

Boundary value problems for Dirac operators and Maxwell's equations in nonsmooth domains

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Abstract

We study the well-posedness of the half-Dirichlet and Poisson problems for Dirac operators in three-dimensional Lipschitz domains, with a special emphasis on optimal Lebesgue and Sobolev-Besov estimates. As an application, an elliptization procedure for the Maxwell system is devised.

1 Introduction

In this paper we derive sharp estimates for the ‘half’-Dirichlet problem as well as the Poisson problem for Dirac operators in Lipschitz domains in the three dimensional setting. We shall work with boundary data from Lebesgue and Sobolev-Besov spaces, well adapted to the problems under discussion. Recall next that $\Omega \subset \mathbb{R}^m$ is called a *Lipschitz* domain provided $\partial\Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions.

The Poisson problem in Lipschitz domains for the Laplace operator with Dirichlet boundary conditions and optimal estimates on classical Sobolev-Besov spaces has been solved by D. Jerison and C. Kenig [10], using harmonic measure techniques. In [6], [22], a new approach was developed which relies on singular integral operators. In particular, this worked well in the treatment of Neumann boundary conditions, and in the case of certain (low dimensional) systems. Similar problems for higher order operators have been addressed in [1].

Quite recently, more progress in the direction of extending these techniques to systems of PDE's has been made in [21], where optimal estimates for the Poisson problem for the three dimensional Maxwell system

$$\begin{cases} \operatorname{curl} E - ikH = 0 \text{ in } \Omega, \\ \operatorname{curl} H + ikE = 0 \text{ in } \Omega, \\ \nu \times E = h \text{ on } \partial\Omega. \end{cases} \quad (1.1)$$

in a Lipschitz domain $\Omega \subset \mathbb{R}^3$ with compact boundary (appropriate radiation conditions must be imposed when Ω is unbounded) have been established on Sobolev-Besov scales.

*Supported in part by NSF and a UMC Summer Research Fellowship
1991 *Math. Subject Classification.* 30G35, 35F15, 35J55, 35Q60, 42B20, 45E05, 46E35
Key words. Dirac operators, Sobolev spaces, Besov spaces, Maxwell's equations, Lipschitz domains.

Now, it has been long understood that there are basic connections between the Dirichlet-Laplacian, the Neumann-Laplacian and the Maxwell system on the one hand, and the Hodge-Dirac operator $D = d + \delta$ (with d , δ , denoting the exterior derivative and its adjoint, respectively), on the other hand. A classical observation which underscores this point is that Maxwell's equations can be written in the compact form

$$\mathcal{D}_k u = 0, \quad \text{where } \mathcal{D}_k := D + k e_4, \text{ and } u := H - i e_4 E. \quad (1.2)$$

Here \mathcal{D}_k can be thought of as a perturbed Dirac operator, whereas the vector fields $E = (E_j)_j$ and $H = (H_j)_j$ are regarded as suitable Clifford algebra-valued functions, i.e.

$$E = E_1 e_1 + E_2 e_2 + E_3 e_3, \quad H = H_1 e_2 e_3 + H_2 e_3 e_1 + H_3 e_1 e_2. \quad (1.3)$$

Our goal is to present a unified approach to all these problems which relies on the Clifford algebra formalism. Some of the main results are summarized in the theorem below (more detailed statements can be found in §6). To state it, recall that for a (possibly algebra-valued) function u defined in Ω , the *nontangential maximal function* $\mathcal{N}u$ is given by $\mathcal{N}u(x) := \sup \{|u(y)|; y \in \Omega, |x - y| \leq \kappa \text{dist}(y, \partial\Omega)\}$, $x \in \partial\Omega$, where $\kappa > 1$ is some fixed, large constant. Also, 'wedge', the exterior product, is introduced in §2 where Clifford algebra rudiments are collected. The Sobolev and Besov spaces, L_s^p , $B_s^{p,p}$, are defined in §4.

Theorem 1.1 *For each three-dimensional Lipschitz domain Ω , with unit normal ν , there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ with the following significance. For any $k \in \mathbb{C}$, possibly except for a discrete subset of \mathbb{R} , and for any Clifford algebra-valued function u defined in Ω which satisfies $\mathcal{D}_k u = 0$, there holds,*

$$\begin{aligned} \|\mathcal{N}(u)\|_{L^p(\partial\Omega)} + \|\mathcal{N}(du)\|_{L^p(\partial\Omega)} + \|\mathcal{N}(\delta u)\|_{L^p(\partial\Omega)} \\ \leq \kappa(\|\nu \wedge u\|_{L^p(\partial\Omega)} + \|\nu \wedge du\|_{L^p(\partial\Omega)}), \end{aligned} \quad (1.4)$$

whenever $1 < p < 2 + \varepsilon$, where $\kappa = \kappa(\partial\Omega, k, p) > 0$ is independent of u . Furthermore,

$$\|\mathcal{N}(u)\|_{L^p(\partial\Omega)} \leq \kappa \|\nu \wedge u\|_{L^p(\partial\Omega)} \quad (1.5)$$

if $2 - \varepsilon < p < 2 + \varepsilon$, and

$$\begin{aligned} \|u\|_{L_s^p(\Omega)} + \|du\|_{L_s^p(\Omega)} + \|\delta u\|_{L_s^p(\Omega)} \\ \leq \kappa(\|\nu \wedge u\|_{B_{s-1/p}^{p,p}(\partial\Omega)} + \|\nu \wedge du\|_{B_{s-1/p}^{p,p}(\partial\Omega)}), \end{aligned} \quad (1.6)$$

for each $p \in (1, \infty)$ and $s \in (-1 + 1/p, 1/p)$ such that $\frac{1}{3} - \varepsilon < \frac{1}{p} - \frac{s}{3} < \frac{2}{3} + \varepsilon$.

All three estimates above are sharp in the class of Lipschitz domains (but extend to $1 < p < \infty$, $1 < p < \infty$ and $-1 < -1 + 1/p < s < 1/p < 1$, respectively, if the unit normal to $\partial\Omega$ has vanishing mean oscillations; in particular, this is the case when $\partial\Omega \in C^1$).

We shall employ singular integral operators of Cauchy type, which are briefly reviewed in §3. A systematic presentation of the Sobolev-Besov spaces involved can be found in §4-§5. The actual proof of the above theorem is then essentially carried out in §6. Finally, in §7, we explore connections between boundary value problems for the (perturbed) Hodge-Dirac operator on the one hand, and the Helmholtz operator with Dirichlet and Neumann boundary conditions and Maxwell's system, on the other hand.

For more background material and further general references, the interested reader is referred to the monographs [3], [7], [19], [24], [8], [12]; see also the articles [14], [13], [15], [2], for harmonic and Fourier analysis methods in the context of Clifford algebras. An excellent survey of progress in the area of harmonic analysis techniques for nonsmooth elliptic problems until early 1990's, can be found in [11].

2 The Clifford algebra structure

Recall that *the (complex) Clifford algebra* associated with \mathbb{R}^m endowed with the usual Euclidean metric is the minimal enlargement of \mathbb{R}^m to a unitary complex algebra \mathcal{A}_m , which is not generated (as an algebra) by any proper subspace of \mathbb{R}^m and such that $x^2 = -|x|^2$, for any $x \in \mathbb{R}^m$. This identity readily implies that, if $\{e_j\}_{j=1}^m$ is the standard orthonormal basis in \mathbb{R}^m , then

$$e_j^2 = -1 \text{ for any } 1 \leq j \leq m, \text{ and } e_j e_k = -e_k e_j \text{ for any } 1 \leq j \neq k \leq m. \quad (2.1)$$

In particular, we have the embedding $\iota : \mathbb{R}^m \hookrightarrow \mathcal{A}_m$, defined by $\iota(x) := \sum_j x_j e_j$ for each $x = (x_j)_j \in \mathbb{R}^m$. Also, any element $u \in \mathcal{A}_m$ can be uniquely represented in the form

$$u = \sum_{l=0}^m \sum'_{|I|=l} u_I e_I, \quad u_I \in \mathbb{C}, \quad (2.2)$$

where e_I stands for the product $e_{i_1} e_{i_2} \dots e_{i_l}$ if $I = (i_1, i_2, \dots, i_l)$ (we make the convention that $e_\emptyset := 1$). For each such multi-index I we call l the *length* of I and denote it by $|I|$. Hereafter, \sum' indicates that the sum is performed only over strictly increasing multi-indices.

A natural inner product $\langle \cdot, \cdot \rangle$ on \mathcal{A}_m is defined by agreeing that $\{e_I\}_{0 \leq |I| \leq m}$, indexed by increasing multiindices, is an orthonormal basis. Throughout the paper, $\langle \cdot, \cdot \rangle$ will stand for either some (pointwise) inner product (also occasionally denoted by dot), or some natural pairing between a space and its dual.

With u as in (2.2), define $\Pi_l u := \sum'_{|I|=l} u_I e_I$ and denote by Λ^l the range of $\Pi_l : \mathcal{A}_m \rightarrow \mathcal{A}_m$. Elements in Λ^l will be referred to as l -vectors or differential forms of degree l . The exterior and interior product of forms are defined for $\alpha \in \Lambda^1$, $u \in \Lambda^l$, respectively, by

$$\alpha \wedge u := \Pi_{l+1}(\alpha u) \quad \text{and} \quad \alpha \vee u := -\Pi_{l-1}(\alpha u). \quad (2.3)$$

By linearity, these operations extend to arbitrary $u \in \mathcal{A}_m$ and we have $\alpha u = \alpha \wedge u - \alpha \vee u$.

We shall work with the Dirac operator $D := \sum_{j=1}^m e_j \partial_j$. As is well known, $D^2 = -\Delta$, the negative of the Laplacian in \mathbb{R}^m . The exterior derivative operator and its formal transpose act on a Λ^l -valued function u by

$$du := \Pi_{l+1}(Du), \quad \delta u := \Pi_{l-1}(Du), \quad (2.4)$$

and then extend by linearity to arbitrary \mathcal{A}_m -valued functions. It follows that

$$D = d + \delta, \quad d^2 = 0, \quad \delta^2 = 0, \quad d\delta + \delta d = -\Delta. \quad (2.5)$$

The Hodge star operator can be defined as the unique linear mapping $*$: $\Lambda^l \rightarrow \Lambda^{m-l}$ such that $e_I(*e_I) = e_1 e_2 \dots e_m$. Thus, possibly except for a sign, $*$ intertwines \wedge with \vee , and d with δ .

For reasons which will become more apparent shortly, we shall find it useful to further embed \mathcal{A}_m into a larger Clifford algebra, say $\mathbb{R}^m \subseteq \mathcal{A}_m \subseteq \mathcal{A}_{m+1}$. In the sequel, it will be important to remember that even though we may work in the larger environment \mathcal{A}_{m+1} , the spaces Λ^l along with the operators Π_l , d , δ and $*$ are all considered relative to the *original* algebra \mathcal{A}_m .

Finally, for $k \in \mathbb{C}$ we set

$$\mathbb{D}_k := D + ke_{m+1}, \quad (2.6)$$

i.e., if u is a \mathcal{A}_{m+1} -valued function defined in a subregion of \mathbb{R}^m then

$$\mathbb{D}_k u := \sum_{j=1}^m e_j \partial_j u + ke_{m+1} u. \quad (2.7)$$

It follows that $-\mathbb{D}_k^2 = \Delta + k^2$. In particular, null solutions of $\mathbb{D}_k u = 0$ are also annihilated by the Helmholtz operator $\Delta + k^2$.

3 Singular integral operators

For each $k \in \mathbb{C}$, we let Φ_k stand for the standard radial fundamental solution for the Helmholtz operator $\Delta + k^2$ in \mathbb{R}^m , that is,

$$\Phi_k(x) := \frac{1}{4i} \left(\frac{k}{2\pi} \right)^{(m-2)/2} \frac{1}{|x|^{(m-2)/2}} H_{\frac{m-2}{2}}^{(1)}(k|x|), \quad x \in \mathbb{R}^m \setminus \{0\}. \quad (3.1)$$

where $H_\alpha^{(1)}$ denotes the Hankel function of first kind and order α .

Consider now Ω a Lipschitz subdomain of \mathbb{R}^m . The associated single layer acoustic potential operator is defined by

$$\mathcal{S}_k f(x) := \int_{\partial\Omega} \Phi_k(x-y) f(y) d\sigma_y, \quad x \in \Omega. \quad (3.2)$$

We shall also need the boundary version of (3.2), i.e. $S_k f := (\mathcal{S}_k f)|_{\partial\Omega}$. A convenient Cauchy type operator is then obtained by setting

$$\mathcal{C}_k := \mathbb{D}_k \mathcal{S}_k = d\mathcal{S}_k + \delta\mathcal{S}_k + ke_{m+1}\mathcal{S}_k. \quad (3.3)$$

Thus, if $C_k := d\mathcal{S}_k + \delta\mathcal{S}_k + ke_{m+1}\mathcal{S}_k$ is the (principal value) boundary version of (3.3), we formally have (cf. [20], [9])

$$-\frac{1}{2}\nu f + C_k f = (C_k f)|_{\partial\Omega} \quad \text{and} \quad \frac{1}{2}(\nu \wedge \nu \vee f) + \nu \wedge C_k f = \nu \wedge (C_k f)|_{\partial\Omega}. \quad (3.4)$$

From now on we shall restrict attention to the physically most relevant case, i.e. $m = 3$, and work with $\mathcal{A}_3 \leftrightarrow \mathcal{A}_4$. Employing standard three-dimensional notation, we note that for any Λ^1 -valued functions u, v ,

$$\operatorname{div} u = -\delta u, \quad \langle u, v \rangle = u \vee v, \quad \operatorname{curl} u = *du, \quad \text{and} \quad u \times v = *(u \wedge v). \quad (3.5)$$

Also, the fundamental solution (3.1) becomes $\Phi_k(x) = -\frac{e^{ik|x|}}{4\pi|x|}$, for $x \in \mathbb{R}^3 \setminus \{0\}$.

In this context, we shall also find it useful to work with the so-called (principal value) double layer acoustic potential operator

$$K_k f(x) := \text{p.v.} \frac{1}{4\pi} \int_{\partial\Omega} \frac{\langle \nu(y), y - x \rangle}{|x - y|^3} e^{ik|x-y|} (1 - ik|x-y|) f(y) d\sigma_y, \quad x \in \partial\Omega, \quad (3.6)$$

and its formal transpose K_k^* . Note that $K_k f = -\operatorname{div} S_k(\nu f)$ and $K_k^* f = \nu \cdot \nabla S_k f$.

Finally, the (principal value) magnetic dipole operator is given by

$$M_k f(x) := \nu(x) \times \left(\text{p.v.} \int_{\partial\Omega} \operatorname{curl}_x \{ \Phi_k(x-y) f(y) \} d\sigma_y \right), \quad x \in \partial\Omega, \quad (3.7)$$

where, this time, f is a vector field defined on $\partial\Omega$. Again, formally, $M_k f = \nu \times (\operatorname{curl} S_k f)$ and $-\frac{1}{2}\nu \times (\nu \times f) + M_k f = \nu \times (\operatorname{curl} S_k f)|_{\partial\Omega}$.

4 Function spaces

Let Ω be a Lipschitz domain and, for each $1 < p < \infty$, denote by $L^p(\partial\Omega)$ the Lebesgue space of p -power integrable functions with respect to the surface measure $d\sigma$. Also, let ∇_{\tan} stand for the tangential gradient on $\partial\Omega$.

We denote by $L_1^p(\partial\Omega)$ the Sobolev space of functions in $L^p(\partial\Omega)$ with tangential gradients in $L^p(\partial\Omega)$, $1 < p < \infty$. Spaces with fractional smoothness can then be defined via complex interpolation, i.e. $L_\theta^p(\partial\Omega) := [L^p(\partial\Omega), L_1^p(\partial\Omega)]_\theta$, $0 < \theta < 1$, $1 < p < \infty$. We also set $L_{-s}^p(\partial\Omega) := (L_s^{p'}(\partial\Omega))^*$ for $0 \leq s \leq 1$, $1 < p, p' < \infty$, $1/p + 1/p' = 1$. Besov spaces with positive smoothness on $\partial\Omega$ can then be introduced via real interpolation, i.e.

$$B_\theta^{p,q}(\partial\Omega) := (L^p(\partial\Omega), L_1^p(\partial\Omega))_{\theta,q}, \quad \text{with } 0 < \theta < 1, 1 < p, q < \infty. \quad (4.1)$$

Also, for $-1 < s < 0$ and $1 < p, q < \infty$, we set

$$B_s^{p,q}(\partial\Omega) := (B_{-s}^{p',q'}(\partial\Omega))^*, \quad 1/p + 1/p' = 1, 1/q + 1/q' = 1. \quad (4.2)$$

We now briefly discuss the case of Sobolev classes in the interior of a Lipschitz domain $\Omega \subset \mathbb{R}^3$ and their traces. First, $L_s^p(\mathbb{R}^3) := \{(I - \Delta)^{s/2} f; f \in L^p(\mathbb{R}^3)\}$ for $1 < p < \infty$, $s \in \mathbb{R}$, and for $s \geq 0$ we denote by $L_s^p(\Omega)$ the restriction of elements in $L_s^p(\mathbb{R}^3)$ to Ω . Following [10], for $s \in \mathbb{R}$ we define the space $L_{s,0}^p(\Omega)$ to consist of distributions in $L_s^p(\mathbb{R}^3)$ supported in $\overline{\Omega}$ (with the norm inherited from $L_s^p(\mathbb{R}^3)$). Recall (cf. [10]) that the trace operator

$$\text{Tr} : L_s^p(\Omega) \longrightarrow B_{s-\frac{1}{p}}^{p,p}(\partial\Omega) \quad (4.3)$$

is well defined, bounded and onto if $1 < p < \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$. Also, this operator has a bounded right inverse, and its null space is precisely $L_{s,0}^p(\Omega)$; cf. [10, Proposition 3.3]. For positive s , $L_{-s}^p(\Omega)$ is defined as the space of distributions in Ω such that

$$\|f\|_{L_{-s}^p(\Omega)} := \sup \left\{ |\langle f, \varphi \rangle|; \varphi \in C_{\text{comp}}^\infty(\Omega), \|\tilde{\varphi}\|_{L_s^{p'}(\mathbb{R}^3)} \leq 1 \right\} < \infty, \quad (4.4)$$

where tilde denotes the extension by zero outside Ω and $\frac{1}{p} + \frac{1}{p'} = 1$. A more detailed exposition of these and other related matters can be found in [10], [23], [22]. Here we only want to point out that $L_s^p(\Omega, \mathbb{R}^3) := L_s^p(\Omega) \otimes \mathbb{R}^3$, $L_s^p(\Omega, \mathcal{A}_3) := L_s^p(\Omega) \otimes \mathcal{A}_3$, $L_s^p(\Omega, \mathcal{A}_4) := L_s^p(\Omega) \otimes \mathcal{A}_4$. A similar convention is in place for spaces defined on $\partial\Omega$, and for the Besov scale.

Next, we discuss the surface divergence operator. First, for $1 < p < \infty$, we set

$$L_{\text{tan}}^p(\partial\Omega) := \{f \in L^p(\partial\Omega, \mathbb{R}^3); \langle \nu, f \rangle = 0 \text{ a.e. on } \partial\Omega\}, \quad (4.5)$$

and introduce

$$\text{Div} : L_{\text{tan}}^p(\partial\Omega) \rightarrow L_{-1}^p(\partial\Omega), \quad \int_{\partial\Omega} g \text{Div} f \, d\sigma = - \int_{\partial\Omega} \langle f, \nabla_{\text{tan}} g \rangle \, d\sigma, \quad (4.6)$$

for each $f \in L_{\text{tan}}^p(\partial\Omega)$, and $g \in L_1^{p'}(\partial\Omega) = (L_{-1}^p(\partial\Omega))^*$, $1/p + 1/p' = 1$. For $1 < p < \infty$, a space which is going to be important for us in the sequel is

$$L_{\text{tan}}^{p,\text{Div}}(\partial\Omega) := \{f \in L_{\text{tan}}^p(\partial\Omega); \text{Div} f \in L^p(\partial\Omega)\}, \quad (4.7)$$

equipped with $\|f\|_{L_{\text{tan}}^{p,\text{Div}}(\partial\Omega)} := \|f\|_{L^p(\partial\Omega, \mathbb{R}^3)} + \|\text{Div} f\|_{L^p(\partial\Omega)}$.

5 Distributional tangential and normal traces

Let $\Omega \subset \mathbb{R}^3$ be an arbitrary Lipschitz domain with outward unit normal ν . We shall need to work with some Sobolev-like spaces which are naturally adapted to the type of differential operators we intend to study. Specifically, for $1 < p < \infty$, $s \in \mathbb{R}$, introduce

$$H^{s,p}(\Omega; \text{curl}) := \{u \in L_s^p(\Omega, \mathbb{R}^3); \text{curl} u \in L_s^p(\Omega, \mathbb{R}^3)\}, \quad (5.1)$$

equipped with the natural graph norm. Throughout the paper, all derivatives are taken in the sense of distributions. If $u \in H^{s,p}(\Omega; \text{curl})$ for some $1 < p < \infty$ and $-1 + 1/p < s < 1/p$, then we can define $\nu \times u \in B_{s-\frac{1}{p}}^{p,p}(\partial\Omega, \mathbb{R}^3)$ by

$$\langle \nu \times u, \text{Tr} \varphi \rangle := \iint_{\Omega} [\langle \text{curl} u, \varphi \rangle - \langle u, \text{curl} \varphi \rangle], \quad (5.2)$$

for any $\varphi \in L_{1-s}^{p'}(\Omega, \mathbb{R}^3)$, $1/p + 1/p' = 1$. It follows from (5.1), (5.2) that the operator

$$\nu \times : H^{s,p}(\Omega; \text{curl}) \longrightarrow B_{s-\frac{1}{p}}^{p,p}(\partial\Omega, \mathbb{R}^3) \quad (5.3)$$

is bounded as long as $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. Note that if, e.g., $u \in C^1(\overline{\Omega}, \mathbb{R}^3)$ then $\nu \times u$ coincides with the usual (pointwise) cross product with ν .

We now proceed to describe the image of the operator (5.3). Following [21], for $1 < p < \infty$ and $-1 + 1/p < s < 1/p$, we introduce the space

$$TH_s^p(\partial\Omega) := \left\{ f \in B_{s-\frac{1}{p}}^{p,p}(\partial\Omega, \mathbb{R}^3); f = \nu \times u \text{ for some } u \in H^{s,p}(\Omega; \text{curl}) \right\}, \quad (5.4)$$

endowed with the norm

$$\|f\|_{TH_s^p(\partial\Omega)} := \inf \left\{ \|u\|_{L_s^p(\Omega, \mathbb{R}^3)} + \|\text{curl } u\|_{L_s^p(\Omega, \mathbb{R}^3)}; f = \nu \times u \right\}. \quad (5.5)$$

As such, $TH_s^p(\partial\Omega)$ is a reflexive Banach space, continuously and densely embedded into the Besov space $B_{s-\frac{1}{p}}^{p,p}(\partial\Omega, \mathbb{R}^3)$; cf. [21].

The extension of the definition of the surface divergence in the context of (5.4), i.e.

$$\text{Div} : TH_s^p(\partial\Omega) \longrightarrow B_{s-\frac{1}{p}}^{p,p}(\partial\Omega), \quad 1 < p < \infty, \quad -1 + 1/p < s < 1/p, \quad (5.6)$$

is obtained by setting

$$\text{Div } f := -\nu \cdot (\text{curl } u), \quad (5.7)$$

if $f \in TH_s^p(\partial\Omega)$ is of the form $f = \nu \times u$, for some $u \in H^{s,p}(\Omega; \text{curl})$. One can check that the definitions (5.6)-(5.7) and (4.6) are compatible. The following result has been recently proved in [21].

Proposition 5.1 *If $1 < p < \infty$, $-1 + 1/p < s < 1/p$, then for each $f \in TH_s^p(\partial\Omega)$,*

$$\|f\|_{TH_s^p(\partial\Omega)} \approx \|f\|_{B_{s-\frac{1}{p}}^{p,p}(\partial\Omega, \mathbb{R}^3)} + \|\text{Div } f\|_{B_{s-\frac{1}{p}}^{p,p}(\partial\Omega)}. \quad (5.8)$$

Analogously to (5.1), we introduce

$$H^{s,p}(\Omega; \text{div}) := \{u \in L_s^p(\Omega, \mathbb{R}^3); \text{div } u \in L_s^p(\Omega)\}, \quad (5.9)$$

equipped with the natural graph norm. If $1 < p < \infty$, $-1 + 1/p < s < 1/p$, and $u \in H^{s,p}(\Omega; \text{div})$, then we can define $\nu \cdot u \in B_{s-\frac{1}{p}}^{p,p}(\partial\Omega)$ by setting

$$\langle \nu \cdot u, \text{Tr } \psi \rangle := \iint_{\Omega} [\psi \text{div } u + \langle u, \nabla \psi \rangle] \quad (5.10)$$

for each $\psi \in L_{1-s}^{p'}(\Omega)$, $1/p + 1/p' = 1$. It follows that the operator

$$\nu \cdot : H^{s,p}(\Omega; \text{div}) \longrightarrow B_{s-\frac{1}{p}}^{p,p}(\partial\Omega) \quad (5.11)$$

is bounded for $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. The following has been proved in [21].

Proposition 5.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then the operator (5.11) is onto for each $1 < p < \infty$, $-1 + 1/p < s < 1/p$.*

Another operator of interest for us is

$$\nu \times \nabla_{\tan} : B_{1-\frac{1}{p}+s}^{p,p}(\partial\Omega) \longrightarrow TH_s^p(\partial\Omega), \quad 1 < p < \infty, \quad -1 + 1/p < s < 1/p, \quad (5.12)$$

defined by $(\nu \times \nabla_{\tan})f := \nu \times (\nabla u)$ for $f \in B_{1-\frac{1}{p}+s}^{p,p}(\partial\Omega)$, if $u \in L_{s+1}^p(\Omega)$ is so that $f = \text{Tr } u$.

In analogy with (5.1)-(5.4), for $1 < p < \infty$, $-1 + 1/p < s < 1/p$, we next introduce a distinguished subspace of $B_{s-1/p}^{p,p}(\partial\Omega, \mathcal{A}_4)$. More concretely, if

$$H_s^{p,d}(\Omega, \mathcal{A}_4) := \{u \in L_s^p(\Omega, \mathcal{A}_4); du \in L_s^p(\Omega, \mathcal{A}_4)\} \quad (5.13)$$

(equipped with the natural graph norm), we set

$$\langle \nu \wedge u, \varphi \rangle := \int \int_{\Omega} \langle du, v \rangle - \int \int_{\Omega} \langle u, \delta v \rangle, \quad (5.14)$$

for any $u \in H_s^{p,d}(\Omega, \mathcal{A}_4)$ and any $\varphi \in L_{1-s}^{p'}(\Omega, \mathcal{A}_4)$, $1/p + 1/p' = 1$, with $\text{Tr } v = \varphi$. It follows that $\nu \wedge u \in B_{s-1/p}^{p,p}(\partial\Omega, \mathcal{A}_4)$ and we define

$$\mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4) := \{\nu \wedge u; u \in H_s^{p,d}(\Omega, \mathcal{A}_4)\}, \quad (5.15)$$

equipped with the natural (infimum type) norm. We also define the (bounded) operator

$$d_{\partial} : \mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4) \longrightarrow \mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4), \quad d_{\partial}(\nu \wedge u) := -\nu \wedge (du), \quad (5.16)$$

and notice that $d_{\partial}d_{\partial} = 0$. Going further, we set

$$L_{\text{nor}}^p(\partial\Omega, \mathcal{A}_4) := \{f \in L^p(\partial\Omega, \mathcal{A}_4); \nu \wedge f = 0 \text{ a.e. on } \partial\Omega\} \quad (5.17)$$

and extend the operator d_{∂} to this context. In concrete terms, for $f \in L_{\text{nor}}^p(\partial\Omega, \mathcal{A}_4)$, we define the distribution $d_{\partial}f$ by requiring that

$$\int_{\partial\Omega} \langle d_{\partial}f, \psi \rangle d\sigma = \int_{\partial\Omega} \langle f, \delta\psi \rangle d\sigma \quad \text{for any } \psi \in C^1(\mathbb{R}^3, \mathcal{A}_4). \quad (5.18)$$

Then we define

$$L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4) := \{f \in L_{\text{nor}}^p(\partial\Omega, \mathcal{A}_4); d_{\partial}f \in L^p(\partial\Omega, \mathcal{A}_4)\}, \quad (5.19)$$

and equip it with the norm $\|f\|_{L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4)} := \|f\|_{L^p(\partial\Omega, \mathcal{A}_4)} + \|d_{\partial}f\|_{L^p(\partial\Omega, \mathcal{A}_4)}$.

Since $d\mathcal{S}_k = \mathcal{S}_k d_{\partial}$ (cf. [9]), the techniques in [4] can be adapted to produce the following.

Proposition 5.3 *There holds $d\mathcal{C}_k = -\mathcal{C}_k d_{\partial}$ both on $\mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4)$ and on $L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4)$. Also,*

$$\|\mathcal{N}(\mathcal{C}_k f)\|_{L^p(\partial\Omega)} + \|\mathcal{N}(d\mathcal{C}_k f)\|_{L^p(\partial\Omega)} \leq \kappa \|f\|_{L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4)}, \quad (5.20)$$

$$\|\mathcal{C}_k f\|_{H_s^{p,d}(\Omega, \mathcal{A}_4)} \leq \kappa \|f\|_{\mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4)}, \quad (5.21)$$

for some $\kappa = \kappa(\partial\Omega, k) > 0$. In particular, $\nu \wedge \mathcal{C}_k$ is a bounded endomorphism of $\mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4)$ and $L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4)$. In both cases, $d_{\partial}(\nu \wedge \mathcal{C}_k) = (\nu \wedge \mathcal{C}_k)d_{\partial}$.

6 Boundary value problems

The main results of this section are Theorems 6.3-6.4, dealing with half-Dirichlet problems for the Dirac operator, with data in Lebesgue, Sobolev and Besov type spaces. As a preliminary matter, in Theorems 6.1-6.2, we study invertibility properties of the (boundary) Cauchy operators on the relevant spaces.

Theorem 6.1 *For each Lipschitz domain $\Omega \subset \mathbb{R}^3$ with compact boundary, there exist a discrete subset \mathcal{U} of \mathbb{R} and some $\varepsilon > 0$ with the following significance. Whenever $k \in \mathbb{C} \setminus \mathcal{U}$, the Cauchy operators*

$$\pm \frac{1}{2}I + \nu \wedge C_k : L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4) \rightarrow L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4) \quad (6.1)$$

are invertible for $1 < p < 2 + \varepsilon$. Also, $\pm \frac{1}{2}I + \nu \wedge C_k : L_{\text{nor}}^p(\partial\Omega, \mathcal{A}_4) \rightarrow L_{\text{nor}}^p(\partial\Omega, \mathcal{A}_4)$ are invertible if $2 - \varepsilon < p < 2 + \varepsilon$ and $k \in \mathbb{C} \setminus \mathcal{U}$.

Proof. When $2 - \varepsilon < p < 2 + \varepsilon$ this is proved in [20]. Thus, it suffices to show that, whenever $1 < p < 2 + \varepsilon$, the operators (6.1) are Fredholm with index zero for each $k \in \mathbb{C}$. Our first observation is that $\mathcal{C}_k e_4 = -e_4 \mathcal{C}_{-k}$, which entails $(\pm \frac{1}{2}I + \nu \wedge C_k) e_4 = -e_4 (\mp \frac{1}{2}I + \nu \wedge C_{-k})$. Also, $L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4) = L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_3) \oplus e_4 [L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_3)]$, and

$$L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_3) = \nu L_1^p(\partial\Omega) \oplus \left[*L_{\text{tan}}^{p,\text{Div}}(\partial\Omega) \right] \oplus [*L^p(\partial\Omega)]. \quad (6.2)$$

This last identity follows from the fact that if $f \in L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_3)$ decomposes as $f = \nu f_0 + *(f_1) + *(f'_0)$ then, at the L^p -level,

$$d_{\partial} f = - * (\nu \times \nabla_{\text{tan}} f_0) - *(\text{Div } f_1). \quad (6.3)$$

Thus, if $f = F + e_4 \tilde{F}$, where both F, \tilde{F} are \mathcal{A}_3 -valued, we have

$$(\pm \frac{1}{2}I + \nu \wedge C_k) f = (\pm \frac{1}{2}I + \nu \wedge C_k) F - e_4 (\mp \frac{1}{2}I + \nu \wedge C_{-k}) \tilde{F}. \quad (6.4)$$

Now, in general,

$$\mathcal{C}_k f = d\mathcal{S}_k f + \delta\mathcal{S}_k f + k e_4 \mathcal{S}_k f = \delta\mathcal{S}_k f + \mathcal{S}_k(d_{\partial} f) + k e_4 \mathcal{S}_k f, \quad (6.5)$$

from which it follows that whenever f is \mathcal{A}_3 -valued,

$$\begin{aligned} (\pm \frac{1}{2}I + \nu \wedge C_k) f &= \nu [(\pm \frac{1}{2}I + K_k) f_0] + *[(\pm \frac{1}{2}I + M_k) f_1] + *[(\pm \frac{1}{2}I - K_k^*) f'_0] \\ &\quad + \nu \wedge \mathcal{S}_k(d_{\partial} f) + k e_4 \nu \wedge \mathcal{S}_k f. \end{aligned} \quad (6.6)$$

The crux of the matter is now that $\pm \frac{1}{2}I + K_k, \pm \frac{1}{2}I + M_k, \pm \frac{1}{2}I + K_k^*$ are all Fredholm with index zero on $L_1^p(\partial\Omega), L_{\text{tan}}^{p,\text{Div}}(\partial\Omega)$ and $L^p(\partial\Omega)$, respectively, for each $1 < p < 2 + \varepsilon$. See [5], [25], [17]. Consequently, since the last two operators in (6.6) are compact on $L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4)$ for any $1 < p < \infty$, it follows from this and (6.4) that (6.1) are Fredholm with index zero whenever $1 < p < 2 + \varepsilon$. \square

Theorem 6.2 *For each bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ there exist a discrete subset \mathcal{U} of \mathbb{R} and some $\varepsilon > 0$ with the following significance. Whenever*

$$1 < p < \infty, \quad -1 + \frac{1}{p} < s < \frac{1}{p}, \quad \frac{1}{3} - \varepsilon < \frac{1}{p} - \frac{s}{3} < \frac{2}{3} + \varepsilon, \quad (6.7)$$

and $k \in \mathbb{C} \setminus \mathcal{U}$, the operators

$$\pm \frac{1}{2}I + \nu \wedge C_k : \mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4) \rightarrow \mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4) \quad (6.8)$$

are invertible.

Proof. Again, the main step is to show that, for the indicated range of pairs (s, p) , the operators (6.8) are Fredholm with index zero for each $k \in \mathbb{C}$. The starting point is the decomposition

$$\mathbb{H}_s^p(\partial\Omega, \mathcal{A}_3) = \nu B_{1+s-1/p}^{p,p}(\partial\Omega) \oplus [*TH_s^p(\partial\Omega)] \oplus \left[*B_{s-1/p}^{p,p}(\partial\Omega) \right]. \quad (6.9)$$

This is analogous to (6.2) and relies on Proposition 5.2. Next, we make the observation that (6.6) continues to hold in this setting. The heart of the matter is now that $\pm \frac{1}{2}I + K_k$, $\pm \frac{1}{2}I + M_k$, $\pm \frac{1}{2}I + K_k^*$ are all Fredholm with index zero on $B_{1+s-1/p}^{p,p}(\partial\Omega)$, $TH_s^p(\partial\Omega)$ and $B_{s-1/p}^{p,p}(\partial\Omega)$, respectively, as long as (6.7) is satisfied. See [6], [22], [21]. Furthermore, the last two operators in (6.6) are compact on $\mathbb{H}_s^p(\partial\Omega, \mathcal{A}_4)$ for any $1 < p < \infty$, $-1 + 1/p < s < 1/p$. We may therefore conclude as before that the operators (6.8) are Fredholm with index zero for the range of indices specified in the problem. \square

We now turn attention to boundary value problems. In the theorem below, emphasis is placed on nontangential maximal function estimates.

Theorem 6.3 *Let $\Omega \subset \mathbb{R}^3$ be an arbitrary Lipschitz domain with compact boundary. Then there exist $\varepsilon = \varepsilon(\Omega) > 0$ and a sequence of real, nonnegative numbers $\{k_j\}_j$ such that for each $1 < p < 2 + \varepsilon$ and $k \in \mathbb{C} \setminus \{\pm k_j\}_j$ the boundary value problem*

$$\begin{cases} u \in C^1(\Omega, \mathcal{A}_4), \\ \mathcal{D}_k u = 0 \text{ in } \Omega, \\ \mathcal{N}(u), \mathcal{N}(du), \mathcal{N}(\delta u) \in L^p(\partial\Omega), \\ \nu \wedge u = f \in L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4), \end{cases} \quad (6.10)$$

has a unique solution. Moreover, there exists $\kappa = \kappa(\partial\Omega, k) > 0$ so that the solution satisfies

$$\|\mathcal{N}(u)\|_{L^p(\partial\Omega)} + \|\mathcal{N}(du)\|_{L^p(\partial\Omega)} + \|\mathcal{N}(\delta u)\|_{L^p(\partial\Omega)} \leq \kappa \|f\|_{L_{\text{nor}}^{p,d}(\partial\Omega, \mathcal{A}_4)}. \quad (6.11)$$

Furthermore, if $2 - \varepsilon < p < 2 + \varepsilon$, then for each $k \in \mathbb{C} \setminus \{\pm k_j\}_j$ the boundary problem

$$\begin{cases} u \in C^1(\Omega, \mathcal{A}_4), \\ \mathcal{D}_k u = 0 \text{ in } \Omega, \\ \mathcal{N}(u) \in L^p(\partial\Omega), \\ \nu \wedge u = f \in L_{\text{nor}}^p(\partial\Omega, \mathcal{A}_4), \end{cases} \quad (6.12)$$

is well-posed. In particular, $\|\mathcal{N}(u)\|_{L^p(\partial\Omega)} \leq \kappa\|f\|_{L^p_{\text{nor}}(\partial\Omega, \mathcal{A}_4)}$.

Proof. The case of (6.10) and (6.12) is all dimensions for $2 - \varepsilon < p < 2 + \varepsilon$ has been treated in §7 of [20]. Thus, we shall concentrate on (6.10) when $1 < p \leq 2$. Let $\{k_j\}_j$ be a sequence of nonnegative numbers such that the operators (6.1) are invertible for each $k \notin \{\pm k_j\}_j$, when $1 < p \leq 2$. Then a solution obeying the estimate (6.11) can be found in the form

$$u := \mathcal{C}_k[(\tfrac{1}{2}I + \nu \wedge C_k)^{-1}f] \quad \text{in } \Omega. \quad (6.13)$$

There remains uniqueness to which we now turn. As mentioned before, when $2 - \varepsilon < p < 2 + \varepsilon$ this has been handled in [20] and, so we claim, matters can always be reduced to this case. To see this, notice that if u is a null-solution of (6.10) then Cauchy's reproducing formula gives $u = \mathcal{C}_k(\nu u)$ in Ω . Going to the boundary and taking $\nu \vee$ of both sides then yields $(-\tfrac{1}{2}I + \nu \vee C_k)(\nu \vee u) = 0$. Applying Hodge's star isomorphism further transforms this identity into $(-\tfrac{1}{2}I + \nu \wedge C_k)[*(\nu \vee u)] = 0$. The bottom line is that

$$*(\nu \vee u) \in \text{Ker}(-\tfrac{1}{2}I + \nu \wedge C_k; L^p_{\text{nor}}(\partial\Omega, \mathcal{A}_4)). \quad (6.14)$$

However, thanks to Theorem 6.1, the null-space space in (6.14) is independent of $p \in (1, 2 + \varepsilon)$; cf. Lemma 1.2.4 in [16]. Thus, $\nu \vee u \in L^2(\partial\Omega, \mathcal{A}_4)$ and, ultimately, $\mathcal{N}u \in L^2(\partial\Omega)$ from Cauchy's reproducing formula. At this stage, the conclusion follows from the L^2 -uniqueness result alluded to above. \square

Let us point out that for any $k \in \{\pm k_j\}_j$ both (6.10) and (6.12) are Fredholm solvable, i.e. existence and uniqueness hold modulo finite dimensional spaces.

Next, we deal with the Dirac Poisson problem in Sobolev-Besov spaces.

Theorem 6.4 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exist $\varepsilon = \varepsilon(\Omega) > 0$ and a sequence of real, nonnegative numbers $\{k_j\}_j$ such that if $1 < p < \infty$ and $s \in (-1 + 1/p, 1/p)$ satisfy (6.7), then the boundary value problem*

$$\begin{cases} u \in L^p_s(\Omega, \mathcal{A}_4), \\ \mathbb{D}_k u = \eta \in L^p_s(\Omega, \mathcal{A}_4), \\ du, \delta u \in L^p_s(\Omega, \mathcal{A}_4), \\ \nu \wedge u = f \in \mathbb{H}^p_s(\partial\Omega, \mathcal{A}_4), \end{cases} \quad (6.15)$$

has a unique solution whenever $k \in \mathbb{C} \setminus \{\pm k_j\}_j$. Moreover, the solution satisfies

$$\|u\|_{L^p_s(\Omega, \mathcal{A}_4)} + \|du\|_{L^p_s(\Omega, \mathcal{A}_4)} + \|\delta u\|_{L^p_s(\Omega, \mathcal{A}_4)} \leq \kappa\|f\|_{\mathbb{H}^p_s(\partial\Omega, \mathcal{A}_4)}. \quad (6.16)$$

Proof. Let $(\Delta + k^2)^{-1}$ stand for the (acoustic) Newtonian potential, i.e. $(\Delta + k^2)^{-1}w(x) := \iint_{\Omega} \Phi_k(x - y)w(y) dy$, $x \in \Omega$. It has been proved in [21] that $(\Delta + k^2)^{-1} : L^p_s(\Omega) \rightarrow L^p_{s+2}(\Omega)$ is well defined and bounded if $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Thus, a solution for (6.15) is given by

$$u := \mathcal{C}_k[(\tfrac{1}{2}I + \nu \wedge C_k)^{-1}(f + \nu \wedge \mathbb{D}_k(\Delta + k^2)^{-1}\eta)] - \mathbb{D}_k(\Delta + k^2)^{-1}\eta, \quad (6.17)$$

provided $k \in \mathbb{C}$ is such that the inverse in (6.17) exists. This, however, is taken care by invoking Theorem 6.2. Note that (6.17) automatically satisfies the estimate (6.16). Uniqueness can then be proved by arguing much as in the proof of Theorem 6.3. \square

Once again, if $k \in \{\pm k_j\}_j$ then the problem (6.15) becomes Fredholm solvable.

7 Connections with Maxwell's equations

Let us rephrase $\mathcal{D}_k u = \eta$ exclusively in terms of homogeneous vectors. To this effect, assume

$$u = U - ie_4 \tilde{U}, \quad \eta = \Upsilon + ie_4 \tilde{\Upsilon}, \quad (7.1)$$

where

$$U = \sum_{j=0}^1 [U_j + *U'_j], \quad \tilde{U} = \sum_{j=0}^1 [\tilde{U}_j + *\tilde{U}'_j], \quad \Upsilon = \sum_{j=0}^1 [\Upsilon_j + *\Upsilon'_j], \quad \tilde{\Upsilon} = \sum_{j=0}^1 [\tilde{\Upsilon}_j + *\tilde{\Upsilon}'_j], \quad (7.2)$$

and each function carrying the subscript j is Λ^j -valued. Then

$$\mathcal{D}_k u = \eta \Leftrightarrow \mathcal{P}_k((U_j)_j, (U'_j)_j, (\tilde{U}_j)_j, (\tilde{U}'_j)_j) = ((\Upsilon_j)_j, (\Upsilon'_j)_j, (\tilde{\Upsilon}_j)_j, (\tilde{\Upsilon}'_j)_j), \quad (7.3)$$

where

$$\mathcal{P}_k := \begin{pmatrix} 0 & -\text{div} & 0 & 0 & ik & 0 & 0 & 0 \\ \nabla & 0 & 0 & \text{curl} & 0 & ik & 0 & 0 \\ 0 & \text{curl} & -\nabla & 0 & 0 & 0 & 0 & ik \\ 0 & 0 & \text{div} & 0 & 0 & 0 & ik & 0 \\ -ik & 0 & 0 & 0 & 0 & -\text{div} & 0 & 0 \\ 0 & -ik & 0 & 0 & \nabla & 0 & 0 & \text{curl} \\ 0 & 0 & 0 & -ik & 0 & \text{curl} & -\nabla & 0 \\ 0 & 0 & -ik & 0 & 0 & 0 & 0 & \text{div} \end{pmatrix}. \quad (7.4)$$

Also, if

$$f = F + ie_4 \tilde{F}, \quad F = \sum_{j=0}^1 [F_j + *F'_j], \quad \tilde{F} = \sum_{j=0}^1 [\tilde{F}_j + *\tilde{F}'_j], \quad (7.5)$$

where, once again, functions labeled with the subscript j are Λ^j -valued, then

$$\nu \wedge u = f \text{ on } \partial\Omega \Leftrightarrow \begin{cases} F_0 = 0, & F_1 = \nu U_0|_{\partial\Omega}, & F'_1 = \nu \times U_1|_{\partial\Omega}, & F'_0 = \langle \nu, U'_1|_{\partial\Omega} \rangle, \\ \tilde{F}_0 = 0, & \tilde{F}_1 = \nu \tilde{U}_0|_{\partial\Omega}, & \tilde{F}'_1 = \nu \times \tilde{U}_1|_{\partial\Omega}, & \tilde{F}'_0 = \langle \nu, \tilde{U}'_1|_{\partial\Omega} \rangle, \end{cases} \quad (7.6)$$

It is also not too hard to check from definitions that

$$\mathcal{D}_k u = 0 \text{ in } \Omega, \text{ and } \nu \wedge u = f \text{ on } \partial\Omega \Rightarrow \begin{cases} \partial_\nu U'_0 = ik\tilde{F}'_0 - \text{Div } F'_1 \\ \partial_\nu \tilde{U}'_0 = -ikF'_0 - \text{Div } \tilde{F}'_1. \end{cases} \quad (7.7)$$

In particular, there are two Dirichlet problems for the Helmholtz operator, i.e.

$$\begin{cases} (\Delta + k^2)U_0 = 0 \text{ in } \Omega, \\ U_0|_{\partial\Omega} = \langle \nu, F_1 \rangle, \end{cases} \quad \begin{cases} (\Delta + k^2)\tilde{U}_0 = 0 \text{ in } \Omega, \\ \tilde{U}_0|_{\partial\Omega} = \langle \nu, F_1 \rangle, \end{cases} \quad (7.8)$$

and two Neumann problems for the Helmholtz operator, i.e.

$$\begin{cases} (\Delta + k^2)U'_0 = 0 \text{ in } \Omega, \\ \partial_\nu U'_0|_{\partial\Omega} = ikF'_0 - \text{Div } \tilde{F}'_1, \end{cases} \quad \begin{cases} (\Delta + k^2)\tilde{U}'_0 = 0 \text{ in } \Omega, \\ \partial_\nu \tilde{U}'_0|_{\partial\Omega} = -ik\tilde{F}'_0 - \text{Div } F'_1, \end{cases} \quad (7.9)$$

which are implicit in the equation $\mathcal{D}_k u = 0$.

Theorem 7.1 *For each Ω , Lipschitz domain in \mathbb{R}^3 with compact boundary, there exist $\varepsilon > 0$ and a sequence of nonnegative numbers $\{k_j\}_j$ such that the following are true.*

- (i) *For each $1 < p < 2 + \varepsilon$, $k \in \mathbb{C} \setminus \{\pm k_j\}_j$ the boundary problem (6.10), with u, f written componentwise as in (7.1)-(7.2) and (7.5), respectively, reduces to two Maxwell systems (with opposite wave numbers), i.e.*

$$\begin{cases} \text{curl } \tilde{U}_1 - ikU'_1 = 0 \text{ in } \Omega, \\ \text{curl } U'_1 + ik\tilde{U}_1 = 0 \text{ in } \Omega, \\ \nu \times \tilde{U}_1 = \tilde{F}'_1 \text{ on } \partial\Omega, \end{cases} \quad \begin{cases} \text{curl } U_1 + ik\tilde{U}'_1 = 0 \text{ in } \Omega, \\ \text{curl } \tilde{U}'_1 - ikU_1 = 0 \text{ in } \Omega, \\ \nu \times U_1 = F'_1 \text{ on } \partial\Omega, \end{cases} \quad (7.10)$$

with boundary data in $L_{\text{tan}}^{p, \text{Div}}(\partial\Omega)$, if and only if

$$d_\partial f + ke_4 f \text{ is } (\Lambda^2 + e_4 \Lambda^2)\text{-valued.} \quad (7.11)$$

- (ii) *For each $2 - \varepsilon < p < 2 + \varepsilon$, $k \in \mathbb{C} \setminus \{\pm k_j\}_j$, the boundary problem (6.12), reduces to the two Maxwell systems (7.10) with boundary data in $L_{\text{tan}}^{p, \text{Div}}(\partial\Omega)$ if and only if $f \in L_{\text{nor}}^{p, d}(\partial\Omega, \mathcal{A}_4)$ and (7.11) holds.*

- (iii) *For each $k \in \mathbb{C} \setminus \{\pm k_j\}_j$ and s, p so that (6.7) holds, the boundary problem (6.15) with $\eta = 0$ reduces to the two Maxwell systems (7.10) with boundary data in $TH_s^p(\partial\Omega)$ if and only if (7.11) holds.*

In all cases, the connection between the boundary data for the Dirac equation and that for the Maxwell systems (7.10) reads

$$- * [(ik)^{-1}(d_\partial f + ke_4 f)] = \tilde{F}'_1 + ie_4 F'_1. \quad (7.12)$$

Proof. The point (ii) has been proved in [20]. As for (i) and (iii), the desired conclusion follows from the observation that (7.11) is equivalent to

$$F_1 = \tilde{F}_1 = 0, \quad ikF'_0 - \text{Div } \tilde{F}'_1 = 0, \quad -ik\tilde{F}'_0 - \text{Div } F'_1 = 0, \quad (7.13)$$

in concert with the well-posedness of the following PDE's: the Dirichlet problems (7.8) with boundary data either in $L_1^p(\partial\Omega)$ if $1 < p < 2 + \varepsilon$, or in $B_{1+s-1/p}^{p,p}(\partial\Omega)$ if (6.7) is satisfied; the Neumann problems (7.9) with boundary data either in $L^p(\partial\Omega)$ if $1 < p < 2 + \varepsilon$, or in $B_{s-1/p}^{p,p}(\partial\Omega)$ if (6.7) is satisfied; the Maxwell systems (7.10) with boundary data either in $L_{\tan}^{p,\text{Div}}(\partial\Omega)$ if $1 < p < 2 + \varepsilon$, or in $TH_s^p(\partial\Omega)$ if (6.7) is satisfied. \square

Making $F_1' = 0$ in the previous theorem we obtain the following.

Corollary 7.2 *In each of the well-posedness instances described in §6, the problem $\mathbb{D}_k u = 0$ in Ω , $\nu \wedge u = f$ on $\partial\Omega$ reduces to just one Maxwell system of the type (1.1) if and only if the boundary datum f is $(\Lambda^3 + e_4 \Lambda^2)$ -valued and satisfies $d_\partial f - k e_4 \wedge f = 0$.*

*In particular, the aforementioned Dirac half-Dirichlet boundary problem reduces precisely to (1.1) if and only if $f = (ik)^{-1} * \text{Div } h + i e_4(*h)$.*

Below we collect a few closing remarks.

- (i) Essentially, Theorem 7.1 can be regarded as an ‘elliptization’ method for the original Maxwell system. Roughly speaking, the Maxwell system is ‘embedded’ into a more general, elliptic system via a procedure which also identifies the (more specialized) type of boundary data for which the two systems are actually equivalent.
- (ii) Via Hodge duality, it follows that boundary value problems similar to (6.10), (6.12), (6.15), but in which $\nu \vee u$ is prescribed on the boundary, are also well-posed.
- (iii) All our main results continue to hold in the context of variable coefficient Hodge-Dirac operators and, more generally, on Lipschitz subdomains of three dimensional Riemannian manifolds. Moreover, for $2 - \varepsilon < p < 2 + \varepsilon$, our results are valid in *all* space dimensions. For related developments, see [22], [18], [20], [21].

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