MAXIMAL OPERATOR FOR MULTILINEAR SINGULAR INTEGRALS WITH NON-SMOOTH KERNELS

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Abstract. In this article we prove Cotlar's inequality for the maximal singular integrals associated with operators whose kernels satisfy regularity conditions weaker than those of the standard m-linear Calderón-Zygmund kernels. The present study is motivated by the fundamental example of the maximal mth order Calderón commutators whose kernels are not regular enough to fall under the scope of the m-linear Calderón-Zygmund theory; the Cotlar inequality is a new result even for these operators.

Contents

1. Introduction and main results

Multilinear Calderón-Zygmund singular integral operators originated in the work of Coifman and Meyer [CM1], [CM2] in the seventies; see also [CM3]. In recent years the study of these operators has made significant advances. At present, several features of the theory of multilinear operators are developed but certain other aspects remain unexplored.

In this work, we prove Cotlar's inequality (see [C]) for maximal singular integral operators associated with m-linear operators whose kernels satisfy regularity conditions significantly weaker than those of the standard Calderón-Zygmund kernels. We

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use this inequality to obtain multilinear weighted norm inequalities for these operators. An important example of singular multilinear operators whose kernels satisfy our weak regularity conditions but do not satisfy the standard Calderón-Zygmund kernel regularity are the mth order commutators of Calderón. In Proposition 4.1 and Remark 4.2 we obtain new weighted estimates for the mth order commutators of Calderón.

We begin by recalling the notation in [GT1], [GT2] and [DGY]. These articles, and the references therein, contain background material on this subject.

Let $K(x, y_1, \cdots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, and let $T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \mapsto \mathcal{S}'(\mathbb{R}^n)$ be an mth linear operator associated with the kernel $K(x, y_1, \cdots, y_m)$ in the following way:

$$\begin{aligned} &\langle T(f_1, \cdots, f_m), g\rangle \\ &= \int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1)\cdots f_m(y_m) g(x)\, dy_1 \cdots dy_m dx, \end{aligned} \tag{1.1}$$

where $f_1, \cdots, f_m, g$ in $\mathcal{S}(\mathbb{R}^n)$ with $\cap_{j=1}^m \mathrm{supp}\, f_j \cap \mathrm{supp}\, g = \emptyset$. Throughout the paper, we assume the following *size estimate* on the kernel K,

$$\big|K(x, y_1, \cdots, y_j, \cdots, y_m)\big| \le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \tag{1.2}$$

for some $A > 0$ and all $(x, y_1, \cdots, y_j, \cdots, y_m)$ with $x \neq y_j$ for some j.

We work with a class of integral operators $\{A_t\}_{t>0}$, that play the role of an approximation to the identity as in [DM]. We assume that the operators A_t are associated with kernels $a_t(x, y)$ in the sense that

$$A_t f(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y)\, dy$$

for every function $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and that the kernels $a_t(x, y)$ satisfy the following size conditions

$$|a_t(x, y)| \le h_t(x, y) = t^{-n/s} h\Big(\frac{|x-y|^s}{t}\Big), \tag{1.3}$$

where s is a positive fixed constant and h is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\eta} h(r^s) = 0 \tag{1.4}$$

for some $\eta > 0$. These conditions imply that for some $C' > 0$ and all $0 < \eta' \le \eta$, the kernels $a_t(x, y)$ obey the estimate

$$|a_t(x, y)| \le C' t^{-n/s}(1 + t^{-1/s}|x-y|)^{-n-\eta'}.$$

Now, let T be a multilinear operator associated with a kernel $K(x, y_1, \cdots, y_m)$ in the sense in (1.1). The first of the basic assumptions we impose on T is the following:

Assumption (H1). Assume that for each $j = 1, 2, \cdots, m$, there exist operators $\{A_t^{(j)}\}_{t>0}$ with kernels $a_t^{(j)}(x, y)$ that satisfy conditions (1.3) and (1.4) with constants s and η, and there exist kernels $K_t^{(j)}(x, y_1, \cdots, y_m)$ such that

$$\begin{aligned}&\langle T(f_1, \cdots, A_t^{(j)} f_j, \cdots, f_m), g\rangle \\ &= \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)^m} K_t^{(j)}(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) g(x) dy_1 \cdots dy_m dx,\end{aligned} \tag{1.5}$$

for all $f_1, \cdots, f_m, g$ in $\mathcal{S}(\mathbb{R}^n)$ with $\cap_{j=1}^m \mathrm{supp}\, f_j \cap \mathrm{supp}\, g = \emptyset$, and there exist a function $\phi \in C(\mathbb{R})$ with supp$\phi \subset [-1, 1]$ and a constant $\epsilon > 0$ such that for all $x, y_1, \cdots, y_m \in \mathbb{R}^n$ and $t > 0$ we have

$$\begin{aligned}&\big|K(x, y_1, \cdots, y_m) - K_t^{(j)}(x, y_1, \cdots, y_m)\big| \\ &\le \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \sum_{\substack{k=1\\k\ne j}}^m \phi\Big(\frac{|y_j - y_k|}{t^{1/s}}\Big) \\ &+ \frac{A t^{\epsilon/s}}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\epsilon}}\end{aligned} \tag{1.6}$$

for some $A > 0$, whenever $t^{1/s} \le |x - y_j|/2$.

Kernels satisfying (1.5) and (1.6) with parameters m, A, s, η, ϵ are called generalized Calderón-Zygmund kernels; their collection is denoted by m-$GCZK_0(A, s, \eta, \epsilon)$.

We recall the standard Calderón-Zygmund kernel condition for multilinear operators. For some $\epsilon > 0$ we have the *smoothness estimates*

$$\begin{aligned}&|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x', y_1, \cdots, y_j, \cdots, y_m)| \\ &\le \frac{A|x - x'|^\epsilon}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\epsilon}}\end{aligned} \tag{1.7}$$

whenever $|x - x'| \le \frac{1}{2}\max_{1\le j\le m}|x - y_j|$ and also for each j,

$$\begin{aligned}&|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x, y_1, \cdots, y_j', \cdots, y_m)| \\ &\le \frac{A|y_j - y_j'|^\epsilon}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\epsilon}}\end{aligned} \tag{1.8}$$

whenever $|y_j - y_j'| \le \frac{1}{2}\max_{1\le j\le m}|x - y_j|$.

We refer reader to Proposition 2.1 in [DGY] where it was proved that the condition (1.6) is weaker than, and indeed a consequence of, the Calderón-Zygmund kernel condition (1.8). The main reason for the introduction of the condition (1.6) is that it is sufficient for the weak type endpoint estimate. The next result is contained in [DGY] (Theorem 1.1); the analogous result in the multilinear Calderón-Zygmund theory is contained in [GT1] and [GT2].

Theorem A. *Let T be a multilinear operator with kernel in m-$GCZK_0(A, s, \eta, \epsilon)$. Assume that for some $1 \le q_1, q_2, \cdots, q_m < \infty$ and some $0 < q < \infty$ with*

$$\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = \frac{1}{q}, \tag{1.9}$$

T maps $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$. Then T has a bounded extension from the m-fold product $L^1(\mathbb{R}^n)\times\cdots\times L^1(\mathbb{R}^n)$ to $L^{1/m,\infty}(\mathbb{R}^n)$.

Moreover, for some constant $C_{n,m}$ (that depends only on the parameters indicated) we have that

$$\|T\|_{L^1\times\cdots\times L^1\to L^{1/m,\infty}}\le C_{n,m}(A+\|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}}). \tag{1.10}$$

In this article, our goal is to study the maximal truncated operator

$$T_*(f_1,\cdots,f_m)(x)=\sup_{\delta>0}\Big|T_\delta(f_1,\cdots,f_m)(x)\Big|,$$

where (with the notation $\vec{y}=(y_1,\cdots,y_m)$ and $d\vec{y}=dy_1\cdots dy_m$),

$$T_\delta(f_1,\cdots,f_m)(x)=\int_{|x-y_1|^2+\cdots+|x-y_m|^2\ge\delta^2}K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y}.$$

We note that if $f_j\in L^{p_j}(\mathbb{R}^n)$ with $1\le p_j\le\infty$, then $T_\delta(f_1,\cdots,f_m)$ is given by an absolutely convergent integral and thus is well defined (see [GT3] for a proof). Thus $T_*(f_1,\cdots,f_m)(x)$ is also pointwise well-defined when $f_j\in L^{p_j}(\mathbb{R}^n)$ with $1\le p_j\le\infty$. In Theorem 1.1 below we prove a pointwise estimate for T_* when the f_j's lie in suitable Lebesgue spaces.

The next assumption (**H2**) is similar to (**H1**) with the main difference being that the role of y_j $(j=1,...,m)$ in (**H1**) is assumed by x in (**H1**).

Assumption (**H2**). Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x,y)$ that satisfy conditions (1.3) and (1.4) with constants s and η and also that there exist kernels $K_t^{(0)}(x,y_1,\cdots,y_m)$ such that the representation is valid

$$K_t^{(0)}(x,y_1,\cdots,y_m)=\int_{\mathbb{R}^n}K(z,y_1,\cdots,y_m)b_t(x,z)dz\,; \tag{1.11}$$

assume also that there exist a function $\phi\in C(\mathbb{R})$ with supp$\phi\subset[-1,1]$ and a constant $\epsilon>0$ such that

$$\begin{aligned}\big|K(x,y_1,\cdots,y_m)-K_t^{(0)}(x,y_1,\cdots,y_m)\big|&\\ \le\ \frac{A}{(|x-y_1|+\cdots+|x-y_m|)^{mn}}\sum_{\substack{k=1\\k\ne j}}^{m}\phi\Big(\frac{|x-y_k|}{t^{1/s}}\Big)&\\ +\ \frac{At^{\epsilon/s}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\epsilon}}&\end{aligned} \tag{1.12}$$

for some $A>0$, whenever $2t^{1/s}\le\max_{1\le j\le m}|x-y_j|$.

By a proof similar to that of Proposition 2.1 in [DGY], we can see that (**H2**) is weaker than the Hölder continuity condition (1.7) for $K(x,y_1,\cdots,y_m)$.

We next introduce assumption (**H3**); this assumption is discussed in Section 2.2.

Assumption (H3). Assume that there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x,y)$ that satisfy conditions (1.3) and (1.4) with constants s and η and there exist kernels $K_t^{(0)}(x,y_1,\cdots,y_m)$ such that (1.11) holds. Also assume that there exist positive constants A and ϵ such that

$$\big|K_t^{(0)}(x,y_1,\cdots,y_m)\big| \le \frac{A}{\big(|x-y_1|+\cdots+|x-y_m|\big)^{mn}} \tag{1.13}$$

whenever $2t^{1/s} \le \min_{1\le j\le m}|x-y_j|$, and

$$\begin{aligned}\big|K_t^{(0)}(x,y_1,\cdots,y_m) - K_t^{(0)}(x',y_1,\cdots,y_m)\big| \\ \le \frac{At^{\epsilon/s}}{\big(|x-y_1|+\cdots+|x-y_m|\big)^{mn+\epsilon}}\end{aligned} \tag{1.14}$$

whenever $2|x-x'| \le t^{1/s}$ and $2t^{1/s} \le \min_{1\le j\le m}|x-y_j|$.

Our first main result is the following improvement of the multilinear Cotlar inequality:

Theorem 1.1. *Let T be a multilinear operator with a kernel $K(x,y_1,\cdots,y_m)$ in the sense in (1.1) that satisfies assumptions* (**H1**), (**H2**) *and* (**H3**). *Assume that for some $1\le q_1,q_2\cdots,q_m<\infty$ and some $0<q<\infty$ with*

$$\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}=\frac{1}{q}, \tag{1.15}$$

T maps $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$. Then for all $\eta>0$, there exists a constant $C_\eta = C(\eta,n)<\infty$ such that for all $\vec{f}$ in any product of $L^{p_j}(\mathbb{R}^n)$ spaces, with $1\le p_j<\infty$, the following inequality holds for all x in $\mathbb{R}^n$

$$T_*(\vec{f})(x) \le C_\eta\Big(\big(\mathcal{M}(\big|T(\vec{f})\big|^\eta)(x)\big)^{\frac{1}{\eta}} + (A+W)\prod_{j=1}^m \mathcal{M}(f_j)(x)\Big), \tag{1.16}$$

where W denotes the norm of $\|T\|_{L^{q_1}\times\cdots\times L^{q_m}\to L^{q,\infty}}$, and $\mathcal{M}$ denotes the Hardy-Littlewood maximal function with respect to balls on $\mathbb{R}^n$.

An immediate consequence of Theorem 1.1 is that if T is given by a principle value integral of the form

$$\begin{aligned}&T(f_1,\cdots,f_m)(x)\\ &= \lim_{\delta\to 0}\int_{|x-y_1|^2+\cdots+|x-y_m|^2\ge\delta^2} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y},\end{aligned} \tag{1.17}$$

when the functions f_j are in the Schwartz class, then the integrals in (1.17) converge almost everywhere for all f_j in $L^{p_j}(\mathbb{R}^n)$; see [GT3].

The second main result of the paper is the weighted estimates of mth commutator of Calderón in Theorem 4.3 in Section 4.

The layout of the paper is as follows. In Section 2, we give a proof of Theorem 1.1, and check that for suitable "approximations of the identity", the regularity conditions (1.12) and (1.14) are significantly weaker than those of the standard Calderón-Zygmund kernels. We then apply Theorem 1.1 to obtain multilinear weighted norm inequalities in Section 3 and deduce the corresponding results for the mth commutators in Section 4.

Throughout, the letter "C" denotes (possibly different) constants that are independent of the essential variables.

2. Cotlar's inequality for multilinear singular integrals

In the section, we use the vector notation $\vec{f}=(f_1,\cdots,f_m)$ for brevity. For a given $x\in\mathbb{R}^n$ and $\delta>0$ we denote by $S_\delta(x)=\big\{\vec{y}:\ \sup_{1\le j\le m}|y_j-x|<\delta\big\}$, and

$$U_\delta(x)=\Big\{\vec{y}:\ \sum_{j=1}^m|y_j-x|^2<\delta^2\Big\}\quad\text{and}\quad V_\delta(x)=\Big\{\vec{y}:\ \inf_{1\le j\le m}|y_j-x|\ge\delta\Big\}.$$

2.1. **Proof of Theorem 1.1.** It suffices to prove the theorem for η arbitrarily small; in fact we provide an argument for $0<\eta<1/m$. Fix x in $\mathbb{R}^n$ and let $\delta>0$. From condition (1.2), we have

$$\begin{aligned}&\sup_{\delta>0}\Big|\int_{(\mathbb{R}^n)^m\setminus\big(U_\delta(x)\cup V_\delta(x)\big)}K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y}\Big|\\ &\le\ CA\sup_{\delta>0}\int_{(\mathbb{R}^n)^m\setminus\big(U_\delta(x)\cup V_\delta(x)\big)}\frac{\prod_{j=1}^m|f_j(y_j)|}{\big(\delta+|x-y_1|+\cdots+|x-y_m|\big)^{mn}}d\vec{y}.\end{aligned}\tag{2.1}$$

Note that (2.1) can be written as a sum of integrals over sets $R_{j_1,\cdots,j_\ell}$ in $(\mathbb{R}^n)^m$ for some $\{j_1,\cdots,j_\ell\}\subseteq\{1,\cdots,m\}$ and $\ell\le m$, such that for $\vec{y}=(y_1,\cdots,y_m)$ in $R_{j_1,\cdots,j_\ell}$ we have that $|x-y_j|\le\delta$ if and only if $j\in\{j_1,\cdots,j_\ell\}$. It follows that for every $\delta>0$,

$$\begin{aligned}&\int_{(\mathbb{R}^n)^m\setminus\big(U_\delta(x)\cup V_\delta(x)\big)}\frac{\prod_{j=1}^m|f_j(y_j)|}{\big(\delta+|x-y_1|+\cdots+|x-y_m|\big)^{mn}}d\vec{y}\\ &\le\ C\prod_{j\in\{j_1,\cdots,j_\ell\}}\frac{1}{\delta^n}\int_{|x-y_j|<\delta}|f_j(y_j)|dy_j\prod_{j\notin\{j_1,\cdots,j_l\}}\int_{|x-y_j|\ge\delta}\frac{\delta^{\frac{\ell n}{m-\ell}}}{|x-y_j|^{\frac{mn}{m-\ell}}}|f_j(y_j)|dy_j\\ &\le\ C\prod_{j\in\{j_1,\cdots,j_\ell\}}\mathcal{M}f_j(x)\prod_{j\notin\{j_1,\cdots,j_l\}}\int_{|x-y_j|\ge\delta}\frac{\delta^{\frac{\ell n}{m-\ell}}}{|x-y_j|^{\frac{mn}{m-\ell}}}|f_j(y_j)|dy_j\\ &\le\ C\prod_{j=1}^m\mathcal{M}f_j(x).\end{aligned}$$

It suffices therefore to show (1.16) with $T_*(\vec{f})(x)$ replaced by

$$\widetilde{T}_*(\vec{f})(x):=\sup_{\delta>0}\big|\big(\widetilde{T}\big)_\delta(\vec{f})(x)\big|,\tag{2.2}$$

where

$$(\widetilde{T})_\delta(\vec{f})(x) := \int_{\substack{|y_1-x|\geq\delta \\ \cdots\cdots \\ |y_m-x|\geq\delta}} K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y}.$$

Fix $\delta>0$ and let $B(x,\delta/2)$ be the ball of center x and radius $\delta/2$. Note that, since $\vec{f}$ lies in a product of Lebesgue spaces and T is a Calderón-Zygmund operator, $T(\vec{f})$ is in some L^p space and hence it is finite almost everywhere. We use the linearity of T to obtain, for $z\in B(x,\delta/2)$, that

$$\begin{aligned}(\widetilde{T})_\delta(\vec{f})(z) &= T(\vec{f})(z) - T(\vec{f_0})(z) \\ &\quad - \int_{(\mathbb{R}^n)^m\setminus\left(U_\delta(z)\cup V_\delta(z)\right)} K(z,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y},\end{aligned} \tag{2.3}$$

where $\vec{f_0}=(f_1\chi_{B(z,\delta)},\cdots,f_m\chi_{B(z,\delta)})$.

Observe that for every $j=1,\cdots,m$, the conditions $z\in B(x,\delta/2)$ and $|y_j-z|>\delta$ imply that $|y_j-z|/2\leq|y_j-x|\leq 2|y_j-z|$. We then use an argument as that in (2.1) to deduce

$$\begin{aligned}&\Big|\int_{(\mathbb{R}^n)^m\setminus\left(U_\delta(z)\cup V_\delta(z)\right)} K(z,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y}\Big| \\ &\quad \leq CA\prod_{j=1}^m \mathcal{M}f_j(x).\end{aligned} \tag{2.4}$$

Consider the term $\left|(\widetilde{T})_\delta(\vec{f})(x)-(\widetilde{T})_\delta(\vec{f})(z)\right|$ when $z\in B(x,\delta/2)$. Let $t=\left(\frac{\delta}{4}\right)^s$ where s is a constant in (1.3). We decompose it into

$$\begin{aligned}(\widetilde{T})_\delta(\vec{f})(x)-(\widetilde{T})_\delta(\vec{f})(z) &= \left(\widetilde{B_tT}\right)_\delta(\vec{f})(z) - (\widetilde{T})_\delta(\vec{f})(z) \\ &\quad + \left(\widetilde{B_tT}\right)_\delta(\vec{f})(x) - \left(\widetilde{B_tT}\right)_\delta(\vec{f})(z) \\ &\quad + (\widetilde{T})_\delta(\vec{f})(x) - \left(\widetilde{B_tT}\right)_\delta(\vec{f})(x) \\ &=: \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned} \tag{2.5}$$

For term I we use assumption (**H2**). First note that $|z-y_j|\geq\delta=4t^{1/s}$. This, in combination with the fact that $\mathrm{supp}\phi\subset[-1,1]$, yields that $\phi\Big(\frac{|z-y_j|}{t^{1/s}}\Big)=0$. By assumption (**H2**) we have

$$
\begin{aligned}
|\mathrm{I}| &\leq CA\int_{\substack{|y_1-z|\geq\delta\\ \cdots\cdots\\ |y_m-z|\geq\delta}} \frac{\delta^\epsilon}{(|z-y_1|+\cdots+|z-y_m|)^{mn+\epsilon}}\prod_{j=1}^m |f_j(y_j)|d\vec{y}\\
&\leq CA\int_{\substack{|y_1-x|\geq\delta/2\\ \cdots\cdots\\ |y_m-x|\geq\delta/2}} \frac{\delta^\epsilon}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\epsilon}}\prod_{j=1}^m |f_j(y_j)|d\vec{y},
\end{aligned}
$$

where we made use of the facts that $z\in B(x,\delta/2)$ and $|y_j-z|\geq\delta$, and consequently $|y_j-z|/2\leq|y_j-x|\leq 2|y_j-z|$ again. Hence,

$$
|\mathrm{I}| \leq CA\prod_{j=1}^m\int_{\mathbb{R}^n}\frac{\delta^{\epsilon/m}|f_j(y_j)|}{(\delta+|x-y_j|)^{n+\frac{\epsilon}{m}}}dy_j\leq CA\prod_{j=1}^m\mathcal{M}f_j(x).
$$

In particular, we also have that $|\mathrm{III}|\leq CA\prod_{j=1}^m\mathcal{M}f_j(x)$.

We now turn to term II. We rewrite this term as

$$
\begin{aligned}
\mathrm{II} &= \int_{V_\delta(x)\backslash V_\delta(z)} K_t^{(0)}(x,y_1,\cdots,y_m)\prod_{j=1}^m f_j(y_j)d\vec{y}\\
&\quad+\int_{V_\delta(z)}\Big(K_t^{(0)}(x,y_1,\cdots,y_m)-K_t^{(0)}(z,y_1,\cdots,y_m)\Big)\prod_{j=1}^m f_j(y_j)d\vec{y}\\
&=: \mathrm{II}_1+\mathrm{II}_2.
\end{aligned}
$$

Since $z\in B(x,\delta/2)$, we have that $V_{3\delta/2}(x)\subset V_\delta(z)$. This shows that $V_\delta(x)\backslash V_\delta(z)\subset V_\delta(x)\backslash V_{3\delta/2}(x)\subset(\mathbb{R}^n)^m\backslash\big(U_\delta(x)\cup V_{3\delta/2}(x)\big)$. Using (1.13) of assumption (**H3**) and an argument similar to that in (2.1), we obtain

$$
\begin{aligned}
|\mathrm{II}_1| &\leq \int_{(\mathbb{R}^n)^m\backslash\left(U_\delta(x)\cup V_{3\delta/2}(x)\right)}\big|K_t^{(0)}(x,y_1,\cdots,y_m)\big|\prod_{j=1}^m|f_j(y_j)|d\vec{y}\\
&\leq CA\int_{(\mathbb{R}^n)^m\backslash\left(U_\delta(x)\cup V_{3\delta/2}(x)\right)}\frac{\prod_{j=1}^m|f_j(y_j)|}{\big(\delta+|x-y_1|+\cdots+|x-y_m|\big)^{mn}}d\vec{y}\\
&\leq CA\prod_{j=1}^m\mathcal{M}f_j(x).
\end{aligned}
$$

For term II_2, we use property (1.14) of assumption (**H3**), and the argument employed in the handling of term I to deduce that for $z\in B(x,\delta/2)$ we have

$$
\begin{aligned}
|\mathrm{II}_2| &\le CA\int_{\substack{|y_1-z|\ge\delta\\ \cdots\cdots\\ |y_m-z|\ge\delta}} \frac{\delta^{\epsilon}}{\big(|z-y_1|+\cdots+|z-y_m|\big)^{mn+\epsilon}}\big|f_1(y_1)\cdots f_m(y_m)\big|d\vec{y}\\
&\le CA\prod_{j=1}^{m}\int_{\mathbb{R}^n}\frac{\delta^{\epsilon/m}\big|f_j(y_j)\big|}{\big(\delta+|x-y_j|\big)^{n+\frac{\epsilon}{m}}}dy_j\\
&\le CA\prod_{j=1}^{m}\mathcal{M}f_j(x).
\end{aligned}
$$

Therefore, we obtain

$$
|\mathrm{II}|\le CA\prod_{j=1}^{m}\mathcal{M}f_j(x).
$$

Collecting estimates for terms I, II and III, it follows from (2.3), (2.4) and (2.5) that for $z\in B(x,\delta/2)$,

$$
\Big|\big(\widetilde{T}\big)_{\delta}(\vec{f})(x)\Big|\le CA\prod_{j=1}^{m}\mathcal{M}f_j(x)+\big|T(\vec{f})(z)-T(\vec{f_0})(z)\big|. \tag{2.6}
$$

Fix now $0<\eta<1/m$. Raising (2.6) to the power η, integrating over $z\in B=B(x,\delta/2)$, and dividing by $|B|$ we obtain

$$
\begin{aligned}
&\Big|\big(\widetilde{T}\big)_{\delta}(\vec{f})(x)\Big|^{\eta}\\
&\le \Big(CA\prod_{j=1}^{m}\mathcal{M}f_j(x)\Big)^{\eta}+\mathcal{M}\big(\big|T(\vec{f})\big|^{\eta}\big)(x)+\frac{1}{|B|}\int_B\big|T(\vec{f_0})(z)\big|^{\eta}dz. \qquad (2.7)
\end{aligned}
$$

Next we estimate the last term in (2.7). First, we note that from Theorem A (see also Theorem 1.1, [DGY]), T is bounded from $L^1(\mathbb{R}^n)\times\cdots\times L^1(\mathbb{R}^n)$ to $L^{1/m,\infty}(\mathbb{R}^n)$. Hence,

$$
\begin{aligned}
\int_B\big|T(\vec{f_0})(z)\big|^{\eta}dz &= m\eta\int_0^{\infty}\lambda^{m\eta-1}\big|\{z\in B:\big|T(\vec{f_0})(z)\big|^{1/m}>\lambda\}\big|d\lambda\\
&\le m\eta\int_0^{\infty}\lambda^{m\eta-1}\min\Big\{|B|,\frac{W^{\frac{1}{m}}}{\lambda}\Big(\prod_{j=1}^{m}\big\|f_j\chi_{B(x,\delta)}\big\|_{L^1(\mathbb{R}^n)}\Big)^{\frac{1}{m}}\Big\}d\lambda.
\end{aligned}
$$

Letting

$$
R=W^{1/m}\Big(\prod_{j=1}^{m}\big\|f_j\chi_{B(x,\delta)}\big\|_{L^1(\mathbb{R}^n)}\Big)^{1/m},
$$

we use the condition $m\eta<1$ to obtain

$$\begin{aligned}\frac{1}{|B|}\int_B \big|T(\vec{f}_0)(z)\big|^\eta dz &\leq \frac{m\eta}{|B|}\Big(\int_0^{R/|B|}\lambda^{m\eta-1}|B|d\lambda+\int_{R/|B|}^\infty \lambda^{m\eta-2}Rd\lambda\Big)\\ &\leq C_\eta R^{m\eta}|B|^{1-m\eta}\\ &\leq C_\eta W^\eta |B|^{-m\eta}\Big(\prod_{j=1}^m \big\|f_j\chi_{B(x,\delta)}\big\|_{L^1(\mathbb{R}^n)}\Big)^\eta\\ &\leq C_\eta W^\eta\Big(\prod_{j=1}^m \mathcal{M}f_j(x)\Big)^\eta\end{aligned}$$

and if we insert this estimate into (2.7) and raise to the power $1/\eta$, we obtain the desired estimate (1.16). This concludes the proof of Theorem 1.1.

Before we state the following corollary, we recall that an m-linear operator $T : \mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)\mapsto\mathcal{S}'(\mathbb{R}^n)$ is linear in every entry and consequently it has m formal transposes. The jth transpose T^{*j} of T is defined via

$$\langle T^{*j}(f_1,\cdots,f_m),h\rangle=\langle T(f_1,\cdots,f_{j-1},h,f_{j+1},\cdots,f_m),f_j\rangle, \tag{2.8}$$

for all $f_1,\cdots,f_m,g$ in $\mathcal{S}(\mathbb{R}^n)$.

It is easy to check that the kernel K^{*j} of T^{*j} is related to the kernel K of T via the identity

$$K^{*j}(x,y_1,\cdots,y_{j-1},y_j,y_{j+1},\cdots,y_m)=K(y_j,y_1,\cdots,y_{j-1},x,y_{j+1},\cdots,y_m). \tag{2.9}$$

Note that if a multilinear operator T maps a product of a Banach spaces $X_1\times\cdots\times X_m$ to another Banach space X, then the transpose $T^{*,j}$ maps the product of Banach spaces $X_1\times\cdots\times X_{j-1}\times X^*\times X_{j+1}\times\cdots\times X_m$ to X_j^*. Moreover, the norms of T and $T^{*,j}$ are equal. For notational convenience, we may occasionally denote T by T^{*0} and K by K^{*0}.

Assumption (H4). Assume that for all $i=1,\cdots,m$, there exist operators $\{A_t^{(i)}\}_{t>0}$ with kernels $a_t^{(i)}(x,y)$ that satisfy conditions (1.3) and (1.4) with constants s and η and that for every $j=0,1,2,\cdots,m$, there exist kernels $K_t^{*,j,(i)}(x,y_1,\cdots,y_m)$ such that

$$\begin{aligned}&\langle T^{*j}(f_1,\cdots,A_t^{(i)}f_i,\cdots,f_m),g\rangle\\ &=\int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^m}K_t^{*j,(i)}(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)g(x)dy_1\cdots dy_mdx,\end{aligned} \tag{2.10}$$

for all $f_1,\cdots,f_m$ in $\mathcal{S}(\mathbb{R}^n)$ with $\cap_{k=1}^m\text{supp}\, f_k\cap\text{supp}\, g=\emptyset$. Assume also that there exist a function $\phi\in C(\mathbb{R})$ with $\text{supp}\,\phi\subset[-1,1]$ and a constant $\epsilon>0$ such that for every $j=0,1,\ldots,m$ and every $i=1,2,\cdots,m$, we have

$$\begin{aligned}\big|K^{*j}(x,y_1,\cdots,y_m) - K_t^{*j,(i)}(x,y_1,\cdots,y_m)\big| \\ \leq \quad \frac{A}{(|x-y_1|+\cdots+|x-y_m|)^{mn}} \sum_{\substack{k=1\\k\neq i}}^{m} \phi\Big(\frac{|y_i-y_k|}{t^{1/s}}\Big) \\ + \frac{At^{\epsilon/s}}{(|x-y_1|+\cdots+|x-y_m|)^{mn+\epsilon}}\end{aligned} \tag{2.11}$$

whenever $t^{1/s} \leq |x-y_i|/2$.

Under assumption (**H4**) we will say that T is an m-linear operator with *generalized Calderón-Zygmund kernel* K. The collection of functions K satisfying (2.10) and (2.11) with parameters m, A, s,η and ϵ is denoted by m-$GCZK(A,s,\eta,\epsilon)$. A kernel K belongs to m-$GCZK(A,s,\eta,\epsilon)$ exactly when it belongs to m-$GCZK_0(A,s,\eta,\epsilon)$ and all of its adjoints also belong to m-$GCZK_0(A,s,\eta,\epsilon)$. For operators with such kernels we have the following result (see Theorem 3.1 in [DGY]).

Proposition 2.1. *Assume that T is a multilinear operator with kernel K in m-$GCZK(A,s,\eta,\epsilon)$. Let $1<q_1,q_2,\cdots,q_m,q<\infty$ be given numbers satisfying*

$$\frac{1}{q_1}+\frac{1}{q_2}+\cdots+\frac{1}{q_m}=\frac{1}{q}.$$

Assume that T maps $L^{q_1}(\mathbb{R}^n)\times\cdots\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Let p,p_j be numbers satisfying $1/m\leq p<\infty$, $1\leq p_j\leq\infty$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}$. Then all statements below are valid:

(i) when all $p_j>1$, then T can be extended to be a bounded operator from the m-fold product $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$;

(ii) when some $p_j=1$, then T can be extended to be a bounded operator from the m-fold product $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$.

When all the previous continuity properties hold, we say that T is an m-linear generalized Calderón-Zygmund operator. Some fundamental results concerning these operators are obtained in [DGY]. Concerning the maximal operator T_*, we have the following analogous result.

Proposition 2.2. *Let T be an m-linear generalized Calderón-Zygmund operator with kernel K satisfying assumptions (**H3**) and (**H4**). Then for all exponents $p_1,\cdots,p_m$ and p satisfying $1/p_1+1/p_2+\cdots+1/p_m=1/p$, we have*

$$T_*: L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)$$

when $1<p_1,p_2,\cdots,p_m<\infty$ and $p<\infty$. We also have

$$T_*: L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^{p,\infty}(\mathbb{R}^n)$$

when at least one p_j is equal to one.

Proof. The proof is based on the idea of Corollary 1 in [GT3]. First, the strong estimates follow directly from (1.16) (with any $0<\eta<1/m$), Proposition 2.1, and the boundedness of $\mathcal{M}$. For the weak estimates we just observe, for instance if $q=1/m$, that by picking $\eta<1/m$, we have

$$\begin{aligned}\big\|\mathcal{M}\big(|T(\vec{f})|^{\eta}\big)^{1/\eta}\big\|_{L^{1/m,\infty}(\mathbb{R}^n)} &= \big\|\mathcal{M}\big(|T(\vec{f})|^{\eta}\big)\big\|^{1/\eta}_{L^{1/(m\eta),\infty}(\mathbb{R}^n)}\\ &\leq C\big\||T(\vec{f})|^{\eta}\big\|^{1/\eta}_{L^{1/(m\eta),\infty}(\mathbb{R}^n)}\\ &= C\big\|T(\vec{f})\big\|_{L^{1/m,\infty}(\mathbb{R}^n)},\end{aligned}$$

since $\mathcal{M}$ maps $L^{p,\infty}(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. □

2.2. **Further study on assumption (H3).** A natural question concerning Theorem 1.1 is the strength of the imposed condition (1.14) of assumption (**H3**), and its relation with the following *smoothness estimates:* For some $\epsilon > 0$

$$\big|K(x, y_1, \cdots, y_m) - K(x', y_1, \cdots y_m)\big| \leq \frac{A|x-x'|^{\epsilon}}{(|x-y_1| + \cdots + |x-y_m|)^{mn+\epsilon}}\,, \tag{2.12}$$

whenever $|x - x'| \leq \frac{1}{2}\max_{1\leq j\leq m}|x - y_j|$.

Theorem 1 in [GT3] says that condition (2.12) combined with the assumption that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$, for some $1 \leq q_1, q_2, \cdots, q_m < \infty$ and $0 < q < \infty$ with $1/q_1 + 1/q_2 + \cdots + 1/q_m = 1/q$, yields inequality (1.16) for the maximal operator T_*. This result was proved in [GT3] via the m-linear Cotlar inequality; the latter generalizes the linear Cotlar inequality (see [C], [G], and [St]).

In the following, we show that, for suitably chosen $\{B_t\}_{t>0}$, condition (1.14) of assumption (**H3**) is actually a consequence of (2.12). This implies that Theorem 1.1 is a strengthening of the analogous theorem in [GT3]. Precisely, following Proposition 2 of [DM], we construct $b_t(x, y)$ with the following properties

$$b_t(x, y) = 0, \quad \text{when } |x - y| \geq t^{1/s}, \tag{2.13}$$

$$\int_{\mathbb{R}^n} b_t(x, y)dx = 1 \tag{2.14}$$

for all $y \in \mathbb{R}^n, t > 0$. This can be achieved by choosing

$$b_t(x, y) = t^{-n/s}\chi_{B(y,t^{1/s})}(x),$$

where $\chi_{B(y,t^{1/s})}$ denotes the characteristic function of the ball $B(y, t^{1/s})$. Then let B_t be the linear operators whose kernels are $b_t(x, y)$.

Proposition 2.3. *Assume that T has an associated kernel $K(x, y_1, \cdots, y_m)$ that satisfies condition (2.12) for some $\epsilon > 0$. Let $\{B_t\}_{t>0}$ be approximations to the identity represented by kernels $b_t(x, y)$ satisfying (2.13) and (2.14). Then the kernels $K_t^{(0)}(x, y)$ of (1.11) satisfy assumption (**H3**). More precisely, there exists positive constant C such that*

$$\big|K_t^{(0)}(x, y_1, \cdots, y_m)\big| \leq \frac{A}{\big(|x-y_1| + \cdots + |x-y_m|\big)^{mn}} \tag{2.15}$$

whenever $2t^{1/s} \le \min_{1\le j\le m}|x-y_j|$, *and*

$$\big|K^{(0)}(x,y_1,\cdots,y_m)-K_t^{(0)}(x',y_1,\cdots,y_m)\big| \le C\frac{At^{\epsilon/s}}{\big(|x-y_1|+\cdots+|x-y_m|\big)^{mn+\epsilon}} \tag{2.16}$$

whenever $2|x-x'|\le t^{1/s}$ *and* $2t^{1/s}\le \min_{1\le j\le m}|x-y_j|$.

Proof. We first estimate (2.15). It follows from the conditions

$$2t^{1/s} \le \min_{1\le j\le m}|x-y_j|,$$

and the fact that $b_t(x,z)=0$ when $|x-z|\ge t^{1/s}$, that

$$\sum_{i=1}^m |z-y_i| \sim \sum_{i=1}^m |x-y_i|.$$

This gives

$$\begin{aligned}
\big|K_t^{(0)}(x,y_1,\cdots,y_2)\big| &= \Big|\int_{|x-z|<t^{1/s}} K(z,y_1,\cdots,y_m)b_t(x,z)dz\Big| \\
&\le \int_{|x-z|<t^{1/s}} \frac{A}{\big(|z-y_1|+\cdots+|z-y_m|\big)^{mn}}|b_t(x,z)|dz \\
&\le \frac{A}{\big(|x-y_1|+\cdots+|x-y_m|\big)^{mn}}\int_{|x-z|<t^{1/s}}|b_t(x,z)|dz \\
&\le \frac{A}{\big(|x-y_1|+\cdots+|x-y_m|\big)^{mn}}.
\end{aligned}$$

Let us now estimate (2.16). We may write

$$\begin{aligned}
&\big|K_t^{(0)}(x,y_1,\cdots,y_m)-K_t^{(0)}(x',y_1,\cdots,y_m)\big| \\
&\quad\le \big|K_t^{(0)}(x,y_1,\cdots,y_m)-K(x,y_1,\cdots,y_m)\big| \\
&\qquad+\big|K(x,y_1,\cdots,y_m)-K(x',y_1,\cdots,y_m)\big| \\
&\qquad+\big|K(x',y_1,\cdots,y_m)-K_t^{(0)}(x',y_1,\cdots,y_m)\big| \\
&\quad=: \mathrm{I}+\mathrm{II}+\mathrm{III}. \qquad (2.17)
\end{aligned}$$

Since $2|x-x'|\le t^{1/s}$ and $2t^{1/s}\le \min_{1\le j\le m}|x-y_j|$, we have

$$\begin{aligned}
&\big|K_t^{(0)}(x,y_1,\cdots,y_2)-K(x,y_1,\cdots,y_2)\big|\\
&\qquad= \Big|\int_{\mathbb{R}^n} K(z,y_1,\cdots,y_m)b_t(x,z)dz-K(x,y_1,\cdots,y_m)\Big|\\
&\qquad= \Big|\int_{|x-z|<t^{1/s}} K(z,y_1,\cdots,y_m)b_t(x,z)dz\\
&\qquad\qquad\qquad -K(x,y_1,\cdots,y_2)\int_{|x-z|<t^{1/s}} b_t(x,z)dz\Big|\\
&\qquad\le \int_{|x-z|<t^{1/s}} \big|K(z,y_1,\cdots,y_m)-K(x,y_1,\cdots,y_m)\big||b_t(x,z)|dz\\
&\qquad\le \frac{A\,t^{\epsilon/s}}{\big(|x-y_1|+\cdots+|x-y_m|\big)^{mn+\epsilon}}\int_{|x-z|<t^{1/s}} |b_t(x,z)|dz\\
&\qquad\le \frac{A\,t^{\epsilon/s}}{\big(|x-y_1|+\cdots+|x-y_m|\big)^{mn+\epsilon}}.
\end{aligned}$$

Note that the second equality was a consequence of condition (2.12) while the last inequality of condition (2.14). Similarly, we have that

$$\mathrm{III}\le \frac{A\,t^{\epsilon/s}}{\big(|x-y_1|+\cdots+|x-y_m|\big)^{mn+\epsilon}}.$$

Collecting these estimates and using (2.12), we obtain the desired estimate (2.16). This completes the proof of Proposition 2.2. □

3. Weighted norm inequalities

Recall that a weight w is in the class A_∞ if and only if there exists $C,\theta>0$ such that for every cube Q and every measurable set $E\subset Q$,

$$\frac{w(E)}{w(Q)}\le C\Big(\frac{|E|}{|Q|}\Big)^{\theta}, \tag{3.1}$$

where, for a measurable set $F, w(F)=\int_F w(x)dx$.

Let $\widetilde{T_*}$ be the modified maximal truncated singular integral defined in (2.2). Then we have the following result.

Theorem 3.1. *Let T be an m-linear generalized Calderón-Zygmund operator with kernel K satisfying assumptions* (**H3**) *and* (**H4**) *and let W denote the norm in (1.10). Let $\vec f$ be in any product of $L^{q_j}(\mathbb{R}^n)$ spaces, with $1\le q_j\le\infty$. Also let $w\in A_\infty$ and θ be as in (3.1). Then there exists a positive constant C such that for all $\alpha>0$ and all $\gamma>0$ sufficiently small we have*

$$w\Big(\Big\{\widetilde{T_*}(\vec f)>2^{m+1}\alpha\Big\}\cap\Big\{\prod_{j=1}^m \mathcal{M}f_j\le\gamma\alpha\Big\}\Big)\le C(A+W)^{\frac{\theta}{m}}\gamma^{\frac{\theta}{m}}w\Big(\Big\{\widetilde{T_*}(\vec f)>\alpha\Big\}\Big).$$

Proof. The proof is based on a multilinear adaptation of the good-lambda inequality maximal singular integrals developed in [CF]; see also Theorem 2 in [GT3]. One writes

$$\Omega = \big\{x : \widetilde{T_*}(\vec{f})(x) > \alpha\big\} = \bigcup_{\nu} Q_\nu,$$

where Q_ν are Whitney cubes. In view of (3.1), it suffices to show that for all Whitney cubes Q_ν we have the estimate

$$\Big| Q_\nu \cap \Big\{\widetilde{T_*}(\vec{f}) > 2^{m+1}\alpha\Big\} \cap \Big\{\prod_{j=1}^m \mathcal{M}f_j \le \gamma\alpha\Big\}\Big| \le C(A+W)^{1/m}\gamma^{1/m}|Q_\nu|. \tag{3.2}$$

For each Whitney cube Q_ν fix a large multiple of it Q_ν^* and a point y_ν in ${}^c\Omega \cap Q_\nu^*$ with the property that

$$\max_{z\in Q_\nu} |y_\nu - z| \le \frac{1}{2}\operatorname{dist}\big(y_\nu, {}^c(Q_\nu^*)\big). \tag{3.3}$$

In order to prove (3.2), for a given cube Q_ν we may assume that there exists a point ξ_ν in Q_ν such that

$$\mathcal{M}f_1(\xi_\nu)\cdots\mathcal{M}f_m(\xi_\nu) \le \gamma\alpha, \tag{3.4}$$

otherwise there is nothing to prove.

Given $\vec{f} = (f_1, \cdots, f_m)$, define $f_j^0 = f_j\chi_{Q_\nu^*}$ and $f_j^\infty = f_j - f_j^0$ for $j = 1, \cdots, m$. The set

$$Q_\nu \cap \Big\{\widetilde{T_*}(\vec{f}) > 2^{m+1}\alpha\Big\} \cap \Big\{\prod_{j=1}^m \mathcal{M}f_j \le \gamma\alpha\Big\}$$

is contained in the union of 2^m sets of the form

$$Q_\nu \cap \Big\{\widetilde{T_*}(f_1^{r_1}, \cdots, f_m^{r_m}) > 2\alpha\Big\} \cap \Big\{\prod_{j=1}^m \mathcal{M}f_j \le \gamma\alpha\Big\}, \tag{3.5}$$

where $r_j \in \{0, \infty\}$ for all $j = 1, \cdots, m$.

The argument in Theorem 2 of [GT3] shows that by picking γ small enough, we make the set in (3.5) empty when $r_1 = \cdots = r_\ell = \infty$ and $r_{\ell+1} = \cdots = r_m = 0$. Likewise with all the remaining sets where at least one r_j is infinity. We are now left with the set in (3.5) where all the r_j's are equal to infinity, that is, the set

$$Q_\nu \cap \Big\{\widetilde{T_*}(f_1^\infty, \cdots, f_m^\infty) > 2\alpha\Big\} \cup \Big\{\prod_{j=1}^m \mathcal{M}f_j \le \gamma\alpha\Big\}. \tag{3.6}$$

In the following, we set $\vec{f}^\infty = (f_1^\infty, \cdots, f_m^\infty)$. We claim that for every $\delta > 0$ and for all $x \in Q_\nu$, we have

$$(\widetilde{T})_\delta(\vec{f}^\infty)(x) - (\widetilde{T})_\delta(\vec{f}^\infty)(y_\nu) \le CA\prod_{j=1}^m \mathcal{M}f_j(\xi_\nu), \tag{3.7}$$

where ξ_v is defined in (3.4).

Let us prove the claim (3.7). Let $t = \big(\frac{\ell(Q_\nu)}{4}\big)^s$ where $\ell(Q_\nu)$ is the side length of the cube Q_ν and s is a constant in (1.3). We may decompose it as

$$\begin{aligned}(\widetilde{T})_\delta(\vec{f}^\infty)(x) - (\widetilde{T})_\delta(\vec{f}^\infty)(y_\nu) &= \Big(\widetilde{B_tT}\Big)_\delta(\vec{f}^\infty)(y_\nu) - (\widetilde{T})_\delta(\vec{f}^\infty)(y_\nu)\\ &\quad+\Big(\widetilde{B_tT}\Big)_\delta(\vec{f}^\infty)(x) - \Big(\widetilde{B_tT}\Big)_\delta(\vec{f}^\infty)(y_\nu)\\ &\quad+(\widetilde{T})_\delta(\vec{f}^\infty)(x) - \Big(\widetilde{B_tT}\Big)_\delta(\vec{f}^\infty)(x)\\ &=: \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned} \tag{3.8}$$

We estimate term I, using assumption (**H2**). First note that $|y_\nu - y_j| > 2t^{1/s}$. This, in combination with the fact that supp$\phi \subset [-1,1]$, gives that $\phi\Big(\frac{|y_\nu - y_j|}{t^{1/s}}\Big) = 0$. By assumption (**H2**),

$$\begin{aligned}|\mathrm{I}| &\le CA\int_{(\mathbb{R}^n)^m} \frac{\ell(Q_\nu)^\epsilon}{(|y_\nu - y_1| + \cdots + |y_\nu - y_m|)^{mn+\epsilon}}\prod_{j=1}^m \big|f_j^\infty(y_j)\big|d\vec{y}\\ &\le CA|Q_\nu|^{\epsilon/n}\prod_{j=1}^m\int_{^c(Q_\nu^*)}\frac{\big|f_j(y_j)\big|}{|y_\nu - y_j|^{\frac{mn+\epsilon}{m}}}dy_j\\ &\le CA\prod_{j=1}^m \mathcal{M}f_j(\xi_\nu).\end{aligned}$$

Similarly, we also have that $|\mathrm{III}| \le CA\prod_{j=1}^m \mathcal{M}f_j(\xi_\nu)$.

We now turn to the estimate for term II. Using assumption (**H3**) and the argument used in estimate of term II in Section 2.1, we have that for $y_\nu \in {}^c\Omega \cap Q_\nu^*$,

$$\begin{aligned}|\mathrm{II}| &\le CA\int_{(\mathbb{R}^n)^m\setminus\big(U_\delta(x)\cup V_{3\delta/2}(x)\big)} \frac{1}{(|x - y_1| + \cdots + |x - y_m|)^{mn}}\prod_{j=1}^m \big|f_j^\infty(y_j)\big|d\vec{y}\\ &\quad+CA\int_{(\mathbb{R}^n)^m} \frac{\ell(Q_\nu)^\epsilon}{(|y_\nu - y_1| + \cdots + |y_\nu - y_m|)^{mn+\epsilon}}\prod_{j=1}^m \big|f_j^\infty(y_j)\big|d\vec{y}\\ &\le CA\prod_{j=1}^m \mathcal{M}f_j(\xi_\nu) + CA|Q_\nu|^{\epsilon/n}\prod_{j=1}^m\int_{^c(Q_\nu^*)}\frac{\big|f_j(y_j)\big|}{|y_\nu - y_j|^{\frac{n+\epsilon}{m}}}dy_j\\ &\le CA\prod_{j=1}^m \mathcal{M}f_j(\xi_\nu),\end{aligned}$$

which proves (3.7). On the other hand, following the proof of (22) in [GT3], we easily derive that

$$|(\widetilde{T})_\delta(\vec{f}^\infty)(y_\nu)| \leq (\widetilde{T})_\delta(\vec{f})(y_\nu) + CA\prod_{j=1}^m \mathcal{M}f_j(\xi_\nu).$$

This inequality combined with (3.7), gives

$$\begin{aligned}|(\widetilde{T})_\delta(\vec{f}^\infty)(x)| &\leq (\widetilde{T})_\delta(\vec{f})(y_\nu) + CA\prod_{j=1}^m \mathcal{M}f_j(\xi_\nu)\\ &\leq \alpha + CA\gamma\alpha < 2\alpha,\end{aligned}$$

if γ is small enough because y_ν is in ${}^c\Omega$. For these γ's the set (3.6) is empty and then the proof of the theorem is complete. □

We then have the following proposition:

Proposition 3.2. *Let $1 \leq p_1, \cdots, p_m < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots \frac{1}{p_m}$, and $\omega \in A_\infty$. Let T be an m-linear generalized Calderón-Zygmund operator with kernel satisfying assumptions* (**H3**) *and* (**H4**). *Then there is a constant $C_{p,n} < \infty$ such that for all $\vec{f} = (f_1, \cdots, f_m)$ satisfying $\|T_*(\vec{f})\|_{L^p(w)} < \infty$ we have*

$$\left\|T_*(\vec{f})\right\|_{L^p(w)} \leq C_{p,n}(A+W)\prod_{j=1}^m \left\|\mathcal{M}f_j\right\|_{L^{p_j}(w)}.$$

Moreover, if $p_0 = \min(p_1, \cdots, p_m) > 1$, and $\omega \in A_{p_0}$, then

$$\left\|T_*(\vec{f})\right\|_{L^p(w)} \leq C_{p,n}(A+W)\prod_{j=1}^m \left\|f_j\right\|_{L^{p_j}(w)}.$$

Proof. The proof is identical to that of Corollary 2 of [GT3]. □

Note: The hypothesis $\|T_*(\vec{f})\|_{L^p(w)} < \infty$ is always satisfied if each component of $\vec{f}$ is a bounded function with compact support and ω is in $A_{p_0}, p_0 > 1$ as above.

We now extend the previous weighted norm inequalities to a generalized Calderón-Zygmund operator T.

Proposition 3.3. *Fix exponents $1 < p_1, \cdots, p_m < \infty$, and p such that $1/p_1 + 1/p_2 + \cdots + 1/p_m = 1/p$ and let $\omega \in A_\infty$. Let T be an m-linear generalized Calderón-Zygmund operator with kernel satisfying assumptions* (**H3**) *and* (**H4**). *Then there is a constant $C_{p,n} < \infty$ such that for all $\vec{f} = (f_1, \cdots, f_m)$ with f_j bounded and compactly supported we have*

$$\|T(\vec{f})\|_{L^p(w)} \leq C_{p,n}(A+W)\prod_{j=1}^m \|\mathcal{M}f_j\|_{L^{p_j}(w)}.$$

Moreover, if $\omega \in A_{p_0}$ *with* $p_0 = \min(p_1, \cdots, p_m) > 1$*, then*

$$\|T(\vec{f})\|_{L^p(w)} \le C_{p,n}(A+W) \prod_{j=1}^m \|f_j\|_{L^{p_j}(w)},$$

and, in particular, T *extends to a bounded operator from* $L^{p_1}(w) \times \cdots \times L^{p_m}(w)$ *to* $L^p(w)$.

Proof. The proof is obtained just as that of Corollary 3 in [GT3]. □

4. Applications: commutators of singular integrals

In this section we apply Theorem 1.1 to deduce nontrivial bounds for the maximal singular integral associated with the commutators of A.P. Calderón. These commutators first appeared in the study of the Cauchy integral along Lipschitz curves and led to the first proof of the L^2-boundedness of the latter. The first order commutator is defined as

$$\mathcal{C}_2(a,f)(x) := \int_{\mathbb{R}} \frac{A(x)-A(y)}{(x-y)^2} f(y)dy, \quad \text{where} \quad A' = a. \tag{4.1}$$

It is known that $\mathcal{C}_2(a,f)$ extends as a bounded operator from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^r(\mathbb{R})$, when $1 < p,\ q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $r \ne \infty$. Moreover, if either p or q equals 1, then $\mathcal{C}_2(a,f)$ maps $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^{r,\infty}(\mathbb{R})$ and, in particular, it maps $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ to $L^{1/2,\infty}(\mathbb{R})$. We refer to the articles of [AC], [CC] and [CM1] for these results.

In this section we obtain boundedness of a maximal function concerning the mth order commutator of Calderón using the techniques developed in the previous sections. Recall that the mth commutator is given by

$$\mathcal{C}_{m+1}(a_1,\dots,a_m,f)(x) = \int_{\mathbb{R}} \frac{\prod_{j=1}^m \big(A_j(x)-A_j(y)\big)}{(x-y)^{m+1}} f(y)\,dy \qquad x \in \mathbb{R}\,, \tag{4.2}$$

where $A_j' = a_j$. It is a well-known fact that $f \to \mathcal{C}_{m+1}(a_1,\dots,a_m,f)$ is a bounded operator on $L^2(\mathbb{R})$. Moreover $\mathcal{C}_{m+1}$ can be viewed as an $(m+1)$-linear operator that satisfies an $\big(L^\infty(\mathbb{R})\big)^m \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ estimate for each number m, that is, there is a constant $C_m > 0$ such that for all suitable functions $a_1,\dots,a_m,f$ we have

$$\|\mathcal{C}_{m+1}(a_1,\dots,a_m,f)\|_{L^\infty\times\ldots\times L^\infty\times L^2\to L^2} \le C_m\, \|f\|_{L^2(\mathbb{R})} \Big(\prod_{j=1}^m \|A_j'\|_{L^\infty(\mathbb{R})}\Big).$$

There are various estimates for C_{m+1}. For example, there is an absolute constant $C > 0$ such that $C_{m+1} \le C(2+m)^3$ for every m. See Theorem 3, page 68 of [CJ].

Define

$$e(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Since $A_j' = a_j$, the multilinear operator $\mathcal{C}_{m+1}(f,a_1,\dots,a_m)$ can be written as

$$\begin{aligned}&\mathcal{C}_{m+1}(a_1,\ldots,a_m,f)(y_0)\\&\quad:=\int_{\mathbb{R}^{m+1}}K(y_0,\ldots,y_{m+1})a_1(y_1)\ldots a_m(y_m)f(y_{m+1})dy_1\ldots dy_{m+1},\end{aligned}$$

where the kernel K is

$$K(x,y_1,\ldots,y_{m+1})=\frac{(-1)^{e(y_{m+1}-x)m}}{(x-y_{m+1})^{m+1}}\prod_{\ell=1}^{m}\chi_{\big(\min(x,y_{m+1}),\max(x,y_{m+1})\big)}(y_\ell). \tag{4.3}$$

We have the following result concerning $\mathcal{C}_{m+1}$.

Proposition 4.1. *The mth order commutator $\mathcal{C}_{m+1}$ satisfies assumption* (**H3**). *Precisely, there exist operators $\{B_t\}_{t>0}$ with kernels $b_t(x,y)$ that satisfy conditions (1.3) and (1.4) with constants s and η and there exist kernels $K_t^{(0)}(x,y_1,\cdots,y_{m+1})$ such that*

$$K_t^{(0)}(x,y_1,\cdots,y_{m+1})=\int_{\mathbb{R}}K(z,y_1,\cdots,y_{m+1})b_t(x,z)dz \tag{4.4}$$

holds and

$$\big|K_t^{(0)}(x,y_1,\cdots,y_{m+1})\big|\le\frac{A}{\big(|x-y_1|+\cdots+|x-y_{m+1}|\big)^{m+1}} \tag{4.5}$$

whenever $2t^{1/s}\le\min_{1\le j\le m+1}|x-y_j|$, and there exists some $\epsilon>0$ such that

$$\begin{aligned}&\big|K_t^{(0)}(x,y_1,\cdots,y_{m+1})-K_t^{(0)}(x',y_1,\cdots,y_{m+1})\big|\\&\qquad\le\;C\frac{t^{\epsilon/s}}{(|x-y_1|+\cdots+|x-y_{m+1}|)^{m+1+\epsilon}}\end{aligned} \tag{4.6}$$

whenever $2|x-x'|\le t^{1/s}$ and $2t^{1/s}\le\min_{1\le j\le m+1}|x-y_j|$.

Proof. Let $\phi\in C^\infty(\mathbb{R})$ be even, $0\le\phi\le1$, $\phi(0)=1$ and supp $(\phi)\subset[-1,1]$. We set $\Phi=\phi'$ and $\Phi_t(x)=t^{-1}\Phi(x/t)$. Define

$$B_tf(x)=\int_{\mathbb{R}}b_t(x,y)f(y)dy\quad\text{where}\quad b_t(x,y)=\Phi_t\big(x-y\big)\chi_{(-\infty,x)}(y).$$

Then the kernels $b_t(x,y)$ satisfy the conditions (1.1) and (1.2) with constants $s=\eta=1$. From (4.4), we have

$$\begin{aligned}&K_t^{(0)}(x,y_1,\cdots,y_{m+1})\\&\quad=\int_{-\infty}^{x}\frac{(-1)^{e(y_{m+1}-z)m}}{(z-y_{m+1})^{m+1}}\prod_{\ell=1}^{m}\chi_{\big(\min(z,y_{m+1}),\,\max(z,y_{m+1})\big)}(y_\ell)\Phi\Big(\frac{x-z}{t}\Big)\frac{dz}{t}.\end{aligned} \tag{4.7}$$

First, it follows from (2.15) of Proposition 2.3 that estimate (4.5) holds. Let us now verify (4.6). There are only two subcases to consider: $x<y_{m+1}$ and $y_{m+1}<x$.

Case 1. $x<y_{m+1}$.

Since $|x-y_j|\geq 2t$ for all $j=1,2,\cdots,m+1$ and $|x-x'|<t/2$, we have that $|y_j-x'|>3t/2$. It follows from identity (4.4) that if $K_t^{(0)}(x,y_1,\cdots,y_{m+1})\neq 0$, we then have that $x<y_j<y_{m+1}$ for all $j=1,\cdots,m$, and thus $|x-y_{m+1}|\sim\sum_{i=1}^{m+1}|x-y_i|$. Hence,

$$\begin{aligned}
&\Big|K_t^{(0)}(x,y_1,\cdots,y_{m+1})-K_t^{(0)}(x',y_1,\cdots,y_{m+1})\Big|\\
&\quad\leq\ t^{-1}\Big|\int_{-\infty}^{x}\frac{1}{(z-y_{m+1})^{m+1}}\Phi\Big(\frac{x-z}{t}\Big)dz-\int_{-\infty}^{x'}\frac{1}{(z-y_{m+1})^{m+1}}\Phi\Big(\frac{x'-z}{t}\Big)\Big|\\
&\quad\leq\ \Big|\frac{1}{(x-y_{m+1})^{m+1}}-\frac{1}{(x'-y_{m+1})^{m+1}}\Big|+C\Big|\int_{-\infty}^{x}\frac{1}{(z-y_{m+1})^{m+2}}\phi\Big(\frac{x-z}{t}\Big)dz\Big|\\
&\qquad\qquad+C\Big|\int_{-\infty}^{x'}\frac{1}{(z-y_{m+1})^{m+2}}\phi\Big(\frac{x'-z}{t}\Big)dz\Big|\\
&\quad\leq\ \frac{Ct}{\big(|x-y_1|+\cdots+|x-y_{m+1}|\big)^{m+2}}.
\end{aligned}$$

Case 2. $y_{m+1}<x$.

In this case, an argument as in **Case 1** shows that if $K_t^{(0)}(x,y_1,\cdots,y_{m+1})\neq 0$, we then have that $y_{m+1}<y_j<x$ for all $j=1,\cdots,m$ and $|x-y_{m+1}|\sim\sum_{i=1}^{m+1}|x-y_i|$. Hence,

$$\begin{aligned}
&\Big|K_t^{(0)}(x,y_1,\cdots,y_{m+1})-K_t^{(0)}(x',y_1,\cdots,y_{m+1})\Big|\\
&\quad\leq\ t^{-1}\Big|\int_{-\infty}^{x}\frac{1}{(z-y_{m+1})^{m+1}}\Phi\Big(\frac{x-z}{t}\Big)dz-\int_{-\infty}^{x'}\frac{1}{(z-y_{m+1})^{m+1}}\Phi\Big(\frac{x'-z}{t}\Big)\Big|\\
&\quad\leq\ \Big|\frac{1}{(x-y_{m+1})^{m+1}}-\frac{1}{(x'-y_{m+1})^{m+1}}\Big|+C\Big|\int_{-\infty}^{x}\frac{1}{(z-y_{m+1})^{m+2}}\phi\Big(\frac{x-z}{t}\Big)dz\Big|\\
&\qquad\qquad+C\Big|\int_{-\infty}^{x'}\frac{1}{(z-y_{m+1})^{m+2}}\phi\Big(\frac{x'-z}{t}\Big)dz\Big|\\
&\quad\leq\ \frac{Ct}{\big(|x-y_1|+\cdots+|x-y_{m+1}|\big)^{m+2}}.
\end{aligned}$$

This proves (4.6), and concludes the proof of Proposition 4.1. □

Remark 4.2. *It follows from Proposition 4.1 of* [DGY] *and Proposition 4.1 of this article that the mth order commutator* $\mathcal{C}_{m+1}$ *satisfies assumptions* (**H1**), (**H2**) *and* (**H3**) *of Theorem 1.1. However, it follows from (4.3) that if* $x<y_{m+1}$*, then*

$$\begin{aligned} K(x,y_1,\dots,y_{m+1}) &= \frac{(-1)^m}{(x-y_{m+1})^{m+1}}\prod_{\ell=1}^{m}\chi_{(x,y_{m+1})}(y_\ell) \\ &= \begin{cases} \frac{(-1)^m}{(x-y_{m+1})^{m+1}}, & \text{if } y_\ell\in\big(x,y_{m+1}\big) \text{ for all } \ell=1,2,\cdots,m \\ 0, & \text{otherwise.}\end{cases} \end{aligned}$$

Hence, for any fixed x, y_{m+1} and $\epsilon>0$, we can choose $y_1,\cdots,y_m$ and y_1' such that $y_1'<x<y_\ell<y_{m+1}$, $\ell=1,2,\cdots,m$, and $|y_1-y_1'|<\epsilon$. Then we have

$$K(x,y_1',\dots,y_{m+1})=0 \quad\text{and}\quad K(x,y_1,\dots,y_{m+1})=\frac{(-1)^m}{(x-y_{m+1})^{m+1}},$$

hence

$$\Big|K(x,y_1',\dots,y_{m+1})-K(x,y_1,\dots,y_{m+1})\Big|=\frac{1}{|x-y_{m+1}|^{m+1}}.$$

As $|y_1-y_1'|$ can be chosen to be arbitrarily small, the kernel $K(x,y_1,\dots,y_{m+1})$ does not satisfy the standard Calderón-Zygmund kernel regularity conditions (1.7) and (1.8).

Consider the *maximal commutator operator*

$$\mathcal{C}^*_{m+1}(a_1,\dots,a_m,f)(x):=\sup_{\delta>0}\Big|\int_{|x-y|\ge\delta}\frac{\prod_{j=1}^m\big(A_j(x)-A_j(y)\big)}{(x-y)^{m+1}}f(y)\,dy\Big|, \tag{4.8}$$

where $A_j'=a_j$. Recall the kernel K in (4.3). Define

$$\begin{aligned}&\widetilde{\mathcal{C}}^*_{m+1}(a_1,\dots,a_m,f)(x):=\\ &\sup_{\delta>0}\Big|\int_{|x-y_1|^2+\cdots+|x-y_{m+1}|^2\ge\delta^2}K(x,y_1,\cdots,y_m)a(y_1)\cdots a(y_m)f(y_{m+1})dy_1\cdots dy_{m+1}\Big|.\end{aligned}$$

It is clearly that for all $x\in\mathbb{R}$,

$$\mathcal{C}^*_{m+1}(a_1,\dots,a_m,f)(x)\le\widetilde{\mathcal{C}}^*_{m+1}(a_1,\dots,a_m,f)(x).$$

Applying Theorem 1.1 and Proposition 3.2, we deduce the following new result concerning the mth order maximal commutators $\widetilde{\mathcal{C}}^*_{m+1}(a_1,\dots,a_m,f)$.

Theorem 4.3. *Fix exponents $1\le p_1,\cdots,p_{m+1}<\infty$ and p such that $1/p=1/p_1+\cdots+1/p_{m+1}$. Let $\mathcal{C}^*_{m+1}(a_1,\dots,a_m,f)$ be an operator as in (4.8). Then for all $\eta>0$, there exists a constant $C_\eta=C(\eta,m)<\infty$ such that for all $(a_1,\cdots,a_m,f)$ in any product of $L^{q_j}(\mathbb{R})$ spaces, the following inequality holds for all x in $\mathbb{R}$,*

$$\begin{aligned}&\mathcal{C}^*_{m+1}(a_1,\dots,a_m,f)(x)\\ &\quad\le\ C_\eta\Big(\Big(\mathcal{M}(\big|\mathcal{C}_{m+1}(a_1,\dots,a_m,f)\big|^\eta)(x)\Big)^{\frac{1}{\eta}}+\Big(\prod_{j=1}^m\mathcal{M}(a_j)(x)\Big)\mathcal{M}(f)(x)\Big).\end{aligned}$$

As a consequence, if $\omega\in A_{p_0}$ with $p_0=\min(p_1,\cdots,p_{m+1})>1$, then

$$\big\|\mathcal{C}_{m+1}^*(a_1,\dots,a_m,f)\big\|_{L^p(w)} \le C_p\Big(\prod_{j=1}^m \big\|a_j\big\|_{L^{p_j}(w)}\Big)\big\|f\big\|_{L^{p_{m+1}}(w)},$$

and, in particular, $\mathcal{C}_{m+1}^*$ *extends as a bounded operator from* $L^{p_1}(w)\times\cdots\times L^{p_{m+1}}(w)$ *into* $L^p(w)$.

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