OFF-DIAGONAL MULTILINEAR INTERPOLATION BETWEEN ADJOINT OPERATORS

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Abstract. We extend a theorem by Grafakos and Tao [4] on multilinear interpolation between adjoint operators to an off-diagonal situation. We provide an application.

1. Introduction and the Main Result

Multilinear interpolation has proved to be a powerful and indispensable tool in analysis. The two main linear interpolation theorems, the Marcinkiewicz and Riesz-Thorin theorems, have well-established multilinear analogs. The works [9], [5], [2], [3] provide multilinear extensions of the Marcinkiewicz interpolation theorem. The Riesz-Thorin theorem is easily adapted in the multilinear case in [11, 21, Chapter XII, (3.3)] and [1, Theorem 4.4.2]; related versions of this result have appeared in [10], [7], [8].

A different type of interpolation is that between adjoint operators. In the linear case a typical result would be as follows: if an operator and its adjoint are of weak type \((1, 1)\), then the operator is \(L^p\) bounded for all \(p \in (1, \infty)\). A multilinear version of this result was obtained in [4]. This theorem says that, under an initial condition similar to (1.1) below, if an \(m\)-linear operator and all of its \(m\) adjoints are of restricted weak type \((1, 1, \ldots, 1, 1/m)\), then the operator is bounded from \(L^{q_1} \times \cdots \times L^{q_m}\) to \(L^q\) for all \(1 < q_1, \ldots, q_m < \infty\) and \(1/m < q < \infty\). In this context, an \(m\)-linear operator is called of restricted weak type \((q_1, q_1, \ldots, q_m, q)\) if it maps \(L^{q_1} \times \cdots \times L^{q_m}\) to \(L^q\) when restricted to characteristic functions of sets of finite measure. In this note we obtain the following off-diagonal version of this result in which the diagonal case \(t = 1\) and \(s = 1/m\) was previously considered in [4].

**Theorem 1.1.** Let \(1 \leq t < \infty\), \(0 < s \leq 1\), \(1 < p < t'\), and \(t < p_1, \ldots, p_m < \infty\) be such that \(1/p_1 + \cdots + 1/p_m - 1/p = m/t - 1/s\). Let \((X, \mu), (X, \mu_1), \ldots, (X, \mu_m)\) be \(\sigma\)-finite measure spaces. Suppose that an \(m\)-linear operator \(T\) is defined on the space of simple functions on \(X_1 \times \cdots \times X_m\) and takes values in the space of measurable functions on \(X\). We assume that \(T\) satisfies

\[
\sup_{A_0, A_1, \ldots, A_m} \frac{1}{\mu(A_0)^{1/p} \mu_1(A_1)^{1/p_1} \cdots \mu_m(A_m)^{1/p_m}} \left| \int_{A_0} T(\chi_{A_1}, \ldots, \chi_{A_m}) d\mu \right| < \infty, \tag{1.1}
\]

where the supremum is taken over all measurable subsets \(A_i\) of \(X_i\) with nonzero finite measure. Suppose that for each \(j \in \{0, 1, \ldots, m\}\), \(T^j\) is of restricted weak type \((t, t, \ldots, t, s)\) with constant \(B_j\). Then there is a constant \(C = C(p_1, \ldots, p_m, p, t, s)\) such that \(T\) is of restricted weak type \((p_1, \ldots, p_m, p)\) with norm at most

\[
CB_0^{\theta(t-1)} B_1^{\theta(t-1/p_1)} \cdots B_m^{\theta(t-1/p_m)}. \tag{1.2}
\]

where \(\theta = (1/t + 1/s - 1)^{-1}\).

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The following well-known characterization of weak $L^p$ will be used in the proof.

**Proposition 1.2.** Let $0 < p < \infty$, $A, B > 0$, and let $f$ be a measurable function on a \( \sigma \)-finite measure space \((X, \mu)\).

(i) Suppose that \( \|f\|_{L^{p, \infty}} \leq A \). Then for every measurable set $E$ of finite measure there exists a measurable subset $E'$ of $E$ with $\mu(E') \geq \mu(E)/2$ such that $f$ is bounded on $E'$ and

\[
\left| \int_{E'} f \, d\mu \right| \leq 2^{\frac{1}{p}} A \mu(E)^{1 - \frac{1}{p}}.
\]

(ii) Suppose that a measurable function $f$ on $X$ has the property that for any measurable subset $E$ of $X$ with $\mu(E) < \infty$ there is a measurable subset $E'$ of $E$ with $\mu(E') \geq \mu(E)/2$ such that $f$ is integrable on $E'$ and

\[
\left| \int_{E'} f \, d\mu \right| \leq B \mu(E)^{1 - \frac{1}{p}}.
\]

Then we have that

\[
\|f\|_{L^{p, \infty}} \leq B 2^{\frac{2}{p} + \frac{3}{2}}.
\]

2. The Proof

**Proof.** First consider the case where

\[
\frac{\mu(A_0)}{B_0^p} = \max \left( \frac{\mu_1(A_1)}{B_1^p}, \ldots, \frac{\mu_m(A_m)}{B_m^p} \right).
\]  

(2.1)

Let $M$ be the supremum given in (1.1). It will be enough to show that $M$ is bounded above by the constant in (1.2), from which Proposition 1.2(ii) gives the desired boundedness.

Since $T$ is restricted weak type \((t, \ldots, t, s)\), Proposition 1.2 gives a subset $A'_0$ of $A_0$ with measure $\mu(A'_0) \geq \mu(A_0)/2$ so that

\[
\left| \int_{A'_0} T(\chi_{A_1}, \ldots, \chi_{A_m}) \, d\mu \right| \leq KB_0 \mu_1(A_1)^{\frac{1}{p_1}} \cdots \mu_m(A_m)^{\frac{1}{p_m}} \mu(A_0)^{1 - \frac{1}{p}}
\]

for some constant $K = K(s)$. It then follows that

\[
\left| \int_{A_0} T(\chi_{A_1}, \ldots, \chi_{A_m}) \, dx \right| \leq \left| \int_{A'_0} T(\chi_{A_1}, \ldots, \chi_{A_m}) \, dx \right| + \left| \int_{A_0 \setminus A'_0} T(\chi_{A_1}, \ldots, \chi_{A_m}) \, dx \right|
\]

\[
:= I + II
\]

where

\[
I \leq KB_0 \mu_1(A_1)^{\frac{1}{p_1}} \cdots \mu_m(A_m)^{\frac{1}{p_m}} \mu(A_0)^{1 - \frac{1}{p}}
\]

\[
= \mu(A_0)^{\frac{1}{p'}} \mu_1(A_1)^{\frac{1}{p_1}} \cdots \mu_m(A_m)^{\frac{1}{p_m}} \left( KB_0 \left( \frac{B_1^p}{B_0^p} \right)^{\frac{1}{p_1} - \frac{1}{p}} \cdots \left( \frac{B_m^p}{B_0^p} \right)^{\frac{1}{p_m} - \frac{1}{p}} \right)
\]

and

\[
II \leq M \mu_1(A_1)^{\frac{1}{p_1}} \cdots \mu_m(A_m)^{\frac{1}{p_m}} \left( \frac{1}{2} \mu(A_0) \right)^{\frac{1}{p'}}
\]

\[
\leq \mu(A_0)^{\frac{1}{p'}} \mu_1(A_1)^{\frac{1}{p_1}} \cdots \mu_m(A_m)^{\frac{1}{p_m}} \left( M 2^{-\frac{1}{p}} \right).
\]
Consequently,
\[ M \leq KB_0 \left( \frac{B_0^\theta}{B_0^\theta} \right)^{\frac{1}{p} - \frac{1}{p_i}} \cdots \left( \frac{B_m^\theta}{B_0^\theta} \right)^{\frac{1}{p} - \frac{1}{p_m}} + M 2^{-\frac{1}{p'}} \]
and hence
\[ M \leq \frac{K}{1 - 2^{-\frac{1}{p'}}} B_0^\theta \left( \frac{1}{p} - \frac{1}{p_i} \right) B_0^\theta \left( \frac{1}{p} - \frac{1}{p} \right) \cdots B_m^\theta \left( \frac{1}{p} - \frac{1}{p_m} \right). \]
The preceding inequality is an implication of the fact that
\[ 1 - \theta \left( \frac{m}{t} - \frac{1}{p_1} - \cdots - \frac{1}{p_m} \right) = 1 - \theta \left( \frac{1}{s} - \frac{1}{p} \right) = \theta \left( \frac{1}{\theta} - \frac{1}{s} + \frac{1}{p} \right) = \theta \left( \frac{1}{t} - \frac{1}{p'} \right). \]
There are \( m \) more cases left in each of which \( \mu_j(A_j)/B_j^\theta \) is interchanged with \( \mu(A_0)/B_0^\theta \) in (2.1) for some \( j \in \{1, \ldots, m\} \). Fixing such a \( j \) we recall the assumption that the \( j \)th adjoint \( T^{\ast j} \) of \( T \) is also of restricted weak type \((t, \ldots, t, s)\). Setting \( p_0 = p' \) and \( \mu_0 = \mu \), we notice that (1.1) can be written as
\[ \sup_{A_0, A_1, \ldots, A_m} \frac{1}{\mu_j(A_j)^{\frac{1}{p_j}}} \mu_0(A_0)^{\frac{1}{p_0}} \prod_{i \neq j} \mu_i(A_i)^{\frac{1}{p_i}} \left| \int_{A_j} T^{\ast j}(\chi_{A_1}, \ldots, \chi_{A_0}, \ldots, \chi_{A_m})d\mu_j \right| < \infty, \]
the \((m + 1)\)-tuple \((p_1, \ldots, p_m, p)\) transforms to \((p_1, \ldots, p_{j-1}, p_0, \ldots, p_{j+1}, p_j')\), and the identity \( \frac{1}{p_1} + \cdots + \frac{1}{p_m} - \frac{1}{p} = \frac{m}{t} - \frac{1}{s} \) transforms to \( \frac{1}{p_0} + \sum_{i \neq j} \frac{1}{p_i} - \frac{1}{p_j'} = \frac{m}{t} - \frac{1}{s} \). The argument in this case follows by an identical repetition of the argument in the preceding case, under this change of notation and concludes the proof in all cases.

\[ \square \]

Remark 1. One may wonder if hypothesis (1.1) weakens the statement of the main theorem. As in [4], it is an essential element of the proof, but in most applications it does not present any significant restriction. In fact, in most cases, one may work with truncated versions of an operator \( T \) for which (1.1) holds with constants depending on the truncation. Then boundedness is obtained for truncated operators with bounds independent of the truncation and a limiting argument implies the same conclusion for the original operator \( T \).

Remark 2. It is also worth noting that if (1.1) holds for every point \((p_1, \ldots, p_m)\) with \( 1 < p < t' \), \( t < p_1, \ldots, p_m < \infty \) and \( 1/p_1 + \cdots + 1/p_m - 1/p = m/t - 1/s \), then one obtains restricted weak type estimates at every point in the open convex hull \( H \) of these points combined with the point \((\frac{1}{t}, \ldots, \frac{1}{t}, \frac{1}{s})\). Then by the multilinear Marcinkiewicz interpolation theorem (see for instance [3]), it follows that \( T \) satisfies strong type bounds in \( H \).

3. An Application

Let \( 0 < \alpha < n \). Consider the bilinear fractional integral
\[ J_\alpha(f, g)(x) = \int_{\mathbb{R}^n} f(x + t)g(x - t)|t|^\alpha dt, \quad (3.1) \]
defined for positive functions \( f, g \) on \( \mathbb{R}^n \). It was shown in [2] and [6] that \( J_\alpha \) maps \( L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \) to \( L^r(\mathbb{R}^n) \) whenever \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{n}{n-\alpha} \) and \((\frac{1}{p}, \frac{1}{q}, \frac{1}{r})\) lies in the open convex hull of the points \((\frac{n}{\alpha}, \infty, \infty), (\infty, \frac{n}{\alpha}, \infty), (1, \infty, \frac{n}{n-\alpha}), (\infty, 1, \frac{n}{n-\alpha})\), and \((1, 1, \frac{n}{2n-\alpha})\). The proof is achieved in two steps: (a) first we prove restricted weak type estimates at the aforementioned five points; (b) then we use multilinear interpolation to obtain boundedness in the open convex hull \( H \) of these five points.
In this note we simplify the proof of boundedness of $I_\alpha$ in $H$ by reducing it to a restricted weak type estimate at only the point $(1, 1, \frac{n}{2n-\alpha})$ for $I_\alpha$ and its two adjoints, which look very similar (for instance $I^*\alpha$ is defined with $f(x-t)$ in place of $f(x+t)$ and $g(x-2t)$ in place of $g(x-t)$ in (3.1)). To use Theorem 1.1 we introduce the truncated versions of $I_\alpha$

$$I^{\epsilon,N,M}_\alpha(f,g)(x) = \chi_{|x| \leq M} \int_{\epsilon \leq |t| \leq N} f(x+t)g(x-t)|t|^{\alpha-n}dt,$$

and we verify condition (1.1) for $I^{\epsilon,N,M}_\alpha$. But for $1 < p < \infty$ it is easy to see that

$$\|I^{\epsilon,N,M}_\alpha(\chi_{A_1}, \chi_{A_2})\|_{L^p} \leq C_{\epsilon,N,M} \min(1, |A_1|)^{\frac{1}{p'}} \min(1, |A_2|) \min(1, |A_1|, |A_2|)^{\frac{1}{p}}$$

and from this (1.1) follows for $I^{\epsilon,N,M}_\alpha$ via Hölder’s inequality. Here $p' = \frac{p}{p-1}$ and $0 < \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} < 1$. Then Theorem 1.1 and Remark 2 yield boundedness for $I^{\epsilon,N,M}_\alpha$ in $H$ with bounds as in (1.2) i.e., independent of $\epsilon, N, M$. Letting $\epsilon \downarrow 0$ and $N, M \uparrow \infty$ we obtain the same conclusion for $I_\alpha$, via the Lebesgue monotone theorem.

**References**


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