Sharp Hodge Decompositions, Maxwell's Equations, and Vector Poisson Problems on Non-Smooth, Three-Dimensional Riemannian Manifolds

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Dedicated to the memory of José F. Escobar

Abstract

We solve three basic potential theoretic problems: Hodge decompositions for vector fields, Poisson problems for the Hodge-Laplacian, and inhomogeneous Maxwell equations, in arbitrary Lipschitz subdomains of a smooth, compact, three dimensional, Riemannian manifold. In each case we derive sharp estimates on Sobolev-Besov scales and establish integral representation formulas for the solution. The proofs rely on tools from harmonic analysis and algebraic topology, such as Calderón-Zygmund theory and the de Rham theory.

1 Introduction

In this paper we solve three basic potential theoretic problems: **(I)** Hodge decompositions for vector fields, **(II)** Poisson problems for the Hodge-Laplacian, and **(III)** inhomogeneous Maxwell equations, in Lipschitz subdomains of a smooth, compact, boundaryless, *three* dimensional, Riemannian manifold $\mathcal{M}$. They are all considered in the context of Sobolev-Besov spaces, i.e. when the global smoothness of both the data and the solutions is measured on these scales.

In hindsight, the problems **(I)-(III)** above turn out to be closely related. A manifestation of this is that they share a common, (asymptotically) sharp 'well-posedness region', stemming from necessary limitations on the indices s (smoothness) and p (integrability), of the Sobolev-Besov spaces for which these PDE's have unique solutions, continuously dependent on the given data.

In turn, this region, call it $\mathcal{R}_\Omega$, is entirely determined by the geometric characteristics of the underlying domain $\Omega \subset \mathcal{M}$. More concretely,

$$(s, 1/p) \in \mathcal{R}_\Omega \Longleftrightarrow \left\{ \begin{array}{l} 0 < \frac{1}{p} < 1, \quad -1 + \frac{1}{p} < s < \frac{1}{p}, \\ \frac{2}{3}\left(1 - \frac{1}{p_\Omega}\right) < \frac{1}{p} - \frac{s}{3} < \frac{1}{3}\left(\frac{2}{p_\Omega} + 1\right). \end{array} \right. \tag{1.1}$$

Here, p_Ω is further defined in terms of the critical exponents intervening in the (regular) Dirichlet and Neumann problems for the Laplace-Beltrami operator in Ω (as well as its complement), when optimal L^p estimates for the associated nontangential maximal function are sought. A precise definition is given in §4. For the purpose of this introduction, we note that, generally speaking, $1 \leq p_\Omega < 2$; cf. **[11]**, **[12]**, **[31]**, **[14]**, **[65]**, for the flat, Euclidean setting and **[46]**-**[49]** for Lipschitz subdomains of Riemannian manifolds.

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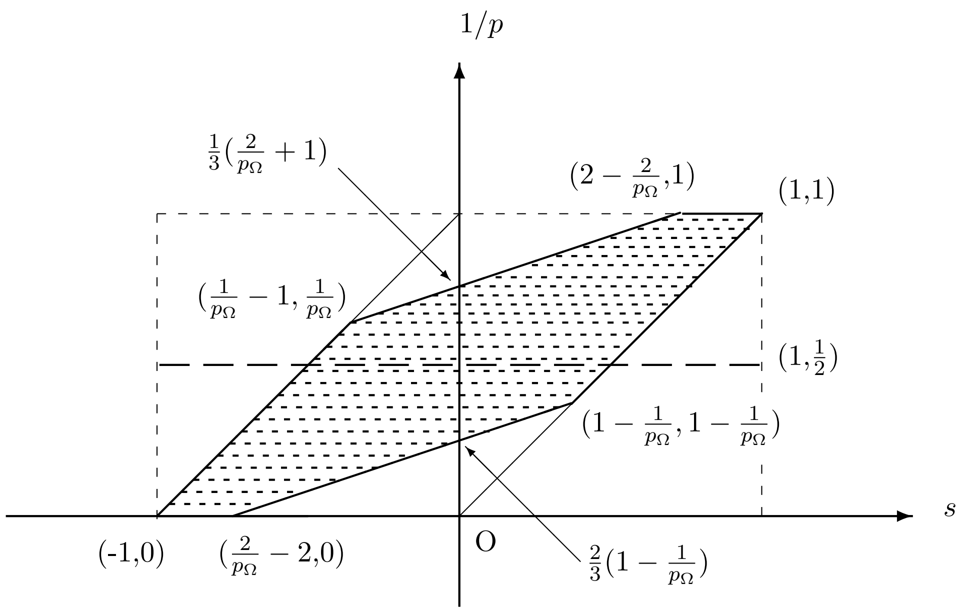
One feature of Ω which influences the size of p_Ω is the local oscillations of the unit conormal ν to $\partial\Omega$, i.e. $\lim_{\varepsilon\to 0}\sup\{\mathrm{dist}\,(\nu(x),\nu(y));\ \mathrm{dist}\,(x,y)<\varepsilon\}$ (cf. the comments in [**5**]). In fact, p_Ω tends to 1 as these local oscillations tend to zero. Moreover, $p_\Omega=1$ when $\partial\Omega\in C^1$ ([**19**]), and this continues to be the case even when the unit conormal only has vanishing mean oscillations. Finally, for a Lipschitz polyhedron in the Euclidean setting, p_Ω can be estimated in terms of the dihedral angles involved; cf. [**24**].

The picture below represents the two-dimensional region $\mathcal{R}_\Omega$ alluded to above, when considered in the $(s,1/p)$-plane (i.e., smoothness vs. reciprocal integrability):



The (interior of the) dashed hexagon represents the well-posedness region for the Hodge decompositions, Maxwell's equations, and vector Poisson problems.

We now proceed to describe our main results in greater detail (complete definitions – including those for the standard Sobolev-Besov spaces L^p_s, $B^{p,q}_s$, as well as various 'nonstandard' versions such as $H^{s,p}$, TH^p_s, etc. – are given in §2).

I. Hodge Decompositions. Let Ω be an arbitrary Lipschitz domain in $\mathcal{M}$ with outward unit normal ν. Call a 1-form *monogenic*, if it is both curl and divergence free. For $1<p<\infty$, $s\in\mathbb{R}$, let L^p_s stand for the usual Sobolev scale and, for $-1+1/p<s<1/p$, consider

$$\mathcal{H}^{s,p}_\bullet(\Omega):=\{u\in L^p_s(\Omega,\Lambda^1T\mathcal{M});\ \mathrm{div}\,u=0,\mathrm{curl}\,u=0,\ \nu\cdot u=0\},\tag{1.2}$$

$$\mathcal{H}^{s,p}_\times(\Omega):=\{u\in L^p_s(\Omega,\Lambda^1T\mathcal{M});\ \mathrm{div}\,u=0,\mathrm{curl}\,u=0,\ \nu\times u=0\},\tag{1.3}$$

i.e., monogenic 1-forms in L^p_s, with vanishing normal and tangential components on $\partial\Omega$.

Theorem 1.1 *For each $(s,1/p)\in\mathcal{R}_\Omega$, the following Hodge-type decompositions hold:*

$$L^p_s(\Omega,\Lambda^1 T\mathcal{M}) = \nabla[L^p_{1+s,0}(\Omega)] \oplus \mathrm{curl}[H^{s,p}(\Omega;\mathrm{curl})] \oplus \mathcal{H}^{s,p}_\times(\Omega), \tag{1.4}$$

$$L^p_s(\Omega,\Lambda^1 T\mathcal{M}) = \nabla[L^p_{1+s}(\Omega)] \oplus \mathrm{curl}[H^{s,p}_\times(\Omega;\mathrm{curl})] \oplus \mathcal{H}^{s,p}_\bullet(\Omega), \tag{1.5}$$

where the direct sums (of closed subspaces) are topological.

II. Poisson problems for the Hodge-Laplacian on differential forms. Recall that the vector Hodge-Laplacian on $\mathcal{M}$ is defined as $\Delta := -\,\mathrm{curl}\,\mathrm{curl} + \nabla\mathrm{div}$.

Theorem 1.2 *Let $\Omega \subseteq \mathcal{M}$ be an arbitrary Lipschitz domain. Then for each $(s,1/p)$ in $\mathcal{R}_\Omega$, the boundary value problem*

$$(\textit{1st Poisson Problem}) \begin{cases} \Delta u = \eta \in L^p_s(\Omega,\Lambda^1 T\mathcal{M}), \\ u \in H^{s,p}(\Omega;\mathrm{curl}),\ \mathrm{curl}\ u \in H^{s,p}(\Omega;\mathrm{curl}),\ \mathrm{div}\, u \in L^p_{s+1}(\Omega), \\ \nu\cdot u = f \in B^{p,p}_{s-1/p}(\partial\Omega), \\ \nu\times \mathrm{curl}\, u = g \in TH^p_s(\partial\Omega), \end{cases} \tag{1.6}$$

is Fredholm solvable of index zero. More specifically, (1.6) has a solution if and only if the data satisfy the linear constraints

$$\langle \eta, h\rangle = \langle g, \nu\times(\nu\times h)\rangle, \qquad \forall\, h\in \mathcal{H}^{-s,p'}_\bullet(\Omega), \quad 1/p+1/p'=1, \tag{1.7}$$

and the space of null solutions is $\mathcal{H}^{s,p}_\bullet(\Omega)$. The latter is a finite dimensional space, whose dimension is $b_1(\Omega)$, the first Betti number of Ω.

Similar results are valid for

$$(\textit{2nd Poisson Problem}) \begin{cases} \Delta u = \eta \in L^p_s(\Omega,\Lambda^1 T\mathcal{M}), \\ u \in H^{s,p}(\Omega;\mathrm{curl}),\ \mathrm{curl}\ u \in H^{s,p}(\Omega;\mathrm{curl}),\ \mathrm{div}\, u \in L^p_{s+1}(\Omega), \\ \mathrm{Tr}\,(\mathrm{div}\, u) = f \in B^{p,p}_{s+1-1/p}(\partial\Omega), \\ \nu\times u = g \in TH^p_s(\partial\Omega). \end{cases} \tag{1.8}$$

In this case, the necessary compatibility conditions read

$$\langle \eta, h\rangle = \langle f, \nu\cdot h\rangle, \qquad \forall\, h\in \mathcal{H}^{-s,p'}_\times(\Omega), \quad 1/p+1/p'=1, \tag{1.9}$$

and the space of null solutions is precisely $\mathcal{H}^{s,p}_\times(\Omega)$, whose dimension is $b_2(\Omega)$, the second Betti number of Ω.

III. Maxwell's equations. Consider the time-harmonic version of the Maxwell system, governing the propagation of electromagnetic waves in a Lipschitz domain $\Omega\subset\mathcal{M}$. See, e.g., [**7**], [**9**], [**10**].

Theorem 1.3 *For any Lipschitz domain $\Omega\subset\mathcal{M}$ and any $k\in\mathbb{C}\setminus\{0\}$, the inhomogeneous Maxwell system*

$$(\textit{Inhomogeneous Maxwell}) \begin{cases} E, H \in H^{s,p}(\Omega;\mathrm{curl}), \\ \mathrm{curl}\, E - ikH = K \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\ \mathrm{curl}\, H + ikE = J \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\ \nu \times E = f \in TH^p_s(\partial\Omega), \end{cases} \tag{1.10}$$

is Fredholm solvable, of index zero. Moreover, there exists an unbounded, non-decreasing sequence of real, nonnegative numbers $\{k_j\}_j$ *such that for each* $k \in \mathbb{C} \setminus \{\pm k_j\}_j$*, the boundary value problem (1.10) is well-posed, as long as* $(s, 1/p) \in \mathcal{R}_\Omega$*.*

It is important to point out that the validity range for Theorems 1.1-1.3 when $s = 0$ (i.e. for $L^p(\Omega, \Lambda^1 T\mathcal{M})$ fields) becomes $3(\frac{2}{p_\Omega} + 1)^{-1} < p < \frac{3}{2}(1 - \frac{1}{p_\Omega})^{-1}$, as is visible from the above picture. In particular, $\frac{3}{2} \leq p \leq 3$ will do in *any* Lipschitz domain (when $s = 0$). The counterexamples in **[20]** can then be used to prove that the range of indices s, p described in Theorem 1.1 is in the nature of best possible.

To put matters in the proper perspective, let us now briefly discuss the origins of our results. The crowning achievement of the classical theory of elliptic PDE's (sometimes referred to as the Shift Theorem; cf. [**26**], [**61**]) is that all regular elliptic problems $Lu = f +$ *boundary conditions*, in C^∞ domains are Fredholm solvable (i.e. existence and uniqueness hold modulo finite dimensional spaces) on *all* Sobolev-Besov scales and any solution u is *deg* L units smoother than the datum f.

The situation is radically different in less smooth domains. For instance, Dahlberg **[13]** has constructed a domain Ω in $\mathbb{R}^n$ with a C^1-boundary and $f \in C^\infty(\bar{\Omega})$ such that

$$\Delta u = f,\ u \in L^2_1(\Omega),\ \mathrm{Tr}\, u = 0 \Longrightarrow \frac{\partial^2 u}{\partial x_j \partial x_k} \notin L^p(\Omega),\ \forall\, j, k = 1, 2, ..., n,\ 1 < p < \infty. \tag{1.11}$$

In other words, for Dahlberg's domain Ω and each $1 < p < \infty$, the Poisson problem for the Laplacian with homogeneous Dirichlet boundary conditions fails to be well-posed for data in $L^p(\Omega)$ (in the sense that it is no longer reasonable to expect solutions with two derivatives in $L^p(\Omega)$). This raises the fundamental issue of identifying those Sobolev-Besov spaces within which the natural correlation between the smoothness of the data and that of the solutions is preserved when the domain in question is allowed to have a minimally smooth boundary (in the sense of **[60]**).

In their ground breaking work **[32]**, Jerison and Kenig were able to produce such an optimal well-posedness region for the Poisson problem with Dirichlet boundary conditions for the scalar, flat space Laplacian in bounded, Euclidean Lipschitz domains in the context of Sobolev-Besov spaces. The basic estimate proved in **[32]** is that

$$\|u\|_{L^p_{s+1/p}(\Omega)} \approx \|\mathrm{Tr}\, u\|_{B^{p,p}_s(\partial\Omega)}, \tag{1.12}$$

uniformly for *u harmonic* in Ω, for an (asymptotically) optimal range of indices s, p. The methods in **[32]**, though beautiful in their elegance and sharpness, rely in an essential fashion on harmonic measure estimates (and, by extension, on positivity and maximum principles) and, as such, do not readily adapt to *systems* of PDE's, or to other natural boundary conditions, e.g. of *Neumann type.* In fact, the latter issue makes the object of one of the open problems listed in Kenig's book **[36]**; cf. # 3.2.21, p. 121. In the author's words (cf. Introduction, *loc. cit.*), these have been singled out as "*problems which we find particularly challenging, and which we feel will lead to further important developments in the subject.*"

Relatively recently, in **[20]**, **[48]**, we have developed an alternative strategy to proving (1.12) which relies on a systematic use of boundary integral methods. This approach, in principle, does not distinguish between Dirichlet and Neumann type conditions, or between a single equation and systems. At the heart of the matter is the fact that, with K standing for the harmonic (principal value) double layer potential operator on $\partial\Omega$ and I denoting the identity operator (see §4 for a discussion),

$$(s+1/p-1,1/p)\in\mathcal{R}_\Omega \Longrightarrow \tfrac{1}{2}I+K : B^{p,p}_s(\partial\Omega) \xrightarrow{\sim} B^{p,p}_s(\partial\Omega) \text{ is an isomorphism.} \tag{1.13}$$

The chief goal of this paper is to continue this line of work and initiate the study of *systems* of PDE's in Sobolev-Besov spaces in Lipschitz domains. For the purpose of this introduction, let us point out that one of the main estimates we establish in this paper is that whenever $(s,1/p)\in\mathcal{R}_\Omega$ and k is not an eigenvalue, then

$$\|E\|_{L^p_s(\Omega,\Lambda^1 T\mathcal{M})}+\|H\|_{L^p_s(\Omega,\Lambda^1 T\mathcal{M})}\approx\|\nu\times E\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}+\|\nu\cdot H\|_{B^{p,p}_{s-1/p}(\partial\Omega)}, \tag{1.14}$$

uniformly for vector fields E,H solving (the homogeneous) Maxwell's equations

$$\mathrm{curl}\,E-ikH=0,\qquad \mathrm{curl}\,H+ikE=0 \tag{1.15}$$

in the Lipschitz domain Ω.

Our approach to the three problems listed at the beginning of this section is constructive in nature, as is based on singular integral operators, and we momentarily digress in order to explain some of the difficulties which arise in this scenario.

The essence of Hodge decompositions is the identity $I=\nabla\mathrm{div}\,G-\mathrm{curl}\,\mathrm{curl}\,G+P$, where G is the Green operator associated with the Hodge-Laplacian Δ (i.e., the solution operator Δ^{-1} for a vector Poisson problems with appropriate boundary conditions), and P is the projection onto monogenic forms. Consequently, the success of decomposing a vector field as in (1.4)-(1.5), where the individual summands have the same amount of regularity as the original field, depends to a large extent on the mapping properties of operators like $\mathrm{curl}\,\mathrm{curl}\,\Delta^{-1}$ and $\nabla\mathrm{div}\,\Delta^{-1}$. On a smooth domain, these are classical zero-order pseudodifferential operators, but in the presence of boundary irregularities they fail even to be of Calderón-Zygmund type. In this latter scenario, one can only describe them in terms of boundary layer potentials, typically of *vector* character, and their inverses (whenever meaningful).

It has been understood for some time (cf. the discussion in **[40]**) that, in the context of differential forms on a subdomain Ω of an arbitrary Riemannian manifold $\mathcal{M}$, there exists a natural *hierarchy* of boundary layer potentials $\{M^\ell\}_\ell$, $0\leq\ell\leq\dim\mathcal{M}$, (acting on boundary ℓ-forms), so that $M^0\equiv K$ is simply the usual harmonic double layer acting on scalar functions (viewed as differential forms of degree zero). In the setting of $L^p(\partial\Omega)$, the invertibility of the entire scale $\{\frac{1}{2}I+M^\ell\}_\ell$ is now understood, thanks to the work in **[41]**, **[40]**. There we have shown that $2-\varepsilon<p<2+\varepsilon$ will do, and this is asymptotically the best range if one insists on allowing *arbitrary* ℓ's. On the technical side, the novel step we take here is clarifying what the analogue of (1.13) is, both in terms of the range of indices s, p, as well as the nature of the spaces which should replace the scalar Besov scale in (1.13), at the 'next level up' on this hierarchy, i.e. when $\ell=1$. This is achieved by exploiting some remarkable *intertwining identities* involving, on the one hand, the operators M^ℓ at the scalar level ($\ell=0$) in concert with those at the vector level ($\ell=1$) and, on the other hand, some natural first order (boundary) differential operators.

In this connection, a major ingredient required to carry out this program is understanding the Fredholm properties of certain surface differential operators. More specifically, recall that there are two natural first-order operators on the Lipschitz manifold $\partial\Omega$: the *tangential gradient* and its adjoint, the *surface divergence*. As their composition is zero, they give rise to a complex

$$0 \hookrightarrow \text{Scalar Functions} \xrightarrow{\textit{tangential grad}} \text{Vector Fields} \xrightarrow{\textit{surface div}} \text{Scalar Functions} \longrightarrow 0. \quad (1.16)$$

The technical accomplishment alluded to above is to find suitable smoothness spaces guaranteeing that the complex (1.16) is *exact*, and then to identify its cohomology groups. This is done via tools from algebraic topology and harmonic analysis.

It should be pointed out that the intertwining identities, referred to two paragraphs above, are most useful when the dimension of the ambient space is *three*, a restriction which appears inherent to our method. What the corresponding situation is in arbitrary space dimensions and for other values of ℓ remains an open problem at the moment.

In broad outline, our plan is to prove first a simpler version of Theorem 1.2, dealing with vector Poisson problems with *homogeneous* boundary conditions (cf. §7), by employing layer potentials (plus other tools developed in §4-§5). This, in turn, leads to a constructive approach to Theorem 1.1. These Hodge decompositions are key ingredients used in §8-9, in the solution of the inhomogeneous Maxwell system in Theorem 1.2. With this in hand, the cycle is then completed by returning to (and finishing the proof of) Theorem 1.2 in §10. The (asymptotic) sharpness of our main results is a consequence of the counterexamples in [**32**], [**20**].

Let us conclude with a few historical notes. The subject of Hodge decompositions in the C^∞ context can be traced back to the work of Hodge, Kodaira and de Rham ([**27**], [**37**], [**2**], [**16**]) starting in the 1930's. A modern account can be found in, e.g., [**61**], [**58**]. Subsequent developments, emphasizing regularity aspects and/or allowing less smooth structures can be found in [**50**], [**51**], [**52**], [**62**], [**28**], [**59**], [**42**], [**40**], [**56**]. The methods employed by these authors do not work in the present context. The problem **II** that we solve here goes back to a question posed to us by Eugene Fabes and Michael Taylor in the mid 90's. Natural boundary problems for the Hodge-Laplacian in Lipschitz domains for L^p-boundary data have been systematically studied in [**40**], where optimal nontangential maximal function estimates have been proved. The study of the Maxwell system via integral equation methods has a long tradition, originating in the pioneering work of A. P. Calderón [**4**] and C. Müller [**53**], [**54**]. L^p estimates for the nontangential maximal function operator (denoted in the sequel by $\mathcal{N}$) have been produced in [**45**], [**44**]. There it is proved that for solutions of the homogeneous Maxwell's equations (1.15), there holds

$$\begin{aligned} \|\mathcal{N}(E)\|_{L^2(\partial\Omega,\Lambda^1 T\mathcal{M})} &+ \|\mathcal{N}(H)\|_{L^2(\partial\Omega,\Lambda^1 T\mathcal{M})} \\ &\approx \|E\|_{L^2_{1/2}(\Omega,\Lambda^1 T\mathcal{M})} + \|H\|_{L^2_{1/2}(\Omega,\Lambda^1 T\mathcal{M})} \\ &\approx \|\nu\times E\|_{L^2(\partial\Omega,\Lambda^1 T\mathcal{M})} + \|\nu\cdot H\|_{L^2(\partial\Omega)}, \end{aligned} \quad (1.17)$$

whenever k is not an eigenvalue for the Lipschitz domain Ω; this can be regarded as the limiting case $p = 2$, $s = 1/2$ of (1.14). An extension to higher dimensions is in [**30**]; cf. also [**40**]. The novelty here is producing sharp estimates in Sobolev-Besov spaces for an optimal range of indices.

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2 Preliminary results

In this section, which is further divided into four subsections, we review basic notation, collect definitions and prove some preparatory results.

2.1 The geometrical setting

Let $\mathcal{M}$ be a smooth, compact, boundaryless, connected, oriented manifold of real dimension $\dim \mathcal{M} = 3$. As is customary, we denote by $\Lambda^\ell T\mathcal{M}$, $\ell = 0,1,2,3$, the exterior power bundles (i.e. differential forms of degree ℓ), and by d and $\wedge$, respectively, the exterior derivative operator and the exterior product of forms. When acting on scalar-valued functions, we denote d by ∇.

Assume next that $\mathcal{M}$ is endowed with a metric tensor $g = \sum_{j,k} g_{jk} dx_j \otimes dx_k$, whose coefficients g_{jk} are of class $C^{1,1}$. This induces a volume element $d\mathrm{Vol}$ and, further, a pointwise inner product $(u,v) \mapsto u \cdot v = \langle u, v\rangle$ in each $\Lambda^\ell T\mathcal{M}$, $0 \leq \ell \leq 3$. Recall that the Hodge star operator is the unique vector bundle morphism $* : \Lambda^\ell T\mathcal{M} \to \Lambda^{3-\ell} T\mathcal{M}$ such that $u \wedge (*u) = |u|^2\, d\mathrm{Vol}$. We also denote by δ the formal adjoint of d and set

$$\operatorname{div} u := -\delta\, u, \quad \operatorname{curl} u := *du \;\text{ and }\; u \times v := *(u \wedge v), \quad \forall\, u, v \in \Lambda^1 T\mathcal{M}. \tag{2.1}$$

In particular, the fact that $d^2 = 0$ entails $\operatorname{curl} \nabla = 0$.

Recall that the Laplace-Beltrami operator Δ on $\mathcal{M}$ is given in local coordinates, where the metric tensor reads $g = \sum g_{jk} dx_j \otimes dx_k$, by

$$\Delta u := \operatorname{div}(\nabla u) = (\det(g_{jk}))^{-1/2} \sum_j \partial_j \Big(\sum_k g^{jk} (\det(g_{jk}))^{1/2}\, \partial_k u \Big), \tag{2.2}$$

where we take (g^{jk}) to be the matrix inverse of (g_{jk}). This operator fits naturally into the framework of the Hodge Laplacian $\Delta := -d\delta - \delta d$ on ℓ-forms, when $\ell = 0$. Here (and in the sequel), we make no special notational distinction between the scalar Laplace-Beltrami operator and the Hodge-Laplacian on 1-forms.

As is well known,

$$\operatorname{curl}\operatorname{curl} = -\Delta + \nabla \operatorname{div} \text{ on 1-forms, and } \operatorname{div} \nabla = \Delta \text{ on scalar functions.} \tag{2.3}$$

Also,

$$\operatorname{curl} \Delta = \Delta \operatorname{curl} \qquad \text{and} \qquad \operatorname{div} \Delta = \Delta \operatorname{div}. \tag{2.4}$$

Recall next that $\Omega \subset \mathcal{M}$ is called a *Lipschitz* domain provided $\partial\Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions. For each $1 \leq p \leq \infty$, we denote by $L^p(\partial\Omega)$ the Lebesgue space of p-power integrable functions with respect to the surface measure $d\sigma$ (with the usual convention when $p = \infty$). If $\nu \in T^*\mathcal{M}$ is the unit outward conormal to $\partial\Omega$, we let $\nabla_{\tan} := -\nu \times (\nu \times \nabla)$ stand for the tangential gradient on $\partial\Omega$. Finally, throughout the paper, $\langle \cdot, \cdot \rangle$ will stand for either a pointwise inner product, or the natural pairing between a space and its dual. Also, if $X \subset Y$ are Banach spaces, we will let $X^\circ \subset Y^*$ denote the annihilator of X.

2.2 Scalar Sobolev and Besov spaces in Lipschitz domains

Denote by $L^p_1(\partial\Omega)$ the Sobolev space of functions in $L^p(\partial\Omega)$ with tangential gradients in $L^p(\partial\Omega)$, $1 < p < \infty$. Spaces with fractional smoothness can then be defined via complex interpolation, i.e.

$L^p_\theta(\partial\Omega) := [L^p(\partial\Omega), L^p_1(\partial\Omega)]_\theta$, $0<\theta<1$, $1<p<\infty$. We also set $L^p_{-s}(\partial\Omega) := \Big(L^{p'}_s(\partial\Omega)\Big)^*$ for $0\le s\le 1$, $1<p,p'<\infty$, $1/p+1/p'=1$.

Next, Besov spaces with positive smoothness on $\partial\Omega$ can then be introduced via real interpolation, i.e.

$$B^{p,q}_\theta(\partial\Omega) := (L^p(\partial\Omega), L^p_1(\partial\Omega))_{\theta,q}\,, \quad \text{with } 0<\theta<1,\ 1<p,q<\infty. \tag{2.5}$$

An intrinsic definition for membership to $B^{p,q}_s(\mathbb{R}^2)$, $1\le p,q<\infty$, $0<s<1$, is obtained by requiring that

$$\|f\|_{B^{p,q}_s(\mathbb{R}^2)} := \|f\|_{L^p(\mathbb{R}^2)} + \left(\iint_{\mathbb{R}^2} \frac{\|f(\cdot+t)-f(\cdot)\|^q_{L^p(\mathbb{R}^2)}}{|t|^{2+sq}}\,dt\right)^{1/q} < +\infty. \tag{2.6}$$

For the same range of indices, when Ω is the region from $\mathbb{R}^3$ above the graph of a Lipschitz function $\phi:\mathbb{R}^2\to\mathbb{R}$, we define $B^{p,q}_s(\partial\Omega)$ as the space of functions f for which the assignment $x\mapsto f(x,\phi(x))$ belongs to $B^{p,q}_s(\mathbb{R}^2)$. This definition then readily extends to the case of arbitrary Lipschitz subdomains of $\mathcal{M}$ via a standard partition of unity argument.

The case when $p=q=\infty$ corresponds to the usual (non-homogeneous) Hölder spaces. Also, for $-1<s<0$ and $1<p,q<\infty$ or $p=q=\infty$, we set

$$B^{p,q}_s(\partial\Omega) := (B^{p',q'}_{-s}(\partial\Omega))^*, \qquad 1/p+1/p'=1,\ 1/q+1/q'=1. \tag{2.7}$$

In the sequel, we shall also need to work with the Besov spaces $B^{1,1}_{-s}(\partial\Omega)$, $s\in(0,1)$. Inspired by the corresponding atomic characterization from **[21]**, we set

$$B^{1,1}_{-s}(\partial\Omega) := L^q(\partial\Omega) + \Big\{f=\sum\lambda_j\vartheta_j;\ \vartheta_j\text{-atom},\ (\lambda_j)_j\in\ell^1\Big\}, \tag{2.8}$$

where the series converges in the sense of distributions, and $q>1$ is arbitrary (different choices yield isomorphic spaces). In this context, a $B^{1,1}_{-s}(\partial\Omega)$-atom, $0<s<1$, is a function $\vartheta\in L^\infty(\partial\Omega)$ with support contained in a surface ball $B_r(x_0)\cap\partial\Omega$, $x_0\in\partial\Omega$, $0<r<\operatorname{diam}\Omega$, and satisfying

$$\int_{\partial\Omega}\vartheta\,d\sigma = 0, \qquad \|\vartheta\|_{L^\infty(\partial\Omega)}\le r^{-s-2}. \tag{2.9}$$

Furthermore, for $f\in B^{1,1}_{-s}(\partial\Omega)$, $0<s<1$,

$$\|f\|_{B^{1,1}_{-s}(\partial\Omega)} := \inf\Big\{\|g\|_{L^q(\partial\Omega)} + \sum|\lambda_j|;\ f=g+\sum\lambda_j\vartheta_j\Big\}, \tag{2.10}$$

where $g\in L^q(\partial\Omega)$, $q>1$ (different q's yield equivalent norms), ϑ_j's and $(\lambda_j)_j$ are as in (2.8). Let us also point out that $B^{1,1}_{-s}(\partial\Omega)$ is 'local', in the sense that this space is a module over $B^{\infty,\infty}_\alpha(\partial\Omega)$, for each $\alpha\in(s,1)$, and that $(B^{1,1}_{-s}(\partial\Omega))^* = B^{\infty,\infty}_s(\partial\Omega)$, for $0<s<1$.

We now briefly discuss the case of Sobolev and Besov classes on an open subset Ω of $\mathcal{M}$. First, the Sobolev (or potential) scale $L^p_s(\mathcal{M})$, $1<p<\infty$, $s\ge 0$, is obtained by lifting $L^p_s(\mathbb{R}^3) := \{(I-\Delta)^{s/2}f;\ f\in L^p(\mathbb{R}^3)\}$ to $\mathcal{M}$, and we denote by $L^p_s(\Omega)$ the restriction of elements in $L^p_s(\mathcal{M})$ to Ω. Similarly, for $1\le p,q\le\infty$, $s>0$, the Besov space $B^{p,q}_s(\mathcal{M})$ is defined by localizing and transporting via local charts its Euclidean counterpart, i.e., $B^{p,q}_s(\mathbb{R}^3)$ (the latter is defined analogously to (2.5)-(2.6); see, e.g., **[55]**, **[1]**, **[63]**, **[33]**). Then, $B^{p,q}_s(\Omega)$, $1\le p,q\le\infty$, $s>0$, consists of restrictions to Ω of functions from $B^{p,q}_s(\mathcal{M})$.

Both scales in Ω are equipped with natural norms, defined by taking the infimum of the corresponding norms of all possible extensions to $\mathcal{M}$. Using Stein's extension operator and then invoking well known real interpolation results (cf., e.g., [**1**]), it follows that for any Lipschitz domain $\Omega \subset \mathcal{M}$, $L^p_s(\Omega)$ and $B^{p,q}_s(\Omega)$ are complex interpolation scales, i.e.

$$[L^{p_0}_{s_0}(\Omega), L^{p_1}_{s_1}(\Omega)]_\theta = L^p_s(\Omega) \tag{2.11}$$

if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $s = (1-\theta)s_0 + \theta s_1$, $0 < \theta < 1$, $1 < p_0, p_1 < \infty$, $s_0, s_1 \geq 0$, and

$$[B^{p_0,q_0}_{s_0}(\Omega), B^{p_1,q_1}_{s_1}(\Omega)]_\theta = B^{p,q}_s(\Omega) \tag{2.12}$$

if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $s = (1-\theta)s_0 + \theta s_1$, $0 < \theta < 1$, $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, $s_0, s_1 > 0$, $s_0 \neq s_1$. As is well known, the Besov and Sobolev spaces on the domain are also related via real interpolation. For instance, we have the formula

$$\big(L^p(\Omega), L^p_k(\Omega)\big)_{s,q} = B^{p,q}_{sk}(\Omega) \tag{2.13}$$

when $0 < s < 1$, $1 < p, q < \infty$ and k is a positive integer. In this connection, let us also point out that

$$\big(B^{p,q_0}_{s_0}(\Omega), B^{p,q_1}_{s_1}(\Omega)\big)_{s,q} = B^{p,q}_s(\Omega) \tag{2.14}$$

if $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $s = (1-\theta)s_0 + \theta s_1$, $0 < \theta < 1$, $1 \leq p, q_0, q_1 \leq \infty$, $s_0, s_1 > 0$, and $s_0 \neq s_1$.

For the remainder of this subsection we assume that Ω is a Lipschitz subdomain of $\mathcal{M}$. Following [**32**], for $1 < p < \infty$, $s \in \mathbb{R}$ we define the space $L^p_{s,0}(\Omega)$ to consist of distributions in $L^p_s(\mathcal{M})$ supported in $\overline{\Omega}$ (with the norm inherited from $L^p_s(\mathcal{M})$). It is known that $C^\infty_o(\Omega)$ is dense in $L^p_{s,0}(\Omega)$ for all values of s and p.

Recall (cf. [**33**]) that the trace operator

$$\mathrm{Tr} : L^p_s(\Omega) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \tag{2.15}$$

is well-defined, bounded and onto if $1 < p < \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$. This also has a bounded right inverse whose operator norm is controlled exclusively in terms of p, s and the Lipschitz character of Ω. Similar results are valid for $\mathrm{Tr} : B^{p,q}_s(\Omega) \to B^{p,q}_{s-\frac{1}{p}}(\partial\Omega)$. In this latter case we may allow $1 \leq p, q \leq \infty$; cf. [**1**]. If $1 < p < \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$, the space $L^p_{s,0}(\Omega)$ is the kernel of the trace operator Tr acting on $L^p_s(\Omega)$. This follows from the Euclidean result (Proposition 3.3 in [**32**]). In fact, for the same range of indices, $L^p_{s,0}(\Omega)$ is the closure of $C^\infty_o(\Omega)$ in the $L^p_s(\Omega)$ norm.

For positive s, $L^p_{-s}(\Omega)$ is defined as the space of distributions in Ω such that

$$\|f\|_{L^p_{-s}(\Omega)} := \sup\Big\{|\langle f, \varphi\rangle|;\, \varphi \in C^\infty_o(\Omega),\ \|\tilde{\varphi}\|_{L^q_s(\mathcal{M})} \leq 1\Big\} < +\infty, \tag{2.16}$$

where tilde denotes the extension by zero outside Ω and $\frac{1}{p} + \frac{1}{q} = 1$. For all values of p and s, $C^\infty(\overline{\Omega})$ is dense in $L^p_s(\Omega)$. Also, $C^\infty_o(\Omega)$ is dense in $L^p_s(\Omega)$ if $s \leq 0$. Next, for any $s \in \mathbb{R}$,

$$L^q_{-s,0}(\Omega) = (L^p_s(\Omega))^* \quad \text{and} \quad L^p_{-s}(\Omega) = \Big(L^q_{s,0}(\Omega)\Big)^*. \tag{2.17}$$

For later reference, let us point out that for each $1 < p < \infty$, $-1 + 1/p < s < 1/p$, there exists a bounded, linear extension operator

$$L^p_s(\Omega) \ni u \mapsto \tilde{u} \in L^p_s(\mathcal{M}), \tag{2.18}$$

with the property that $\operatorname{supp} \tilde{u} \subseteq \bar{\Omega}$; cf. Theorem 3.5 in [**64**]. (Of course, when $s \geq 0$, this is just the extension by zero outside Ω.) Its image is precisely $L^p_{s,0}(\Omega)$, allowing the identification

$$L^p_s(\Omega) = L^p_{s,0}(\Omega), \qquad \forall\, p \in (1,\infty),\ \forall\, s \in (-1+1/p, 1/p). \tag{2.19}$$

Thus, if $p, p' \in (1,\infty)$ are such that $1/p + 1/p' = 1$, then

$$(L^p_s(\Omega))^* = L^{p'}_{-s}(\Omega), \qquad \forall\, s \in (-1+1/p, 1/p). \tag{2.20}$$

Also, the restriction operator to Ω, denoted by $\cdot|_\Omega$ in the sequel, has a linear, bounded realization

$$\cdot|_\Omega : L^p_s(\mathcal{M}) \longrightarrow L^p_s(\Omega), \qquad \langle u|_\Omega, \varphi\rangle := \langle u, \tilde{\varphi}\rangle, \quad \forall\, \varphi \in C^\infty_o(\Omega), \tag{2.21}$$

for each $1 < p < \infty$, $s > -1 + 1/p$. Furthermore, the extension operator to $\mathcal{M}$ by zero outside the natural domain −denoted in the sequel by tilde− extends to a bounded map from $L^p_s(\Omega)$ into $L^p_s(\mathcal{M})$ for any $1 < p < \infty$, $-1 + 1/p < s < 1/p$; cf. **[64]**. In particular,

$$\langle u|_\Omega, v\rangle = \langle u, \tilde{v}\rangle, \quad \forall\, u \in L^p_s(\mathcal{M}),\ \forall\, v \in L^{p'}_{-s}(\Omega), \tag{2.22}$$

if $1 < p, p' < \infty$, $1/p + 1/p' = 1$, and $-1 + 1/p < s < 1/p$.

Another important aspect (which follows from Corollary 13.5 in **[21]**, or Proposition 3.5 in **[32]**; cf. also **[64]**) is that multiplication by χ_Ω, the characteristic function of Ω, maps $L^p_s(\mathcal{M})$ boundedly onto $L^p_{s,0}(\Omega) = L^p_s(\Omega)$, for each $-1 + 1/p < s < 1/p$, $1 < p < \infty$.

We also refer to **[57]**, **[63]**, **[1]**, **[17]**, **[32]**, **[64]**, **[48]**, for a more detailed exposition of these and other related matters, such as embedding theorems (frequently used in the paper). Here we only want to alert the reader that $L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ will stand for $L^p_s(\Omega) \otimes \Lambda^1 T\mathcal{M}$, i.e. the collection of 1-forms with coefficients in $L^p_s(\Omega)$. A similar convention is in place for spaces defined on $\partial\Omega$, and for the Besov scale. E.g., $B^{p,q}_s(\partial\Omega, \Lambda^1 T\mathcal{M}) := B^{p,q}_s(\partial\Omega) \otimes \Lambda^1 T\mathcal{M}$, etc.

2.3 Vector Sobolev spaces. Tangential and normal traces

In this paper we shall work with certain nonstandard Sobolev spaces which are naturally adapted to the type of differential operators we intend to study. Specifically, if Ω is an open subset of $\mathcal{M}$ and if $1 < p < \infty$, $s \in \mathbb{R}$, we introduce

$$H^{s,p}(\Omega; \mathrm{div}) := \{u \in L^p_s(\Omega, \Lambda^1 T\mathcal{M});\ \mathrm{div}\, u \in L^p_s(\Omega)\}, \tag{2.23}$$

$$H^{s,p}(\Omega; \mathrm{curl}) := \{u \in L^p_s(\Omega, \Lambda^1 T\mathcal{M});\ \mathrm{curl}\, u \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})\}, \tag{2.24}$$

equipped with the natural graph norms. Throughout the paper, all derivatives are taken in the sense of distributions.

Let us now assume (as we shall do for the remainder of this subsection) that $\Omega \subseteq \mathcal{M}$ be an arbitrary Lipschitz domain with outward unit conormal $\nu \in T^*\mathcal{M}$. If $1 < p < \infty$, $-1 + 1/p < s < 1/p$, and $u \in H^{s,p}(\Omega; \mathrm{div})$ then we can then define $\nu \cdot u \in B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)$ by setting

$$\langle \nu \cdot u, \mathrm{Tr}\ \psi\rangle := \iint_\Omega [\psi\, \mathrm{div}\, u + \langle u, \nabla\psi\rangle]\, d\mathrm{Vol}, \tag{2.25}$$

for each $\psi \in L^{p'}_{1-s}(\Omega)$, $1/p + 1/p' = 1$. It follows that the operator

$$\nu\cdot\ : H^{s,p}(\Omega; \mathrm{div}) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \tag{2.26}$$

is bounded, that is,

$$\|\nu\cdot u\|_{B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)}\leq C(\|u\|_{L^p_s(\Omega,\Lambda^1T\mathcal{M})}+\|\mathrm{div}\,u\|_{L^p_s(\Omega)}),\tag{2.27}$$

granted that $1<p<\infty$ and $-1+1/p<s<1/p$. Similarly, if $u\in H^{s,p}(\Omega;\mathrm{curl})$ for some $1<p<\infty$ and $-1+1/p<s<1/p$ then we can define $\nu\times u\in B^{p,p}_{s-\frac{1}{p}}(\partial\Omega,\Lambda^1T\mathcal{M})$ by

$$\langle\nu\times u,\mathrm{Tr}\ \varphi\rangle:=\iint_\Omega[\langle\mathrm{curl}\,u,\varphi\rangle-\langle u,\mathrm{curl}\,\varphi\rangle]\,d\mathrm{Vol},\tag{2.28}$$

for any $\varphi\in L^{p'}_{1-s}(\Omega,\Lambda^1T\mathcal{M})$, $1/p+1/p'=1$. Furthermore, it follows from (2.24), (2.28) that the operator

$$\nu\times\ :H^{s,p}(\Omega;\mathrm{curl})\longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega,\Lambda^1T\mathcal{M})\tag{2.29}$$

is bounded, i.e.

$$\|\nu\times u\|_{B^{p,p}_{s-\frac{1}{p}}(\partial\Omega,\Lambda^1T\mathcal{M})}\leq C(\|u\|_{L^p_s(\Omega,\Lambda^1T\mathcal{M})}+\|\,\mathrm{curl}\ u\|_{L^p_s(\Omega,\Lambda^1T\mathcal{M})})\tag{2.30}$$

as long as $1<p<\infty$ and $-1+1/p<s<1/p$. Parenthetically, let us also point out that if, e.g., $u\in C^1(\overline{\Omega},\Lambda^1T\mathcal{M})$ then $\nu\times u$, $\nu\cdot u$ coincide with the usual (pointwise) cross product and dot product with ν, respectively.

Our immediate goal is to describe the images of the operators (2.26) and (2.29).

Proposition 2.1 *Let $\Omega\subset\mathcal{M}$ be a connected Lipschitz domain and fix $1<p<\infty$, $-1+1/p<s<1/p$. Then for any $f\in B^{p,p}_{s-1/p}(\partial\Omega)$ with $\int_{\partial\Omega}f\,d\sigma=0$ there exists $u\in H^{s,p}(\Omega;\mathrm{div})$ with $\mathrm{div}\,u=0$ and $\nu\cdot u=f$. Furthermore, matters can be arranged so that $\|u\|_{L^p_s(\Omega,\Lambda^1T\mathcal{M})}\leq C\|f\|_{B^{p,p}_{s-1/p}(\partial\Omega)}$ for some constant $C=C(\partial\Omega,p,s)>0$.*

As a corollary, the operator in (2.26) is always onto.

In order to prove the above result we need a lemma of independent interest.

Lemma 2.2 *Let $\Omega\subset\mathcal{M}$ be Lipschitz and suppose that $1<p<\infty$. Then, for each $s>-1+1/p$, the gradient*

$$\nabla:\ L^p_s(\Omega)\longrightarrow L^p_{s-1}(\Omega,\Lambda^1T\mathcal{M})\tag{2.31}$$

is well-defined, bounded, and has closed range.

Proof. The well-definiteness and boundedness when $s\geq 1$ are known (cf. Proposition 2.18, p. 173 in **[32]**). Furthermore, when $0\leq s\leq 1$, we can interpolate between the standard cases $s=0$ and $s=1$. Thus, as far as the first part of the claim in the lemma is concerned, we are left with analyzing the situation when $-1+1/p<s<0$. In this case, for each $u\in L^p_s(\Omega)=(L^{p'}_{-s}(\Omega))^*$, with $1/p+1/p'=1$, $\psi\in C^\infty_o(\Omega,\Lambda^1T\mathcal{M})$, and with the pairing $\langle\cdot,\cdot\rangle$ considered in the distributional sense, we write

$$\begin{aligned}|\langle\nabla u,\psi\rangle| &= |\langle u,\mathrm{div}\,\psi\rangle|\\ &\leq \|u\|_{(L^{p'}_{-s}(\Omega))^*}\|\mathrm{div}\,\psi\|_{L^{p'}_{-s}(\Omega)}\\ &\leq C\|u\|_{(L^{p'}_{-s}(\Omega))^*}\|\psi\|_{L^{p'}_{1-s}(\Omega,\Lambda^1T\mathcal{M})}.\end{aligned}\tag{2.32}$$

Since $C^\infty_o(\Omega,\Lambda^1 T\mathcal{M})\hookrightarrow L^{p'}_{1-s,0}(\Omega,\Lambda^1 T\mathcal{M})$ densely and −for the range of indices we are considering− this latter space is the closure of the former in the $L^{p'}_{1-s}$-norm, we conclude from (2.32) that ∇u belongs to the space $(L^{p'}_{1-s,0}(\Omega,\Lambda^1 T\mathcal{M}))^*=L^p_{s-1}(\Omega,\Lambda^1 T\mathcal{M})$, as desired.

Turning our attention to the last claim in the lemma, it suffices to treat the case when $-1+1/p<s\leq 1$ (cf., e.g., p. 173 in **[32]** for $s\geq 1$). Now, since $\dim \mathrm{Ker}\,(\nabla;L^p_s(\Omega))<\infty$, the closedness of the range of (2.31) is equivalent to the validity of the estimate

$$\|u\|_{L^p_s(\Omega)}\leq C\|\nabla u\|_{L^p_{s-1}(\Omega,\Lambda^1 T\mathcal{M})}+\|\mathrm{Comp}\,u\|, \tag{2.33}$$

uniformly for $u\in L^p_s(\Omega)$, where Comp denotes a compact operator on $L^p_s(\Omega)$. In turn, this estimate localizes so there is no loss of generality assuming that Ω is a bounded, Euclidean Lipschitz domain, which is starlike with respect to some ball B. In this context, as it follows from the discussion in the Appendix, if $\theta\in C^\infty_o(B)$, $\int\!\!\int\theta=1$, then there exists a linear, bounded operator $\mathcal{J}:L^q_{\alpha,0}(\Omega)\to L^q_{\alpha+1,0}(\Omega)$, for $1<q<\infty$, $-2+1/q<\alpha<1/q$, such that $\mathcal{J}\varphi\in C^\infty_o(\Omega,\Lambda^1\mathbb{R}^3)$ for any $\varphi\in C^\infty_o(\Omega)$ and $\mathrm{div}\,\mathcal{J}\varphi=\varphi-\theta(\int\!\!\int\varphi)$ for any $\varphi\in C^\infty_o(\Omega)$. Note that $1/p+1/p'=1$ and $-1+1/p<s<1+1/p$ ensures that $\mathcal{J}:L^{p'}_{-s,0}(\Omega)\to L^{p'}_{1-s,0}(\Omega)$. Then, for any $\varphi\in C^\infty_o(\Omega)$, we have

$$\begin{aligned}
|\langle u,\varphi\rangle| &\leq |\langle u,\mathrm{div}\,\mathcal{J}\varphi\rangle|+|\langle u,\theta\rangle|\,|\langle\varphi,1\rangle|\\
&\leq |\langle\nabla u,\mathcal{J}\varphi\rangle|+\|\mathrm{Comp}\,u\|\,\|\varphi\|_{L^{p'}_{-s,0}(\Omega)}\\
&\leq \|\nabla u\|_{L^p_{s-1}(\Omega,\Lambda^1 T\mathcal{M})}\|\mathcal{J}\varphi\|_{L^{p'}_{1-s,0}(\Omega,\Lambda^1 T\mathcal{M})}+\|\mathrm{Comp}\,u\|\,\|\varphi\|_{L^{p'}_{-s,0}(\Omega)}\\
&\leq C(\|\nabla u\|_{L^p_{s-1}(\Omega,\Lambda^1 T\mathcal{M})}+\|\mathrm{Comp}\,u\|)\|\varphi\|_{L^{p'}_{-s,0}(\Omega)}.
\end{aligned} \tag{2.34}$$

Since $C^\infty_o(\Omega)$ is dense in $L^{p'}_{-s,0}(\Omega)$, we see that $u\in\Big(L^{p'}_{s-1,0}(\Omega)\Big)^*=L^p_s(\Omega)$ and (2.33) holds. The desired conclusion follows. □

For further use, let us point out that the above reasoning also shows that for any distribution u in the Lipschitz domain Ω, the following implication holds:

$$1<p<\infty,\ s>-1+\tfrac{1}{p}\ \text{ and }\ \nabla u\in L^p_{s-1}(\Omega,\Lambda^1 T\mathcal{M})\Longrightarrow u\in L^p_s(\Omega). \tag{2.35}$$

Now we are ready for the

Proof of Proposition 2.1. Since $f\in B^{p,p}_{s-1/p}(\partial\Omega)=\Big(B^{p',p'}_{1-s-1/p'}(\partial\Omega)\Big)^*$ with $1/p+1/p'=1$, and $\mathrm{Tr}:L^{p'}_{1-s}(\Omega)\to B^{p',p'}_{1-s-1/p'}(\partial\Omega)$, it follows that $\Phi:=f\circ\mathrm{Tr}$ belongs to the space $\Big(L^{p'}_{1-s}(\Omega)\Big)^*=L^p_{s-1,0}(\Omega)$ and satisfies $\langle\Phi,1\rangle=0$. In other words, Φ is a bounded linear functional on $L^{p'}_{1-s}(\Omega)/\mathbb{R}$. In turn, this can be thought of as a closed subspace in $L^{p'}_{-s}(\Omega,\Lambda^1 T\mathcal{M})$ via the identification $u\equiv\nabla u$. That this works is guaranteed by Lemma 2.2. Thus, by Hahn-Banach theorem, there exists $u\in\Big(L^{p'}_{-s}(\Omega)\Big)^*=L^p_s(\Omega)$ such that

$$\langle u,\nabla\varphi\rangle=\langle\Phi,\varphi\rangle=\langle f,\mathrm{Tr}\,\varphi\rangle,\qquad\forall\,\varphi\in L^{p'}_{1-s}(\Omega). \tag{2.36}$$

Choosing $\varphi\in C^\infty_o(\Omega)$ in the above identity entails $\mathrm{div}\,u=0$, so $u\in H^{s,p}(\Omega;\mathrm{div})$ is divergence-free. Then another look at (2.36) reveals that $\nu\cdot u=f$, as desired. □

Later on, it is going to be important to extend elements from $H^{s,p}(\Omega;\mathrm{div})$, $H^{s,p}(\Omega;\mathrm{curl})$ to distributions in $H^{s,p}(\mathcal{M};\mathrm{div})$ and $H^{s,p}(\mathcal{M};\mathrm{curl})$, respectively. For the time being, we prove the following.

Lemma 2.3 *Let Ω, D be two arbitrary Lipschitz subdomains of $\mathcal{M}$ and assume that $1<p<\infty$, $-1+1/p<s<1/p$. Then*

$$u\in H^{s,p}(\Omega;\mathrm{div}),\ \nu\cdot u=0 \text{ on } D\cap\partial\Omega\Longrightarrow(\mathrm{div}\,\tilde{u})|_D=(\widetilde{\mathrm{div}\,u})|_D, \tag{2.37}$$

$$u\in H^{s,p}(\Omega;\mathrm{curl}),\ \nu\times u=0 \text{ on } D\cap\partial\Omega\Longrightarrow(\mathrm{curl}\,\tilde{u})|_D=(\widetilde{\mathrm{curl}\,u})|_D. \tag{2.38}$$

Proof. This follows from (2.25), (2.28). □

While the operator in (2.26) is onto, this is clearly not the case for (2.29). Since the range of this latter operator is going to be of fundamental importance for us in the sequel, for $1<p<\infty$ and $-1+1/p<s<1/p$, we introduce

$$TH^p_s(\partial\Omega):=\Big\{f\in B^{p,p}_{s-\frac{1}{p}}(\partial\Omega,\Lambda^1T\mathcal{M});\ f=\nu\times u \text{ for some } u\in H^{s,p}(\Omega;\mathrm{curl})\Big\}. \tag{2.39}$$

Now, if we define

$$H^{s,p}_{\bullet}(\Omega;\mathrm{div}):=\{u\in H^{s,p}(\Omega;\mathrm{div});\ \nu\cdot u=0\}, \tag{2.40}$$

$$H^{s,p}_{\times}(\Omega;\mathrm{curl}):=\{u\in H^{s,p}(\Omega;\mathrm{curl});\ \nu\times u=0\}, \tag{2.41}$$

(i.e. the null spaces of (2.26) and (2.29), respectively) then, at the level of quotient spaces,

$$\nu\times\ :H^{s,p}(\Omega;\mathrm{curl})\Big/H^{s,p}_{\times}(\Omega;\mathrm{curl})\longrightarrow TH^p_s(\partial\Omega) \tag{2.42}$$

becomes an algebraic isomorphism. Thus, it is natural to endow the space (2.39) with the norm for which (2.42) is also topological, i.e.

$$\|f\|_{TH^p_s(\partial\Omega)}:=\inf\Big\{\|u\|_{L^p_s(\Omega,\Lambda^1T\mathcal{M})}+\|\mathrm{curl}\,u\|_{L^p_s(\Omega,\Lambda^1T\mathcal{M})};\ f=\nu\times u,\ u\in H^{s,p}(\Omega;\mathrm{curl})\Big\}. \tag{2.43}$$

Some of the basic properties of the spaces just introduced are summarized in the following proposition.

Proposition 2.4 *Let Ω be an arbitrary Lipschitz subdomain of $\mathcal{M}$ and assume that $1<p<\infty$, $-1+1/p<s<1/p$. Also, let $1<p'<\infty$ be the conjugate exponent of p. Then the following hold.*

(i) $TH^p_s(\partial\Omega)$ is a reflexive Banach space, continuously embedded into $B^{p,p}_{s-\frac{1}{p}}(\partial\Omega,\Lambda^1T\mathcal{M})$.

(ii) $\nu\times C^\infty(\bar{\Omega},\Lambda^1T\mathcal{M})\Big|_{\partial\Omega}\hookrightarrow TH^p_s(\partial\Omega)$ continuously and densely.

(iii) *The mapping*

$$\nu\times\ : TH^{p'}_{-s}(\partial\Omega) \longrightarrow (TH^p_s(\partial\Omega))^* \tag{2.44}$$

defined by

$$\langle \nu\times f, g\rangle := \langle \operatorname{curl} u, w\rangle - \langle u, \operatorname{curl} w\rangle \tag{2.45}$$

for $u\in H^{-s,p'}(\Omega;\operatorname{curl})$ *with* $f=\nu\times u$*, and* $w\in H^{s,p}(\Omega;\operatorname{curl})$ *with* $g=\nu\times w$ *is well-defined and bounded, in fact an isomorphism.*

(iv) *The inverse of the operator (2.44) is* $\nu\times\cdot\ : (TH^p_s(\partial\Omega))^*\to TH^{p'}_{-s}(\partial\Omega)$*, defined by* $\nu\times\Phi := -f$ *if* $\Phi\in(TH^p(\partial\Omega))^*$ *is of the form* $\Phi=\nu\times f$*,* $f\in TH^{p'}_{-s}(\partial\Omega)$*. In particular, we have* $(\nu\times\cdot)^{-1}=-\nu\times\cdot$ *and* $(\nu\times\cdot)^*=-\nu\times\cdot$ *both on the scale* $(TH^p_s(\partial\Omega))^*$ *as well as on the scale* $TH^p_s(\partial\Omega)$*. Finally, with the above conventions,*

$$\langle \operatorname{curl} u, v\rangle = \langle u, \operatorname{curl} v\rangle - \langle \nu\times u, \nu\times(\nu\times v)\rangle \tag{2.46}$$

for any $u\in H^{s,p}(\Omega;\operatorname{curl})$*,* $v\in H^{-s,p'}(\Omega;\operatorname{curl})$*.*

Proof. That (2.39) is Banach follows from (2.42) and definitions, whereas the fact that (2.39) is reflexive will be a consequence of *(iii)*.

Going further, *(ii)* is a direct consequence of the following approximation result:

$$\begin{aligned} &\forall\, u\in H^{s,p}(\Omega;\operatorname{curl}) \Longrightarrow \exists\, u_\varepsilon\in C^\infty(\bar\Omega,\Lambda^1 T\mathcal{M})\\ &\text{with } u_\varepsilon\to u \text{ in } L^p_s(\Omega,\Lambda^1 T\mathcal{M}) \text{ and } \operatorname{curl} u_\varepsilon\to\operatorname{curl} u \text{ in } L^p_s(\Omega,\Lambda^1 T\mathcal{M}).\end{aligned} \tag{2.47}$$

This, so we claim, is valid as long as $1<p<\infty$ and $-1+1/p<s<1/p$. In order to see this, observe that, by using a partition of unity argument, there is no loss of generality to assume that $\Omega\subset\mathbb{R}^3$ is a bounded, Euclidean Lipschitz domain and $\operatorname{supp} u\cap\partial\Omega$ is small enough so that it is possible to construct an open cone Γ centered at the origin for which

$$\Gamma + (\partial\Omega\cap\operatorname{supp} u)\subseteq \mathbb{R}^3\setminus\overline{\Omega}. \tag{2.48}$$

Next, let φ be a smooth, compactly supported function in $\mathbb{R}^3$, having integral one and such that $\operatorname{supp}\varphi\subseteq\Gamma$. As usual, set $\varphi_\varepsilon := \varepsilon^{-3}\varphi(\cdot\varepsilon^{-1})$ for $\varepsilon>0$, and let $U\in L^p_s(\mathbb{R}^3,\Lambda^1\mathbb{R}^3)$ be so that $U|_\Omega=u$, $\operatorname{supp} U\subseteq\bar\Omega$. Finally, introduce $u_\varepsilon := (\varphi_\varepsilon * U)|_\Omega\in C^\infty(\bar\Omega,\Lambda^1\mathbb{R}^3)$. Relying on the boundedness of the operator (2.21), it follows that $u_\varepsilon\to u$ in $L^p_s(\Omega,\Lambda^1\mathbb{R}^3)$. Now, if $W\in L^p_s(\mathbb{R}^3,\Lambda^1\mathbb{R}^3)$ is such that $W|_\Omega=\operatorname{curl} u$, $\operatorname{supp} W\subseteq\bar\Omega$, then $\operatorname{curl} U = W+\Sigma$, where Σ is a distribution supported on $\partial\Omega$. Thus, by (2.48),

$$\operatorname{supp}(\Sigma*\varphi_\varepsilon)\subseteq\Gamma+(\partial\Omega\cap\operatorname{supp} u)\subseteq\mathbb{R}^3\setminus\overline{\Omega}. \tag{2.49}$$

Consequently, $\operatorname{curl} u_\varepsilon = (\varphi_\varepsilon*\operatorname{curl} U)|_\Omega = (\varphi_\varepsilon * W)|_\Omega\to\operatorname{curl} u$ in $L^p_s(\Omega,\Lambda^1\mathbb{R}^3)$, once again thanks to the boundedness of (2.21) for the range of indices we are considering here. This, in concert with (2.19), justifies (2.47) and finishes the proof of *(ii)*.

As for *(iii)*, it is clear that the map (2.44) is well-defined, linear, bounded and one-to-one. There remains to shows that it is also onto. With this goal in mind, let $\theta\in(TH^p_s(\partial\Omega))^*$ be arbitrary and consider $\hat\theta : H^{s,p}(\Omega;\operatorname{curl})\to\mathbb{R}$ defined by $\hat\theta(u) := \theta(\nu\times u)$. Via the identification

$$H^{s,p}(\Omega;\mathrm{curl}) \ni u \mapsto (u, \mathrm{curl}\, u) \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}) \oplus L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \tag{2.50}$$

we can regard $H^{s,p}(\Omega;\mathrm{curl})$ as a (closed) subspace of $L^p_s(\Omega, \Lambda^1 T\mathcal{M}) \oplus L^p_s(\Omega, \Lambda^1 T\mathcal{M})$. In this scenario, the Hahn-Banach theorem and (2.50) allow us to conclude that there exist $v_1, v_2 \in L^{p'}_{-s}(\Omega, \Lambda^1 T\mathcal{M})$ such that

$$\hat{\theta}(u) = \langle v_1, \mathrm{curl}\, u\rangle - \langle v_2, u\rangle, \qquad \forall\, u \in H^{s,p}(\Omega;\mathrm{curl}). \tag{2.51}$$

Note that choosing $u \in C^\infty_o(\Omega, \Lambda^1 T\mathcal{M})$ yields $\mathrm{curl}\, v_1 = v_2$. In particular $v := v_1 \in H^{-s,p'}(\Omega;\mathrm{curl})$. Utilizing this back in (2.51) gives that $\theta(\nu \times u) = \langle \nu \times (\nu \times v_1), \nu \times u\rangle$, for each u which, in turn, entails $\nu \times (\nu \times v_1) = \theta$. Hence, the map (2.44) is onto and this finishes the proof of (iii). Finally, (iv) is implicit in what we have proved so far. □

An important issue whose discussion is postponed at the moment is whether the space $TH^p_s(\partial\Omega)$ ultimately depends only on $\partial\Omega$ (and not on Ω itself, as (2.39) may initially suggest). This aspect will be addressed in §3.3.

2.4 The surface divergence and related operators

In this subsection, we discuss the surface divergence operator. First, at the L^p-level with $1 < p < \infty$, we set

$$L^p_{\tan}(\partial\Omega) := \{f \in L^p(\partial\Omega, \Lambda^1 T\mathcal{M});\ \langle \nu, f\rangle = 0 \text{ a.e. on } \partial\Omega\}, \tag{2.52}$$

and introduce

$$\mathrm{Div} : L^p_{\tan}(\partial\Omega) \to L^p_{-1}(\partial\Omega), \tag{2.53}$$

by requiring

$$\int_{\partial\Omega} g\,(\mathrm{Div}\, f)\, d\sigma = -\int_{\partial\Omega} \langle f, \nabla_{\tan} g\rangle\, d\sigma, \tag{2.54}$$

for each $f \in L^p_{\tan}(\partial\Omega)$, and $g \in L^{p'}_1(\partial\Omega) = (L^p_{-1}(\partial\Omega))^*$, $1/p + 1/p' = 1$. A space which is going to be important for us in the sequel is

$$L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) := \{f \in L^p_{\tan}(\partial\Omega);\ \mathrm{Div}\, f \in L^p(\partial\Omega)\}, \tag{2.55}$$

$1 < p < \infty$, which we equip with the natural norm

$$\|f\|_{L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)} := \|f\|_{L^p(\partial\Omega, \Lambda^1 T\mathcal{M})} + \|\mathrm{Div}\, f\|_{L^p(\partial\Omega)}. \tag{2.56}$$

A closed subspace of (2.55) is

$$L^{p,o}_{\tan}(\partial\Omega) := \{f \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega);\ \mathrm{Div}\, f = 0\}. \tag{2.57}$$

A convenient way to extend the definition of the surface divergence in context of (2.39) is to let

$$\mathrm{Div} : TH^p_s(\partial\Omega) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega), \quad 1 < p < \infty,\ -1 + 1/p < s < 1/p, \tag{2.58}$$

act according to

$$\operatorname{Div} f := -\nu \cdot (\operatorname{curl} u), \tag{2.59}$$

if $f \in TH^p_s(\partial\Omega)$ is of the form $f = \nu \times u$, for some $u \in H^{s,p}(\Omega; \operatorname{curl})$. From definitions, it is immediate that the operator Div above is well-defined, linear and bounded. Moreover, Div defined in (2.53)-(2.54) is compatible with Div defined in (2.58)-(2.59).

Another operator of interest for us is

$$\nu \times \nabla_{\tan} : B^{p,p}_{1-\frac{1}{p}+s}(\partial\Omega) \longrightarrow TH^p_s(\partial\Omega), \quad 1 < p < \infty, \ -1 + 1/p < s < 1/p, \tag{2.60}$$

defined by

$$(\nu \times \nabla_{\tan}) f := \nu \times (\nabla u), \tag{2.61}$$

if $f \in B^{p,p}_{1-\frac{1}{p}+s}(\partial\Omega)$, and if $u \in L^p_{s+1}(\Omega)$ is so that $f = \operatorname{Tr} u$ on $\partial\Omega$. Again, one can see that $\nu \times \nabla_{\tan}$ in (2.61) is well-defined, linear and bounded. Also, from (2.54), we have the factorization $\operatorname{Div} = -(\nu \times \nabla_{\tan})^* \circ (\nu \times \cdot)$. The kernel of the operator (2.58)-(2.59) is the space

$$TH^{p,o}_s(\partial\Omega) := \Big\{ f \in TH^p_s(\partial\Omega);\ \operatorname{Div} f = 0 \Big\}. \tag{2.62}$$

Clearly, since curl $\nabla = 0$, we get from (2.59) and (2.61) that

$$\operatorname{Div} \circ (\nu \times \nabla_{\tan}) = 0 \tag{2.63}$$

and, further,

$$\nu \times \nabla_{\tan} \left(B^{p,p}_{1-\frac{1}{p}+s}(\partial\Omega) \right) \subseteq TH^{p,o}_s(\partial\Omega). \tag{2.64}$$

However, a deeper issue is that of computing the *index* of the inclusion in (2.64); we plan to address this, along with other related problems, in the next section.

3 Fredholm first order differential operators

The main aim of this section is to identify certain convenient functional analytic settings in which some of the (first order, differential) operators introduced in §2 become Fredholm. We start by reviewing some concepts and results from algebraic topology.

3.1 Singular homology and sheaf theory

For a topological space $\mathcal{X}$, we set $H^\ell_{\rm sing}(\mathcal{X}; \mathbb{R})$ for the ℓ-th *singular homology group* of $\mathcal{X}$ over the reals, $\ell = 0, 1, ...$ (cf., e.g., **[38]**). Then $b_\ell(\mathcal{X})$, the ℓ-th *Betti number* of $\mathcal{X}$, is defined as the dimension of $H^\ell_{\rm sing}(\mathcal{X}; \mathbb{R})$. As is well known, $b_\ell(\mathcal{X})$, $\ell = 0, 1, ...$, are topological invariants of $\mathcal{X}$. In fact, $b_0(\mathcal{X})$ is simply the number of connected components of $\mathcal{X}$. The most important case for us is when $\mathcal{X}$ is a Lipschitz subdomain Ω of the (three dimensional) manifold $\mathcal{M}$, or its boundary. Then $b_\ell(\Omega)$, $b_\ell(\partial\Omega)$ are all finite, nonnegative integers and, according to Poincaré's duality theorem, $b_0(\partial\Omega) = b_2(\partial\Omega)$.

Next, we include a brief synopsis of some basic terminology together with some fundamental results from sheaf theory. Recall that a *sheaf* $\mathcal{F}$ on a topological space $\mathcal{X}$ is a double collection $\{\mathcal{F}(U), \rho^U_V\}_{V \subseteq U \subseteq \mathcal{X}}$, indexed by open subsets of $\mathcal{X}$, such that:

1. For each U open subset of $\mathcal{X}$, $\mathcal{F}(U)$ is a vector space (over the reals) whose elements are called *sections of $\mathcal{F}$ over U*;

2. For each pair $V \subseteq U$ of open subsets of $\mathcal{X}$, we have that $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ is a vector space homomorphism, called the *restriction map*, subject to the following two axioms. Firstly, ρ_U^U is the identity homomorphism of $\mathcal{F}(U)$, for any open set U. Secondly, for any triplet $W \subseteq V \subseteq U$ of open sets in $\mathcal{X}$,

$$\rho_W^U = \rho_V^U \circ \rho_W^V. \tag{3.1}$$

   In order to lighten notation, for each $\omega \in \mathcal{F}(U)$ and any $V \subseteq U$ open, we may write $\omega|_V$ in place of $\rho_V^U(\omega)$. By virtue of (3.1), this is without loss of information.

3. For each U, open subset of $\mathcal{X}$, any open covering $\{U_i\}_{i\in I}$ of U, and any family $\{\omega_i\}_{i\in I}$, $\omega_i \in \mathcal{F}(U_i)$, satisfying the compatibility condition

$$\omega_i|_{U_i\cap U_j} = \omega_j|_{U_i\cap U_j}, \text{ for any } i,j \in I \tag{3.2}$$

   there exists a unique section $\omega \in \mathcal{F}(U)$ such that $\omega|_{U_i} = \omega_i$ for any $i \in I$.

Given two sheaves $\mathcal{F}$, $\mathcal{G}$ over $\mathcal{X}$, a *sheaf homomorphism* $\vartheta : \mathcal{F} \to \mathcal{G}$ is a collection of vector space homomorphisms $\{\vartheta(U) : \mathcal{F}(U) \to \mathcal{G}(U)\}_{U\subseteq\mathcal{X}}$, indexed by open subsets of $\mathcal{X}$, which commute (in a natural way) with the restriction mappings. We define *supp* (ϑ) as the smallest closed set outside of which ϑ is the null sheaf homeomorphism.

A sheaf $\mathcal{F}$ over $\mathcal{X}$ is said to be *fine* if for each open, locally finite cover $\{U_i\}_{i\in I}$ of $\mathcal{X}$ there exists a family of sheaf homomorphisms $\vartheta_i : \mathcal{F} \to \mathcal{F}$, $i \in I$, such that

$$\operatorname{supp}(\vartheta_i) \subseteq U_i, \ \forall\, i \in I, \qquad \sum_i \vartheta_i = \text{identity}_{\mathcal{F}}. \tag{3.3}$$

Next, assume that $\mathcal{F}^0, \mathcal{F}^1, ...$ are sheaves over the topological space $\mathcal{X}$ and that, for $\ell = 0, 1, ...$, the mappings $\vartheta_\ell : \mathcal{F}^\ell \to \mathcal{F}^{\ell+1}$ are sheaf homomorphisms. Then

$$0 \longrightarrow \mathcal{F}^0 \xrightarrow{\ \vartheta_0\ } \mathcal{F}^1 \xrightarrow{\ \vartheta_1\ } \mathcal{F}^2 \xrightarrow{\ \vartheta_2\ } \cdots \tag{3.4}$$

is called *an exact complex* provided the following two conditions are true:

1. (*the complex condition*) $\vartheta_{\ell+1} \circ \vartheta_\ell = 0$ for $\ell = 0, 1, ...$;

2. (*the exactness condition*) for each fixed index $\ell = 1, 2, ...$, each point $x_0 \in \mathcal{X}$, each open neighborhood U of x_0 and any section $\omega \in \mathcal{F}^\ell(U)$ such that $\vartheta_\ell(U)(\omega) = 0$, there exist $V \subseteq U$, open neighborhood of x_o and a section $\omega' \in \mathcal{F}^{\ell-1}(V)$ for which $\vartheta_{\ell-1}(V)(\omega') = \omega|_V$.

To each sheaf $\mathcal{F}$ over a topological space $\mathcal{X}$ one can associate the so called *cohomology groups* $H^\ell(\mathcal{X}; \mathcal{F})$, $\ell = 0, 1, ...$ For a precise definition as well as for more extensive discussion we refer the reader to, e.g., **[66]**, **[25]**. In this respect, two results are going to be of basic importance for us in the sequel. To state the first one, for each open set $\mathcal{O} \subseteq \mathcal{X}$ consider

$$\mathbb{R}_{\mathcal{O}} := \{f : \mathcal{O} \to \mathbb{R};\ \text{locally constant function}\}, \tag{3.5}$$

and introduce the sheaf of locally constant functions on $\mathcal{X}$

$$\mathrm{LCF}_{\mathcal{X}} := \Big\{\mathbb{R}_{\mathcal{O}}\Big\}_{\mathcal{O}\ \text{open in}\ \mathcal{X}}. \tag{3.6}$$

Then, for any reasonable topological space $\mathcal{X}$, the ℓ-th cohomology group of $\mathrm{LCF}_{\mathcal{X}}$ is isomorphic to the $H^{\ell}_{\mathrm{sing}}(\mathcal{X};\mathbb{R})$, the classical ℓ-th singular homology group of $\mathcal{X}$ over the reals.

The other result referred to above is the deep theorem of de Rham which we present below in an abstract form, well suited for our purposes.

Theorem 3.1 (The Abstract de Rham Theorem) *Let $\mathcal{X}$ be a Hausdorff, para-compact topological space, and let $\mathcal{F}$ be a sheaf over $\mathcal{X}$. Also, let $\mathcal{L}^0, \mathcal{L}^1, ..$ be fine sheaves over $\mathcal{X}$ and, for $\ell = 0, 1, ...$, let $\vartheta_\ell : \mathcal{L}^\ell \to \mathcal{L}^{\ell+1}$ be sheaf homomorphisms such that the following is an exact complex:*

$$0 \longrightarrow \mathcal{F} \overset{\iota}{\hookrightarrow} \mathcal{L}^0 \xrightarrow{\vartheta_0} \mathcal{L}^1 \xrightarrow{\vartheta_1} \mathcal{L}^2 \xrightarrow{\vartheta_2} \cdots \tag{3.7}$$

(hereafter, ι denotes inclusion). Then

$$H^\ell(\mathcal{X};\mathcal{F}) \cong \frac{\mathrm{Ker}\,(\vartheta_\ell : \mathcal{L}^\ell(\mathcal{X}) \longrightarrow \mathcal{L}^{\ell+1}(\mathcal{X}))}{\mathrm{Im}\,(\vartheta_{\ell-1} : \mathcal{L}^{\ell-1}(\mathcal{X}) \longrightarrow \mathcal{L}^{\ell}(\mathcal{X}))}, \quad \ell = 1, 2, ... \tag{3.8}$$

See **[66]**, Theorem 5.25, pp. 185 for a proof; cf. also **[22]**.

3.2 The calculation of the index: boundary operators

Here we prove that, when considered between appropriate spaces, Div and $\nu \times \nabla_{\tan}$ are Fredholm operators with indices depending exclusively on the topology of Ω (and its boundary). We debut with the following theorem.

Theorem 3.2 *Assume that Ω is an arbitrary Lipschitz domain in $\mathcal{M}$ and suppose that $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Then*

$$\mathrm{Div} : \frac{TH^p_s(\partial\Omega)}{TH^{p,o}_s(\partial\Omega)} \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega), \tag{3.9}$$

$$\nu \times \nabla_{\tan} : B^{p,p}_{1+s-\frac{1}{p}}(\partial\Omega) \longrightarrow TH^{p,o}_s(\partial\Omega), \tag{3.10}$$

are Fredholm operators with indices $-b_0(\partial\Omega)$ and $b_0(\partial\Omega) - b_1(\partial\Omega)$, respectively.

Proof. First, we localize the definition of Div. More specifically, if $1 < p < \infty$, $-1 + 1/p < s < 1/p$, for U an arbitrary, fixed open subset of $\partial\Omega$, we define $TH^p_s(U)$ as the subspace of functions f locally belonging to $B^{p,p}_s(U, \Lambda^1 T\mathcal{M})$ and which enjoy the following property: for each $x \in U$ there exists $u \in H^{s,p}(\Omega; \mathrm{curl})$ such that f and $\nu \times u$ coincide in some open neighborhood of x in U. Clearly, each $TH^p_s(U)$ is an additive Abelian group and also a module over $\mathrm{Lip}(\partial\Omega)$, the algebra of Lipschitz functions on $\partial\Omega$. It follows that the family $TH^p_s := (TH^p_s(U))_U$, indexed by open subsets in $\partial\Omega$, is a sheaf on the topological space $\partial\Omega$. Given its local character and the existence of a (smooth) partition of unity, this sheaf is fine.

We shall also work with the fine sheaf $B^{p,p}_s := (B^{p,p}_s(U))_U$, again indexed by open subsets U of $\partial\Omega$, and where $1 < p < \infty$, $0 < |s| < 1$. This time, the group $B^{p,p}_s(U)$ is the collection of distributions $f \in (\mathrm{Lip}(\partial\Omega))^*$ such that $f\varphi \in B^{p,p}_s(\partial\Omega)$ for each $\varphi \in \mathrm{Lip}(\mathcal{M})$ with $\partial\Omega \cap \mathrm{supp}\,\varphi \subset U$. Thus, if $-1 + 1/p < s < 1/p$, $1 < p < \infty$, we can then define

$$\mathrm{Div}: TH^p_s(U) \longrightarrow B^{p,p}_{s-1/p}(U) \tag{3.11}$$

by setting $\mathrm{Div}\, f := -\nu \times \mathrm{curl}\, u$ near $x \in U$, if $f \in TH^p_s(U)$ is locally given by $\nu \wedge u$ near x, for some $u \in H^{s,p}(\Omega; \mathrm{curl})$. In the same context, we can also introduce

$$\nu \times \nabla_{\tan} : B^{p,p}_{1+s-1/p}(U) \longrightarrow TH^p_s(U) \tag{3.12}$$

by asking that $(\nu \times \nabla_{\tan})g := \nu \times (\nabla w)$ near $x \in U$, if $g \in B^{p,p}_{1+s-1/p}(U)$ is locally given by $\mathrm{Tr}\, w$ near x, for some $w \in L^p_{s+1}(\Omega)$.

Going further, observe that we have a natural sequence of sheaf morphisms

$$0 \longrightarrow \mathrm{LCF}_{\partial\Omega} \xrightarrow{\ \iota\ } B^{p,p}_{1+s-1/p} \xrightarrow{\nu\times\nabla_{\tan}} TH^p_s \xrightarrow{\mathrm{Div}} B^{p,p}_{s-1/p} \longrightarrow 0. \tag{3.13}$$

Here $\mathrm{LCF}_{\partial\Omega}$ stands for the sheaf of germs of locally constant functions on $\partial\Omega$, and ι is the natural inclusion operator.

Since $\mathrm{Div} \circ (\nu \times \nabla_{\tan}) = 0$, the above is a *complex*. In fact, so we claim, (3.13) provides a *fine resolution* of the sheaf $\mathrm{LCF}_{\partial\Omega}$. The essential ingredient in the proof of this claim is the *acyclicity* of the complex (3.13). Granted this, de Rham's theorem applies to our context and gives that

$$\frac{TH^{p,o}_s(\partial\Omega)}{\nu \times \nabla_{\tan}\Big(B^{p,p}_{1+s-1/p}(\partial\Omega)\Big)} \cong H^1_{\mathrm{sing}}(\partial\Omega; \mathbb{R}), \qquad \frac{B^{p,p}_{s-1/p}(\partial\Omega)}{\mathrm{Div}(TH^p_s(\partial\Omega))} \cong H^2_{\mathrm{sing}}(\partial\Omega; \mathbb{R}), \tag{3.14}$$

for each $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. With this at hand, we may conclude that the operators (3.9)-(3.10) have finite dimensional kernels and cokernels. To show that they also have closed ranges, we rely on a general functional analytic result (perhaps folklore), to the effect that *if* $T : \mathrm{Dom}\,(T) \subseteq X \to Y$ *is a closed operator between two Banach spaces, with the property that* $\mathrm{Im}\, T$*, the image of* T*, has finite codimension in* Y*, then* $\mathrm{Im}\, T$ *is a closed subspace of* Y*.*

Thus, we are left with proving the acyclicity of the sheaf (3.13). It is not hard to see that this is equivalent to the following two claims:

$$\begin{aligned} &\forall\, x_o \in \partial\Omega \text{ and for any } f \in B^{p,p}_{s-1/p}(\partial\Omega) \Longrightarrow \\ &\exists\, u \in H^{s,p}(\Omega; \mathrm{curl}) \text{ with } \nu \cdot \mathrm{curl}\, u = f \text{ near } x_o, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\forall\, x_o \in \partial\Omega \text{ and } u \in H^{s,p}(\Omega; \mathrm{curl}) \text{ with } \nu \cdot \mathrm{curl}\, u = 0 \text{ near } x_o, \\ &\Longrightarrow \exists\, \varphi \in L^p_{1+s}(\Omega) \text{ so that } \nu \times u = \nu \times (\nabla\varphi) \text{ near } x_o. \end{aligned} \tag{3.16}$$

Consider first (3.15). Since the statement is local, there is no loss of generality assuming that $\int_{\partial\Omega} f\, d\sigma = 0$. In this context, Proposition 2.1 works and gives some $w \in H^{s,p}(\Omega; \mathrm{div})$ such that $\mathrm{div}\, w = 0$ in Ω and $f = \nu \cdot w$ on $\partial\Omega$. Let now $D \subset \Omega$ be a sufficiently small Lipschitz subdomain so that $\partial D \cap \partial\Omega$ is an open neighborhood of x_o in $\partial\Omega$. Furthermore, we can assume that the Poincaré type results proved in the Appendix are valid on D. This plus an application of the Hodge star isomorphism, and in concert with (2.35), then give that there exists $u \in L^p_{s+1}(D, \Lambda^1 T\mathcal{M}) \hookrightarrow H^{s,p}(D; \mathrm{curl})$ such that $\mathrm{curl}\, u = w|_D$. In particular, $\nu \cdot \mathrm{curl}\, u = \nu \cdot w = f$ near x_o, which justifies (3.15).

As for (3.16), assume that $u \in H^{s,p}(\Omega; \mathrm{curl})$ satisfies $\nu \cdot \mathrm{curl}\, u = 0$ near $x_o \in \partial\Omega$. Fix now a suitably small Lipschitz domain D in $\mathcal{M}$ such that $x_o \in D$ and, in fact, $\nu \cdot \mathrm{curl}\, u = 0$ on $D \cap \partial\Omega$.

Lemma 2.3 and (2.18) ensure that $(\widetilde{\operatorname{curl} u})|_D \in H^{s,p}(D;\operatorname{div})$ and $\operatorname{div}(\widetilde{\operatorname{curl} u})|_D = 0$. Going further, the Poincaré type results – with preservation of smoothness and support – proved in the Appendix (together with an application of the Hodge star operator) give, much as before, that there exists $\omega \in L^p_{s+1}(D,\Lambda^1 T\mathcal{M})$ with $\operatorname{supp}\omega \subset D \cap \bar{\Omega}$ such that $\widetilde{\operatorname{curl} u} = \operatorname{curl}\omega$ in D. Thus, $\operatorname{Tr}\omega = 0$ on $D \cap \partial\Omega$ so that, further,

$$\nu \times \omega = 0 \quad \text{on} \quad D \cap \partial\Omega. \tag{3.17}$$

Observing that $\operatorname{curl}(u-\omega) = 0$ in $D \cap \Omega$ and, once again appealing to the results proved in the Appendix, yields that there exists $\varphi \in L^p_{s+1}(D \cap \Omega)$ with $u - \omega = \nabla\varphi$ in $D \cap \Omega$. Hence, by invoking (3.17), we see that $\nu \times u = \nu \times (\nabla\varphi)$ near x_o, which proves the claim (3.16). □

Theorem 3.3 *Assume that Ω is an arbitrary Lipschitz domain in $\mathcal{M}$. Then, for each $1 < p < \infty$, the operators*

$$\operatorname{Div} : \frac{L^{p,\operatorname{Div}}_{\tan}(\partial\Omega)}{L^{p,o}_{\tan}(\partial\Omega)} \longrightarrow L^p(\partial\Omega), \quad \nu \times \nabla_{\tan} : L^p_1(\partial\Omega) \longrightarrow L^{p,o}_{\tan}(\partial\Omega) \tag{3.18}$$

are Fredholm with indices $-b_0(\partial\Omega)$ and $b_0(\partial\Omega) - b_1(\partial\Omega)$, respectively. Also, for $1 < p < \infty$, the operator

$$\operatorname{Div} : \frac{L^p_{\tan}(\partial\Omega)}{L^{p,o}_{\tan}(\partial\Omega)} \longrightarrow L^p_{-1}(\partial\Omega) \tag{3.19}$$

is Fredholm with index $-b_0(\partial\Omega)$.

Proof. Once again, the treatment of the operators (3.18) relies on a convenient application of de Rham theory (cf. Theorem 3.1). The arguments closely parallel those in the proof of Theorem 3.2 and are somewhat simpler since, this time, the necessary Poincaré lemma is at the level of L^p spaces; cf. Appendix. We omit the details of this step.

Consider next the operator in (3.19). Clearly, this is one-to-one. To compute its range, observe that it suffices to consider the situation when the original domain is replaced by the space $L^p_{\tan}(\partial\Omega)/\nu \times \nabla_{\tan}(L^p_1(\partial\Omega))$. Assuming that this is the case, we claim that Div in (3.19) factorizes as

$$\operatorname{Div} : \frac{L^p_{\tan}(\partial\Omega)}{\nu \times \nabla_{\tan}(L^p_1(\partial\Omega))} \xrightarrow{\Phi} (L^{q,o}_{\tan}(\partial\Omega))^* \xrightarrow{(\nu\times\nabla_{\tan})^*} \left(\frac{L^q_1(\partial\Omega)}{\mathbb{R}_{\partial\Omega}}\right)^*, \tag{3.20}$$

where $1/p + 1/q = 1$, and Φ is the operator defined by

$$\langle \Phi([f]), g\rangle := -\int_{\partial\Omega} \langle \nu \times f, g\rangle \, d\sigma, \tag{3.21}$$

for any

$$[f] \in \frac{L^p_{\tan}(\partial\Omega)}{\nu \times \nabla_{\tan}(L^p_1(\partial\Omega))}, \quad g \in L^{q,o}_{\tan}(\partial\Omega). \tag{3.22}$$

Note that Φ is well-defined since, by (2.54), the right side of (3.21) always vanishes when one pairs elements of the form $f = \nu \times \nabla_{\tan} h$, for $h \in L^p_1(\partial\Omega)$, with $g \in L^{q,o}_{\tan}(\partial\Omega)$.

To justify the factorization (3.20), for $f \in L^p_{\tan}(\partial\Omega)$ and $g \in L^q_1(\partial\Omega)$ we write

$$
\begin{aligned}
\langle \mathrm{Div}([f]), [g] \rangle &= -\int_{\partial\Omega} \langle f, \nabla_{\tan} g \rangle \, d\sigma = \langle \Phi([f]), \nu \times \nabla_{\tan} g) \rangle \\
&= \langle (\nu \times \nabla_{\tan})^* \Phi([f]), [g] \rangle, \qquad (3.23)
\end{aligned}
$$

i.e., $\mathrm{Div} = (\nu \times \nabla_{\tan})^* \circ \Phi$, and (3.20) is proved.

Next, we claim that Φ is surjective. To see this, fix an arbitrary functional $\xi \in (L^{q,o}_{\tan}(\partial\Omega))^*$. By Hahn-Banach's extension theorem and Riesz's representation theorem there exists $f \in L^p_{\tan}(\partial\Omega)$ which satisfies $\xi(h) = \int_{\partial\Omega} \langle f, h \rangle \, d\sigma$ for any $h \in L^{q,o}_{\tan}(\partial\Omega)$. Thus, Φ sends $[-\nu \times f]$ into ξ and, hence, Φ is onto.

We now examine the second arrow in (3.20). Recall the definition (3.6). Thanks to what we know already from the first part of the current theorem, the operator

$$
\nu \times \nabla_{\tan} : \frac{L^q_1(\partial\Omega)}{\mathbb{R}_{\partial\Omega}} \longrightarrow L^{q,o}_{\tan}(\partial\Omega) \qquad (3.24)
$$

is one-to-one and Fredholm. Thus, its adjoint is onto.

Combining all these facts, we see that the image of Div in (3.20) is $(L^q_1(\partial\Omega)/\mathbb{R}_{\partial\Omega})^*$, i.e. the space $\{f \in L^p_{-1}(\partial\Omega);\ \langle f, \chi \rangle = 0,\ \forall \chi \in \mathbb{R}_{\partial\Omega}\}$. The desired conclusion follows. □

3.3 An intrinsic description of the tangential Sobolev spaces

The aim of this subsection is to provide an alternative description of the space (2.39) which is intrinsic, in the sense that it only depends on p, s and $\partial\Omega$ (and *not* on Ω itself). The starting point is the following Fredholmness result.

Proposition 3.4 *Let $\Omega \subset \mathcal{M}$ be an arbitrary Lipschitz domain, and assume that $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Then the operator*

$$
\nu \times \nabla_{\tan} : B^{p,p}_{s-1/p+1}(\partial\Omega) \longrightarrow B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^1 T\mathcal{M}) \qquad (3.25)
$$

has closed range and finite dimensional kernel; in particular, it is semi-Fredholm.

Proof. We shall show that (3.25) is bounded from below, modulo compact operators. Given that we are dealing with a first-order differential operator (so that the commutator between this and multiplication by a smooth cut-off function is a compact mapping in the current context), makes this particular problem local in nature so there is no loss of generality in assuming that Ω has trivial topology. In this latter context, the proof of (3.15) shows that if $g \in (B^{p,p}_{s-1/p+1}(\partial\Omega))^* = B^{p',p'}_{-s-1/p'}(\partial\Omega)$, $1/p + 1/p' = 1$, is such that $\langle g, 1 \rangle = 0$, then there exists $u \in L^{p'}_{1-s}(\Omega, \Lambda^1 T\mathcal{M})$ so that $\nu \cdot \mathrm{curl}\, u = g$ and $\|u\|_{L^{p'}_{1-s}(\Omega,\Lambda^1 T\mathcal{M})} \leq C \|g\|_{B^{p',p'}_{-s-1/p'}(\partial\Omega)}$. Thus, if $f \in B^{p,p}_{s-1/p+1}(\partial\Omega)$ is arbitrary and $\phi \in L^p_{s+1}(\Omega)$ is so that $\mathrm{Tr}\, \phi = f$, we may write

$$
\langle f, g \rangle = \langle f, \nu \cdot \mathrm{curl}\, u \rangle = \langle \nabla\phi, \mathrm{curl}\, u \rangle = \langle (\nu \times \nabla_{\tan}) f, \mathrm{Tr}\, u \rangle. \qquad (3.26)
$$

Consequently, for each g as above,

$$
|\langle f, g \rangle| \leq C \|(\nu \times \nabla_{\tan}) f\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})} \|g\|_{(B^{p,p}_{s-1/p+1}(\partial\Omega))^*}, \qquad (3.27)
$$

which, since g is arbitrary, further entails

$$\|f\|_{B^{p,p}_{s-1/p+1}(\partial\Omega)/\mathbb{R}} \leq C\|(\nu\times\nabla_{\tan})f\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}. \tag{3.28}$$

Thus, $\nu\times\nabla_{\tan}: B^{p,p}_{s-1/p+1}(\partial\Omega)/\mathbb{R}\to B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})$ is bounded from below, which proves that the operator (3.25) has closed range. Since its kernel is precisely $\mathbb{R}_{\partial\Omega}$, the claim in the proposition follows. □

Proposition 3.5 *For any $\Omega\subset\mathcal{M}$, arbitrary Lipschitz domain, and any indices $1<p<\infty$, $-1+1/p<s<1/p$, the space $TH^{p,o}_s(\partial\Omega)$ is closed in $B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})$. In particular, the norms $\|\cdot\|_{TH^p_s(\partial\Omega)}$ and $\|\cdot\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}$ are equivalent on $TH^{p,o}_s(\partial\Omega)$.*

Proof. The first claim follows from Proposition 3.4 and the first isomorphism in (3.14), in concert with a general functional analytic result, according to which *if X, Y are subspaces of a larger Banach space Z so that X is closed and Y/X is finite dimensional, then Y is also closed.* In turn, this can be readily handled using the fact that the sum between two subspaces of a Banach space, one of which is closed while the other is finite dimensional, is always closed.

As for the last claim in the proposition, we may use the fact that *if X, Y are Banach spaces so that the inclusion $X\hookrightarrow Y$ is continuous with closed range, then $\|\cdot\|_X\approx\|\cdot\|_Y$ on X* (itself, a simple consequence of the Inverse Mapping Theorem). □

Theorem 3.6 *Consider $\Omega\subset\mathcal{M}$ an arbitrary Lipschitz domain, and assume that $1<p<\infty$, $-1+1/p<s<1/p$. Then*

$$\|f\|_{TH^p_s(\partial\Omega)}\approx\|f\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}+\|\mathrm{Div}\, f\|_{B^{p,p}_{s-1/p}(\partial\Omega)}, \tag{3.29}$$

uniformly for $f\in TH^p_s(\partial\Omega)$. Thus, by (ii) in Proposition 2.4, the space $TH^p_s(\partial\Omega)$ can be described as the completion of $\nu\times C^\infty(\mathcal{M},\Lambda^1 T\mathcal{M})|_{\partial\Omega}$ in the norm $f\mapsto\|f\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}+\|\mathrm{Div}\, f\|_{B^{p,p}_{s-1/p}(\partial\Omega)}$.

Proof. The left-to-right inequality in (3.29) is straightforward from definitions, so we concentrate on the opposite one. To this end, given the local character of the problem, we may assume that Ω has trivial topology. With this in mind, for an arbitrary $f\in TH^p_s(\partial\Omega)$, the proof of (3.15) shows that we can find $u\in L^p_{1+s}(\Omega,\Lambda^1 T\mathcal{M})$ so that $\nu\cdot\mathrm{curl}\, u=\mathrm{Div}\, f$ and $\|u\|_{L^p_{1+s}(\Omega,\Lambda^1 T\mathcal{M})}\leq C\|\mathrm{Div}\, f\|_{B^{p,p}_{s-1/p}(\partial\Omega)}$. In particular,

$$\begin{aligned}\|\nu\times u\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}&\leq C\|\nu\times u\|_{TH^p_s(\partial\Omega)}\\ &\leq C\|u\|_{H^{s,p}(\Omega;\mathrm{curl})}\leq C\|\mathrm{Div}\, f\|_{B^{p,p}_{s-1/p}(\partial\Omega)}.\end{aligned} \tag{3.30}$$

Next, introduce $\phi:=f+\nu\times u\in TH^p_s(\partial\Omega)$ so that, in fact, $f\in TH^{p,o}_s(\partial\Omega)$ as $\mathrm{Div}\, f=0$. We may therefore write $\|f\|_{TH^p_s(\partial\Omega)}\leq\|\phi\|_{TH^p_s(\partial\Omega)}+\|\nu\times u\|_{TH^p_s(\partial\Omega)}$. On the other hand,

$$\begin{aligned}\|\phi\|_{TH^p_s(\partial\Omega)}&\approx\|\phi\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}\\ &\leq C(\|f\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}+\|\nu\times u\|_{B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})}),\end{aligned} \tag{3.31}$$

where the first equivalence is provided by Proposition 3.5. All in all, the right-to-left inequality in (3.29) follows, and the proof of the theorem is finished. □

Building on Lemma 2.3, we are now in a position to prove an extension theorem similar in spirit to known results in the case of ordinary Sobolev spaces. In the latter case, there holds

$$L^p_s(\Omega) = L^p_s(\mathcal{M})|_\Omega \quad \text{whenever } 1 < p < \infty,\ -1 + 1/p < s. \tag{3.32}$$

See **[64]** for a discussion.

Corollary 3.7 *Assume that $\Omega \subset \mathcal{M}$ is a Lipschitz domain and $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Then*

$$H^{s,p}(\Omega; \mathrm{curl}) = H^{s,p}(\mathcal{M}; \mathrm{curl})|_\Omega \quad \textit{and} \quad H^{s,p}(\Omega; \mathrm{div}) = H^{s,p}(\mathcal{M}; \mathrm{div})|_\Omega. \tag{3.33}$$

Proof. Set $\Omega_+ := \Omega$ and $\Omega_- := \mathcal{M} \setminus \bar{\Omega}$. It follows from Theorem 3.6 that

$$TH^p_s(\partial\Omega_+) = TH^p_s(\partial\Omega_-) \tag{3.34}$$

in the sense that they coincide as sets and their respective norms are equivalent. In particular, if $u \in H^{s,p}(\Omega; \mathrm{curl})$ then there exits $v \in H^{s,p}(\Omega_-; \mathrm{curl})$ with $\nu \times u = \nu \times v$. Using (2.22) and the distributional definition of curl, it is then straightforward to check that $\mathrm{curl}(\tilde{u} - \tilde{v}) = \widetilde{\mathrm{curl}\, u} - \widetilde{\mathrm{curl}\, v}$ on $\mathcal{M}$. Thus, $U := \tilde{u} - \tilde{v} \in H^{s,p}(\mathcal{M}; \mathrm{curl})$ is the desired extension of u, in the sense that $U|_\Omega = u$.

The case of the divergence operator is similar and simpler since, this time, we invoke Proposition 2.1 in lieu of Theorem 3.6. □

3.4 The calculation of the index: domain operators

This is a continuation of the theme in §3.2 except that the current focus is on the operators *curl*, *div*. Our main result is as follows.

Theorem 3.8 *Assume that Ω is an arbitrary Lipschitz domain in $\mathcal{M}$ and suppose that $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Then the operator*

$$\mathrm{curl} : \frac{H^{s,p}(\Omega; \mathrm{curl})}{\nabla\Big(L^p_{s+1}(\Omega)\Big)} \longrightarrow \Big\{u \in H^{s,p}(\Omega; \mathrm{div});\ \mathrm{div}\, u = 0\Big\} \tag{3.35}$$

is Fredholm with index $b_1(\Omega) - b_2(\Omega)$. Moreover, the operator

$$\mathrm{div} : \frac{H^{s,p}(\Omega; \mathrm{div})}{\mathrm{curl}\, H^{s,p}(\Omega; \mathrm{curl})} \longrightarrow L^p_s(\Omega) \tag{3.36}$$

is also Fredholm, with index $b_2(\Omega) - b_0(\Omega)$.

Proof. Consider the sheaf $H^{s,p}_{\mathrm{curl}}$ on the compact topological space $\bar{\Omega}$ which, informally speaking, consists of distributions which, together with their curls, are *locally* in L^p_s. In rigorous terms, this is defined as follows. For each open set $\mathcal{O} \subseteq \mathcal{M}$, the group $H^{s,p}_{\mathrm{curl}}(\mathcal{O} \cap \bar{\Omega})$ consists of (1-form valued) distributions u in $\mathcal{O} \cap \Omega$ with the following two properties: (i) $\varphi\, u \in H^{s,p}(\mathcal{O}; \mathrm{curl})$ for any $\varphi \in C^\infty_o(\mathcal{O})$, if $\mathcal{O} \subseteq \bar{\Omega}$, and (ii) for each $x_o \in \partial\Omega \cap \mathcal{O}$ there exists $\omega \subset \mathcal{O}$, open neighborhood of x_o, along with $U \in H^{s,p}(\omega; \mathrm{curl})$, so that $U|_{\omega\cap\Omega} = u|_{\omega\cap\Omega}$. Due to the local nature of these conditions, the sheaf we have just constructed is fine (in the sense of §3.1). Analogously, we define the fine sheaf $H^{s,p}_{\mathrm{div}}$.

Finally, for $-1 + 1/p < s$ we introduce the sheaf $L^p_{s,\mathrm{loc}}$, by defining the group $L^p_{s,\mathrm{loc}}(\mathcal{O} \cap \bar{\Omega})$ as the collection of all distributions u in $\mathcal{O} \cap \Omega$ such that: (i) $\varphi\, u \in L^p_s(\mathcal{O})$ for any $\varphi \in C^\infty_o(\mathcal{O})$, if $\mathcal{O} \subseteq \bar{\Omega}$;

(ii) for every $x_o \in \partial\Omega \cap \mathcal{O}$ there exists $\omega \subset \mathcal{O}$, open neighborhood of x_o and some $U \in L^p_s(\omega)$ so that $U|_{\omega\cap\Omega} = u|_{\omega\cap\Omega}$. Again, this is a fine sheaf.

These sheaves naturally give rise to the complex

$$0 \longrightarrow \mathrm{LCF}_{\bar{\Omega}} \xrightarrow{\ \iota\ } L^p_{s+1,\mathrm{loc}} \xrightarrow{\ \nabla\ } H^{s,p}_{\mathrm{curl}} \xrightarrow{\ \mathrm{curl}\ } H^{s,p}_{\mathrm{div}} \xrightarrow{\ \mathrm{div}\ } L^p_{s,\mathrm{loc}} \longrightarrow 0 \tag{3.37}$$

which, by the versions of Poincaré's lemma discussed in the Appendix is exact. In order to continue, it is important to observe that, thanks to Corollary 3.7 and (3.32), the *global* sections in $L^p_{s+1,\mathrm{loc}}$, $H^{s,p}_{\mathrm{curl}}$, $H^{s,p}_{\mathrm{div}}$, $L^p_{s,\mathrm{loc}}$ are, respectively, $L^p_{s+1}(\Omega)$, $H^{s,p}(\Omega;\mathrm{curl})$, $H^{s,p}(\Omega;\mathrm{div})$ and $L^p_s(\Omega)$. Consequently, by Theorem 3.1,

$$\dim\left[\frac{\{u \in H^{s,p}(\Omega;\mathrm{curl});\ \mathrm{curl}\, u = 0\}}{\nabla\Big(L^p_{s+1}(\Omega)\Big)}\right] = b_1(\Omega), \quad \dim\left[\frac{\{u \in H^{s,p}(\Omega;\mathrm{div});\ \mathrm{div}\, u = 0\}}{\mathrm{curl}\, H^{s,p}(\Omega;\mathrm{curl})}\right] = b_2(\Omega). \tag{3.38}$$

At this stage, the desired conclusion about (3.35) follows from (3.38). The case of div is similar. □

We conclude this section with a corollary of the above theorem, whose relevance is most apparent in §8. To state it, consider the realization of curl as a closed, unbounded operator on $L^p_s(\Omega, \Lambda^1 T\mathcal{M})$, i.e.

$$\mathrm{curl} : L^p_s(\Omega, \Lambda^1 T\mathcal{M}) \longrightarrow L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \quad \mathrm{Dom}\,(\mathrm{curl}) := H^{s,p}(\Omega;\mathrm{curl}). \tag{3.39}$$

Proposition 3.9 *Assume that $\Omega \subset \mathcal{M}$ is a Lipschitz domain and $1 < p < \infty$, $-1+1/p < s < 1/p$. Then the operator (3.39) has closed range. Its adjoint is*

$$\begin{aligned} &(\mathrm{curl})^* : L^{p'}_{-s}(\Omega, \Lambda^1 T\mathcal{M}) \longrightarrow L^{p'}_{-s}(\Omega, \Lambda^1 T\mathcal{M}), \qquad 1/p + 1/p' = 1, \\ &\mathrm{Dom}\,(\mathrm{curl})^* = H^{-s,p'}_{\times}(\Omega;\mathrm{curl}), \quad (\mathrm{curl})^* u = \mathrm{curl}\, u,\ \forall\, u \in H^{-s,p'}_{\times}(\Omega;\mathrm{curl}). \end{aligned} \tag{3.40}$$

In particular, the operator (3.40) has a closed range as well.

Proof. The fact that the operator (3.39) has closed range follows from Theorem 3.8, given that $\{u \in H^{s,p}(\Omega;\mathrm{div});\ \mathrm{div}\, u = 0\}$ is a closed subspace of $L^p_s(\Omega, \Lambda^1 T\mathcal{M})$.

That (3.40) is the dual of (3.39) is a consequence of (2.46). Finally, the last part in the statement of the proposition follows from what we have proved so far and standard functional analysis (cf. [**35**], [**22**]). □

4 Scalar layer potential operators

For an arbitrary, fixed $V \geq 0$, $V \in L^\infty(\mathcal{M})$, the operator

$$\Delta - V : L^2_1(\mathcal{M}) \longrightarrow L^2_{-1}(\mathcal{M}) \tag{4.1}$$

is invertible provided V is not identically zero (cf. [**46**]). When $V = 0$, then Δ^{-1} should be understood as

$$\Delta^{-1} : \{u \in L^2_{-1}(\mathcal{M});\ \langle u, 1\rangle = 0\} \longrightarrow L^2_1(\mathcal{M})/\mathbb{R}. \tag{4.2}$$

In either case, we denote by $E_V(x,y)$ the Schwartz kernel of $(\Delta - V)^{-1}$. It follows that $E_V(y,x) = E_V(x,y)$. Also, it has been proved in **[47]** that in local coordinates in which the metric tensor is given by $\sum g_{jk} dx_j \otimes dx_k$, we have the asymptotic expansion

$$E_V(x,y) = (\det(g_{jk}(y)))^{-1/2}[e_0(x-y,y) + e_1(x,y)], \tag{4.3}$$

where the main term is given by

$$e_0(z,y) := -(4\pi)^{-1}\Big(\sum g_{jk}(y) z_j z_k\Big)^{-1/2}, \tag{4.4}$$

and the remainder satisfies

$$|\nabla_x^j \nabla_y^k e_1(x,y)| \le C|x-y|^{-j-k}, \quad 0 \le j,k \le 1. \tag{4.5}$$

Next, for Ω arbitrary Lipschitz domain in $\mathcal{M}$, introduce the single and double layer potential operators, by

$$\mathcal{S}_V f(x) := \int_{\partial\Omega} E_V(x,y) f(y)\, d\sigma_y, \qquad x \notin \partial\Omega, \tag{4.6}$$

$$\mathcal{D}_V f(x) := \int_{\partial\Omega} \frac{\partial E_V}{\partial \nu_y}(x,y)\, f(y)\, d\sigma_y, \qquad x \notin \partial\Omega. \tag{4.7}$$

The boundary versions of these operators are

$$S_V f(x) := \int_{\partial\Omega} E_V(x,y) f(y)\, d\sigma_y, \qquad x \in \partial\Omega, \tag{4.8}$$

$$K_V f(x) := \text{p.v.} \int_{\partial\Omega} \frac{\partial E_V}{\partial \nu_y}(x,y)\, f(y)\, d\sigma_y, \qquad x \in \partial\Omega, \tag{4.9}$$

where p.v. indicates that the integral is taken in the principal value sense (i.e., removing small geodesic balls and passing to the limit). Moreover, we denote by K_V^* the formal transpose of K_V and by Π_V the volume potential associated with $\Delta - V$, i.e.

$$\Pi_V u(x) := \iint_\Omega E_V(x,y)\, u(y)\, d\text{Vol}_y, \qquad x \in \Omega. \tag{4.10}$$

Some of the main properties of the operators above are collected in the theorem below; for proofs, the reader is referred to **[40]**, **[47]**, **[48]**. To state this result we need one more piece of notation. Specifically, if u is defined in Ω, then $\mathcal{N}(u)$, the nontangential maximal function of u, is defined at boundary points by

$$\mathcal{N}(u)(x) := \sup\{|u(y)| : y \in \gamma(x)\}, \qquad x \in \partial\Omega, \tag{4.11}$$

where for some fixed, large $\kappa = \kappa(\partial\Omega) > 0$ and each $x \in \partial\Omega$, the nontangential approach region is defined by

$$\gamma(x) := \{y \in \Omega;\ \text{dist}\,(y,x) \le \kappa\, \text{dist}\,(y,\partial\Omega)\}. \tag{4.12}$$

Cf. **[14]**, **[65]**, **[46]** for more details. Here, we only want to mention that all restrictions to the boundary of $\partial\Omega$ are taken in the pointwise nontangential sense. That is,

$$u\Big|_{\partial\Omega}(x) := \lim_{y \in \gamma(x),\, y \to x} u(y), \quad x \in \partial\Omega. \tag{4.13}$$

Theorem 4.1 *Let $\Omega \subset \mathcal{M}$ be a Lipschitz domain. Then, for each potential $V \geq 0$, $V \in L^\infty(\mathcal{M})$,*

$$\Pi_V : L^p_s(\Omega) \longrightarrow L^p_{s+2}(\Omega), \quad 1 < p < \infty,\ -1 + 1/p < s < 1/p, \tag{4.14}$$

is well-defined and bounded. Also,

$$\mathcal{S}_V : L^p_{-s}(\partial\Omega) \longrightarrow B^{p,p\vee 2}_{-s+1+1/p}(\Omega), \qquad \mathcal{D}_V : L^p_s(\partial\Omega) \longrightarrow B^{p,p\vee 2}_{s+1/p}(\Omega), \tag{4.15}$$

are bounded operators for each $1 < p < \infty$, $0 \leq s \leq 1$. Hereafter, $p \vee 2 := \max\{p, 2\}$. Moreover, if $1 \leq p \leq \infty$ and $0 < s < 1$, then

$$\mathcal{S}_V : B^{p,p}_{-s}(\partial\Omega) \longrightarrow B^{p,p}_{-s+1+1/p}(\Omega), \qquad \mathcal{D}_V : B^{p,p}_s(\partial\Omega) \longrightarrow B^{p,p}_{s+1/p}(\Omega), \tag{4.16}$$

are also bounded operators. In fact, the same conclusion applies to

$$\mathcal{S}_V : B^{p,p}_{-s}(\partial\Omega) \longrightarrow L^p_{-s+1+1/p}(\Omega), \qquad \mathcal{D}_V : B^{p,p}_s(\partial\Omega) \longrightarrow L^p_{s+1/p}(\Omega), \tag{4.17}$$

provided $1 < p < \infty$ and $0 < s < 1$. Also, for $1 < p < \infty$,

$$\|\mathcal{N}(\nabla \mathcal{S}_V f)\|_{L^p(\partial\Omega)} + \|\mathcal{N}(\mathcal{D}_V f)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p)\|f\|_{L^p(\partial\Omega)}, \tag{4.18}$$

uniformly for $f \in L^p(\partial\Omega)$, while $\|\nabla\mathcal{N}(\mathcal{D}_V g)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p)\|g\|_{L^p_1(\partial\Omega)}$, uniformly for $g \in L^p_1(\partial\Omega)$. Next, for each $0 \leq s \leq 1$, $1 < p < \infty$, the operators

$$\begin{aligned} K_V &: L^p_s(\partial\Omega) \longrightarrow L^p_s(\partial\Omega), \\ K^*_V &: L^p_{-s}(\partial\Omega) \longrightarrow L^p_{-s}(\partial\Omega), \\ S_V &: L^p_{-s}(\partial\Omega) \longrightarrow L^p_{1-s}(\partial\Omega), \end{aligned} \tag{4.19}$$

are well-defined and bounded, and so are

$$\begin{aligned} K_V &: B^{p,p}_s(\partial\Omega) \longrightarrow B^{p,p}_s(\partial\Omega), \\ K^*_V &: B^{p,p}_{-s}(\partial\Omega) \longrightarrow B^{p,p}_{-s}(\partial\Omega), \\ S_V &: B^{p,p}_{-s}(\partial\Omega) \longrightarrow B^{p,p}_{1-s}(\partial\Omega), \end{aligned} \tag{4.20}$$

for $0 < s < 1$, $1 \leq p \leq \infty$. Also, with $\Omega_+ := \Omega$, $\Omega_- := \mathcal{M} \setminus \bar{\Omega}$ and ∂_ν denoting the boundary normal derivative,

$$\begin{aligned} \partial_\nu \mathcal{S}_V f|_{\partial\Omega_\pm} &= (\mp\tfrac{1}{2}I + K^*_V)f, \\ \mathcal{D}_V f|_{\partial\Omega_\pm} &= (\pm\tfrac{1}{2}I + K_V)f, \\ \mathcal{S}_V f|_{\partial\Omega_\pm} &= S_V f, \end{aligned} \tag{4.21}$$

for each $f \in L^p(\partial\Omega)$, $1 < p < \infty$.

Consider next the (regular) Dirichlet problem for the Laplace-Beltrami operator,

$$\Delta u = 0 \text{ in } \Omega,\ \mathcal{N}(\nabla u) \in L^q(\partial\Omega),\ u|_{\partial\Omega} = f \in L^q_1(\partial\Omega), \tag{4.22}$$

as well as the corresponding Neumann problem with L^p boundary data

$$\begin{aligned} &\Delta u = 0 \text{ in } \Omega,\ \mathcal{N}(\nabla u) \in L^q(\partial\Omega),\ \partial_\nu u = f \in L^q(\partial\Omega), \\ &\langle f, \chi|_{\partial\Omega}\rangle = 0,\ \langle u, \chi\rangle = 0,\ \forall\, \chi \in \mathbb{R}_\Omega. \end{aligned} \tag{4.23}$$

Following the work in **[65]**, **[14]**, it has been shown in **[46]**-**[49]** that (4.22) and (4.23) can be solved in the form

$$u = \mathcal{D}_V[(\tfrac{1}{2}I + K_V)^{-1} f] \ \text{ and } \ u = \mathcal{S}_V[(-\tfrac{1}{2}I + K_V^*)^{-1} f]\ , \text{ respectively,} \tag{4.24}$$

if $1 < q < 2 + \varepsilon$ for some $\varepsilon = \varepsilon(\Omega) > 0$. Here, we can take any $V \in L^\infty(\mathcal{M})$, $V \geq 0$, not identically zero in each component of Ω_-, and such that $\operatorname{supp} V \cap \bar{\Omega} = \emptyset$. In this connection, two parameters which are going to play an important role for us in the sequel are

$$\begin{aligned} q_\Omega := &\text{ the supremum of all } q\text{'s for which both} \\ &(4.22) \text{ and } (4.23) \text{ are well-posed in } \Omega_\pm, \end{aligned} \tag{4.25}$$

and

$$p_\Omega := \text{the Hölder conjugate exponent of } q_\Omega. \tag{4.26}$$

Our next theorem in this section deals with *invertibility* results for the harmonic layer potential operators on Sobolev-Besov spaces. For each Lipschitz domain $\Omega \subset \mathcal{M}$, let $\mathcal{P}_\Omega$ be the subregion of the unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$ consisting of points with coordinates $(s, 1/p)$, where $0 \leq s \leq 1$ and $1 \leq p \leq \infty$, satisfy either of the following three conditions:

$$\begin{aligned} &p_\Omega < p < (1 - \tfrac{1}{p_\Omega})^{-1} \quad \text{and} \quad 0 < s < 1; \\ &1 \leq p < p_\Omega \quad \text{and} \quad \tfrac{2}{p} - \tfrac{2}{p_\Omega} < s < 1; \\ &(1 - \tfrac{1}{p_\Omega})^{-1} < p \leq \infty \quad \text{and } 0 < s < \tfrac{2}{p} + \tfrac{2}{p_\Omega} - 1. \end{aligned} \tag{4.27}$$

Theorem 4.2 *Let Ω be an arbitrary Lipschitz domain in $\mathcal{M}$. Then, for each s, p as in (4.27), and with $1 \leq p' \leq \infty$ denoting the conjugate exponent of p, the operators*

(1) $\pm\frac{1}{2}I + K_V : B^{p,p}_s(\partial\Omega) \longrightarrow B^{p,p}_s(\partial\Omega)$,

(2) $\pm\frac{1}{2}I + K_V^* : B^{p',p'}_{-s}(\partial\Omega) \longrightarrow B^{p',p'}_{-s}(\partial\Omega)$,

(3) $S_V : B^{p',p'}_{-s}(\partial\Omega) \longrightarrow B^{p',p'}_{1-s}(\partial\Omega)$,

are Fredholm, of index zero. Moreover, if

$$0 < \tfrac{1}{p} - \tfrac{s}{q_\Omega} < \tfrac{1}{p_\Omega}, \quad 0 \leq s \leq 1, \quad 1 < p, p' < \infty, \quad \tfrac{1}{p} + \tfrac{1}{p'} = 1, \tag{4.28}$$

then so are

(4) $\pm\frac{1}{2}I + K_V : L^p_s(\partial\Omega) \longrightarrow L^p_s(\partial\Omega)$,

(6) $\pm\frac{1}{2}I + K_V^* : L^{p'}_{-s}(\partial\Omega) \longrightarrow L^{p'}_{-s}(\partial\Omega)$,

(6) $S_V : L^{p'}_{-s}(\partial\Omega) \longrightarrow L^{p'}_{1-s}(\partial\Omega)$.

Furthermore, assuming that $V > 0$ on a set of positive measure both in each connected component of Ω, as well as in each connected component of $\mathcal{M}\setminus\overline{\Omega}$, then all the above operators are in fact isomorphisms. In particular, that is the case when V is a positive constant.

Proof. We shall only outline the main steps. Let us first assume that $V \geq \kappa > 0$ on $\mathcal{M}$. From the definitions (4.25)-(4.26) and jump relations, it follows that for $s \in \{0,1\}$ and $p_\Omega < p < q_\Omega$ the operators $(4) - (6)$ are bounded from below. In fact, by virtue of Proposition 4.3 below and the theory for p near 2 from **[47]**, these operators are actually isomorphisms for the range of indices under discussion.

The next step is to consider the action of the operators in question on atomic Hardy spaces. To this end, recall that $\vartheta \in L^\infty(\partial\Omega)$ is called a $\mathfrak{H}^p_{\rm at}(\partial\Omega)$-atom if $\operatorname{supp}\vartheta \subseteq B_r(x_o)\cap\partial\Omega$ for some $x_o \in \partial\Omega$, $0 < r < \operatorname{diam}\Omega$, and $\int_{\partial\Omega}\vartheta\,d\sigma = 0$, $\|\vartheta\|_{L^\infty(\partial\Omega)} \leq r^{-2/p}$. Now, f is said to belong to $\mathfrak{H}^p_{\rm at}(\partial\Omega)$ provided it can be written in the form

$$f = \sum \lambda_j\vartheta_j, \quad \vartheta_j \text{ atom}, \quad \sum|\lambda_j|^p < \infty. \tag{4.29}$$

We also introduce

$$\|f\|_{\mathfrak{H}^p_{\rm at}(\partial\Omega)} := \inf\Big\{(\sum|\lambda_j|^p)^{1/p};\ f = \sum\lambda_j\vartheta_j,\ \vartheta_j \text{ atom}\Big\}. \tag{4.30}$$

This corresponds to the approach in **[8]** considering $\partial\Omega$ equipped with the measure $d\sigma$ and the geodesic distance as a space of homogeneous type. Then we can set

$$\mathfrak{h}^p_{\rm at}(\partial\Omega) := \mathfrak{H}^p_{\rm at}(\partial\Omega) + \mathbb{R}_{\partial\Omega} = \mathfrak{H}^p_{\rm at}(\partial\Omega) + L^q(\partial\Omega), \quad \forall\, q \in (1,\infty], \tag{4.31}$$

and equip it with the natural quasi-norm. The space $\mathfrak{h}^p_{\rm at}(\partial\Omega)$ is "local" in the sense that, under the action $f \mapsto \varphi f$, it is a module over $B^{\infty,\infty}_s(\partial\Omega)$, for any $2/p - 2 < s < 1$. As is well known, if $2/3 < p < 1$ then $\mathfrak{h}^p_{\rm at}(\partial\Omega)$ is only a quasi-Banach space and

$$(\mathfrak{h}^p_{\rm at}(\partial\Omega))^* = B^{\infty,\infty}_{2/p-2}(\partial\Omega). \tag{4.32}$$

Next, a (regular) $\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)$-atom, for $2/3 < p \leq 1$, is a function $\vartheta \in \mathrm{Lip}$ satisfying $\operatorname{supp}\vartheta \subseteq B_r(x_o)\cap\partial\Omega$ for some $x_o \in \partial\Omega$, $r \in (0, \operatorname{diam}\Omega]$, and $\|\partial_{\tan}\vartheta\|_{L^\infty(\partial\Omega)} \leq r^{-2/p}$. Then the space $\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)$ is defined as the ℓ^p-span of (regular) atoms, and is equipped with the natural quasi-norm $\|\cdot\|_{\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)}$. The crux of the matter is the existence of a finite constant $C = C(\partial\Omega) > 0$ such that, whenever $(\Delta - V)u = 0$ in Ω, then

$$\Big(\int_{\partial\Omega}|\mathcal{N}(\nabla u)|^p\,d\sigma\Big)^{1/p} \leq C\min\{\|\partial_\nu u\|_{\mathfrak{h}^p_{\rm at}(\partial\Omega)},\ \|u\|_{\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)}\}, \tag{4.33}$$

as long as

$$(\tfrac{1}{p_\Omega} + \tfrac{1}{2})^{-1} < p \leq 1. \tag{4.34}$$

A proof of (4.33) when $1 - \varepsilon < p \leq 1$, for some small $\varepsilon = \varepsilon(\partial\Omega) > 0$, can be found in **[47]** which, in turn, builds on **[14]**. This type of result is true in all space dimensions. For three-dimensional Lipschitz domains, the more precise form of the range of p's for which (4.33) holds, claimed in (4.34), can be obtained by reasoning as in Proposition 3.2 in **[43]** (cf. especially the estimate (3.30) *loc. cit.*), where the estimate $\left(\int_{\partial\Omega}|\mathcal{N}(\nabla u)|^p\,d\sigma\right)^{1/p}\leq C\,\|u\|_{\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)}$ is derived in the two-dimensional case. The range (4.34) follows by carefully keeping track of how the critical Dirichlet exponent, p_Ω, (referred to as $2-\varepsilon$ in **[43]**) intervenes in Proposition 3.1 of **[43]**, and by keeping in mind that, in the present setting, the ambient space is three-dimensional. No essentially new ideas are otherwise required to carry out this computation. It should, however, be noted that **[43]** has been inspired by the fundamental work in **[15]** where the case of the three-dimensional Lamé system is treated.

Next, (4.33)-(4.34) readily imply that $\|\partial_\nu u\|_{\mathfrak{h}^p_{\rm at}(\partial\Omega)}\approx\|u\|_{\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)}$ if $(\Delta-V)u=0$ and p is as in (4.34). Utilizing this for u expressed as single and double layer potentials (considered both in Ω_+ and Ω_-) leads, in concert with the jump relations (4.21), to the conclusion that

$$\begin{aligned}
&\pm\tfrac{1}{2}I+K_V:\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)\longrightarrow\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega),\\
&\pm\tfrac{1}{2}I+K_V^*:\mathfrak{h}^{p}_{\rm at}(\partial\Omega)\longrightarrow\mathfrak{h}^{p}_{\rm at}(\partial\Omega),\\
&S_V:\mathfrak{h}^{p}_{\rm at}(\partial\Omega)\longrightarrow\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)
\end{aligned}\tag{4.35}$$

are bounded from below, provided p is as in (4.34). At this point, Proposition 4.3 below applies and gives that the operators in (4.35) are in fact isomorphisms. Then, interpolation between (4.35) with $p=1$ and the operators $(4)-(6)$ with $s\in\{0,1\}$, $p_\Omega<p<q_\Omega$ yields the range (4.28).

Passing from atomic Hardy spaces to Besov spaces is then accomplished by taking Banach envelopes, a concept which we now briefly recall. In general, if X is a quasi-Banach space with a separating dual, then the completion of X in the Mackey topology (i.e. the strongest locally convex topology on X which produces the same topological dual as the original topology on X), is denoted by $\hat{X}$ and is called the Banach envelope of X. A key property is that, for a linear and bounded operator, the quality of being an isomorphism is preserved by applying "hat". Quite recently, it has been proved in **[39]** that if $\Omega\subset\mathcal{M}$ is a (three-dimensional) Lipschitz domain then

$$\widehat{\mathfrak{h}^{p}_{\rm at}(\partial\Omega)}=B^{1,1}_{2-2/p}(\partial\Omega),\qquad\widehat{\mathfrak{H}^{1,p}_{\rm at}(\partial\Omega)}=B^{1,1}_{3-2/p}(\partial\Omega),\qquad\forall\,p\in(\tfrac{2}{3},1).\tag{4.36}$$

This and duality then gives that

$$\begin{aligned}
&\pm\tfrac{1}{2}I+K_V:B^{1,1}_{3-2/p}(\partial\Omega)\longrightarrow B^{1,1}_{3-2/p}(\partial\Omega),\\
&\pm\tfrac{1}{2}I+K_V:B^{\infty,\infty}_{2/p-2}(\partial\Omega)\longrightarrow B^{\infty,\infty}_{2/p-2}(\partial\Omega),\\
&S_V:B^{1,1}_{2-2/p}(\partial\Omega)\longrightarrow B^{1,1}_{3-2/p}(\partial\Omega),\\
&S_V:(B^{1,1}_{3-2/p}(\partial\Omega))^*\longrightarrow B^{\infty,\infty}_{2/p-2}(\partial\Omega)
\end{aligned}\tag{4.37}$$

are isomorphisms, as long as p is as in (4.34). Then the full range (4.27) is covered based on what we have proved so far and interpolation.

Finally, for general $V\geq 0$, the Fredholmness of the operators $(1)-(6)$ is a consequence of what we have just proved and (4.3)-(4.5). □

Here is the auxiliary result used in the proof of Theorem 4.2.

Proposition 4.3 *Let $T:X_\theta\to Y_\theta$ be an operator acting on two complex interpolation scales of quasi-Banach spaces $\{X_\theta\}_{0<\theta<1}$, $\{Y_\theta\}_{0<\theta<1}$, with the property that $\cup_{0<\theta<1}X_\theta$ and $\cup_{0<\theta<1}Y_\theta$ are* *rich, in the sense of* **[34]** *(this latter property is automatically satisfied for Banach spaces). Also, suppose that either T is onto, or injective with closed range, for each $\theta \in (0,1)$, and that there exists $\theta_o \in (0,1)$ so that $T : X_{\theta_o} \to Y_{\theta_o}$ is an isomorphism. Then, $T : X_\theta \to Y_\theta$ is an isomorphism for any $0 < \theta < 1$.*

Proof. This is a version of Theorem 2.10 in **[34]**. □

Recall that $\mathcal{P}_\Omega$ refers to the region described as the collection of points $(s, 1/p)$ whose coordinates satisfy either of the three conditions in (4.27). Much as in **[20]**, **[48]**, from Theorem 4.1 and Theorem 4.2 we have the following immediate corollary (extending work in **[32]**).

Corollary 4.4 *Let $\Omega \subset \mathcal{M}$ be an arbitrary Lipschitz domain. Then for each point $(s, 1/p)$ in the region $\mathcal{P}_\Omega$ the Poisson problem with Dirichlet boundary condition*

$$u \in L^p_{s+\frac{1}{p}}(\Omega),\ \Delta u = f \in L^p_{s+\frac{1}{p}-2}(\Omega),\ \mathrm{Tr}\, u = g \in B^{p,p}_s(\partial\Omega), \tag{4.38}$$

is well-posed. Moreover, if $1/p + 1/p' = 1$, then

$$u \in L^{p'}_{1+\frac{1}{p'}-s}(\Omega),\ \Delta u = f \in \left(L^p_{s+\frac{1}{p}}(\Omega)\right)^*,\ \partial_\nu u = g \in B^{p',p'}_{-s}(\partial\Omega), \tag{4.39}$$

is uniquely (modulo locally constant functions) solvable if and only if $\langle f, \chi\rangle = \langle g, \mathrm{Tr}\,\chi\rangle$ for any χ locally constant function in Ω.

In each case, natural integral representation formulas are valid for the solutions.

We close this section with a simple but useful observation. To state it, recall the region $\mathcal{R}_\Omega$ introduced in (1.1). It is then straightforward to check that

$$(s - 1/p + 1, 1/p) \text{ belongs to the interior of } \mathcal{P}_\Omega \Longleftrightarrow (s, 1/p) \in \mathcal{R}_\Omega. \tag{4.40}$$

5 Vector layer potential operators

In this section we take up the task of introducing and studying layer potential operators at the next level up in the hierarchy of differential forms on $\mathcal{M}$.

5.1 The fundamental solution of the Hodge-Laplacian

Recall from §2.1 the Hodge-Laplacian Δ on 1-forms, and let $V \geq 0$ be a bounded, scalar-valued function. Under the current assumptions,

$$\Delta - V : L^2_1(\mathcal{M}, \Lambda^1 TM) \longrightarrow L^2_{-1}(\mathcal{M}, \Lambda^1 T\mathcal{M}) \tag{5.1}$$

is a bounded, negative, formally self-adjoint operator, which is invertible whenever V is not identically zero. In fact, the same is true for $V \equiv 0$ if and only if $b_1(\mathcal{M})$, the first Betti number of $\mathcal{M}$ vanishes.

From now on, unless specifically mentioned otherwise, we shall assume that $V \neq 0$. In particular, $\Delta - V$ in (5.1) has an inverse, $(\Delta - V)^{-1}$, whose Schwartz kernel, $\Gamma_V(x, y)$, is a symmetric double form of bidegree $(1, 1)$. In local coordinates $\Gamma(x, y)$ satisfies (cf. **[40]**)

$$\Gamma_V(x,y) = \frac{-1}{4\pi\sqrt{\det(g_{jk}(y))}}\left(\sum_{j,k} g_{jk}(y)(x_j-y_j)(x_k-y_k)\right)^{-1/2}$$
$$\times\sum_{\alpha,\beta} g_{\alpha\beta}(y)dx_\alpha\otimes dy_\beta + \text{a less singular term.} \tag{5.2}$$

The fact that the Laplacian commutes with curl and div gives, at the level of Schwartz kernels (recall that $E_V(x,y)$ is the kernel of $(\Delta - V)^{-1}$ on scalar functions), the identities

$$\mathrm{div}_x(\Gamma_V(x,y)) = -\nabla_y(E_V(x,y)) + \text{a residual term,} \tag{5.3}$$
$$\mathrm{curl}_x(\Gamma_V(x,y)) = \mathrm{curl}_y(\Gamma_V(x,y)) + \text{a residual term.} \tag{5.4}$$

In each case, the residue is a double form $R_V(x,y)$ satisfying

$$R_V \in C^2_{\mathrm{loc}}((\mathcal{M}\times\mathcal{M}\setminus \mathrm{diag})\cup\{(x,y) : x\notin \mathrm{supp}\,\nabla V\}),$$
$$|\nabla^j_x\nabla^k_y R_V(x,y)| \le C|x-y|^{-j-k}, \quad 0\le j,k\le 1. \tag{5.5}$$

We shall occasionally also use R_V to denote a generic integral operator with kernel satisfying similar properties to $R_V(x,y)$. An important observation (used frequently in the sequel) is that $R_V \equiv 0$ whenever V is a constant potential.

5.2 The single layer and volume potentials on forms

Once again, for a nonnegative, non identically zero $V\in L^\infty(\mathcal{M})$, we let $\Gamma_V(x,y)$ be the Schwartz kernel of $(\Delta - V)^{-1}$ when acting on 1-forms. Then, if $\Omega\subset\mathcal{M}$ is an arbitrary, fixed Lipschitz domain, we denote by $\mathcal{S}_V$ the single layer potential operator on $\partial\Omega$ with kernel $\Gamma_V(x,y)$. That is,

$$\mathcal{S}_V f(x) := \int_{\partial\Omega}\langle\Gamma_V(x,y), f(y)\rangle\, d\sigma_y, \quad x\in\mathcal{M}\setminus\partial\Omega, \tag{5.6}$$

where f is a 1-form on $\partial\Omega$. Note that $(\Delta - V)\mathcal{S}_V f = 0$ in $\mathcal{M}\setminus\partial\Omega$. Also, the corresponding Newtonian potential acts on a 1-form u defined in Ω by

$$\Pi_V u(x) := \iint_\Omega \langle\Gamma_V(x,y), u(y)\rangle\, d\mathrm{Vol}_y, \quad x\in\Omega. \tag{5.7}$$

In the sequel, we shall make no notational distinction between (5.6), (5.7) and their scalar versions introduced in (4.6), (4.10). In fact, the mapping properties for Π_V and $\mathcal{S}_V$ described in (4.14), (4.15) at the scalar level, continue to hold for the current versions as well.

Proposition 5.1 *Let Ω be a Lipschitz domain in $\mathcal{M}$. Then, if $1<p<\infty$, $-1+1/p<s<1/p$, the operator*

$$\mathcal{S}_V : TH^p_s(\partial\Omega) \longrightarrow L^p_{s+1}(\Omega,\Lambda^1 T\mathcal{M}) \tag{5.8}$$

is well-defined and bounded. In particular, the composition $S_V := Tr\circ\mathcal{S}_V$ maps $TH^p_s(\partial\Omega)$ boundedly into $B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1 T\mathcal{M})$. Furthermore,

$$\operatorname{div} \mathcal{S}_V f = \mathcal{S}_V(\operatorname{Div} f) + R_V f, \tag{5.9}$$

for any $f \in TH^p_s(\partial\Omega)$ *(recall that* R_V *has been introduced at the end of §5.1).*

Proof. For an arbitrary $u \in H^{s,p}(\Omega; \operatorname{curl})$ we may write, thanks to (2.28) and (5.4),

$$\begin{aligned}
\mathcal{S}_V(\nu \times u)(x) &= \langle \Gamma_V(x,\cdot), \nu \times u\rangle \\
&= \iint_\Omega \langle \Gamma_V(x,\cdot), \operatorname{curl} u\rangle \, d\mathrm{Vol} - \iint_\Omega \langle \operatorname{curl} \Gamma_V(x,\cdot), u\rangle \, d\mathrm{Vol} \\
&= \iint_\Omega \langle \Gamma_V(x,\cdot), \operatorname{curl} u\rangle \, d\mathrm{Vol} - \iint_\Omega \langle \operatorname{curl}_x \Gamma_V(x,\cdot), u\rangle \, d\mathrm{Vol} + R_V u(x) \\
&= \Pi_V(\operatorname{curl} u)(x) - \operatorname{curl} \Pi_V u(x) + R_V u(x).
\end{aligned} \tag{5.10}$$

From the identity above and the mapping properties of the volume potential it is clear that $\mathcal{S}_V(\nu\times u)$ belongs to $L^p_{s+1}(\Omega, \Lambda^1 T\mathcal{M})$ and $\|\mathcal{S}_V(\nu \times u)\|_{L^p_{s+1}(\Omega,\Lambda^1 T\mathcal{M})} \leq C\|u\|_{H^{s,p}(\Omega;\operatorname{curl})}$. This readily yields (5.8).

To justify (5.9), let us denote by $R_V(x, y)$, $Q_V(x, y)$ the residual kernels in (5.3) and (5.4), respectively. In particular, $\operatorname{curl}_y R_V(x, y) = \operatorname{div}_x Q_V(x, y)$. Thus, if $f \in TH^p_s(\partial\Omega)$ is of the form $f = \nu \times u$ for some $u \in H^{s,p}(\Omega; \operatorname{curl})$, we have

$$\begin{aligned}
\operatorname{div} \mathcal{S}_V f(x) = &\iint_\Omega \langle \operatorname{div}_x \Gamma_V(x, y), \operatorname{curl}\ u(y)\rangle \, d\mathrm{Vol}_y \\
&- \iint_\Omega \langle \operatorname{div}_x \operatorname{curl}_y \Gamma_V(x, y), u(y)\rangle \, d\mathrm{Vol}_y \\
= &- \iint_\Omega \langle \nabla_x E_V(x, y), \operatorname{curl}\ u(y)\rangle \, d\mathrm{Vol}_y + \iint_\Omega \langle R_V(x, y), \operatorname{curl}\ u(y)\rangle \, d\mathrm{Vol}_y \\
&- \iint_\Omega \langle \operatorname{div}_x Q_V(x, y), u(y)\rangle \, d\mathrm{Vol}_y \\
= &-\langle E_V(x,\cdot), \nu \cdot \operatorname{curl} u\rangle + \langle R_V(x,\cdot), \nu \times u\rangle = \mathcal{S}_V(\operatorname{Div} f) + R_V f,
\end{aligned} \tag{5.11}$$

as desired. □

5.3 Green type formulas

Formally writing $u(x) = \iint_\Omega \langle (\Delta_y - V(y))\Gamma_V(x, y), u(y)\rangle \, d\mathrm{Vol}_y$ and then integrating by parts yields

$$\begin{aligned}
u(x) \quad = \quad &-\Pi_V(Vu)(x) - \iint_\Omega \operatorname{div}_y \Gamma_V(x, y)\, (\operatorname{div} u)(y) \, d\mathrm{Vol}_y \\
&- \iint_\Omega \langle \operatorname{curl}_y \Gamma_V(x, y), (\operatorname{curl} u)(y)\rangle \, d\mathrm{Vol}_y + \int_{\partial\Omega} \operatorname{div}_y \Gamma_V(x, y)\, (\nu \cdot u)(y) \, d\sigma_y \\
&+ \int_{\partial\Omega} \langle \operatorname{curl}_y \Gamma_V(x, y), (\nu \times u)(y)\rangle \, d\sigma_y.
\end{aligned} \tag{5.12}$$

Specializing this formula by selecting V to be a strictly positive constant, say $V \equiv \omega > 0$, leads to

$$\begin{aligned} u &= -\omega\Pi_\omega(u) - \operatorname{curl}\Pi_\omega(\operatorname{curl} u) + \nabla\Pi_\omega(\operatorname{div} u) \\ &\quad + \operatorname{curl}\mathcal{S}_\omega(\nu\times u) - \nabla\mathcal{S}_\omega(\nu\cdot u). \end{aligned} \tag{5.13}$$

By further integrating by parts in the second and third Newtonian potential above finally gives

$$\begin{aligned} u &= \Pi_\omega((\Delta-\omega)u) + \operatorname{curl}\mathcal{S}_\omega(\nu\times u) - \nabla\mathcal{S}_\omega(\nu\cdot u) \\ &\quad + \mathcal{S}_\omega(\nu\times\operatorname{curl} u) - \mathcal{S}_\omega(\nu\operatorname{div} u). \end{aligned} \tag{5.14}$$

Whenever used in the sequel, the smoothness assumptions made on u will, each time, allow us to justify these formal algebraic manipulations.

5.4 The magnetostatic integral operator. Part I

We shall now concern ourselves with the (principal value, singular) integral operator

$$M_V f(x) := p.v.\int_{\partial\Omega}\langle\nu(x)\times\operatorname{curl}_x\Gamma_V(x,y), f(y)\rangle\, d\sigma_y, \quad x\in\partial\Omega. \tag{5.15}$$

From **[40]**, we know that M_V is bounded on $L^p_{\tan}(\partial\Omega)$, $1<p<\infty$, and (with I denoting the identity operator),

$$\nu\times\operatorname{curl}\mathcal{S}_V f|_{\partial\Omega_\pm} = (\pm\tfrac{1}{2}I + M_V)f. \tag{5.16}$$

We aim at extending the action of the operator M_V to the space $TH^p_s(\partial\Omega)$. For an arbitrary $f\in TH^p_s(\partial\Omega)$, set $u:=\operatorname{curl}\mathcal{S}_V f$ in Ω. The claim is that $u\in H^{s,p}(\Omega;\operatorname{curl})$ and that

$$\|u\|_{L^p_s(\Omega,\Lambda^1 T\mathcal{M})} + \|\operatorname{curl} u\|_{L^p_s(\Omega,\Lambda^1 T\mathcal{M})} \le C\|f\|_{TH^p_s(\partial\Omega)} \tag{5.17}$$

holds with a constant independent of f. First, from (5.8) we see that $u\in L^p_{s+1}(\Omega,\Lambda^1 T\mathcal{M})$ plus a natural estimate. Furthermore,

$$\begin{aligned} \operatorname{curl} u = \operatorname{curl}\operatorname{curl}\mathcal{S}_V f &= \nabla\operatorname{div}\mathcal{S}_V f + V\mathcal{S}_V f + R_V f \\ &= \nabla\mathcal{S}_V(\operatorname{Div} f) + V\mathcal{S}_V f + R_V f, \end{aligned} \tag{5.18}$$

where, for the last equality, we have used (5.9). With this at hand, (5.17) follows. Next, in analogy with (5.16), we define

$$M_V f := \nu\times(u|_\Omega) - \tfrac{1}{2}f \in TH^p_s(\partial\Omega). \tag{5.19}$$

It follows then from (5.17) that for each $1<p<\infty$, $-1+1/p<s<1/p$ the operator

$$M_V : TH^p_s(\partial\Omega) \longrightarrow TH^p_s(\partial\Omega) \tag{5.20}$$

is well-defined and bounded. Furthermore, one can check that the above definition of M_V is indeed compatible with the one given in (5.15).

Proposition 5.2 *If the potential V is constant, say $V \equiv \omega$, for some positive $\omega \in \mathbb{R}$, then for any $1 < p < \infty$, $-1 + 1/p < s < 1/p$, we have*

$$\mathrm{Div}\, M_\omega f = -\omega\,\langle \nu, S_\omega f\rangle - K_\omega^*(\mathrm{Div}\, f) \tag{5.21}$$

for each $f \in TH^p_s(\partial\Omega)$, and

$$(\nu \times \nabla_{\tan})K_\omega f = -\omega\,\nu \times S_\omega(\nu f) + M_\omega(\nu \times \nabla_{\tan} f) \tag{5.22}$$

for each $f \in B^{p,p}_{s+1-1/p}(\partial\Omega)$.

Proof. If $f \in \nu \times C^\infty(\bar\Omega, \Lambda^1 T\mathcal{M})|_{\partial\Omega} \hookrightarrow TH^p_s(\partial\Omega)$, then

$$\begin{aligned}
\mathrm{Div}\,(\tfrac{1}{2}I + M_\omega)f &= \mathrm{Div}\,(\nu \times \mathrm{curl}\,\mathcal{S}_\omega f|_{\partial\Omega}) = -\langle \nu, \mathrm{curl}\,\mathrm{curl}\,\mathcal{S}_\omega f|_{\partial\Omega}\rangle \\
&= \langle \nu, (-\Delta + \nabla\mathrm{div})\mathcal{S}_\omega f|_{\partial\Omega}\rangle = -\omega\,\langle \nu, S_\omega f\rangle + \langle \nu, \nabla\mathcal{S}_\omega(\mathrm{Div}\, f)|_{\partial\Omega}\rangle \\
&= -\omega\,\langle \nu, S_\omega f\rangle - K_\omega^*(\mathrm{Div}\, f) + \tfrac{1}{2}\mathrm{Div}\, f.
\end{aligned} \tag{5.23}$$

This clearly yields (5.21) in the case we are considering. The general situation, when $f \in TH^p_s(\partial\Omega)$, follows from this and a density argument based on (ii) in Proposition 2.4.

As for (5.22), let us observe first that if $f \in L^p_1(\partial\Omega)$, then

$$\begin{aligned}
\mathrm{curl}\,\mathcal{S}_\omega(\nu f)(x) &= \int_{\partial\Omega} \langle \mathrm{curl}_x\,\Gamma_\omega(x,y), (\nu f)(y)\rangle\, d\sigma_y \\
&= \int_{\partial\Omega} \langle \mathrm{curl}_y\,\Gamma_\omega(x,y), \nu(y)\rangle f(y)\, d\sigma_y = -\int_{\partial\Omega} \mathrm{Div}(\nu \times \Gamma_\omega(x,\cdot))\, f\, d\sigma \\
&= \int_{\partial\Omega} \langle \Gamma_\omega(x,\cdot), \nu \times \nabla_{\tan} f\rangle\, d\sigma = \mathcal{S}_\omega(\nu \times \nabla_{\tan} f)(x).
\end{aligned} \tag{5.24}$$

Also,

$$\mathcal{D}_\omega f = \int_{\partial\Omega} \langle \nabla_y E_\omega, \nu f\rangle\, d\sigma = -\int_{\partial\Omega} \langle \mathrm{div}_x \Gamma_\omega, \nu f\rangle\, d\sigma = -\mathrm{div}\,\mathcal{S}_\omega(\nu f) \tag{5.25}$$

holds for any $f \in L^p(\partial\Omega)$, $1 < p < \infty$. Consequently,

$$\begin{aligned}
\nabla\mathcal{D}_\omega f &= -\nabla\mathrm{div}\,\mathcal{S}_\omega(\nu f) = -(\Delta + \mathrm{curl}\,\mathrm{curl})\mathcal{S}_\omega(\nu f) \\
&= -\omega\,\mathcal{S}_\omega(\nu f) + \mathrm{curl}\,\mathcal{S}_\omega(\nu \times \nabla_{\tan} f).
\end{aligned} \tag{5.26}$$

Going to the boundary and taking $\nu\times$ of both sides yields (5.22) when $f \in L^p_1(\partial\Omega)$. As before, the case when $f \in B^{p,p}_{s+1-1/p}(\partial\Omega)$ follows from what we have proved so far and a density argument. □

Proposition 5.3 *Once again assume that the potential V is constant, say $V \equiv \omega$, for some positive $\omega \in \mathbb{R}$. Also, assume that $1 < p < \infty$, $-1 + 1/p < s < 1/p$, and $1/p + 1/p' = 1$. Then, the diagram*

$$
\begin{array}{ccc}
TH^{p'}_{-s}(\partial\Omega) & \xrightarrow[\sim]{\nu\times\cdot} & (TH^p_s(\partial\Omega))^* \\
\Big\downarrow{\scriptstyle -M_\omega} & & \Big\downarrow{\scriptstyle M^*_\omega} \\
TH^{p'}_{-s}(\partial\Omega) & \xrightarrow[\sim]{\nu\times\cdot} & (TH^p_s(\partial\Omega))^*
\end{array}
\tag{5.27}
$$

is commutative, where the superscript $*$ indicates adjunction.

Proof. The commutativity of the diagram (5.27) comes down to checking the identity

$$\langle \nu \times (\tfrac{1}{2}I - M_\omega)f, g\rangle = \langle (\tfrac{1}{2}I + M_\omega)^*(\nu \times f), g\rangle \tag{5.28}$$

for any $f \in TH^{p'}_{-s}(\partial\Omega)$ and $g \in TH^p_s(\partial\Omega)$. By density, it suffices to deal with the case when $f, g \in \nu \times C^\infty(\bar{\Omega}, \Lambda^1 T\mathcal{M})|_{\partial\Omega}$. In this latter case, (5.28) is a consequence of (5.4). □

5.5 The magnetostatic integral operator. Part II

Here we discuss invertibility properties of the operators $\pm\frac{1}{2}I + M_V$. First, we assume that $\mathcal{M}$ is a homology sphere. This extra hypothesis is then removed in §8; cf. Step XI in that section, which is carried out for a general manifold $\mathcal{M}$. Until then, however, we are allowed to use the invertibility of the operators $\pm\frac{1}{2}I + M_V$ (stated below) in reasonings which are *local* in nature (by working on small, open patches $\mathcal{O} \subseteq \mathcal{M}$ whose geometric double is a smooth, compact, boundaryless manifold with trivial topology).

Theorem 5.4 *Assume that $\mathcal{M}$ is such that $H^1_{\rm sing}(\mathcal{M};\mathbb{R})$ is trivial, i.e.*

$$b_1(\mathcal{M}) = 0. \tag{5.29}$$

Fix a nonnegative potential $V \in L^\infty(\mathcal{M})$ and a Lipschitz domain $\Omega \subseteq \mathcal{M}$. Then the operators

$$\pm\tfrac{1}{2}I + M_V : TH^p_s(\partial\Omega) \longrightarrow TH^p_s(\partial\Omega), \qquad (s, 1/p) \in \mathcal{R}_\Omega \tag{5.30}$$

are Fredholm with index zero. In fact, for constant, positive potentials, these operators are isomorphisms.

Proof. The topological assumption (5.29) guarantees the absence of global monogenic 1-forms on $\mathcal{M}$. Consequently, the unperturbed Hodge-Laplacian Δ has a global fundamental solution, i.e., (5.1) remains invertible when $V \equiv 0$. In particular, (5.21)-(5.22) are meaningful and become genuine intertwining identities when $\omega = 0$, i.e.

$$\mathrm{Div}\, M_0 = -K_0^* \,\mathrm{Div} \quad \text{on } TH^p_s(\partial\Omega), \text{ and} \tag{5.31}$$

$$(\nu \times \nabla_{\tan})K_0 = M_0(\nu \times \nabla_{\tan}) \quad \text{on } B^{p,p}_{s+1-1/p}(\partial\Omega). \tag{5.32}$$

In turn, (5.31)-(5.32) imply that the diagrams

$$\begin{array}{ccc} B^{p,p}_{s+1-1/p}(\partial\Omega) & \xrightarrow{\pm\frac{1}{2}I+K_0} & B^{p,p}_{s+1-1/p}(\partial\Omega) \\ \nu\times\nabla_{\tan}\Big\downarrow & & \Big\downarrow\nu\times\nabla_{\tan} \\ TH^{p,o}_s(\partial\Omega) & \xrightarrow{\pm\frac{1}{2}I+M_0} & TH^{p,o}_s(\partial\Omega) \end{array} \tag{5.33}$$

and

$$\begin{array}{ccc} TH^p_s(\partial\Omega)\Big/TH^{p,o}_s(\partial\Omega) & \xrightarrow{\pm\frac{1}{2}I+M_0} & TH^p_s(\partial\Omega)\Big/TH^{p,o}_s(\partial\Omega) \\ \text{Div}\Big\downarrow & & \Big\downarrow\text{Div} \\ B^{p,p}_{s-1/p}(\partial\Omega) & \xrightarrow{\pm\frac{1}{2}I-K_0^*} & B^{p,p}_{s-1/p}(\partial\Omega) \end{array} \tag{5.34}$$

are commutative. From these and Theorem 3.2, Theorem 4.2, it follows that

$$\begin{array}{l} \pm\frac{1}{2}I+M_0 \text{ are Fredholm with index zero on the spaces} \\ TH^{p,o}_s(\partial\Omega) \text{ and } TH^p_s(\partial\Omega)\Big/TH^{p,o}_s(\partial\Omega), \text{ for each } (s,1/p)\in\mathcal{R}_\Omega. \end{array} \tag{5.35}$$

In order to continue, we need a general fact from the theory of Fredholm operators which we now begin to describe.

Let A, B, C be Banach spaces and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{5.36}$$

where the two horizontal sequences are exact. Then, if two vertical arrows are Fredholm operators then so is the third one. Furthermore, the index of the middle arrow is the sum of the indexes of the other two vertical arrows.

To implement this result we take

$$A := TH^{p,o}_s(\partial\Omega), \quad B := TH^p_s(\partial\Omega), \quad C := TH^p_s(\partial\Omega)\Big/TH^{p,o}_s(\partial\Omega). \tag{5.37}$$

Also, take the first two horizontal arrows to be inclusions and the next two to be projections (in each short sequence), while all vertical arrows are taken to be natural manifestations of the operator $\pm\frac{1}{2}I+M_0$ on the spaces listed above.

Thus, at this stage, granted (5.29) and for the choice $V=0$,

$$\begin{array}{l} \pm\frac{1}{2}I+M_0 \text{ are Fredholm with index zero} \\ \text{on } TH^p_s(\partial\Omega) \text{ for each } (s,1/p)\in\mathcal{R}_\Omega. \end{array} \tag{5.38}$$

In fact, (5.38) extends to cover the case $V\not\equiv 0$ as well. Indeed, thanks to (5.2) and interpolation, for any two nonnegative potentials $V_1, V_2\in L^\infty(\mathcal{M})$, the difference

$$M_{V_1}-M_{V_2} : L^p_{-s}(\partial\Omega,\Lambda^1 T\mathcal{M}) \to \bigcap_{\varepsilon>0} L^p_{1-s-\varepsilon}(\partial\Omega,\Lambda^1 T\mathcal{M}) \tag{5.39}$$

is well-defined and bounded for each $1 < p < \infty$, $0 \le s \le 1$. In particular, $M_{V_1} - M_{V_2}$ is compact on $TH^p_s(\partial\Omega)$. Thus, a similar conclusion as in (5.38) continues to hold for any nonnegative potential $V \in L^\infty(\mathcal{M})$ (again, granted (5.29)).

Next, in order to prove that the operators in (5.30) are in fact invertible when V is a positive constant, say $V \equiv \omega$, it suffices to show that these operators have dense ranges for each pair $(s, 1/p) \in \mathcal{R}_\Omega$. This, in turn, will be a consequence of two facts. First, so we claim,

$$(5.29) \Rightarrow \pm\tfrac{1}{2}I + M_\omega \text{ are invertible on } L^{q,\mathrm{Div}}_{\tan}(\partial\Omega),\ \forall\, q \in (1, q_\Omega). \tag{5.40}$$

Second, we shall prove that, irrespective of (5.29),

$$\bigcap_{1<q<q_\Omega} L^{q,\mathrm{Div}}_{\tan}(\partial\Omega) \hookrightarrow TH^p_s(\partial\Omega) \ \text{ densely for each } \ (s, 1/p) \in \mathcal{R}_\Omega. \tag{5.41}$$

Clearly, with these at hand, the desired conclusion follows.

Now, (5.40) is proved by following a program similar in spirit to our approach to (5.35). This time, the relevant commutative diagrams are

$$\begin{array}{ccc} L^q_1(\partial\Omega) & \xrightarrow{\ \pm\frac{1}{2}I + K_0\ } & L^q_1(\partial\Omega) \\ \nu \times \nabla_{\tan} \Big\downarrow & & \Big\downarrow \nu \times \nabla_{\tan} \\ L^{q,o}_{\tan}(\partial\Omega) & \xrightarrow{\ \pm\frac{1}{2}I + M_0\ } & L^{q,o}_{\tan}(\partial\Omega) \end{array} \tag{5.42}$$

and

$$\begin{array}{ccc} L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)\big/L^{q,o}_{\tan}(\partial\Omega) & \xrightarrow{\ \pm\frac{1}{2}I + M_0\ } & L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)\big/L^{q,o}_{\tan}(\partial\Omega) \\ \mathrm{Div} \Big\downarrow & & \Big\downarrow \mathrm{Div} \\ L^q(\partial\Omega) & \xrightarrow{\ \pm\frac{1}{2}I - K^*_0\ } & L^q(\partial\Omega) \end{array} \tag{5.43}$$

and Theorem 3.3 is used in place of Theorem 3.2. The conclusion is that $\pm\frac{1}{2}I + M_\omega$ are Fredholm with index zero on $L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)$ for each $1 < q < q_\Omega$. In the Corollary 5.6 below we shall show (independently of the current proof) that these operators are invertible if $p_\Omega < q < q_\Omega$, irrespective of (5.29). This shows that the operators in question are also one-to-one and, hence, yields (5.40).

As for (5.41), it suffices to prove that the inclusion is well-defined and bounded −the fact that it has dense range is then a consequence of (ii) in Proposition 2.4. In turn, this latter problem is local in character, so we may assume that (5.29) holds. Indeed, matters can be reduced to this case by employing a smooth partition of unity subordinated to a finite cover $\{\mathcal{O}_j\}_j$ of $\bar{\Omega}$ with coordinate patches and by working with the so-called double of each $\mathcal{O}_j$ (itself, a smooth, compact, boundaryless manifold). With the above reduction enforced, observe next that each $f \in L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)$ with $1 < q < q_\Omega$ can be written in the form $f = \nu \times u$ with $u := \mathrm{curl}\,\mathcal{S}_\omega((\frac{1}{2}I + M_\omega)^{-1}f)$ in Ω and that, thanks to (4.15), we have $u, \mathrm{curl}\,u \in B^{q,q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M})$. Our aim is now to show that, in general,

$$\bigcap_{1<q<q_\Omega} B^{q,q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M}) \hookrightarrow L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \quad \forall\, (s, 1/p) \in \mathcal{R}_\Omega. \tag{5.44}$$

In fact, we shall show that for each $(s, 1/p) \in \mathcal{R}_\Omega$ it is possible to select $1 < q < q_\Omega$ such that $B^{q,q\vee 2}_{1/q}(\Omega) \hookrightarrow L^p_s(\Omega)$ continuously and densely. Since this is the case if $1/q \geq s$ and $1/p - s/3 \geq 2/(3q)$, it suffices to prove that the intervals $(s, 3/(2p) - s/2)$ and $(1/q_\Omega, 1)$ have a nonempty intersection if $(s, 1/p)$ are as in the right side of (1.1). It turn, this comes down to verifying the inequality $1 - 1/p_\Omega < 3/(2p) - s/2$, which is readily checked based on definitions. This finishes the proof of the theorem. □

We next present an important invertibility result at the L^p level.

Theorem 5.5 *For each nonnegative potential $V \in L^\infty(\mathcal{M})$, which is not identically zero, and for each Lipschitz domain $\Omega \subset \mathcal{M}$, the operators*

$$\pm\tfrac{1}{2}I + M_V : L^p_{\tan}(\partial\Omega) \longrightarrow L^p_{\tan}(\partial\Omega), \tag{5.45}$$

are Fredholm with index zero for each $p_\Omega < p < q_\Omega$. In fact, for the same range of p's, the operators (5.45) are isomorphisms if V is a constant, positive potential.

Proof. Assume temporarily that (5.29) holds and that $V \equiv 0$. In this scenario, we shall employ the commutative diagram

$$\begin{array}{ccc} L^p_{\tan}(\partial\Omega)\Big/L^{p,o}_{\tan}(\partial\Omega) & \xrightarrow{\pm\frac{1}{2}I + M_0} & L^p_{\tan}(\partial\Omega)\Big/L^{p,o}_{\tan}(\partial\Omega) \\ \Big\downarrow \mathrm{Div} & & \Big\downarrow \mathrm{Div} \\ L^p_{-1}(\partial\Omega) & \xrightarrow{\pm\frac{1}{2}I - K_0^*} & L^p_{-1}(\partial\Omega). \end{array} \tag{5.46}$$

Since the bottom arrow is Fredholm with index zero for $p_\Omega < p < \infty$, it follows from the second part of Theorem 3.3 that the top arrow is also Fredholm with index zero for each $p_\Omega < p < \infty$. With this at hand and relying on the fact that $\pm\frac{1}{2}I + M_0$ are Fredholm with index zero when acting on $L^{p,o}_{\tan}(\partial\Omega)$, $1 < p < q_\Omega$, we may conclude, via a reasoning that mirrors in the proof of Theorem 5.4, that the operators in (5.45) are Fredholm with index zero for each $p_\Omega < p < q_\Omega$, at least if $\omega = 0$ and (5.29) holds.

Now, these extra hypotheses can be dispensed with. First, from (5.39), $\pm\frac{1}{2}I + M_V$ are compact perturbations of $\pm\frac{1}{2}I + M_0$ and, hence, inherit the Fredholm character and the index of these latter operators on $L^p_{\tan}(\partial\Omega)$, $p_\Omega < p < q_\Omega$. Second, in order to get rid of the extra assumption (5.29), we observe that the property of M_V of being bounded from below on L^p spaces, modulo compact operators, is local in nature. This is seen via a partition of unity argument and relies on the fact that the commutator between M_V and the operator of multiplication with a smooth cutoff function has a weakly singular kernel. Thus, at this stage, we may conclude that the operators in (5.45) have closed ranges (and finite dimensional kernels) irrespective of whether (5.29) holds. To finish the proof, we proceed as follows. First, via Rellich identities for differential forms (cf. [**45**], [**40**]), one can prove that $\pm\frac{1}{2}I + M_\omega$ with $\omega > 0$ are invertible on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega)$ for any Lipschitz subdomain Ω of an arbitrary manifold $\mathcal{M}$ (in fact, the same holds at the level $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ for p near 2). Since $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega) \hookrightarrow L^p_{\tan}(\partial\Omega)$ densely for $p_\Omega < p \leq 2$, we have that $\pm\frac{1}{2}I + M_\omega$ are onto $L^p_{\tan}(\partial\Omega)$ for $p_\Omega < p \leq 2$. Furthermore, by duality (cf. (5.27)), $\pm\frac{1}{2}I + M_\omega$ are also one-to-one when acting on $L^p_{\tan}(\partial\Omega)$ for $2 \leq p < q_\Omega$. At this stage, Proposition 4.3 applies and gives that the operators (5.45) are isomorphisms for the whole range $p_\Omega < p < q_\Omega$ when V is constant. The fact that for a general potential V they remain Fredholm with index zero is then a consequence of this and (5.39). □

Corollary 5.6 *Let $\Omega \subset \mathcal{M}$ be a Lipschitz domain. Then, for each constant potential $V \equiv \omega$, $\omega > 0$, the operators*

$$\pm\tfrac{1}{2}I + M_\omega : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \longrightarrow L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \tag{5.47}$$

are invertible whenever $p_\Omega < p < q_\Omega$.

Proof. The fact that the operators (5.47) are one-to-one is clear from Theorem 5.5, since we have $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \hookrightarrow L^p_{\tan}(\partial\Omega)$. As for ontoness, if $f \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$ is arbitrary then, by Theorem 5.5, there exists $g \in L^p_{\tan}(\partial\Omega)$ so that $(\frac{1}{2}I + M_\omega)g = f$. Applying Div to both sides yields, in the light of (5.21), $(\frac{1}{2}I - K^*_\omega)(\mathrm{Div}\, g) = \mathrm{Div}\, f - \omega\,\langle \nu, S_\omega g\rangle \in L^p(\partial\Omega)$. Thus, $\mathrm{Div}\, g \in L^p(\partial\Omega)$, by Theorem 4.2, so that in fact $g \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)$. The case of the operator $-\frac{1}{2}I + M_\omega$ is similar, and the proof is finished. □

It is important to point out that, in contrast to the situation on L^p spaces, the commutator between M_ω and the operator of multiplication with a smooth cutoff function is, generally speaking, *not compact* on $TH^p_s(\partial\Omega)$. Thus, establishing the analogue of (5.38) for $V \neq 0$ requires a different approach which we develop in §8.

6 Monogenic forms in Lipschitz domains

Recall the spaces (1.2) and (1.3). The main result of this section is as follows.

Theorem 6.1 *Let $\Omega \subset \mathcal{M}$ be an arbitrary Lipschitz domain. Then the spaces $\mathcal{H}^{s,p}_\bullet(\Omega)$, $\mathcal{H}^{s,p}_\times(\Omega)$ are independent of $1 < p < \infty$, $-1 + 1/p < s < 1/p$, as long as $(s, 1/p) \in \mathcal{R}_\Omega$. Furthermore,*

$$\dim \mathcal{H}^{s,p}_\bullet(\Omega) = b_1(\Omega), \quad \dim \mathcal{H}^{s,p}_\times(\Omega) = b_2(\Omega), \tag{6.1}$$

and

$$\mathcal{H}^{s,p}_\bullet(\Omega),\ \mathcal{H}^{s,p}_\times(\Omega) \hookrightarrow \bigcap_{1<q<q_\Omega} B^{q,q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M}), \tag{6.2}$$

for each $(s, 1/p) \in \mathcal{R}_\Omega$. Moreover,

$$u \in \mathcal{H}^{s,p}_\bullet(\Omega) \cup \mathcal{H}^{s,p}_\times(\Omega) \Longrightarrow \mathcal{N}(u) \in L^q(\partial\Omega), \tag{6.3}$$

for each $(s, 1/p) \in \mathcal{R}_\Omega$ and $1 < q < q_\Omega$.

A key ingredient in the proof of the above theorem is the regularity result contained in the proposition below. Loosely speaking, this states that a vector field has a 'regular' *tangential* component if and only if it has a 'regular' *normal* component.

Proposition 6.2 *Let $\Omega \subset \mathcal{M}$ be a Lipschitz domain. Fix $(s, 1/p) \in \mathcal{R}_\Omega$, as well as some q with*

$$\max\left\{\tfrac{1}{q_\Omega}\,,\, \tfrac{3}{2}(\tfrac{1}{p} - \tfrac{s}{3}) - \tfrac{1}{2}\right\} < \tfrac{1}{q} < \min\left\{1\,,\, \tfrac{3}{2}(\tfrac{1}{p} - \tfrac{s}{3})\right\}. \tag{6.4}$$

Then, for $u \in H^{s,p}(\Omega; \mathrm{curl}) \cap H^{s,p}(\Omega; \mathrm{div})$, the following two conditions are equivalent:

(i) $\nu \cdot u$, originally considered in $B^{p,p}_{s-1/p}(\partial\Omega)$, actually belongs to $L^q(\partial\Omega)$;

(ii) $\nu \times u$, originally viewed in $B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^1 T\mathcal{M})$, belongs in fact to $L^q(\partial\Omega, \Lambda^1 T\mathcal{M})$.

Furthermore, if any of the two conditions above is satisfied, then also

$$u \in \bigcap_{1<\tau<q} B^{\tau,\tau\vee 2}_{1/\tau}(\Omega, \Lambda^1 T\mathcal{M}). \tag{6.5}$$

Proof. To begin with, since both the hypotheses we make and the conclusions we seek are local in nature, there is no loss of generality in assuming that (5.29) holds. With this reduction in mind, let us deal with the implication $(i) \Rightarrow (ii)$. Thus, assuming that $\nu \cdot u \in L^q(\partial\Omega)$ and taking $V \in C^\infty(\mathcal{M})$, $V \geq 0$, not identically zero and with $\operatorname{supp} V \cap \bar{\Omega} = \emptyset$, we have

$$v := u - \nabla \mathcal{S}_V((-\tfrac{1}{2}I + K_V^*)^{-1}(\nu \cdot u)) \Rightarrow v \in H^{s,p}(\Omega; \operatorname{curl}) \cap H^{s,p}_\bullet(\Omega; \operatorname{div}), \tag{6.6}$$

and $\nu \times u \in L^q(\partial\Omega, \Lambda^1 T\mathcal{M}) \iff \nu \times v \in L^q(\partial\Omega, \Lambda^1 T\mathcal{M})$. This segment in our analysis requires that $1/q_\Omega < 1/q < \min\{1, 3(1/p - s/3)/2\}$, which is covered by (6.4).

Now, writing Green's formula (5.12) for v and taking $\nu\times$ of both sides yields

$$(\tfrac{1}{2}I - M_V)(\nu \times v) = -\nu \times \operatorname{Tr}[\operatorname{curl} \Pi_V(\operatorname{curl} v)] + \nu \times \operatorname{Tr}[\nabla \Pi_V(\operatorname{div} v)]. \tag{6.7}$$

Observe next that $L^p_{s+1}(\Omega) \xrightarrow{\operatorname{Tr}} B^{p,p}_{s+1-1/p}(\partial\Omega) \overset{\iota}{\hookrightarrow} L^{p^*}(\partial\Omega)$ for any index p^* such that the estimate $\max\{\frac{1}{q_\Omega}, \frac{3}{2}(\frac{1}{p} - \frac{s}{3}) - \frac{1}{2}\} < \frac{1}{p^*} < \frac{1}{p_\Omega}$ holds. (Note that this double inequality is meaningful, in that $1/p_\Omega > 3/(2p) - s/2 - 1/2$ is always satisfied for $(s, 1/p) \in \mathcal{R}_\Omega$.) Taking this into account and recalling the mapping properties of the Newtonian potential, we arrive at the conclusion that the right side of (6.7) actually belongs to the space $L^{p^*}_{\tan}(\partial\Omega)$. Thus, for some fixed $\omega > 0$, we have $(\frac{1}{2}I - M_\omega)(\nu \times v) = (M_V - M_\omega)(\nu \times v) + (\frac{1}{2}I - M_V)(\nu \times v) \in L^{p^*}_{\tan}(\partial\Omega)$, thanks to (5.39). Since $p_\Omega < p^* < q_\Omega$, by virtue of Theorem 5.5, we obtain $\nu \times v \in L^{p^*}_{\tan}(\partial\Omega)$ and, further, $\nu \times u \in L^q(\partial\Omega, \Lambda^1 T\mathcal{M})$ if $q < p^*$. This latter inequality, however, can always be arranged by appealing to (6.4). The proof of the implication $(i) \Rightarrow (ii)$ is therefore finished.

The opposite implication utilizes similar ideas and is simpler; we omit the details. Finally, (6.5) follows from the integral representation formula (5.12), the fact that $L^q(\partial\Omega) \subset L^\tau(\partial\Omega)$ as long as $1 < \tau \leq q$, and $L^p_{s+1}(\Omega, \Lambda^1 T\mathcal{M}) \hookrightarrow B^{\tau,\tau\vee 2}_{1/\tau}(\Omega, \Lambda^1 T\mathcal{M})$ for each $\tau \in (1, q)$, plus the mapping properties of the layer potentials involved. □

We are now in a position to present the

Proof of Theorem 6.1. Let us first prove that the (nested) family of spaces $\mathcal{H}^{0,p}_\bullet(\Omega)$ is independent of p as long as $\frac{3p_\Omega}{p_\Omega+2} < p < \frac{3p_\Omega}{2p_\Omega-2}$. Indeed, if $u \in \mathcal{H}^{0,p}_\bullet(\Omega)$, Proposition 6.2 gives that $u \in B^{\tau,\tau\vee 2}_{1/\tau}(\Omega, \Lambda^1 T\mathcal{M})$ for any τ so that $\max\{1/q_\Omega, 3/(2p) - 1/2\} < 1/\tau < 1$. Since $B^{\tau,\tau\vee 2}_{1/\tau}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ if $1/p^* > 2/(3\tau)$, it follows that u also belongs to the space $\mathcal{H}^{0,p^*}_\bullet(\Omega)$ whenever $1/p^* > \max\{2/(3q_\Omega), 1/p - 1/3\}$. Iterating this procedure leads to the conclusion that

$$\mathcal{H}^{0,p}_\bullet(\Omega) \hookrightarrow \mathcal{H}^{0,p^*}_\bullet(\Omega), \qquad \forall\, p^* < 3p_\Omega/(2p_\Omega - 2), \tag{6.8}$$

as desired. In fact, from (6.8), (6.5), and standard embedding results, we also see that

$$\mathcal{H}^{0,p}_\bullet(\Omega) \hookrightarrow \bigcap_{1<q<q_\Omega} B^{q,q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M}) \hookrightarrow \bigcap_{(s,1/q)\in\mathcal{R}_\Omega} \mathcal{H}^{s,q}_\bullet(\Omega). \tag{6.9}$$

Consider, next $\mathcal{H}^{s,p}_\bullet(\Omega)$ with $(s, 1/p) \in \mathcal{R}_\Omega$. Then, if the numbers $\tau, p^* > 1$ are such that $\max\{1/q_\Omega, 3(1/p - s/3)/2 - 1/2\} < 1/\tau < 3/(2p^*)$, it follows that

$$\mathcal{H}^{s,p}_\bullet(\Omega) \hookrightarrow B^{\tau,\tau\vee 2}_{1/\tau}(\Omega, \Lambda^1 T\mathcal{M}) \hookrightarrow L^{p^*}(\Omega, \Lambda^1 T\mathcal{M}). \tag{6.10}$$

In particular, $\mathcal{H}^{s,p}_\bullet(\Omega) \hookrightarrow \mathcal{H}^{0,p^*}_\bullet(\Omega)$ if $\max\{2/(3q_\Omega), 1/p-(s+1)/3\} < 1/p^* < 1$. Now, p^* can always be chosen so that $(0,1/p^*)$ belongs to $\mathcal{R}_\Omega$. Thus, from (5.44), $\mathcal{H}^{s,p}_\bullet(\Omega) \hookrightarrow \mathcal{H}^{r,q}_\bullet(\Omega)$ for each $(r,1/q) \in \mathcal{R}_\Omega$. All in all, $\mathcal{H}^{s,p}_\bullet(\Omega)$ is independent of $(s,1/p) \in \mathcal{R}_\Omega$ and is embedded into the right hand-side of (6.2). Similar results hold for the scale $\mathcal{H}^{s,p}_\times(\Omega)$. Now, (6.1) has been proved in **[40]** when $p=2$, $s=0$, so the general case follows.

Finally, we are left with (6.3), to which we now turn. Fix $u \in \mathcal{H}^{s,p}_\bullet(\Omega) \cup \mathcal{H}^{s,p}_\times(\Omega)$, with $(s,1/p) \in \mathcal{R}_\Omega$. Proposition 6.2 and Theorem 4.1, in concert with (5.12) in which we make the choice $\operatorname{supp} V \cap \bar{\Omega} = \emptyset$, give that $\mathcal{N}(u) \in L^q(\partial\Omega)$ for each q as in (6.4). Since the spaces $\mathcal{H}^{s,p}_\bullet(\Omega)$, $\mathcal{H}^{s,p}_\times(\Omega)$ are actually independent of $(s,1/p) \in \mathcal{R}_\Omega$, it suffices to observe that

$$\bigcup_{(s,1/p)\in\mathcal{R}_\Omega} \Big(\max\{\tfrac{1}{q_\Omega}\,,\,\tfrac{3}{2}(\tfrac{1}{p}-\tfrac{s}{3})-\tfrac{1}{2}\}\,,\,\min\{1\,,\,\tfrac{3}{2}(\tfrac{1}{p}-\tfrac{s}{3})\}\Big) = (\tfrac{1}{q_\Omega},1). \tag{6.11}$$

This gives (6.3) and the proof of the theorem is therefore finished. □

7 Vector Green operators

The aim of this section is to introduce two natural Green operators at the level of 1-forms. These are going to be solution operators for the Hodge-Laplacian with suitable (homogeneous) boundary conditions.

7.1 The first homogeneous Poisson problem

The main result of this section is the following.

Theorem 7.1 *Let $\Omega \subseteq M$ be an arbitrary Lipschitz domain. Then for each $(s,1/p) \in \mathcal{R}_\Omega$, the boundary value problem*

$$\begin{cases} \Delta u = \eta \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\ u \in H^{s,p}(\Omega;\mathrm{curl}),\ \operatorname{div} u \in L^p_{s+1}(\Omega), \\ \nu\cdot u = 0 \ \text{in}\ B^{p,p}_{s-1/p}(\partial\Omega), \\ \nu\times \operatorname{curl} u = 0 \ \text{in}\ B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^1 T\mathcal{M}), \end{cases} \tag{7.1}$$

has a solution if and only if

$$\langle \eta, h\rangle = 0, \qquad \forall\, h \in \mathcal{H}^{-s,p'}_\bullet(\Omega), \quad 1/p + 1/p' = 1. \tag{7.2}$$

Any solution automatically satisfies $\operatorname{curl} u \in H^{s,p}(\Omega;\mathrm{curl})$.

Moreover, the space of null solutions is precisely $\mathcal{H}^{s,p}_\bullet(\Omega)$.

The idea is to treat a version of (7.1) with reversed homogeneities and which emphasizes non-tangential maximal function estimates. The key ingredient in this regard is the proposition below.

Proposition 7.2 *For each $p_\Omega < p < q_\Omega$, the boundary value problem*

$$\begin{cases} w \in C^0_{\mathrm{loc}}(\Omega, \Lambda^1 T\mathcal{M}), \\ \Delta w = 0 \ \text{in}\ \Omega, \\ \mathcal{N}(w), \mathcal{N}(\operatorname{curl} w) \in L^p(\partial\Omega), \\ \nu\cdot w = f \in L^p(\partial\Omega), \\ \nu\times \operatorname{curl} w = g \in L^p_{\tan}(\partial\Omega), \end{cases} \tag{7.3}$$

is solvable if and only if

$$g \in [\{h|_{\partial\Omega};\ h \in \mathcal{H}^{0,p'}_{\bullet}(\Omega)\}]^{\circ}, \quad 1/p + 1/p' = 1. \tag{7.4}$$

The space of null solutions is $\mathcal{H}^{0,p}_{\bullet}(\Omega)$*; in particular,* $\operatorname{curl} w$, $\operatorname{div} w$ *are uniquely determined. Moreover any solution satisfies*

$$w,\ \operatorname{curl} w \in \bigcap_{1<q\leq p} B^{q,q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M}). \tag{7.5}$$

Proof. Problems of this type have been systematically studied in **[40]**, even at the level of differential forms of arbitrary degrees. There it is shown that they are all well-posed for $2-\varepsilon < p < 2+\varepsilon$, for some small $\varepsilon > 0$. This restriction on the parameter p is inherited from the main ingredient in the proof, which is the invertibility of operators like (5.45) for $2-\varepsilon < p < 2+\varepsilon$. In our case, thanks to Theorem 5.5 and Theorem 4.2, the argument in **[40]** goes through and yields the well-posedness of (7.3) for the (possibly larger) range $p_\Omega < p < q_\Omega$.

There remains (7.5) which we consider next. Inspired by (4.24), we rely on a suitable Poisson type integral representation formula. In the current setting this amounts to the fact that a harmonic 1-form w with $\mathcal{N}w$, $\mathcal{N}(\operatorname{curl} w) \in L^p(\partial\Omega)$, for $p_\Omega < p < q_\Omega$, is determined in Ω (uniquely, modulo monogenic forms with vanishing normal components) by the values of $\nu\cdot w$ and $\nu\times\operatorname{curl} w$ on $\partial\Omega$. For us here, it is convenient to record a version which emphasizes the invertible operator $\Delta-\omega$, with $\omega > 0$, in place of Δ (in which case (7.3) becomes well-posed). More concretely, we claim that any harmonic 1-form w with $\mathcal{N}w$, $\mathcal{N}(\operatorname{curl} w) \in L^p(\partial\Omega)$, $p_\Omega < p < q_\Omega$, can be represented in Ω as

$$\begin{aligned} w \;=\; & -\omega\,\Pi_\omega(w) - \nabla\mathcal{S}_\omega(\nu\cdot w) + \mathcal{S}_\omega(\nu\times\operatorname{curl} w) \\ & + \operatorname{curl}\mathcal{S}_\omega\left[(-\tfrac{1}{2}I + M_\omega)^{-1}(-\omega\,\nu\times\operatorname{Tr}(\operatorname{div}\Pi_\omega w) - \nu\times\nabla S_\omega(\nu\cdot w) + \nu\times S_\omega(\nu\times\operatorname{curl}\, w))\right] \\ & -\mathcal{S}_\omega\Big[\nu(-\tfrac{1}{2}I + K_\omega)^{-1}(\omega\operatorname{Tr}[\operatorname{div}\Pi_\omega w] + \omega\, S_\omega(\nu\cdot w) - \operatorname{div} S_\omega(\nu\times\operatorname{curl}\, w))\Big] \\ & -\omega\ \operatorname{curl}\mathcal{S}_\omega\Big[(-\tfrac{1}{2}I + M_\omega)^{-1}(\nu\times S_\omega(\nu(-\tfrac{1}{2}I + K_\omega)^{-1}(\operatorname{Tr}[\operatorname{div}\Pi_\omega w] + S_\omega(\nu\cdot w))))\Big] \\ & + \operatorname{curl}\mathcal{S}_\omega\Big[(-\tfrac{1}{2}I + M_\omega)^{-1}(\nu\times S_\omega(\nu(-\tfrac{1}{2}I + K_\omega)^{-1}(\operatorname{div} S_\omega(\nu\times\operatorname{curl}\, w))))\Big]. \end{aligned} \tag{7.6}$$

That is, modulo volume integrals (regarded here as residual; they all disappear if $\omega = 0$), w is determined by $\nu\cdot w$ and $\nu\times\operatorname{curl}\, w$ (i.e. 'half' the boundary data used in Green's formula (5.14)).

This can be proved by starting with the Green formula (5.14), then create two new identities as follows: (i) apply div to both sides, then take the (nontangential) trace on $\partial\Omega$ of both sides; (ii) take the (nontangential) trace on $\partial\Omega$ of both sides, then apply $\nu\times$ to both sides. Going further, we solve for $\operatorname{div} w|_{\partial\Omega}$ in the first (newly created) identity, and then for $\nu\times w$ in the second. Finally, replacing their respective expressions back in Green's formula (5.14), yields (7.6).

For this program to work, we need to assume initially that w and its derivatives are well-behaved near $\partial\Omega$ – better that the original assumption would warrant. Nonetheless, (7.6) is then seen to hold for forms satisfying our weaker current hypotheses *a posteriori*, via a limiting argument (which involves approximating Ω by a sequence of subdomains $\Omega_j \nearrow \Omega$ as in **[5]**).

Next, it can be shown that, for any function w in the three-dimensional Lipschitz domain Ω,

$$\mathcal{N}w \in L^p(\partial\Omega),\ p > 0 \Longrightarrow w \in L^{3p/2}(\Omega). \tag{7.7}$$

See Lemma 6.1 in **[18]**. From (7.6) and (7.7), the regularity statement (7.5) referring to w follows with the aid of Theorem 4.1 (and its vector counterpart). As for the regularity of $\operatorname{curl} w$ we proceed analogously, relying on the identity (7.26) in §7.2, written for $\operatorname{curl} w$ in place of w. □

Before proceeding with the proof of Theorem 7.1 we define

$$p_\# = p_\#(s,p) := \begin{cases} \infty, & \text{if } p(s+1) \geq 3, \\ \frac{2p}{3-(s+1)p}, & \text{if } p(s+1) < 3. \end{cases} \tag{7.8}$$

A direct computation shows that

$$(s, 1/p) \in \mathcal{R}_\Omega \Longrightarrow p_\# > p_\Omega, \tag{7.9}$$

and

$$(s, 1/p) \in \mathcal{R}_\Omega, \;\; p^* \in (p_\Omega\,,\, \min\{p_\#, q_\Omega\}) \Longrightarrow B^{p,p}_{s+1-\frac{1}{p}}(\partial\Omega) \hookrightarrow L^{p^*}(\partial\Omega). \tag{7.10}$$

Also, much as in (5.44),

$$(s, 1/p) \in \mathcal{R}_\Omega \Longrightarrow \bigcap_{1<q<p_\#} B^{q,q\vee 2}_{1/q}(\Omega) \hookrightarrow L^p_s(\Omega). \tag{7.11}$$

Proof of Theorem 7.1. The fact that (7.2) is a necessary condition follows from integrations by parts. Next, assuming that (7.2) holds, we aim to show that a solution for (7.1) exists. Concretely, the candidate we look for is of the form

$$u := \Pi_\omega \eta + w, \tag{7.12}$$

where w is a solution of the problem

$$\begin{cases} w \in C^0_{\rm loc}(\Omega, \Lambda^1 T\mathcal{M}), \\ \Delta w = 0 \text{ in } \Omega, \\ \mathcal{N}(w), \mathcal{N}(\operatorname{curl} w) \in L^{p^*}(\partial\Omega), \\ \nu \cdot w = -\nu \cdot \operatorname{Tr}[\Pi_\omega \eta] \in L^{p^*}(\partial\Omega), \\ \nu \times \operatorname{curl} w = -\nu \times \operatorname{Tr}[\operatorname{curl} \Pi_\omega \eta] \in L^{p^*}_{\tan}(\partial\Omega), \end{cases} \tag{7.13}$$

for some fixed $p_\Omega < p^* < \min\{p_\#, q_\Omega\}$.

The fact that w exists is a consequence of Proposition 7.2, (7.2), and (7.4). Next, observe that in order to conclude that (7.12) is a solution of the problem (7.1) it is enough to show that

$$u, \; \operatorname{curl} u, \; \operatorname{curl}\operatorname{curl} u \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \quad \operatorname{div} u \in L^p_{s+1}(\Omega). \tag{7.14}$$

Moreover, from the properties of Π_ω and (7.12), plus the first identity in (2.3), we observe that (7.14) will follow once we prove that

$$w, \; \operatorname{curl} w \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \quad \operatorname{div} w \in L^p_{s+1}(\Omega). \tag{7.15}$$

From (7.5), we know that

$$w, \; \operatorname{curl} w \in \bigcap_{1<q<p^*} B^{q,q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M}). \tag{7.16}$$

Relying on this and (7.11), we conclude that $p^* \in (p_\Omega\,, \min\{p_\#, q_\Omega\})$ can be chosen so that w, $\operatorname{curl} w \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})$. There remains the more subtle case of $\operatorname{div} w$ which we consider next.

The departure point is the Poisson integral representation formula (7.6). Applying div to both sides of this identity and taking into account the actual form of the boundary conditions in (7.13) eventually yields

$$
\begin{aligned}
\operatorname{div} w \;=\;& -\omega \operatorname{div} \Pi_\omega(w) + \omega\, \mathcal{S}_\omega(\nu \cdot \operatorname{Tr} \Pi_\omega \eta) \\
&+\omega\, \mathcal{D}_\omega \Big[(-\tfrac{1}{2}I + K_\omega)^{-1}(\operatorname{Tr}\, \operatorname{div} \Pi_\omega w)\Big] \\
&-\mathcal{D}_\omega \Big[(-\tfrac{1}{2}I + K_\omega)^{-1}(\operatorname{Tr}\, \operatorname{div} \mathcal{S}_\omega(\nu \times \operatorname{curl} \Pi_\omega \eta))\Big] \\
&+\omega\, \mathcal{D}_\omega \Big[(-\tfrac{1}{2}I + K_\omega)^{-1}(S_\omega(\nu \cdot \operatorname{Tr} \Pi_\omega \eta))\Big].
\end{aligned} \tag{7.17}
$$

At this stage, we invoke (5.9) in concert with (2.59) in order to write

$$
\operatorname{div} \mathcal{S}_\omega(\nu \times \operatorname{curl} \Pi_\omega \eta) = \mathcal{S}_\omega(\operatorname{Div}(\nu \times \operatorname{curl} \Pi_\omega \eta)) = -\mathcal{S}_\omega(\nu \cdot \operatorname{curl}\operatorname{curl} \Pi_\omega \eta). \tag{7.18}
$$

Note that $\nu \cdot \operatorname{curl}\operatorname{curl} \Pi_\omega \eta \in B^{p,p}_{s-1/p}(\partial\Omega)$. That $\operatorname{div} w$ belongs to $L^p_{s+1}(\Omega)$ now follows from this, the fact that $\eta \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})$, and the mapping properties of the operators involved. With this, the proof of (7.15) and, with it, that of (7.14), is completed.

Next, we show that the space of null solutions for (7.1) is $\mathcal{H}^{s,p}_\bullet(\Omega)$. Clearly, any $u \in \mathcal{H}^{s,p}_\bullet(\Omega)$ is a null solution for the problem in question. Conversely, let $u \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})$, with $(s, 1/p) \in \mathcal{R}_\Omega$, solve the homogeneous version of (7.1). First, notice that $\nu \cdot \nabla \operatorname{div} u = -\operatorname{Div}(\nu \times \operatorname{curl} u) = 0$, so that $v := \operatorname{div} u \in L^p_{s+1}(\Omega)$ solves the homogeneous Neumann problem for the Laplace-Beltrami operator in Ω. Since, by (4.40), $(s, 1/p) \in \mathcal{R}_\Omega$ ensures the well-posedness of this problem, from Corollary 4.4 we further infer that $\nabla \operatorname{div} u = 0$. In fact, $\operatorname{div} u = 0$ since $\operatorname{div} u$ locally constant and $\nu \cdot u = 0$ force $\iint_\Omega |\operatorname{div} u|^2\, d\mathrm{Vol} = 0$, via an integration by parts. Next, this and (2.3) show that $\operatorname{curl} u \in \mathcal{H}^{s,p}_\times(\Omega) = \mathcal{H}^{0,2}_\times(\Omega)$. Finally, using the last boundary condition in (7.1) and integrating by parts gives $\iint_\Omega |\operatorname{curl} u|^2\, d\mathrm{Vol} = 0$, so that $\operatorname{curl} u = 0$. Thus, ultimately, $u \in \mathcal{H}^{s,p}_\bullet(\Omega)$, as wanted. $\square$

7.2 The second homogeneous Poisson problem

Here we treat the second basic boundary problem for the Hodge-Laplacian with homogeneous boundary conditions.

Theorem 7.3 *Let $\Omega \subseteq M$ be an arbitrary Lipschitz domain. Then for each $(s, 1/p) \in \mathcal{R}_\Omega$, the boundary value problem*

$$
\left\{
\begin{array}{l}
\Delta\, u = \eta \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\
u \in H^{s,p}(\Omega; \operatorname{curl}),\ \operatorname{div} u \in L^p_{s+1,0}(\Omega), \\
\nu \times u = 0 \ \text{in}\ B^{p,p}_{s-1/p}(\partial\Omega, \Lambda^1 T\mathcal{M}),
\end{array}
\right. \tag{7.19}
$$

has a solution if and only if

$$
\langle \eta, h \rangle = 0, \qquad \forall\, h \in \mathcal{H}^{-s,p'}_\times(\Omega), \quad 1/p + 1/p' = 1. \tag{7.20}
$$

Any solution automatically satisfies $\operatorname{curl} u \in H^{s,p}(\Omega; \operatorname{curl})$.

Furthermore, the space of null solutions is precisely $\mathcal{H}^{s,p}_\times(\Omega)$.

In the proof of Theorem 7.3 we shall make essential use of an auxiliary boundary value problem, which we now describe.

Proposition 7.4 *For each $p_\Omega < p < q_\Omega$, the boundary value problem*

$$\begin{cases} w \in C^0_{\rm loc}(\Omega, \Lambda^1 T\mathcal{M}), \\ \Delta\, w = 0 \ \textit{in}\ \Omega, \\ \mathcal{N}(w),\, \mathcal{N}({\rm div}\, w) \in L^p(\partial\Omega), \\ ({\rm div}\, w)|_{\partial\Omega} = f \in L^p(\partial\Omega), \\ \nu \times w|_{\partial\Omega} = g \in L^p_{\rm tan}(\partial\Omega), \end{cases} \tag{7.21}$$

is solvable if and only if

$$f \in [\{\langle \nu, h|_{\partial\Omega}\rangle;\, h \in \mathcal{H}^{0,p'}_{\times}(\Omega)\}]^\circ, \quad 1/p + 1/p' = 1. \tag{7.22}$$

The space of null solutions is $\mathcal{H}^{0,p}_{\times}(\Omega)$; in particular, ${\rm curl}\, w$, ${\rm div}\, w$ *are uniquely determined. Also, the following regularity result is true:*

$$\mathcal{N}({\rm curl}\, w) \in L^p(\partial\Omega) \Longleftrightarrow g \in L^{p,{\rm Div}}_{\rm tan}(\partial\Omega). \tag{7.23}$$

Natural accompanying estimates are valid, and any solution satisfies

$$w \in \bigcap_{1<q\leq p} B^{q,q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M}). \tag{7.24}$$

Moreover,

$$g \in L^{p,{\rm Div}}_{\rm tan}(\partial\Omega) \Longrightarrow {\rm curl}\, w \in \bigcap_{1<q\leq p} B^{q,q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M}). \tag{7.25}$$

Proof. The well-posedness part is a consequence of the general theory developed in [**40**], at least if p is close to 2. The range $p_\Omega < p < q_\Omega$ is dealt with in a similar manner, with the help of Theorem 5.5 and Corollary 5.6. As for the regularity statements, we shall invoke a convenient Poisson type integral representation formula for a harmonic form w, which emphasizes the boundary data ${\rm Tr}\,{\rm div}\, w$ and $\nu \times w$. Concretely, if $\omega \in \mathbb{R}$, $\omega > 0$, then for any harmonic 1-form w with $\mathcal{N}w$, $\mathcal{N}({\rm div}\, w) \in L^p(\partial\Omega)$, $p_\Omega < p < q_\Omega$, there holds

$$\begin{aligned} w \;=\; & -\omega\, \Pi_\omega(w) + {\rm curl}\, \mathcal{S}_\omega(\nu \times w) - \mathcal{S}_\omega(\nu\, {\rm Tr}\,{\rm div}\, w) \\ & + \mathcal{S}_\omega \Big[(-\tfrac{1}{2}I + M_\omega)^{-1}(\omega\, \nu \times {\rm Tr}\ {\rm curl}\, \Pi_\omega w + \omega\, \nu \times S_\omega(\nu \times w) + \nu \times {\rm curl}\, S_\omega(\nu\, {\rm Tr}\,{\rm div}\, w))\Big] \\ & + \omega\, \mathcal{S}_\omega \Big[(-\tfrac{1}{2}I + M_\omega)^{-1}(\nu \times S_\omega(\nu(-\tfrac{1}{2}I + K_\omega)^{-1}({\rm Tr}\,{\rm div}\, S_\omega(\nu \times w))))\Big] \\ & - {\rm curl}\, \mathcal{S}_\omega \Big[\nu(-\tfrac{1}{2}I + K_\omega)^{-1}({\rm Tr}\,{\rm div}\, S_\omega(\nu \times w))\Big] \\ & - \nabla \mathcal{S}_\omega \Big[(\tfrac{1}{2}I + K^*_\omega)^{-1}(-\omega\, \nu \cdot {\rm Tr}\, \Pi_\omega w + \nu \cdot {\rm curl}\ S_\omega(\nu \times w) - \nu \cdot S_\omega(\nu\, {\rm Tr}\,{\rm div}\, w))\Big] \\ & + \omega\, \nabla \mathcal{S}_\omega \Big[(\tfrac{1}{2}I + K^*_\omega)^{-1}(\nu \cdot S_\omega(-\tfrac{1}{2}I + M_\omega)^{-1}(\nu \times S_\omega(\nu(-\tfrac{1}{2}I + K_\omega)^{-1}({\rm Tr}\,{\rm div}\, S_\omega(\nu \times w)))))\Big] \\ & - \nabla \mathcal{S}_\omega \Big[(\tfrac{1}{2}I + K^*_\omega)^{-1}(\nu \cdot {\rm curl}\ S_\omega(\nu(-\tfrac{1}{2}I + K_\omega)^{-1}({\rm Tr}\,{\rm div}\, S_\omega(\nu \times w))))\Big]. \end{aligned} \tag{7.26}$$

A sketch of proof for the above identity is as follows. Write Green's formula (5.14) for w and then derive two other new identities by: (i) applying curl to both sides then going (nontangentially) to the boundary, and (ii) taking the (nontangential) trace on $\partial\Omega$ of both sides and then applying $\nu\cdot$ to both sides. The idea is to solve first for $\nu\times\mathrm{curl}\,w$ in the identity arising as a result of (i), then for $\nu\cdot w$ in the identity derived as in (ii). Replacing these back in Green's formula (5.14) (and using (5.22), (5.25), (5.26)), ultimately yields (7.26).

With this at hand, (7.23), (7.24) and (7.25) follow from the mapping properties of the layer potentials involved. □

We are now ready to tackle the

Proof of Theorem 7.3. The fact that (7.20) is a necessary condition follows from straightforward integrations by parts. Assume next that (7.20) holds; the aim is to show that a solution for (7.19) exists in the form

$$u := \Pi_\omega\eta + w, \tag{7.27}$$

where w is a solution of the problem

$$\begin{cases} w\in C^0_{\mathrm{loc}}(\Omega,\Lambda^1 T\mathcal{M}),\\ \Delta w = 0 \text{ in } \Omega,\\ \mathcal{N}(w),\,\mathcal{N}(\mathrm{div}\,w)\in L^{p^*}(\partial\Omega),\\ \nu\times w = -\nu\times \mathrm{Tr}\,[\Pi_\omega\eta]\in L^{p^*}_{\mathrm{tan}}(\partial\Omega),\\ (\mathrm{div}\,w)|_{\partial\Omega} = -\mathrm{Tr}\,[\mathrm{div}\,\Pi_\omega\eta]\in L^{p^*}_{\mathrm{tan}}(\partial\Omega),\end{cases} \tag{7.28}$$

for some fixed $p_\Omega < p^* < \min\{p_\#, q_\Omega\}$.

The fact that w exists is a consequence of Proposition 7.4, (7.19), and (7.22). Next, observe that in order to conclude that (7.27) is indeed a solution of (7.19) it is enough to show that

$$w,\,\mathrm{curl}\,w\in L^p_s(\Omega,\Lambda^1 T\mathcal{M}),\quad \mathrm{div}\,w\in L^p_{s+1}(\Omega). \tag{7.29}$$

That $w\in L^p_s(\Omega,\Lambda^1 T\mathcal{M})$, is visible from (7.26), the mapping properties of the operators involved and embedding results. Next, on account of $\mathrm{Div}(\nu\times\Pi_\omega\eta) = -\nu\cdot\mathrm{curl}\,\Pi_\omega\eta\in B^{p,p}_{s+1-1/p}(\partial\Omega)\hookrightarrow L^{p^*}(\partial\Omega)$, we may conclude that $\nu\times\Pi_\omega\eta\in L^{p^*,\mathrm{Div}}_{\mathrm{tan}}(\partial\Omega)$. Granted this, the regularity statement in Proposition 7.4 guarantees that $\mathrm{curl}\,w\in B^{q,q\vee 2}_{1/q}(\Omega,\Lambda^1 T\mathcal{M})$ for each $1<q<p^*$. Thus, from this and (7.11), $\mathrm{curl}\,w\in L^p_s(\Omega,\Lambda^1 T\mathcal{M})$, indeed.

Finally, the scalar function $v := \mathrm{div}\,w$ solves the boundary value problem $\Delta\, v = 0$, $v|_{\partial\Omega}\in B^{p,p}_{s+1-1/p}(\partial\Omega)$, $\mathcal{N}(v)\in L^{p^*}(\partial\Omega)$. Thus, by (a slight variation of) Corollary 4.4, if $V\in L^\infty(\mathcal{M})$, $V\geq 0$, is supported away from $\bar\Omega$ and is not identically zero, it follows from the integral representation formula $v = \mathcal{D}_V((\frac{1}{2}I+K_V)^{-1}(v|_{\partial\Omega}))$ that $v\in L^p_{s+1}(\Omega)$, as wanted. □

7.3 Green operators

For an arbitrary Lipschitz domain Ω, denote by $\{h_j;\, 1\leq j\leq b_1(\Omega)\}$ a basis for $\mathcal{H}^{s,p}_\bullet(\Omega)$, $(s,1/p)\in\mathcal{R}_\Omega$, and consider the $b_1(\Omega)\times b_1(\Omega)$ matrix $A := \Big[\big(\langle h_j,h_\ell\rangle\big)_{j,\ell}\Big]^{-1}$. Next, consider the (projection) operator

$$\begin{gathered} P_\bullet : L^p_s(\Omega,\Lambda^1 T\mathcal{M})\longrightarrow L^p_s(\Omega,\Lambda^1 T\mathcal{M}),\\ P_\bullet u := \textstyle\sum_j \alpha_j(u)h_j,\quad \text{where}\quad (\alpha_j(u))_j := A(\langle u,h_j\rangle)_j. \end{gathered} \tag{7.30}$$

It is clear that the definition of $P_\bullet$ is independent of the basis $\{h_j\}_j$. The *normal Green operator* for the Hodge-Laplacian is then defined for each $(s, 1/p) \in \mathcal{R}_\Omega$ by

$$\begin{aligned}
&G_\bullet : L^p_s(\Omega, \Lambda^1 T\mathcal{M}) \longrightarrow L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\
&G_\bullet \eta := \text{the unique solution of (7.1) with datum } \eta - P_\bullet \eta \\
&\qquad\qquad \text{which is orthogonal to } \mathcal{H}_\bullet^{-s,p'}(\Omega).
\end{aligned} \tag{7.31}$$

Similarly, if $\{k_j;\ 1 \le j \le b_2(\Omega)\}$ is a basis for $\mathcal{H}^{s,p}_\times(\Omega)$, $(s, 1/p) \in \mathcal{R}_\Omega$, and if we consider the $b_2(\Omega) \times b_2(\Omega)$ matrix $B := \Big[(\langle k_j, k_\ell\rangle)_{j,\ell}\Big]^{-1}$, then the projection operator of $L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ onto the finite dimensional space $\mathcal{H}^{s,p}_\times(\Omega)$ is given by

$$\begin{aligned}
P_\times : L^p_s(\Omega, \Lambda^1 T\mathcal{M}) \longrightarrow L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\
P_\times u := \textstyle\sum_j \beta_j(u) k_j, \quad \text{where} \quad (\beta_j(u))_j := B(\langle u, k_j\rangle)_j.
\end{aligned} \tag{7.32}$$

Then, the *tangential Green operator* for the Hodge-Laplacian is defined for each $(s, 1/p) \in \mathcal{R}_\Omega$ by

$$\begin{aligned}
&G_\times : L^p_s(\Omega, \Lambda^1 T\mathcal{M}) \longrightarrow L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\
&G_\times \eta := \text{the unique solution of (7.19) with datum } \eta - P_\times \eta \\
&\qquad\qquad \text{which is orthogonal to } \mathcal{H}_\times^{-s,p'}(\Omega).
\end{aligned} \tag{7.33}$$

Clearly, form definitions, the identity operator on $L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ can be decomposed as

$$I = \Delta G_\bullet + P_\bullet = \nabla \text{div}\, G_\bullet - \text{curl}\,\text{curl}\, G_\bullet + P_\bullet, \tag{7.34}$$

$$I = \Delta G_\times + P_\times = \nabla \text{div}\, G_\times - \text{curl}\,\text{curl}\, G_\times + P_\times, \tag{7.35}$$

for each $(s, 1/p) \in \mathcal{R}_\Omega$.

8 Maxwell's equations. Part I

Let $k \in i\mathbb{R} \setminus \{0\}$ be fixed for the duration of this section. Hence, $\omega := -k^2$ is a strictly positive real number. From now on, we shall work with the complexified versions of the function spaces introduced so far, and denote by 'bar' the usual complex conjugation.

For $1 < p < \infty$, $-1 + 1/p < s < 1/p$, and $\Omega \subset \mathcal{M}$ Lipschitz, consider the boundary value problem

$$\left\{\begin{array}{l}
E, H \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\
\text{curl}\, E - ikH = K \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\
\text{curl}\, H + ikE = J \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\
\nu \times E = 0 \text{ on } \partial\Omega.
\end{array}\right. \tag{8.1}$$

Note that any solution pair (E, H) automatically satisfies $E, H \in H^{s,p}(\Omega; \text{curl})$.

Our aim is to prove that (8.1) is well-posed for each $(s, 1/p) \in \mathcal{R}_\Omega$. We proceed in a series of steps, starting with

Step I. *For each $(s,1/p)\in\mathcal{R}_\Omega$, the problem (8.1) has at most one solution.*

Proof. When $p=2$ and $s=0$ this is a consequence of the identity

$$\iint_\Omega [|E|^2+|H|^2]\,d\mathrm{Vol}=-\tfrac{1}{ik}\,\langle \nu\times E,\nu\times(\nu\times\bar{H})\rangle \tag{8.2}$$

valid for any $E,H\in H^{0,2}(\Omega;\mathrm{curl})$ satisfying the homogeneous Maxwell's equations in Ω, i.e.

$$(\mathrm{Maxwell})\ \begin{cases}\mathrm{curl}\,E-ikH=0 & \text{in}\quad\Omega,\\ \mathrm{curl}\,H+ikE=0 & \text{in}\quad\Omega.\end{cases} \tag{8.3}$$

The pairing in the right-hand side of (8.2) is in the sense of *(iii)*-*(iv)* in Proposition 2.4.

The general case, when $(s,1/p)\in\mathcal{R}_\Omega$ is arbitrary, can be reduced to this special situation as follows. If $K=J=0$ then $\mathrm{div}\,E=\mathrm{div}\,H=0$ and $\nu\cdot H=-ik^{-1}\,\mathrm{Div}(\nu\times E)=0$. With this information available, Proposition 6.2 gives that $E,\,H\in B^{\tau,\tau\vee 2}_{1/\tau}(\Omega,\Lambda^1T\mathcal{M})$ whenever $1<\tau<q$ and q is as in (6.4). Consequently, via classical embeddings, $E,H\in L^q(\Omega,\Lambda^1T\mathcal{M})$ for each index $q<\min\{3q_\Omega/2\,,\,[1/p-(s+1)/3]_+^{-1}\}$, where $[a]_+:=(a+|a|)/2$. Iterating this process sufficiently many times gives $E,\,H\in L^2(\Omega,\Lambda^1T\mathcal{M})$, and the desired conclusion follows. $\square$

Step II. *For each $(s,1/p)\in\mathcal{R}_\Omega$, the operators $\pm\frac{1}{2}I+M_\omega$ are one-to-one and with dense ranges when acting on $TH^p_s(\partial\Omega)$.*

Proof. With an eye on injectivity, let $f\in TH^p_s(\partial\Omega)$ be such that $(\frac{1}{2}I+M_\omega)f=0$ and set $E:=\mathrm{curl}\,\mathcal{S}_\omega f$, $H:=-ik^{-1}\,\mathrm{curl}\,E=-ik\mathcal{S}_\omega f-ik^{-1}\nabla\mathcal{S}_\omega(\mathrm{Div}\,f)$ in $\Omega_\pm$. It follows that $E,H\in H^{s,p}(\Omega_\pm;\mathrm{curl})$ and (E,H) solves the homogeneous version of (8.1). Thus, by Step I, $E=H=0$ in Ω_+. Next, observe that $\nu\times(H|_{\Omega_+})=\nu\times(H|_{\Omega_-})$ in $B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1T\mathcal{M})$, a claim which can be justified by observing that the mappings $f\mapsto\nu\times(H|_{\Omega_\pm})$ are linear and continuous from $TH^p_s(\partial\Omega)$ into $B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1T\mathcal{M})$, and coincide on a dense subspace, such as $\nu\times C^\infty(\bar\Omega,\Lambda^1T\mathcal{M})|_{\partial\Omega}$. In turn, the identity just proved entails $\nu\times(H|_{\Omega_-})=0$ and, further, $E=H=0$ in Ω_- by Step I, since in this case $(H,-E)$ is a null-solution of (8.1) in Ω_-. The bottom line is that $E=H=0$ in $\Omega_\pm$ and since $f=\nu\times(E|_{\Omega_+})-\nu\times(E|_{\Omega_-})$ in $B^{p,p}_{s-1/p}(\partial\Omega,\Lambda^1T\mathcal{M})$ (again, an identity which is justifiable from the jump relations (5.16) and a density argument), we arrive at the conclusion that $f=0$, as desired. The case when $(-\frac{1}{2}I+M_\omega)f=0$ is similar, proving that the operators $\pm\frac{1}{2}I+M_\omega$ are one-to-one.

Finally, the fact the operators under discussion have dense ranges is seen from a duality argument, based on (5.27), and with the help of what we have proved so far. $\square$

Step III. *For each $1<p<\infty$ and $-1+1/p<s<1/p$, the boundary problem (8.1) is well-posed if and only if*

$$\begin{cases}E,H\in L^p_s(\Omega,\Lambda^1T\mathcal{M}),\\ \mathrm{curl}\,E-ikH=0\ \text{in}\ \Omega,\\ \mathrm{curl}\,H+ikE=0\ \text{in}\ \Omega,\\ \nu\times E=f\in TH^p_s(\partial\Omega),\end{cases} \tag{8.4}$$

is well-posed.

Proof. Assume first that (8.1) is well-posed and fix some $f\in TH^p_s(\partial\Omega)$, say $f=\nu\times U$, with $U\in H^{s,p}(\Omega;\mathrm{curl})$. If we now let (E',H') solve (8.1) with data $K:=\mathrm{curl}\,U$, $J:=-ik\,U$, then $E:=-E'+U$ and $H:=-H'$ solve (8.4) with boundary datum f.

Conversely, assume that (8.4) is well-posed and fix $J, K \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ arbitrary. Next, let $\tilde{J}, \tilde{K} \in L^p_s(\mathcal{M}, \Lambda^1 T\mathcal{M})$ be extensions of J, K to the whole $\mathcal{M}$ (produced with the help of (2.18)), and introduce

$$
\begin{aligned}
\hat{E} &:= -\operatorname{curl} \Pi_\omega \tilde{K} + \tfrac{1}{ik} \nabla \operatorname{div} \Pi_\omega \tilde{J} - ik \Pi_\omega \tilde{J} \in L^p_s(\mathcal{M}, \Lambda^1 T\mathcal{M}), \\
\hat{H} &:= -\operatorname{curl} \Pi_\omega \tilde{J} - \tfrac{1}{ik} \nabla \operatorname{div} \Pi_\omega \tilde{K} + ik \Pi_\omega \tilde{K} \in L^p_s(\mathcal{M}, \Lambda^1 T\mathcal{M}),
\end{aligned}
\tag{8.5}
$$

where $\Pi_\omega = (\Delta - \omega)^{-1}$ on $\mathcal{M}$. Then, if the pair (E, H) solves (8.4) for the boundary datum $f := \nu \times (\hat{E}|_\Omega) \in TH^p_s(\partial\Omega)$, a direct calculation shows that $(\hat{E}|_\Omega - E, \hat{H}|_\Omega - H)$ solves (8.1). □

Step IV. *Let $\Omega \subset \mathcal{M}$ be Lipschitz and fix $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Then, if (8.4) is well-posed, the Rellich type estimate*

$$
\|\nu \times E\|_{TH^p_s(\partial\Omega)} \approx \|\nu \times H\|_{TH^p_s(\partial\Omega)} \tag{8.6}
$$

holds uniformly in $E, H \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ which solve the homogeneous Maxwell equations (8.3).

Proof. We debut with the observation that if $E, H \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ solve (8.3) then

$$
\|\nu \times E\|_{TH^p_s(\partial\Omega)} \approx \|E\|_{L^p_s(\Omega, \Lambda^1 T\mathcal{M})} + \|H\|_{L^p_s(\Omega, \Lambda^1 T\mathcal{M})}. \tag{8.7}
$$

Indeed, the left-to-right inequality is a consequence of the well-posedness of (8.4), while the right-to-left inequality follows from the definition of the norm on $TH^p_s(\partial\Omega)$.

Now, if the pair (E, H) satisfies (8.3) then so does $(-H, E)$. Consequently, we also have the equivalence $\|\nu \times H\|_{TH^p_s(\partial\Omega)} \approx \|E\|_{L^p_s(\Omega, \Lambda^1 T\mathcal{M})} + \|H\|_{L^p_s(\Omega, \Lambda^1 T\mathcal{M})}$. Thus, (8.6) follows from this and (8.7). □

Step V. *Let $\Omega \subset \mathcal{M}$ be Lipschitz and fix $(s, 1/p) \in \mathcal{R}_\Omega$. If (8.1) is well-posed both in Ω_+ and in Ω_-, then $\pm\frac{1}{2}I + M_\omega$ are isomorphisms of $TH^p_s(\partial\Omega)$.*

Proof. For $f \in TH^p_s(\partial\Omega)$ arbitrary, we set in $\Omega_\pm$

$$
E := \operatorname{curl} \mathcal{S}_\omega f \quad \text{and} \quad H := -ik^{-1} \operatorname{curl} E = -ik \mathcal{S}_\omega f - ik^{-1} \nabla \mathcal{S}_\omega (\operatorname{Div} f). \tag{8.8}
$$

Thus (E, H) solves the Maxwell system (8.3) both in Ω_+ and in Ω_-. Consequently, by Step IV and our assumptions,

$$
\|\nu \times (E|_{\Omega_\pm})\|_{TH^p_s(\partial\Omega)} \approx \|\nu \times (H|_{\Omega_\pm})\|_{TH^p_s(\partial\Omega)}. \tag{8.9}
$$

If we now observe that $\nu \times (H|_{\Omega_+}) = \nu \times (H|_{\Omega_-})$ and $\nu \times (E|_{\Omega_\pm}) = (\pm\frac{1}{2}I + M_\omega) f$, then we may write, based on the Rellich estimates (8.9),

$$
\begin{aligned}
\|(\tfrac{1}{2}I + M_\omega) f\|_{TH^p_s(\partial\Omega)} &= \|\nu \times (E|_{\Omega_+})\|_{TH^p_s(\partial\Omega)} \approx \|\nu \times (H|_{\Omega_+})\|_{TH^p_s(\partial\Omega)} = \|\nu \times (H|_{\Omega_-})\|_{TH^p_s(\partial\Omega)} \\
&\approx \|\nu \times (E|_{\Omega_-})\|_{TH^p_s(\partial\Omega)} = \|(-\tfrac{1}{2}I + M_\omega) f\|_{TH^p_s(\partial\Omega)}.
\end{aligned}
\tag{8.10}
$$

Here we have used in an essential way the fact that $TH^p_s(\partial\Omega_+) \equiv TH^p_s(\partial\Omega_-)$; cf. (3.34). Thus, by the triangle inequality,

$$
\begin{aligned}
\|f\|_{TH^p_s(\partial\Omega)} &\le \|(\tfrac{1}{2}I + M_\omega) f\|_{TH^p_s(\partial\Omega)} + \|(-\tfrac{1}{2}I + M_\omega) f\|_{TH^p_s(\partial\Omega)} \\
&\le C \min\Big\{ \|(-\tfrac{1}{2}I + M_\omega) f\|_{TH^p_s(\partial\Omega)}, \|(\tfrac{1}{2}I + M_\omega) f\|_{TH^p_s(\partial\Omega)} \Big\}.
\end{aligned}
\tag{8.11}
$$

All in all, $\pm\frac{1}{2}I + M_\omega$ are bounded from below and, hence, are one-to-one with closed ranges. Furthermore, by Step II, the operators under discussion have also dense ranges. Hence, they are isomorphisms of $TH^p_s(\partial\Omega)$. $\square$

Step VI. *For each $(s, 1/p) \in \mathcal{R}_\Omega$, the boundary value problem (8.1) is well-posed if and only if*

$$\begin{cases} E, H \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}), \\ \operatorname{curl} E - ikH = K \in H^{s,p}_\bullet(\Omega; \operatorname{div}),\ \operatorname{div} K = 0, \\ \operatorname{curl} H + ikE = J \in H^{s,p}(\Omega; \operatorname{div}),\ \operatorname{div} J = 0, \\ \nu \times E = 0, \end{cases} \tag{8.12}$$

is well-posed.

Proof. It suffices to show that (8.1) can always be reduced to solving a problem like (8.12). To this end, given $K, J \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ arbitrary, we use the Green operators (7.31), (7.33) to decompose $K = -ik\nabla\varphi - \operatorname{curl} u + \xi$, with $\varphi := ik^{-1}\operatorname{div} G_\bullet K \in L^p_{1+s}(\Omega)$, $u := \operatorname{curl} G_\bullet K \in H^{s,p}_\times(\Omega; \operatorname{curl})$, $\xi := P_\bullet K \in \mathcal{H}^{s,p}_\bullet(\Omega)$, as well as $J = ik\nabla\psi - \operatorname{curl} w + \zeta$, with $\psi := -ik^{-1}\operatorname{div} G_\times J \in L^p_{1+s,0}(\Omega)$, $w := \operatorname{curl} G_\times J \in H^{s,p}(\Omega; \operatorname{curl})$, $\zeta := P_\times J \in \mathcal{H}^{s,p}_\times(\Omega)$.

Now, the incisive observation is that if (E', H') solve (8.12) for the (admissible) data $(K', J') := (-\operatorname{curl} u + \xi, -\operatorname{curl} w + \zeta)$, then $(E, H) := (E' + \nabla\psi, H' + \nabla\varphi)$ will solve (8.1) for the original data (K, J). $\square$

Step VII. *If the operators $\pm\frac{1}{2}I + M_\omega$ are isomorphisms of $TH^p_s(\partial\Omega)$ for some $1 < p < \infty$, $-1 + 1/p < s < 1/p$, then (8.1) is well-posed both in Ω_+ as well as in Ω_-.*

Proof. By Step III, it is enough to prove that (8.4) is well-posed in $\Omega_\pm$. To see this, for $f \in TH^p_s(\partial\Omega)$ arbitrary, we set $E_\pm := \operatorname{curl} \mathcal{S}_\omega((\pm\frac{1}{2}I + M_\omega)^{-1} f)$, $H_\pm := -ik^{-1} \operatorname{curl} E_\pm$ in $\Omega_\pm$. Then $(E_\pm, H_\pm)$ solve (8.4) in $\Omega_\pm$ with boundary datum f. Uniqueness has been already taken care of in Step I. $\square$

Step VIII. *For any Lipschitz domain Ω, the boundary problem (8.12) is well-posed if $s = 0$ and $2 \le p < \frac{3p_\Omega}{2p_\Omega - 2}$.*

Proof. Uniqueness is guaranteed by Step I, so there remains existence (plus estimates). Consider first the case when $p = 2$. In this situation, the conclusion we seek is a consequence of (9.16), proved in the next section, and the fact that $k \notin \mathbb{R}$.

Suppose next that $2 < p < \frac{3p_\Omega}{2p_\Omega - 2}$ (recall that we are currently assuming $s = 0$). In particular, given $K, J \in L^p(\Omega, \Lambda^1 T\mathcal{M})$ with $\operatorname{div} K = \operatorname{div} J = 0$ and $\nu \cdot K = 0$, it makes sense to talk about the solution (E, H) of (8.12) in $L^2(\Omega, \Lambda^1 T\mathcal{M}) \oplus L^2(\Omega, \Lambda^1 T\mathcal{M})$. Our goal is to show that this L^2-solution is, in fact, a L^p-solution.

As $\operatorname{div} E = \operatorname{div} H = 0$ and $\nu \cdot H = -ik^{-1}[\nu \cdot K - \operatorname{Div}(\nu \times E)] = 0$ are implicit in (8.12), Proposition 6.2 yields $E, H \in \bigcap_{1<q<q_\Omega} B^{q, q\vee 2}_{1/q}(\Omega, \Lambda^1 T\mathcal{M})$. There remains to prove that this intersection embeds further into $L^p(\Omega, \Lambda^1 T\mathcal{M})$. Indeed, this comes down to checking that $1/p > 2(1 - 1/p_\Omega)/3$, which is part of our current assumptions on p. $\square$

Step IX. *The operators $\pm\frac{1}{2}I + M_\omega$ are isomorphisms of $TH^p_s(\partial\Omega)$ if $\frac{3p_\Omega}{p_\Omega + 2} < p < \frac{3p_\Omega}{2p_\Omega - 2}$ and $s = 0$.*

Proof. From Step VIII, Step VI and Step V, we know that $\pm\frac{1}{2}I + M_\omega$ are indeed isomorphisms of $TH^p_s(\partial\Omega)$ if $s = 0$ for each $2 \le p < \frac{3p_\Omega}{2p_\Omega - 2}$. The full range in question is then covered by duality; cf. Proposition 5.3. $\square$

Step X. *In any Lipschitz domain Ω, (8.12) is well-posed for any $(s, 1/p) \in \mathcal{R}_\Omega$, as long as $s \ge 0$.*

Proof. Fix some $(s, 1/p) \in \mathcal{R}_\Omega$ with $s \ge 0$, and set $1/p^* := 1/p - s/3$. It follows that $\frac{3p_\Omega}{p_\Omega + 2} < p^* <$

$\frac{3p_\Omega}{2p_\Omega-2}$ and $L^p_s(\Omega)\hookrightarrow L^{p^*}(\Omega)$. Also, fix some arbitrary $K, J\in H^{s,p}(\Omega;{\rm div})$ with ${\rm div}\,K={\rm div}\,J=0$, $\nu\cdot K=0$. Based on Step IX, Step VII and Step VI we know that, for this data, (8.12) has a solution $E, H\in L^{p^*}(\Omega,\Lambda^1T\mathcal{M})$. Let us introduce $p^{\#}:=p^*$ if $p^*<q_\Omega$, and $p^{\#}:=q_\Omega$ if $q_\Omega\leq p^*<\frac{3p_\Omega}{2p_\Omega-2}$. Then the regularity result proved in Proposition 6.2 gives that

$$E,\, H\in\bigcap_{1<q<p^{\#}}B^{q,q\vee 2}_{1/q}(\Omega,\Lambda^1T\mathcal{M}) \tag{8.13}$$

and the justification of Step X is finished as soon as we show that there exits $1<q<p^{\#}$ such that the inclusion $B^{q,q\vee 2}_{1/q}(\Omega)\hookrightarrow L^p_s(\Omega)$ holds. Checking this comes to verifying that there exists $q\in(1,p^{\#})$ with $s<1/q<(3/2)(1/p-s/3)$ or, equivalently, $(1/p^{\#},1)\cap(s,3/(2p^*))\neq\emptyset$. Ultimately, this claim reduces to proving $p^{\#}>2p^*/3$ which, in turn, is clear from definitions. □

Step XI. *For any Lipschitz domain Ω and for each $(s,1/p)\in\mathcal{R}_\Omega$, the operators $\pm\frac{1}{2}I+M_\omega$ are isomorphisms of $TH^p_s(\partial\Omega)$.*

Proof. When $s\geq 0$ this is immediate from Step X, Step VI and Step V. This and duality (cf. (5.27)), then yield the conclusion we seek. □

Step XII. *The boundary problems (8.1) and (8.4) are well-posed for any $(s,1/p)$ belonging to $\mathcal{R}_\Omega$.*

Proof. It follows from Step XI, Step VII and Step III. □

In closing, let us point out that the Step XI above is a strengthened version of Theorem 5.4, since it applies to a general manifold $\mathcal{M}$ (without having to impose (5.29)).

9 Maxwell's equations. Part II

Consider the realization of the Maxwell operator

$$\mathcal{A}:=\begin{pmatrix} 0 & i\ {\rm curl}\\ -i\ {\rm curl} & 0\end{pmatrix} \tag{9.1}$$

as the closed, densely defined, unbounded operator

$$\mathcal{A}:L^p_s(\Omega,\Lambda^1T\mathcal{M})\oplus L^p_s(\Omega,\Lambda^1T\mathcal{M})\to L^p_s(\Omega,\Lambda^1T\mathcal{M})\oplus L^p_s(\Omega,\Lambda^1T\mathcal{M}), \tag{9.2}$$

$1<p<\infty$, $-1+1/p<s<1/p$, with domain

$${\rm Dom}\,(\mathcal{A}):=H^{s,p}_\times(\Omega;{\rm curl})\oplus H^{s,p}(\Omega;{\rm curl}). \tag{9.3}$$

Since the operators (3.39) and (3.40) are adjoints to each other, it is clear that the dual of $\mathcal{A}$ considered on $(L^p_s(\Omega,\Lambda^1T\mathcal{M}))^2$ is $\mathcal{A}$ acting on $(L^{p'}_{-s}(\Omega,\Lambda^1T\mathcal{M}))^2$, where p' denotes the conjugate exponent of p. Finer functional analytic properties of this operator are contained in the theorem below.

Theorem 9.1 *Let $\Omega\subseteq M$ be an arbitrary Lipschitz domain. Then for each $(s,1/p)\in\mathcal{R}_\Omega$ and $k\in\mathbb{C}\setminus\{0\}$, the operator $\mathcal{A}-kI$ is Fredholm with index zero on the space $L^p_s(\Omega,\Lambda^1T\mathcal{M})\oplus L^p_s(\Omega,\Lambda^1T\mathcal{M})$. In fact, except for a discrete set of values of k (which is real, symmetric with respect to zero and accumulates only at $\pm\infty$) the operator in question is one-to-one and onto.*

Proof. Consider the Banach space

$$W^{s,p}(\Omega) := \{(E,H) \in (L^p_s(\Omega,\Lambda^1 T\mathcal{M}))^2;\ \operatorname{div} E = \operatorname{div} H = 0,\ \nu \cdot H = 0\} \tag{9.4}$$

which is a closed subspace of $L^p_s(\Omega,\Lambda^1 T\mathcal{M}) \oplus L^p_s(\Omega,\Lambda^1 T\mathcal{M})$. Our first claim is that for each $(s,1/p) \in \mathcal{R}_\Omega$, there holds

$$(W^{s,p}(\Omega))^* = W^{-s,p'}(\Omega), \qquad 1/p + 1/p' = 1. \tag{9.5}$$

Indeed, if we consider the natural mapping

$$\Phi : W^{-s,p'}(\Omega) \to (W^{s,p}(\Omega))^*, \quad (\Phi(A,B))(E,H) := \langle E, A\rangle + \langle H, B\rangle, \tag{9.6}$$

then, clearly, this is well-defined and linear. Showing that Φ is one-to-one is readily reduced to proving the implications

$$A \in H^{-s,p'}(\Omega;\operatorname{div}),\ \operatorname{div} A = 0,\ \langle E, A\rangle = 0,$$
$$\forall\, E \in H^{s,p}(\Omega;\operatorname{div}) \text{ with } \operatorname{div} E = 0 \Longrightarrow A = 0; \tag{9.7}$$
$$B \in H^{-s,p'}_\bullet(\Omega;\operatorname{div}),\ \operatorname{div} B = 0,\ \langle H, B\rangle = 0,$$
$$\forall\, H \in H^{s,p}_\bullet(\Omega;\operatorname{div}) \text{ with } \operatorname{div} H = 0 \Longrightarrow B = 0. \tag{9.8}$$

These, in turn, can be checked based on (7.34) and (7.35), respectively; we omit the straightforward details.

To prove that Φ in (9.6) is also onto, fix some arbitrary linear, bounded functional $\xi : W^{s,p}(\Omega) \to \mathbb{R}$. By Hahn-Banach's theorem, ξ extends to an element in $[L^p_s(\Omega,\Lambda^1 T\mathcal{M}) \oplus L^p_s(\Omega,\Lambda^1 T\mathcal{M})]^* = L^{p'}_{-s}(\Omega,\Lambda^1 T\mathcal{M}) \oplus L^{p'}_{-s}(\Omega,\Lambda^1 T\mathcal{M})$. Thus, there exist $A, B \in L^{p'}_{-s}(\Omega,\Lambda^1 T\mathcal{M})$ so that $\xi(E,H) = \langle E, A\rangle + \langle H, B\rangle$ for each $(E,H) \in W^{s,p}(\Omega)$. Introduce next $\hat{A} := -\operatorname{curl}\operatorname{curl} G_\times A + P_\times A$ along with $\hat{B} := -\operatorname{curl}\operatorname{curl} G_\bullet B + P_\bullet B$, so that $(\hat{A},\hat{B}) \in W^{-s,p'}(\Omega)$ and, moreover, $\xi = \Phi(\hat{A},\hat{B})$, via integrations by parts. Hence, Φ in (9.6) is an isomorphism and this proves (9.5).

Next, introduce the unbounded operator

$$\mathcal{A}_o : W^{s,p}(\Omega) \longrightarrow W^{s,p}(\Omega), \quad \mathcal{A}_o := \mathcal{A}|_{\operatorname{Dom}(\mathcal{A}_o)}, \tag{9.9}$$

by setting

$$\operatorname{Dom}(\mathcal{A}_o) := W^{s,p}(\Omega) \cap [H^{s,p}_\times(\Omega;\operatorname{curl}) \oplus H^{s,p}(\Omega;\operatorname{curl})]. \tag{9.10}$$

Our next goal is to show that for each $(s,1/p) \in \mathcal{R}_\Omega$, $1/p + 1/p' = 1$,

$$\text{the dual of } \mathcal{A}_o \text{ considered on } W^{s,p}(\Omega) \text{ is } \mathcal{A}_o \text{ acting on } W^{-s,p'}(\Omega). \tag{9.11}$$

That $\mathcal{A}_o$, acting on $W^{-s,p'}(\Omega)$, is included (in the sense of unbounded operators) into the dual of $\mathcal{A}_o$ acting on $W^{s,p}(\Omega)$ is straightforward from definitions. Conversely, let $(U,V) \in (W^{s,p}(\Omega))^* = W^{-s,p'}(\Omega)$, belong to $\operatorname{Dom}(\mathcal{A}_o^*) \subseteq W^{-s,p'}(\Omega)$, and set $(\hat{U},\hat{V}) := \mathcal{A}_o^*(U,V) \in W^{-s,p'}(\Omega)$. Then, for any $(E,H) \in \operatorname{Dom}(\mathcal{A}_o) \subseteq W^{s,p}(\Omega)$,

$$\langle i\,\operatorname{curl} H, \bar{U}\rangle - \langle i\,\operatorname{curl} E, \bar{V}\rangle = \langle \mathcal{A}_o(E,H), (\bar{U},\bar{V})\rangle = \langle E, \bar{\hat{U}}\rangle + \langle H, \bar{\hat{V}}\rangle. \tag{9.12}$$

In order to continue, we need a general fact to the effect that

$$X \in H^{-s,p'}(\Omega; \mathrm{div}),\ Y \in H^{-s,p'}(\Omega; \mathrm{div}),\ \mathrm{div}\, X = 0,\ \langle \mathrm{curl}\, Z, X \rangle = \langle Z, Y \rangle,$$
$$\forall\, Z \in H^{s,p}_{\times}(\Omega; \mathrm{curl}) \cap H^{s,p}(\Omega; \mathrm{div}) \text{ with } \mathrm{div}\, Z = 0 \Longrightarrow \mathrm{curl}\, X = Y. \tag{9.13}$$

Indeed, had we been able to allow arbitrary forms $Z \in H^{s,p}_{\times}(\Omega; \mathrm{curl})$ in (9.13), this would readily follow from (3.40). That matters can always be reduced to this situation is seen as follows. For an arbitrary $Z \in H^{s,p}_{\times}(\Omega; \mathrm{curl})$ set $\hat{Z} := Z - \nabla \mathrm{div}\, G_{\times} Z$, so that $\hat{Z} \in H^{s,p}_{\times}(\Omega; \mathrm{curl}) \cap H^{s,p}(\Omega; \mathrm{div})$ and $\mathrm{div}\, \hat{Z} = 0$. Then, $\langle \mathrm{curl}\, Z, X \rangle = \langle \mathrm{curl}\, \hat{Z}, X \rangle = \langle Z, Y \rangle - \langle \nabla \mathrm{div}\, G_{\times} Z, Y \rangle = \langle Z, Y \rangle$, as desired. This proves (9.13).

Returning now to the analysis of (9.12) and taking $H \equiv 0$ and $E \in H^{s,p}_{\times}(\Omega; \mathrm{curl}) \cap H^{s,p}(\Omega; \mathrm{div})$ with $\mathrm{div}\, E = 0$ gives, with the help of (9.13), that

$$V \in H^{-s,p'}(\Omega; \mathrm{curl}) \quad \text{and} \quad \mathrm{curl}\, V = -i\, \hat{U} \in L^{p'}_{-s}(\Omega, \Lambda^1 T\mathcal{M}). \tag{9.14}$$

Utilizing this information back in (9.12), and further specializing this identity to the case when $E \equiv 0$ and $H := Z - \nabla \mathrm{div}\, G_{\bullet} Z$, for arbitrary $Z \in H^{s,p}(\Omega; \mathrm{curl})$, forces $\langle i\ \mathrm{curl}\, Z, \bar{U} \rangle = \langle Z, \bar{\hat{V}} \rangle$ for any $Z \in H^{s,p}(\Omega; \mathrm{curl})$. Consequently,

$$U \in H^{-s,p'}_{\times}(\Omega; \mathrm{curl}) \quad \text{and} \quad \mathrm{curl}\, U = i\, \hat{V} \in L^{p'}_{-s}(\Omega, \Lambda^1 T\mathcal{M}). \tag{9.15}$$

Thus, $(U, V) \in \mathrm{Dom}\,(\mathcal{A}_o) \subseteq W^{-s,p'}(\Omega)$ and $\mathcal{A}_o(U, V) = (\hat{U}, \hat{V})$, proving (9.11). In particular,

$$\mathcal{A}_o \text{ is self adjoint when acting on } W^{0,2}(\Omega). \tag{9.16}$$

Furthermore, from Proposition 6.2 this operator has a compact resolvent. Thus, if $\Sigma(\Omega)$ denotes the spectrum of $\mathcal{A}_o$ on $W^{0,2}(\Omega)$, it follows that $\Sigma(\Omega) \subset \mathbb{R}$ is discrete, consists of eigenvalues and, given the specific block structure of $\mathcal{A}_o$, is symmetric with respect to the origin.

From §8, we know that the operator $\mathcal{A}_o - k_o I : \mathrm{Dom}\,(\mathcal{A}_o) \to W^{s,p}(\Omega)$ is one-to-one and onto for each $k_o \in i\mathbb{R} \setminus \{0\}$. Also, the inclusion $\mathrm{Dom}\,(\mathcal{A}_o) \hookrightarrow W^{s,p}(\Omega)$ is compact by Proposition 6.2. Thus, when considered on the space $W^{s,p}(\Omega)$, its inverse

$$(\mathcal{A}_o - k_o I)^{-1} : W^{s,p}(\Omega) \longrightarrow \mathrm{Dom}\,(\mathcal{A}_o) \hookrightarrow W^{s,p}(\Omega) \tag{9.17}$$

is a compact, everywhere defined operator. Next, for each $k \in \mathbb{C}$, the factorization

$$\mathcal{A}_o - kI = (\mathcal{A}_o - k_o I) \Big[I - (k - k_o)(\mathcal{A}_o - k_o I)^{-1} \Big] \tag{9.18}$$

proves (cf. Theorem IV.2.7, p. 103 in **[22]**) that

$$\mathcal{A}_o - kI \text{ is a Fredholm operator with index zero on } W^{s,p}(\Omega). \tag{9.19}$$

Recall that we seek a similar conclusion for the operator $\mathcal{A}$. To this end, for each $(J, K) \in L^p_s(\Omega, \Lambda^1 T\mathcal{M}) \oplus L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ we write:

$$(J, K) \in \mathrm{Im}\,(\mathcal{A} - kI; L^p_s(\Omega, \Lambda^1 T\mathcal{M}) \oplus L^p_s(\Omega, \Lambda^1 T\mathcal{M})) \Leftrightarrow$$
$$(J - \nabla \mathrm{div}\, G_{\times} J, K - \nabla \mathrm{div}\, G_{\bullet} K) \in \mathrm{Im}\,(\mathcal{A} - kI; (L^p_s(\Omega, \Lambda^1 T\mathcal{M}))^2) \Leftrightarrow$$
$$(J - \nabla \mathrm{div}\, G_{\times} J, K - \nabla \mathrm{div}\, G_{\bullet} K) \in \mathrm{Im}\,(\mathcal{A}_o - kI; W^{s,p}(\Omega)) \Leftrightarrow$$

$$
\begin{aligned}
&(J-\nabla \mathrm{div}\, G_\times J, K-\nabla \mathrm{div}\, G_\bullet K)\in \Big[\mathrm{Ker}\,(\mathcal{A}_o-\bar{k}I; W^{-s,p'}(\Omega))\Big]^\circ \Leftrightarrow\\
&\langle J-\nabla \mathrm{div}\, G_\times J, \bar{E}\rangle+\langle K-\nabla \mathrm{div}\, G_\bullet K, \bar{H}\rangle=0, \qquad (9.20)\\
&\forall\,(E,H)\in\ \mathrm{Ker}\,(\mathcal{A}_o-\bar{k}I; W^{-s,p'}(\Omega)) \Longleftrightarrow\\
&\langle J,\bar{E}\rangle+\langle K,\bar{H}\rangle=0,\quad \forall\,(E,H)\in\ \mathrm{Ker}\,(\mathcal{A}-\bar{k}I; (L^p_s(\Omega,\Lambda^1 T\mathcal{M}))^2)\Leftrightarrow\\
&(J,K)\in\Big[\mathrm{Ker}\,(\mathcal{A}-\bar{k}I; L^p_s(\Omega,\Lambda^1 T\mathcal{M})\oplus L^p_s(\Omega,\Lambda^1 T\mathcal{M}))\Big]^\circ.
\end{aligned}
$$

The first equivalence is proved much as in Step VI in §8. Also, the second, fourth and sixth ones follow from definitions, while the third one is implied by (9.11) and (9.19). Finally, the fifth one is justified via integrations by parts.

Let us now observe that, as a consequence of Proposition 6.2, the kernel of the operator $\mathcal{A}-kI$ on $L^p_s(\Omega,\Lambda^1 T\mathcal{M})\oplus L^p_s(\Omega,\Lambda^1 T\mathcal{M})$ is actually independent of $(s,1/p)\in\mathcal{R}_\Omega$ and, in fact, coincides with $\mathrm{Ker}\,(\mathcal{A}_o-kI;W^{0,2}(\Omega))$. In particular, from (9.16), this kernel is trivial whenever $k\notin\mathbb{R}$. This, in concert with the sequence of equivalences (9.20), now shows that when acting on $L^p_s(\Omega,\Lambda^1 T\mathcal{M})\oplus L^p_s(\Omega,\Lambda^1 T\mathcal{M})$, the operator $\mathcal{A}-kI$ is Fredholm with index zero for each $k\in\mathbb{C}\setminus\{0\}$, and is in fact an isomorphism whenever $k\notin\Sigma(\Omega)$. □

10 Proofs of the main results

This section is devoted to presenting the (final details in the) proofs of Theorems 1.1-1.3, from §1.

10.1 Proof of Theorem 1.1.

Given $u\in L^p_s(\Omega,\Lambda^1 T\mathcal{M})$, with $(s,1/p)\in\mathcal{R}_\Omega$, we may write $u=\nabla\varphi+\mathrm{curl}\,w+h$, where $\varphi:=\mathrm{div}\,G_\bullet u\in L^p_{1+s}(\Omega)$, $w:=-\mathrm{curl}\,G_\bullet u\in H^{s,p}_\times(\Omega;\mathrm{curl})$, and $h:=P_\bullet u\in\mathcal{H}^{s,p}_\bullet(\Omega)$. Natural estimates are implicit. To prove that the sums in (1.4) are direct, assume that

$$
0=\nabla\varphi+\mathrm{curl}\,w+h,\ \text{ for some } \varphi\in L^p_{1+s}(\Omega),\ w\in H^{s,p}_\times(\Omega;\mathrm{curl}),\ h\in\mathcal{H}^{s,p}_\bullet(\Omega). \tag{10.1}
$$

It follows that φ solves the homogeneous Neumann problem $\Delta\varphi=0$ in Ω, $\partial_\nu\varphi=0$ on $\partial\Omega$. Thus, from Corollary 4.4, we see that $\nabla\varphi=0$ in Ω. Next, $\langle h,h\rangle=-\langle\mathrm{curl}\,w,h\rangle=0$, as a simple integration by parts shows. Thus, $h=0$ in Ω, which also forces $\mathrm{curl}\,w$ to vanish, as desired.

The fact that $\mathrm{curl}\,H^{s,p}_\times(\Omega;\mathrm{curl})$ is a closed subspace of $L^p_s(\Omega,\Lambda^1 T\mathcal{M})$ has been established in Proposition 3.9. Finally, Lemma 2.2 and its proof (cf. (2.33)) give that $\nabla L^p_{s+1}(\Omega)$ and $\nabla L^p_{s+1,0}(\Omega)$ are closed subspaces of $L^p_s(\Omega,\Lambda^1 T\mathcal{M})$.

The reasoning for (1.5) is similar; this time we employ $G_\times$ and rely on the well-posedness of the scalar Poisson problem with Dirichlet boundary conditions.

10.2 Proof of Theorem 1.2.

The strategy is to reduce (1.6) to the case when $f=0$, $g=0$, i.e. to the problem (7.1). This can be done as follows. Let $\omega\in\mathbb{R}$ be a fixed, positive number and consider a 1-form w so that

$$
\left\{
\begin{array}{l}
(\Delta-\omega)w=0 \text{ in } \Omega,\\
w\in H^{s,p}(\Omega;\mathrm{curl}),\ \mathrm{div}\,w\in L^p_{s+1}(\Omega),\\
\nu\cdot w=f\in B^{p,p}_{s-1/p}(\partial\Omega),\\
\nu\times\mathrm{curl}\,w=g\in TH^p_s(\partial\Omega).
\end{array}
\right. \tag{10.2}
$$

Also, let $\hat{u}$ solve (7.1) for the datum $\hat{\eta} := \eta - \Delta w$ in Ω. Then, so we claim

$$u := \hat{u} + w - P_{\bullet}(\hat{u} + w) \tag{10.3}$$

solves (1.6). For this program to work, we need that w and $\hat{u}$ exist and satisfy natural estimates.

First, the existence of w is assured by taking

$$w := \mathcal{S}_\omega\Big[(\tfrac{1}{2}I + M_\omega)^{-1}g\Big] + \nabla\mathcal{S}_\omega\Big[(-\tfrac{1}{2}I + K^*_\omega)^{-1}[f - \nu\cdot S_\omega(\tfrac{1}{2}I + M_\omega)^{-1}g]\Big]. \tag{10.4}$$

Note that, from (10.4) and the mapping properties of the intervening operators,

$$\|w\|_{H^{s,p}(\Omega;\mathrm{curl})} + \|\mathrm{div}\, w\|_{L^p_{s+1}(\Omega)} \leq C(\|f\|_{B^{p,p}_{s-1/p}(\partial\Omega)} + \|g\|_{TH^p_s(\partial\Omega)}). \tag{10.5}$$

Second, the fact that $\hat{u}$ exists (plus accompanying estimates) is a consequence of the fact that (7.1) is solvable for the datum $\hat{\eta}$, i.e. the compatibility condition $\hat{\eta} \in [\mathcal{H}_{\bullet}^{-s,p'}(\Omega)]^\circ$ is satisfied. In turn, this can be readily justified from (1.7) and integrations by parts. Finally, the space of null solutions for (1.6) is the same as the one corresponding to (7.1), i.e. $\mathcal{H}^{s,p}_{\bullet}(\Omega)$.

To treat (1.8), we need to consider the auxiliary problem

$$\begin{cases} (\Delta - \omega)w = 0 \text{ in } \Omega, \\ w \in H^{s,p}(\Omega;\mathrm{curl}),\ \mathrm{div}\, w \in L^p_{s+1}(\Omega), \\ \mathrm{Tr}\,(\mathrm{div}\, w) = f \in B^{p,p}_{s+1-1/p}(\partial\Omega), \\ \nu \times w = g \in TH^p_s(\partial\Omega), \end{cases} \tag{10.6}$$

for some $\omega > 0$, which we solve in the form

$$\begin{aligned} w := {} & \mathcal{S}_\omega\Big[\nu(\tfrac{1}{2}I + K_\omega)^{-1}f\Big] \\ & + \mathrm{curl}\,\mathcal{S}_\omega\Big[(\tfrac{1}{2}I + M_\omega)^{-1}\Big[g - \nu\times S_\omega[\nu(\tfrac{1}{2}I + K_\omega)^{-1}f]\Big]\Big]. \end{aligned} \tag{10.7}$$

Then, if $\hat{u}$ solves (7.19) for the datum $\hat{\eta} := \eta - \Delta w$ in Ω, the form $u := \hat{u} + w - P_\times(\hat{u} + w)$ is a solution of (1.8). Note that $\hat{u}$ exists, since the compatibility condition (1.9) ultimately translates into (7.20) for $\hat{\eta}$, via integrations by parts. Finally, the space of null solutions for (1.8) is the same as for (7.19), i.e. $\mathcal{H}^{s,p}_\times(\Omega)$.

10.3 Proof of Theorem 1.3.

For each $J, K \in L^p_s(\Omega, \Lambda^1 T\mathcal{M})$ and $f \in TH^p_s(\partial\Omega)$ of the form $f = \nu \times U$, with $U \in H^{s,p}(\Omega;\mathrm{curl})$, we have that $E, H \in H^{s,p}(\Omega;\mathrm{curl})$ solve (1.10) if and only if $(E - U, H) \in H^{s,p}_\times(\Omega;\mathrm{curl}) \oplus H^{s,p}(\Omega;\mathrm{curl})$ is a solution of (8.1) with K, J replaced by $\hat{K} := K - \mathrm{curl}\, U$ and $\hat{J} := J - ik\,U$, respectively. In turn, by (9.20), a necessary and sufficient condition for this to happen is

$$\begin{gathered} \langle K - \mathrm{curl}\, U, \bar{\hat{E}}\rangle + \langle J - ik\,U, \bar{\hat{H}}\rangle = 0, \quad \forall\,(\hat{E},\hat{H}) \\ \text{null solution of (1.10) with } k \text{ replaced by } \bar{k}. \end{gathered} \tag{10.8}$$

Now, an integration by parts shows that the above condition is further equivalent to

$$\begin{gathered} \langle K, \bar{\hat{E}}\rangle + \langle J, \bar{\hat{H}}\rangle = 0, \quad \forall\,(\hat{E},\hat{H}) \\ \text{null solution of (1.10) with } k \text{ replaced by } \bar{k}. \end{gathered} \tag{10.9}$$

From §9, we know that space of null solution for (1.10) is trivial whenever $k \notin \mathbb{R}$. Thus, the full Poisson problem for the Maxwell system (1.10) is always Fredholm solvable, with index zero. In fact, with the notation introduced in Theorem 9.1, this problem is genuinely well-posed if and only if $k \notin \{\pm k_j\}_j$. This proves Theorem 1.3.

11 Appendix

Here we prove a suitable Poincaré type lemma in the context of differential forms with coefficients in Sobolev spaces in Lipschitz domains.

First, let us recall the classical Cartan homotopy operator. Let $\{\varphi_t\}_{0\leq t\leq 1}$ be a family of C^∞ diffeomorphisms of $\mathcal{M}$ which depends smoothly on the parameter t, and introduce the vector field $X_t(\varphi_t(x)) := \frac{d}{dt}\varphi_t(x)$ (which we also identify canonically with a one-form). Set $\vee$ for the interior product of forms and define

$$\mathcal{C}u := \int_0^1 \varphi_t^*(X_t \vee u)\, dt \tag{11.1}$$

for each differential form u. Then, since $d(X_t\vee u)+X_t\vee du = \frac{d}{dt}(\varphi_t^* u)$, it follows that $d(\mathcal{C}u)+\mathcal{C}(du) = \varphi_1^* u - \varphi_0^* u$. See, e.g., **[61]** for more details.

Next we specialize this discussion to the case when $\mathcal{M} = \mathbb{R}^n$ and assume that $\Omega \subset \mathbb{R}^n$ is an open domain which is starlike with respect to some ball $B \subset \Omega$. Fix some $\theta \in C_o^\infty(B)$ with $\int \theta = 1$ and, for each $y \in B$, consider the family $\varphi_{y;t}(x) := y + t(x-y)$, $0 \leq t \leq 1$. In particular, $\varphi_{y;0}(x) \equiv y$ and $\varphi_{y,1}(x) \equiv x$. Next, take $X_{y;t}(x) := (x-y)/t$ and consider the following integral average of (11.1) (an idea also used in **[29]**):

$$\begin{aligned} C_\ell u(x) &:= \iint_B \theta(y)\left[\int_0^1 \varphi_{y;t}^*(X_{y;t} \vee u)(x)\, dt\right] dy \\ &= \iint_B \int_0^1 t^{\ell-1}\theta(y)(x-y) \vee u(y+t(x-y))\, dtdy, \qquad x \in \Omega, \end{aligned} \tag{11.2}$$

where $u : \Omega \to \Lambda^\ell \mathbb{R}^n$. Inheriting the corresponding property from $\mathcal{C}$, it follows that for any smooth ℓ-form u on Ω

$$u = d(C_\ell u) + C_{\ell+1}(du), \quad \text{ in } \Omega, \text{ if } \ell \geq 1. \tag{11.3}$$

For further reference, we also note here that

$$C_1(df)(x) = f(x) - \iint_\Omega \theta(y) f(y)\, dy, \qquad x \in \Omega, \tag{11.4}$$

for any scalar function f.

For reasons which will become clear in a moment, we find it convenient to consider I_ℓ, the *transpose* (in the sense of distributions) of (11.2), meaning

$$\langle C_\ell u, v\rangle = \langle u, I_{\ell-1} v\rangle, \qquad \forall\, u \in C_o^\infty(\Omega, \Lambda^\ell \mathbb{R}^n),\ \forall\, v \in C_o^\infty(\Omega, \Lambda^{\ell-1}\mathbb{R}^n). \tag{11.5}$$

A straightforward calculation (based on a couple of changes of variables) shows that

$$I_\ell u(x) = -\iint_\Omega \int_1^\infty (t-1)^\ell t^{n-\ell-1}\theta(y+t(x-y))(x-y) \wedge u(y)\, dtdy, \qquad x \in \Omega, \tag{11.6}$$

whenever $u \in C^\infty_o(\Omega, \Lambda^\ell \mathbb{R}^n)$. One most notable feature of the operator (11.6) is that $\mathrm{supp}\,(I_\ell u)$ is a subset of $\{\lambda x + (1-\lambda)y;\, x \in \mathrm{supp}\, u,\ y \in \bar{B},\, 0 \leq \lambda \leq 1\}$. In particular, since Ω is assumed to be starlike with respect to the ball B, we may conclude that

$$I_\ell[C^\infty_o(\Omega, \Lambda^\ell \mathbb{R}^n)] \subseteq C^\infty_o(\Omega, \Lambda^{\ell+1}\mathbb{R}^n). \tag{11.7}$$

Going further, we note that the dual of (11.3) then becomes

$$\begin{aligned} u &= \delta(I_\ell u) + I_{\ell-1}(\delta u) \quad \text{ on } \ell\text{-forms in } \Omega,\ \ell \geq 1, & (11.8) \\ u &= \delta(I_0 u) + \theta \iint_\Omega u\, dx \quad \text{ on scalar functions in } \Omega, & (11.9) \end{aligned}$$

where δ is the formal transpose of d. The identity (11.9) has also been derived in [**3**]. However, the operator we are most interested in at the moment is

$$J_\ell u(x) := \iint_\Omega \int_1^\infty (t-1)^\ell t^{n-\ell-1} \theta(y + t(x-y))(x-y) \vee u(y)\, dt dy, \qquad x \in \Omega, \tag{11.10}$$

if $u \in C^\infty_o(\Omega, \Lambda^\ell \mathbb{R}^n)$. With 'star' denoting the standard Hodge isomorphism in $\mathbb{R}^n$, it follows that

$$* I_\ell = (-1)^{\ell+1} J_{n-\ell} * \qquad \text{on } \ell\text{-forms}. \tag{11.11}$$

Since $*\delta = (-1)^\ell d*$ and $*d = (-1)^{\ell+1}\delta*$ on ℓ-forms, it easily follows from (11.11) and (11.8)-(11.9) that

$$\begin{aligned} u &= d(J_\ell u) + J_{\ell+1}(du) \ \text{ in } \Omega,\ \ \forall\, u \in C^\infty_o(\Omega, \Lambda^\ell \mathbb{R}^n),\ \text{ if } 0 \leq \ell \leq n-1, & (11.12) \\ u &= d(J_n u) + \Big(\iint_\Omega u\Big)\theta dx_1 \wedge \ldots \wedge dx_n, \ \text{ in } \Omega,\ \forall\, u \in C^\infty_o(\Omega, \Lambda^n \mathbb{R}^n). & (11.13) \end{aligned}$$

Our aim is to study mapping properties of the operators J_ℓ on Sobolev scales. To this end, it clearly suffices to analyze scalar integral operators of the form

$$T_{\ell,j} f(x) := \iint_\Omega \int_1^\infty (t-1)^\ell t^{n-\ell-1} (x_j - y_j)\theta(y + t(x-y)) f(y)\, dt dy, \qquad x \in \Omega, \tag{11.14}$$

where $1 \leq j \leq n$, $0 \leq \ell \leq n$.

Lemma 11.1 *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded, Lipschitz domain, starlike with respect to a ball $B \subset \Omega$. Fix $\theta \in C^\infty_o(B)$ with $\int \theta = 1$ and recall (11.14). Then for each $1 \leq j \leq n$ and $0 \leq \ell \leq n$ the operator*

$$T_{\ell,j} : L^p_{s,0}(\Omega) \longrightarrow L^p_{1+s,0}(\Omega) \tag{11.15}$$

is well-defined and bounded whenever $1 < p < \infty$ and $-2 + \frac{1}{p} < s < \frac{1}{p}$.

Proof. In a first stage, we shall prove a related version of (11.15), namely that

$$T_{\ell,j} : L^p_{s,0}(\Omega) \longrightarrow L^p_{1+s}(\Omega), \quad 1 < p < \infty,\ s \geq 0. \tag{11.16}$$

To get started, let us first prove the case $s = 0$ of (11.16), i.e. that $T_{\ell,j}$ is a bounded operator from $L^p(\Omega)$ into $L^p_1(\Omega)$. To this end, note that

$$k_{\ell,j}(x,z) := z_j \int_0^\infty \tau^\ell (1+\tau)^{n-\ell-1} \theta(x+\tau z)\, d\tau \tag{11.17}$$

is the integral kernel of $T_{\ell,j}$, in the sense that $T_{\ell,j}f(x) = \iint_\Omega k_{\ell,j}(x, x-y) f(y)\, dy$, for each $x \in \Omega$. For a fixed, arbitrary $k \in \{1, 2, ..., n\}$, consider next $\partial_{z_k}[k_{\ell,j}(x,z)]$, given by

$$\delta_{jk} \int_0^\infty \tau^\ell (1+\tau)^{n-\ell-1} \theta(x+\tau z)\, d\tau + z_j \int_0^\infty \tau^{\ell+1} (1+\tau)^{n-\ell-1} (\partial_k\theta)(x+\tau z)\, d\tau. \tag{11.18}$$

Expanding $(1+\tau)^{n-\ell-1}$ via the Binomial Theorem and changing variables so that z re-scales to a unit vector eventually shows that the top singularity in the above expression corresponds to $\ell = n$, i.e. is given by

$$k^{top}_{\ell,j,k}(x,z) := \frac{\delta_{jk}}{|z|^n} \int_0^\infty \tau^{n-1} \theta(x+\tau \tfrac{z}{|z|})\, d\tau + \frac{z_j}{|z|^{n+1}} \int_0^\infty \tau^n (\partial_k\theta)(x+\tau\tfrac{z}{|z|})\, d\tau. \tag{11.19}$$

At this stage, the key observation is that this top singularity satisfies the homogeneity condition $k^{top}_{\ell,j,k}(x,\lambda z) = \lambda^{-n} k^{top}_{\ell,j,k}(x,z)$, for each $\lambda > 0$, the uniform estimate $\sup_x \sup_{|z|=1} |k^{top}_{\ell,j,k}(x,z)| < +\infty$ and, finally, the cancellation condition $\int_{|z|=1} k^{top}_{\ell,j,k}(x,z)\, d\omega_z = \int_{\mathbb{R}^n} \partial_{z_k}[z_j\theta(x+z)]\, dz = 0$, since θ has compact support. The bottom line is that $\partial_{x_k}[k_{\ell,j}(x,x-y)]$, $\partial_{y_k}[k_{\ell,j}(x,x-y)]$ are, modulo weakly singular kernels, two (non-convolution) variable coefficient Calderón-Zygmund kernels in $\mathbb{R}^n$.

With p' denoting the conjugate exponent of p, the proof of the case $s = 0$ is finished as soon as we show that, for $1 \le k \le n$,

$$\begin{aligned} &\Big|\iint_\Omega \iint_\Omega k_{\ell,j}(x,x-y) f(y) g(x)\, dydx\Big| \le C\|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}, \\ &\Big|\iint_\Omega \iint_\Omega k_{\ell,j}(x,x-y) f(y) \partial_k g(x)\, dydx\Big| \le C\|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}, \end{aligned} \tag{11.20}$$

uniformly in $f, g \in C^\infty_o(\Omega)$. We shall only prove the last estimate above which is the most delicate. For this, it suffices to establish that, for each $1 \le k \le n$,

$$\Big|\iint_\Omega \iint_\Omega k_{\ell,j}(x,x-y) \eta(\tfrac{x-y}{\varepsilon}) f(y) (\partial_k g)(x)\, dydx\Big| \le C\|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}, \tag{11.21}$$

uniformly in $\varepsilon > 0$. Here, $\eta \in C^\infty$ is a fixed function such that $\eta \equiv 0$ near the origin and $\eta \equiv 1$ at infinity.

Integrating by parts in the left side of (11.21) yields two types of terms under the (double) integral sign, call them $A_\varepsilon(x,y)f(y)g(x)$ and $B_\varepsilon(x,y)f(y)g(x)$, corresponding to ∂_{x_k} falling on $\eta(\frac{x-z}{\varepsilon})$ and on $k_{\ell,j}(x,x-y)$, respectively. Now, by a straightforward estimate, $\sup_{\varepsilon>0} \iint_\Omega |A_\varepsilon(x,y)f(y)|dy$ is dominated by a fixed multiple of $Mf(x)$, the Hardy-Littlewood maximal function of f, so that the contribution from the first term is of the right order, thanks to the L^p-boundedness of M, $1 < p < \infty$. As for the contribution from the second term, since the expression $\sup_{\varepsilon>0} |\iint_\Omega B_\varepsilon(x,y)f(y)dy|$ is controlled in terms of the maximal operator associated with smooth truncations of a Calderón-Zygmund operator, the desired conclusion follows from standard Calderón-Zygmund theory (**[6]**).

The above reasoning proves (11.16) when $s=0$. To handle other (positive) integer values of s, for an arbitrary multi-index $\alpha=(\alpha_1,...,\alpha_n)$ with $|\alpha|=s+1$ pick $k\in\{1,2,...,n\}$ such that $\alpha_k\geq 1$ and set $\beta:=(\alpha_1,...,\alpha_k-1,...\alpha_n)$. In particular, $|\beta|=s$ and $\partial^\alpha=\partial^\beta\partial_k$. We then write

$$\begin{aligned}
&\iint_\Omega\iint_\Omega k_{\ell,j}(x,x-y)f(y)(\partial^\alpha g)(x)\,dydx\\
&=(-1)^s\sum_{\gamma\leq\beta}C_{\beta,\gamma}\iint_\Omega\iint_{\mathbb{R}^n}(\partial_x^\gamma k_{\ell,j})(x,z)(\partial^{\beta-\gamma}f)(x-z)(\partial_k g)(x)\,dzdx\\
&=(-1)^s\sum_{\gamma\leq\beta}C_{\beta,\gamma}\iint_\Omega\iint_\Omega(\partial_x^\gamma k_{\ell,j})(x,x-y)(\partial^{\beta-\gamma}f)(y)(\partial_k g)(x)\,dydx,
\end{aligned}\tag{11.22}$$

for each $f,g\in C^\infty_o(\Omega)$. Since $(\partial^\gamma_x k_{\ell,j})(x,z)$ is of the same type of integral kernel as before, the estimate (11.20) may then be invoked in order to bound each term in the last expression above by $C\|f\|_{L^p_s(\Omega)}\|g\|_{L^{p'}(\Omega)}\leq C\|f\|_{L^p_{s,0}(\Omega)}\|g\|_{L^{p'}(\Omega)}$. Since $C^\infty_o(\Omega)$ is dense in $L^p_{s,0}(\Omega)$ for any $1<p<\infty$ and $s\in\mathbb{R}$ (cf. Remark 2.7 in **[32]**), this further proves that $T_{\ell,j}:L^p_{s,0}(\Omega)\to L^p_{s+1}(\Omega)$ for $s=0,1,2,....$ By interpolation, this takes care of (11.16).

The case $0\leq s<1/p$ of (11.15) then follows from (11.16), plus the fact that $T_{\ell,j}[C^\infty_o(\Omega)]\subseteq C^\infty_o(\Omega)$ (compare with (11.7)), by recalling that the closure of $C^\infty_o(\Omega)$ in $L^p_\alpha(\Omega)$ is $L^p_{\alpha,0}(\Omega)$ whenever $1<p<\infty$ and $1/p<\alpha<1+1/p$ (cf. Proposition 3.12 in **[32]**).

As for the case $-2+1/p<s<0$ of (11.15), the starting point is to prove that the adjoint $T^*_{\ell,j}f(x):=\iint_\Omega k_{\ell,j}(y,y-x)f(y)$, $x\in\Omega$ maps $L^p_{s,0}(\Omega)$ boundedly into $L^p_{s+1}(\Omega)$ for $1<p<\infty$ and $s=0,1,2,....$ This, in turn, may be carried out much as before; the details are left to the reader. By interpolation and duality it follows that $T_{\ell,j}:L^p_{s,0}(\Omega)\to L^p_{s+1}(\Omega)$ if $1<p<\infty$ and $s\leq -1$. Further interpolation with what we have already proved eventually covers the full range of indices in (11.15), thus finishing the proof of the lemma. □

Proposition 11.2 *Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain which is starlike with respect to some ball $B\subset\Omega$, and fix $\theta\in C^\infty_o(B)$ such that $\iint\theta=1$. Then for the family of operators (11.10) the following are true.*

(1) There holds

$$J_\ell[C^\infty_o(\Omega,\Lambda^\ell\mathbb{R}^n)]\subseteq C^\infty_o(\Omega,\Lambda^{\ell-1}\mathbb{R}^n).\tag{11.23}$$

(2) For each $1<p<\infty$ and $-2+\frac{1}{p}<s<\frac{1}{p}$,

$$J_\ell:L^p_{s,0}(\Omega,\Lambda^\ell\mathbb{R}^n)\longrightarrow L^p_{s+1,0}(\Omega,\Lambda^{\ell-1}\mathbb{R}^n)\tag{11.24}$$

is well-defined, linear and bounded.

(3) If $1<p<\infty$ and $-1+1/p<s<1/p$, then

$$u=d(J_\ell u)+J_{\ell+1}(du)\ \ \text{in }\Omega,\quad\text{if }\ 0\leq\ell\leq n-1,\tag{11.25}$$

for each form $u\in L^p_s(\Omega,\Lambda^\ell\mathbb{R}^n)$ with $du\in L^p_s(\Omega,\Lambda^{\ell+1}\mathbb{R}^n)$ and such that $\nu\wedge u=0$ on $\partial\Omega$. Furthermore,

$$u = d(J_n u) + \Big(\iint_\Omega u\Big)\theta\, dx_1 \wedge ... \wedge dx_n, \quad in\ \Omega,\ \ \forall\, u \in L^p_s(\Omega, \Lambda^n\mathbb{R}^n). \tag{11.26}$$

Proof. Point (1) follows from (11.7) and (11.11). Point (2) is a direct consequence of Lemma 11.1. Finally, point (3) is going to be a consequence of (11.12) and a limiting argument, in concert with the following approximation result, valid as long as $1 < p < \infty$ and $-1 + 1/p < s < 1/p$:

$$\begin{aligned}&\forall\, u \in L^p_s(\Omega, \Lambda^\ell\mathbb{R}^n) \text{ with } du \in L^p_s(\Omega, \Lambda^{\ell+1}\mathbb{R}^n), \text{ and } \nu \wedge u = 0 \text{ on } \partial\Omega\\ &\Rightarrow \exists\, u_\varepsilon \in C^\infty_o(\Omega, \Lambda^\ell\mathbb{R}^n) \text{ with } u_\varepsilon \to u \text{ in } L^p_s(\Omega, \Lambda^\ell\mathbb{R}^n) \text{ and } du_\varepsilon \to du \text{ in } L^p_s(\Omega, \Lambda^{\ell+1}\mathbb{R}^n).\end{aligned} \tag{11.27}$$

Much as in the proof of (2.47), there is no loss of generality assuming that there is an open cone Γ centered at the origin of $\mathbb{R}^n$ so that $\Gamma + (\partial\Omega \cap \operatorname{supp} u) \subseteq \Omega$. As before, pick $\varphi \in C^\infty_o(\Gamma)$, $\iint \varphi = 1$, and set $\varphi_\varepsilon := \varepsilon^{-n}\varphi(\cdot\varepsilon^{-1})$ for $\varepsilon > 0$. Then, by the higher dimensional analogue of (2.38), the differential form $u_\varepsilon := (\varphi_\varepsilon * \tilde{u})|_\Omega \in C^\infty_o(\Omega, \Lambda^\ell\mathbb{R}^n)$ does the job advertised in (11.27). This justifies (11.27) and finishes the proof of (3). □

Finally, as far as the operators (11.2) are concerned, we have:

Proposition 11.3 *Under the same assumptions as in Proposition 11.2, then the following hold.*

(1) For each ℓ, the operator (11.2) extends as a linear, bounded mapping on forms with distributions coefficients in Ω, i.e.

$$C_\ell : \Big(C^\infty_o(\Omega, \Lambda^\ell\mathbb{R}^n)\Big)' \longrightarrow \Big(C^\infty_o(\Omega, \Lambda^{\ell-1}\mathbb{R}^n)\Big)' \qquad 1 \le \ell \le n, \tag{11.28}$$

and (11.3) continues to hold in this more general setting.

(2) Let $\mathcal{O}$ be an open subset of Ω such that $\Omega \setminus \mathcal{O}$ is starlike with respect to the ball B. Then, for every distribution-valued ℓ-form u in Ω such that $\operatorname{supp} u \subseteq \bar{\mathcal{O}}$ *we have* $\operatorname{supp}[C_\ell u] \subseteq \bar{\mathcal{O}}$.

(3) For each $1 < p < \infty$ and $-2 + \frac{1}{p} < s < \frac{1}{p}$,

$$C_\ell : L^p_s(\Omega, \Lambda^\ell\mathbb{R}^n) \longrightarrow L^p_{s+1}(\Omega, \Lambda^{\ell-1}\mathbb{R}^n), \tag{11.29}$$

is well-defined, linear and bounded.

Proof. Point (1) is a consequence of (11.5), (11.7) and (11.3). Point (2) is easily justified by observing that if $v \in C^\infty_o(\Omega \setminus \bar{\mathcal{O}}, \Lambda^{\ell-1}\mathbb{R}^n)$ then $\operatorname{supp}(I_\ell v) \subset \Omega \setminus \mathcal{O}$, and then invoking (11.5). Finally, (3) follows from (2) in Proposition 11.2, (11.11) and duality. □

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