)/2} A_{D,\xi}^{(k-1)}(0) \le (-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0)$$

but $vol_n(D) > vol_n(L)$.

The case $k = 1$ of Corollaries 5.3 and 5.4 gives exactly Lutwak's connections.

Let us look at k-intersection bodies from a different point of view. Suppose that D is a k-intersection body in $\mathbb{R}^n$. Then $\|\cdot\|_D^{-k}$ is a positive definite distribution. By Schwartz's generalization of Bochner's theorem (see Section 1), the Fourier transform $(\|\cdot\|_D^{-k})^\wedge$ is a tempered measure on $\mathbb{R}^n$. This tempered measure , say μ, is also a homogeneous distribution of degree $-n+k$. The action of such measures on test functions can be written in polar coordinates (see [K10]), i.e. there exists a finite Borel measure μ_0 on S^{n-1} so that for every even test function ϕ,

$$\int_{\mathbb{R}^n} \|x\|_D^{-k} \phi(x)\ dx = \langle \|\cdot\|_D^{-k}, \phi \rangle = \langle \mu, \hat{\phi} \rangle =$$

$$\int_{\mathbb{R}^n} \hat{\phi}(x)\ d\mu(x) = \int_{S^{n-1}} d\mu_0(\xi) \int_0^\infty t^{k-1} \hat{\phi}(t\xi)\ dt.$$

We get the following characterization of k-intersection bodies that will later be used to relate these bodies to subspaces of L_p :

Lemma 5.5. An origin symmetric star body D in $\mathbb{R}^n$ is a k-intersection body if and only if there exists a finite Borel measure μ_0 on S^{n-1} such that for every even test function ϕ,

$$\int_{\mathbb{R}^n} \|x\|_D^{-k} \phi(x)\ dx = \int_{S^{n-1}} d\mu_0(\xi) \int_0^\infty t^{k-1} \hat{\phi}(t\xi)\ dt.$$

Let $p > 0$ and $L_p = L_p([0,1])$. A well-known simple fact is that an n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p if and only if there exists a finite Borel measure μ on S^{n-1} so that for every $x \in \mathbb{R}^n$,

$$\|x\|^p = \int_{S^{n-1}} |(x,\xi)|^p \ d\mu(x). \tag{5.3}$$

Let us write (5.3) in a different form. For any even test function ϕ,

$$\int_{\mathbb{R}^n} \|x\|^p \phi(x) \ dx = \int_{S^{n-1}} d\mu(\xi) \int_{\mathbb{R}^n} |(x,\xi)|^p \phi(x) \ dx =$$

$$\int_{S^{n-1}} d\mu(\xi) \int_{\mathbb{R}} |t|^p \big(\int_{(x,\xi)=t} \phi(x) \ dx \big) \ dt = \int_{S^{n-1}} \langle |t|^p, \mathcal{R}\phi(\xi;t) \rangle \ d\mu(\xi),$$

where $\mathcal{R}\phi(\xi;t)$ is the Radon transform of the function ϕ in the direction of ξ at the point t. We now use the connection between the Radon and Fourier transforms and the formula (1.4) for the Fourier transform of $|t|^p$ to conclude that

$$\int_{\mathbb{R}^n} \|x\|^p \phi(x) \ dx = c_p \int_{S^{n-1}} \langle |z|^{-1-p}, \hat{\phi}(z\xi) \rangle \ d\mu(\xi),$$

where c_p is the constant from (1.4). This representation does not make much sense for $p > 0$, but if we allow p to be negative then the action of the distribution $|z|^{-1-p}$ can be written as an integral. We arrive at the following

Definition 5.6 ([K10]). We say that an n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_{-p}, where $0 < p < n$, if there exists a finite Borel measure μ on S^{n-1} so that for every even test function ϕ

$$\int_{\mathbb{R}^n} \|x\|^{-p} \phi(x) \ dx = \int_{S^{n-1}} \ d\mu(\xi) \int_0^\infty t^{p-1} \hat{\phi}(t\xi) \ dt.$$

Comparing the latter equation with the result of Lemma 5.5, we get

Theorem 5.7. The following are equivalent:

(i) An origin symmetric star body D is a k-intersection body;
(ii) $\|\cdot\|_D^{-k}$ is a positive definite distribution;
(iii) The space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k}.

The advantage of introducing the concept of embedding in L_{-p} is that now one can try to extend properties of L_p-spaces with $p > 0$ to k-intersection bodies. Since the definition is "analytic" with respect to p, one can expect to be able to extend at least isometric properties of regular L_p-spaces. Let us present a few examples.

Example 1. We have proved (see Corollary 4.2) that for every n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ the function $\|\cdot\|^{-n+3}$ represents a positive definite distribution. In view of Theorem 5.7, this means that every n-dimensional normed space embeds in L_{-n+3}.

For $n = 2$ the latter statement amounts to a well-known fact (proved independently in [He],[Fe], [Li]) that every two-dimensional normed space embeds in L_1.

If $n = 3$ we get that every three-dimensional normed space embeds in L_0. The latter means that there exists a measure μ on S^{n-1} so that for every $x \in \mathbb{R}^n$

$$\log \|x\| = \int_{S^{n-1}} \log |(x, \xi)| \, d\mu(\xi).$$

Putting $n = 4$ we get that every four-dimensional normed space embeds in L_{-1} or, in other words, every symmetric convex body in $\mathbb{R}^4$ is an intersection body. This fact solves the BP problem in affirmative in the dimension 4.

Example 2. Another well-known fact from the isometric theory of L_p-spaces (proved in [BDK]) is that L_q embeds isometrically in L_p for every $0 < p < q \le 2$. This means that L_p-spaces become larger when p decreases from 2 to 0. One can expect that L_p is even "larger" when $p < 0$. Our next result proves exactly this.

Theorem 5.8 ([K5]). Every n-dimensional subspace of $L_q,\ 0 < q \le 2$ embeds in L_{-p} for every $0 < p < n$. In particular, the unit ball of any n-dimensional subspace of L_q with $0 < q \le 2$ is a k-intersection body for every $0 < k < n$.

Proof. A classical result of P.Lèvy is that an n-dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in $L_q,\ 0 < q \le 2$ if and only if $\exp(-\|\cdot\|^q)$ is a positive definite function. Then for every $t > 0$ the function $\exp(-t^q\|\cdot\|^q)$ is also positive definite. For every $p \in (0, n)$, writing

$$\|x\|^{-p} = \frac{q}{\Gamma(p/q)} \int_0^\infty t^{p-1} \exp(-t^q \|x\|^q) \, dt$$

we see that $\|\cdot\|^{-p}$ is a positive combination of positive definite functions. Now the fact that $\|\cdot\|^{-p}$ is a positive definite distribution follows from the Fubini theorem. □

Corollary 5.9. The answer to the Busemann-Petty problem is affirmative if K is the unit ball of a subspace of $L_q,\ 0 < q \le 2$.

Example 3. It follows from Theorems 3.4 and 5.7 that for $q > 2$ the space l_q^n does not embed in L_{-p} if $p < n - 3$. This fact extends to negative p the solution to Schoenberg's problem, posed in 1938 in [Sch] and solved in 1991 in [K1]. This problem asks for which $p > 0$ is the function $\exp(-\|\cdot\|_q^p)$ positive definite, where

$q > 2$, or, in other words, for which $0 < p < 2,\ n \geq 2, q > 2$ does the space l_q^n embed isometrically in L_p. The answer is that, for $n \geq 3$, the space $l_q^n,\ q > 2$ does not embed in any $L_p,\ 0 < p < 2$. For $n = 2$, the embedding exists if and only if $0 < p \leq 1$.

Another generalization of the concept of an intersection body was introduced by Zhang [Zh5]. We say that a symmetric star body D in $\mathbb{R}^n$ is a *generalized k-intersection body* if there exists a finite Borel measure μ on the Grassmanian $G(n; n-k)$ of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$ so that for every $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} \|x\|_D^{-k} f(x)\ dx = \int_{G(n;n-k)} d\mu(H) \int_{S^{n-1}\cap H} f(x)\ dx.$$

This class of bodies is related to the following generalized BP problem in the same way, as intersection bodies are related to the original BP problem: the answer to the generalized BP problem is affirmative in $\mathbb{R}^n$ if and only if every origin symmetric convex body in $\mathbb{R}^n$ is a generalized k-intersection body.

Generalized BP problem. Suppose that $0 < k < n$ and D, L are origin symmetric convex bodies in $\mathbb{R}^n$ so that for every $H \in G(n; n-k)$,

$$vol_{n-k}(D \cap H) \leq vol_{n-k}(L \cap H).$$

Does it follow that $vol_n(D) \leq vol_n(L)$?

It was proved in [BZ] that the answer is negative if $n - k > 3$. A different proof of this fact was given in [K8]. This proof is based on

Theorem 5.10 ([K8]). Every generalized k-intersection body is also a k-intersection body.

Now if $n - k > 3$ then $k < n - 3$ and, by Theorems 3.4 and 5.2, the unit balls of $l_q^n,\ q > 2$ are not k-intersection bodies. By Theorem 5.10, these bodies are not generalized k-intersection bodies, and the negative answer to the generalized BP problem for $n - k > 3$ follows.

The question of whether every k-intersection body is a generalized k-intersection body is open. In view of Corollary 4.2, an affirmative answer to this question would imply an affirmative answer to the generalized BP problem for $n - k \leq 3$.

Remarks. (i) Thorem 5.2 is proved in [K7] for infinitely differentiable bodies, and the approximation argument needed for the general case can be found in [K8].

(ii) Meyer [Me] proved that the answer to the Busemann-Petty problem is affirmative if K is the cross-polytope (unit ball of l_1^n) and L is any origin symmetric convex body. He also proved the Busemann-Petty inequality up to a constant, not depending on the dimension, for the case where K is the unit ball of a subspace of L_p, $0 < p \leq 2$ and conjectured that the constant can be removed. Corollary 5.9, in conjunction with the first Lutwak's connection (Corollary 5.3), confirms the conjecture of Meyer.

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