Chapter 3

Calderón-Zygmund theory

3.1 Singular integral operators of Cauchy type

The starting point is the following result addressing the boundedness of the Cauchy operator on Lipschitz graphs. Denote by $\mathcal{L}(L^p \to L^p)$ the Banach space of bounded, linear operators on L^p.

3.1.1 **Theorem 3.1.1.** *Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with $\|A'\|_{L^\infty(\mathbb{R})} \leq M$. Then for each $1 < p < \infty$ there exist a constant C with the following significance. For each $\varepsilon > 0$, the operator*

$$\mathcal{C}_{A,\varepsilon} f(x) := \int_{|x-y|>\varepsilon} \frac{f(y)}{x - y + i(A(x) - A(y))}\, dy \tag{3.1.1}$$ e3.1.1

satisfies

$$\|\mathcal{C}_{A,\varepsilon}\|_{\mathcal{L}(L^p \to L^p)} \leq C(1 + M). \tag{3.1.2}$$ e3.1.2

Proof. For the first part we present a proof given by M. Melnikov and J. Verdera [MeVe]. For starters, we note that the Cauchy integral operator (3.1.1) can be written as

$$C_\varepsilon f(x) = \int_{|y-x|>\varepsilon} \frac{f(y)}{z(y) - z(x)}\, dy \tag{3.1.3}$$ e3.1.3

where $z(x) = x + iA(x)$ is the natural parametrization of the graph of the Lipschitz function $A : \mathbb{R} \to \mathbb{R}$. Consider some interval $I \subset \mathbb{R}$ with characteristic function χ_I. Note that

$$\int_I |C_\varepsilon(\chi_I)|^2 = \int_{T_\varepsilon} \frac{1}{z(y) - z(x)} \overline{\frac{1}{z(t) - z(x)}}\, dxdydt, \tag{3.1.4}$$ e3.1.4

where

$$T_\varepsilon := \{(x, y, t) \in I^3 : \, |y - x| > \varepsilon, \| t - x| > \varepsilon\}.$$

The key ingredient in the proof by Melnikov-Verdera is symmetrizing the above integral, both with respect to the kernel as well as the domain. This, in turn, brings the so-called *Menger curvature* into play, the main novelty in the approach under discussion. Turning to the actual details, consider the symmetric domain

$$S_\varepsilon := \{(x,y,t) \in I^3 :\ |y-x| > \varepsilon,\ |t-x| > \varepsilon,\ |t-y| > \varepsilon\}.$$

Then the integral (3.1.4) over the domain T_ε is equivalent to the one over S_ε in the sense that

$$\int_I |C_\varepsilon(\chi_I)|^2 = \int_{S_\varepsilon} \frac{1}{z(y)-z(x)} \frac{1}{\overline{z(t)-z(x)}}\, dxdydt + \mathcal{O}(|I|). \tag{3.1.5}$$

e3.1.5

This is not hard to see if we split the remaining set $T_\varepsilon \setminus S_\varepsilon$ into

$$U_{\varepsilon,1} := \{(x,y,t) \in I^3 :\ |y-x| > \varepsilon,\ |t-x| > 2\varepsilon,\ |t-y| < \varepsilon\}, \tag{3.1.6}$$

e3.1.6

$$U_{\varepsilon,2} := \{(x,y,t) \in I^3 :\ |y-x| > \varepsilon,\ 2\varepsilon > |t-x| > \varepsilon,\ |t-y| < \varepsilon\}. \tag{3.1.7}$$

e3.1.7

Majorizing the absolute value of the integrand by ε^2 on $U_{\varepsilon,2}$ and by $\frac{2}{|t-x|^2}$ on $U_{\varepsilon,1}$ allows us to bound

$$\int_{U_{\varepsilon,j}} \left| \frac{1}{z(y)-z(x)} \frac{1}{\overline{z(t)-z(x)}} \right| dxdydt \le C|I| \tag{3.1.8}$$

e3.1.8

for $j = 1, 2$ and for some numerical constant C. To symmetrize the kernel, we average over all possible permutations of the variables $x_1 := x$, $x_2 := y$, $x_3 := t$, so that

$$6\int_I |C_\varepsilon(\chi_I)|^2 = \int_{S_\varepsilon} \sum_\sigma \frac{1}{z(x_{\sigma(2)})-z(x_{\sigma(1)})} \frac{1}{\overline{z(x_{\sigma(3)})-z(x_{\sigma(1)})}}\, dx_1dx_2dx_3 + \mathcal{O}(|I|). \tag{3.1.9}$$

e3.1.9

The main geometric observation to be made here is that the symmetrized integrand becomes

$$\sum_\sigma \frac{1}{z(x_{\sigma(2)})-z(x_{\sigma(1)})} \frac{1}{\overline{z(x_{\sigma(3)})-z(x_{\sigma(1)})}} = c^2(z_1,z_2,z_3), \tag{3.1.10}$$

e3.1.10

where $c(z_1, z_2, z_3)$ is the Menger curvature associated with the points $z_1 := z(x_1)$, $z_2 := z(x_2)$, and $z_3 := z(x_3)$. The latter can be defined as

$$c(z_1,z_2,z_3) := \frac{4S(z_1,z_2,z_3)}{|z_2-z_1||z_3-z_1||z_2-z_3|}, \tag{3.1.11}$$

e3.1.11

with $S(z_1, z_2, z_3)$ denoting the area of triangle with vertices at z_1, z_2 and z_3.

The equality (3.1.10) can be justified using a simple argument to the effect that

$$c(z_1,z_2,z_3) = \frac{2\sin\alpha_{ij}}{|z_i-z_j|}, \tag{3.1.12}$$

e3.1.12

where α_{ij} is the angle opposite to the side z_iz_j in the triangle under consideration, and the elementary formula

$$\mathrm{Re}\left(\frac{1}{a\bar{b}}+\frac{1}{\overline{b(b-a)}}-\frac{1}{(b-a)\overline{a}}\right)=2\left(\frac{|a|^2|b|^2-\big(\mathrm{Re}(a\bar{b})\big)^2}{|a|^2|b|^2|b-a|^2}\right) \tag{3.1.13}$$

e3.1.13

applied to the complex numbers $a := z_2 - z_1$ and $b := z_3 - z_1$. Now we may invoke the formula

$$\begin{aligned} 2S(z_1,z_2,z_3) &= \big|\mathrm{Im}\,[(z_2-z_1)(z_3-z_1)]\big| \\ &= \big|(x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1)\big|, \end{aligned} \tag{3.1.14}$$

e3.1.14

(with $z_j := x_j + iy_j$, $j = 1, 2, 3$), in order to conclude that

$$c(z(x),z(y),z(t)) \leq 2\left|\frac{\frac{A(y)-A(x)}{y-x}-\frac{A(t)-A(x)}{t-x}}{t-y}\right|. \tag{3.1.15}$$

e3.1.15

The next claim we would like to make here is that for every locally absolutely continuous function a such that $a' \in L^2(\mathbb{R})$ there holds

$$\int_{\mathbb{R}^3}\left(\frac{\frac{a(y)-a(x)}{y-x}-\frac{a(t)-a(x)}{t-x}}{t-y}\right)^2 dxdydt = C\|a'\|^2_{L^2(\mathbb{R})} \tag{3.1.16}$$

e3.1.16

for some numerical constant C. This can be seen as follows.

Making a change of variables $h = y - x$ and $k = t - x$, we may invoke Plancherel's inequality in x to rewrite the integral in (3.1.16) as

$$\int_{\mathbb{R}^3}\left|\frac{\frac{e^{i\xi h}-1}{\xi h}-\frac{e^{i\xi k}-1)}{\xi k}}{h-k}\right|^2 |a'(\xi)|^2\, d\xi dh dk. \tag{3.1.17}$$

e3.1.17

Next, we set $u = \xi h$, $v = \xi k$ and $E(u) := \frac{e^{iu}-1}{iu}$, so that the last expression becomes

$$\|a'\|^2_{L^2(\mathbb{R})}\int_{\mathbb{R}^2}\left|\frac{E(u+t)-E(u)}{t}\right|^2 dudt. \tag{3.1.18}$$

e3.1.18

There remains to show that the integral in (3.1.18) is finite. To this end, upon noticing that $\hat{E} = 2\pi\chi_{(0,1)}$ (where 'hat' denotes the Fourier transform) Plancherel's formula allows us to bound the double integral above by the numerical constant:

CHECK NORMALIZATION CONSTANTS FOR THE FOURIER TRANSFORM !!!

$$\begin{aligned} \int_{\mathbb{R}^2}\left|\frac{e^{i\xi t}-1}{t}\right|^2 d\xi dt &= 2\pi\int_{-\infty}^{+\infty}\int_0^1\left|\frac{e^{i\xi t}-1}{t}\right|^2 d\xi dt \\ &= 2\pi\int_{|t|<1}\int_0^1\left|\frac{e^{i\xi t}-1}{t}\right|^2 d\xi dt \end{aligned}$$

$$+2\pi \int_{|t|>1} \int_0^1 \left| \frac{e^{i\xi t}-1}{t} \right|^2 d\xi dt < \infty. \tag{3.1.19}$$ e3.1.19

That the last two integrals above are finite follows by estimating $|(e^{i\xi t}-1)/t|^2$ by $|\xi|^2$ and $4/t^2$, respectively.

Now we may use (3.1.16) to prove that, for every finite interval $I \subset \mathbb{R}$,

$$\int_{I^3} c^2(z(x), z(y), z(t))\, dxdydt \le C\|A'\|^2_{L^\infty(\mathbb{R})}|I|. \tag{3.1.20}$$ e3.1.20

On the interval $I = [x_0, x_1]$ consider $P_I(x) = A(x_0) + A'_I(x - x_0)$, where $A'_I = \frac{1}{|I|}\int_I A'$ is the average of A'. Then, invoking (3.1.15), the left hand-side in (3.1.20) can be bounded by

$$4 \int_{\mathbb{R}^3} \left| \frac{\frac{a(y)-a(x)}{y-x} - \frac{a(t)-a(x)}{t-x}}{t-y} \right|^2 dxdydt \tag{3.1.21}$$ e3.1.21

for the choice $a := (A - P_I)\chi_I$. This, in turn, can be controlled by

$$4\|(A' - A'_I)\chi_I\|^2_{L^2(\mathbb{R})} \le 16C\|A'\|^2_{L^\infty(\mathbb{R})}|I|, \tag{3.1.22}$$ e3.1.22

where (3.1.16) has been invoked. As a result, we arrive at the estimate

$$\int_I |C_\varepsilon(\chi_I)|^2 \le C|I| \tag{3.1.23}$$ e3.1.23

for every interval $I \subset \mathbb{R}$.

Having disposed of (3.1.23), we can then relatively painlessly show that

$$C_\varepsilon : L^\infty(\mathbb{R}) \to BMO(\mathbb{R}) \quad \text{and} \quad C_\varepsilon : H^1(\mathbb{R}) \to L^1(\mathbb{R}), \tag{3.1.24}$$ e3.1.24

are bounded operators with norms bounded independently of $\varepsilon > 0$. The argument is then finished via interpolation [FeSt]:

$$C_\varepsilon : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \quad 1 < p < \infty. \tag{3.1.25}$$ e3.1.25

First, consider some real function $b \in L^\infty(\mathbb{R})$ supported in the interval I. Then

$$\begin{aligned} &2\int_I |C_\varepsilon(b)|^2 - 4\,\mathrm{Re}\int_I C_\varepsilon(b)\overline{C_\varepsilon(\chi_I)}b \\ &= \int_{S_\varepsilon} c^2(z(x), z(y), z(t))b(y)b(t)\, dxdydt + \mathcal{O}(\|b\|^2_{L^\infty(\mathbb{R})}|I|). \end{aligned} \tag{3.1.26}$$ e3.1.26

To prove this, observe that

$$
\begin{aligned}
c^2(z(x),z(y),z(t)) = {} & \frac{1}{z(y)-z(x)}\,\frac{1}{\overline{z(t)-z(x)}} + \frac{1}{z(t)-z(x)}\,\frac{1}{\overline{z(y)-z(x)}} \\
& + 2\,\mathrm{Re}\Big(\frac{1}{z(y)-z(t)}\,\frac{1}{\overline{z(x)-z(t)}}\Big) \\
& + 2\,\mathrm{Re}\Big(\frac{1}{z(t)-z(y)}\,\frac{1}{\overline{z(x)-z(y)}}\Big).
\end{aligned} \tag{3.1.27}
$$

e3.1.27

Next, consider separately the terms in right hand-side of (3.1.27). For the first one we may estimate

$$
\begin{aligned}
&\int_{S_\varepsilon} \frac{1}{z(y)-z(x)}\,\frac{1}{\overline{z(t)-z(x)}} b(t)b(y)\,dxdydt \\
&\qquad = \int_{T_\varepsilon(x)} \frac{b(y)}{z(y)-z(x)}\,\frac{b(t)}{\overline{z(t)-z(x)}}\,dxdydt + \mathcal{O}(\|b\|^2_{L^\infty(\mathbb{R})}|I|) \\
&\qquad = \int_I |C_\varepsilon(x)|^2\,dx + \mathcal{O}(\|b\|^2_{L^\infty(\mathbb{R})}|I|),
\end{aligned} \tag{3.1.28}
$$

eq1.39

where $T_\varepsilon(x) = \{(x,y,t)\subset I^3 : |y-x|>\varepsilon,\ |t-x|>\varepsilon\}$. A similar estimate is valid for the second term in the right hand-side of (3.1.27).
The integral of the third term of (3.1.27) can be controlled by observing that

$$
\begin{aligned}
&\int_{S_\varepsilon} \frac{1}{z(y)-z(t)}\,\frac{1}{\overline{z(x)-z(t)}} b(t)b(y)\,dxdydt \\
&\qquad = -\int_{T_\varepsilon(t)} \frac{dx}{\overline{z(t)-z(x)}}\,\frac{b(t)b(y)\,dydt}{z(y)-z(t)} + \mathcal{O}(\|b\|^2_{L^\infty(\mathbb{R})}|I|) \\
&\qquad = -\int_{t\in I}\int_{|y-t|>\varepsilon} \overline{C_\varepsilon\chi_I(t)}\,\frac{b(y)\,dy}{z(y)-z(t)}\,b(t)\,dt + \mathcal{O}(\|b\|^2_{L^\infty(\mathbb{R})}|I|) \\
&\qquad = -\int_I \overline{C_\varepsilon\chi_I(t)}\,C_\varepsilon b(t)\,b(t)\,dt + \mathcal{O}(\|b\|^2_{L^\infty(\mathbb{R})}|I|).
\end{aligned} \tag{3.1.29}
$$

eq1.42

and then taking the real part of all sides. This finishes the proof of (3.1.26).
Then (3.1.26) and (3.1.23) lead to the inequality

$$
\int_I |C_\varepsilon(b)|^2 \le C\|b\|_{L^\infty(\mathbb{R})}|I|^{1/2}\Big(\int_I |C_\varepsilon(b)|^2\Big)^{1/2} + C\|b\|^2_{L^\infty(\mathbb{R})}|I|, \tag{3.1.30}
$$

eq1.46

which, in turn, implies

$$
\int_I |C_\varepsilon(b)| \le C\|b\|_{L^\infty(\mathbb{R})}|I|, \tag{3.1.31}
$$

eq1.47

for every essentially bounded function b supported in the interval I.

The next order of business is to use the last estimate to prove that the Cauchy operator maps $L^\infty(\mathbb{R})$ boundedly into $BMO(\mathbb{R})$. First, we need to define the operator C_ε on an arbitrary function $f \in L^\infty(\mathbb{R})$ (not necessarily compactly supported).

Consider the set of all intervals on the real line with rational endpoints, $\mathcal{I}$. Then for every pair of points $x_1, x_2 \in \mathbb{R} \setminus \mathbb{Q}$ choose $I \in \mathcal{I}$ such that $x_1, x_2 \in I$. Let us put $f_1 := f\chi_{2I}$, $f_2 := f - f_1$, and

$$\begin{aligned} F(x_1,x_2) \; &:= \; (C_\varepsilon f_1)(x_1) - (C_\varepsilon f_1)(x_2) \\ &\quad + \int_{\mathbb{R}} \Big(\frac{1}{z(x_1)-z(t)} - \frac{1}{z(x_2)-z(t)} \Big) f_2(t)\, dt. \end{aligned} \tag{3.1.32}$$

eq1.48,49,50

Note that the definition of $F(x_1,x_2)$ does not depend on I. Also, for a.e. $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned} F(x,x_1) - F(x,x_2) \; &= \; -(C_\varepsilon f_1)(x_1) + (C_\varepsilon f_1)(x_2) \\ &\quad + \int_{\mathbb{R}} \Big(-\frac{1}{z(x_1)-z(t)} + \frac{1}{z(x_2)-z(t)} \Big) f_2(t)\, dt \end{aligned} \tag{3.1.33}$$

eq1.51

is a constant with respect to x.

This justifies defining the action of the operator C_ε on some arbitrary function $f \in L^\infty(\mathbb{R})$ as the equivalence class (modulo constants) of $x \mapsto F(x_1, x)$, for a.e. $x_1 \in \mathbb{R}$.

Now we only need to prove that the Cauchy operator defined in this way is bounded on $BMO(\mathbb{R})$. By (3.1.31)

$$\int_I |C_\varepsilon f_1(x)|\, dx \le \int_{2I} |C_\varepsilon f_1(x)|\, dx \le C\|f\|_{L^\infty(\mathbb{R})}|I|. \tag{3.1.34}$$

eq1.52

On the other hand, with x_0 denoting the center of the interval I, for every $x \in I$

$$\begin{aligned} &\Big| \int_{\mathbb{R}\setminus 2I} \Big(\frac{1}{z(x)-z(t)} - \frac{1}{z(x_0)-z(t)} \Big) f_2(t)\, dt \Big| \\ &\qquad \le \Big| \int_{\mathbb{R}\setminus 2I} \frac{z(x_0)-z(x)}{(z(x)-z(t))(z(x_0)-z(t))} f_2(t)\, dt \Big| \\ &\qquad \le C\|f\|_{L^\infty(\mathbb{R})}. \end{aligned} \tag{3.1.35}$$

eq1.53,54,55

Based on this and the definition of F, we can therefore conclude that

$$\int_I \big| F(x_0,x) - C_\varepsilon f_1(x_0) \big|\, dx \le C\|f\|_{L^\infty(\mathbb{R})}|I|, \tag{3.1.36}$$

eq1.56

for every interval $I \in \mathcal{I}$. This proves the first statement in (3.1.24).

It is not much harder to show that the second assertion in (3.1.24) holds true. Consider a - atom in $H^1(\mathbb{R})$. The goal is to show that

$$\|C_\varepsilon a\|_{L^1(\mathbb{R})} < C \tag{3.1.37}$$

eq1.57

for some constant C independent of the particular atom. Denoting the support of the atom a by $I := [x_0 - r, x_0 + r]$, we split the discussion into two cases. First, if $x \in (2I)^c$,

$$\begin{aligned} |C_\varepsilon a(x)| &\leq \int_I \frac{|z(x_0) - z(y)|}{|z(y) - z(x)||z(x_0) - z(x)|} |a(y)|\, dy \\ &\leq C \frac{r}{|x - x_0|^2} \int_I |a(y)|\, dy \leq C \frac{r}{|x - x_0|^2}, \end{aligned} \tag{3.1.38}$$

eq1.58,59

hence,

$$\int_{(2I)^c} |C_\varepsilon a(x)|\, dx = 2 \int_{x_0 + 2r}^{+\infty} \frac{r}{|x - x_0|^2}\, dx \leq C. \tag{3.1.39}$$

eq1.60

Second, if $x \in 2I$, then by (3.1.31)

$$\int_{2I} |C_\varepsilon a(x)|\, dx \leq C \|a\|_{L^\infty(\mathbb{R})} |I| \leq C < \infty. \tag{3.1.40}$$

eq1.61

This finishes the proof of (3.1.24) and, as it was mentioned before, the conclusion (3.1.25) follows by interpolation.

3.1.2 **Corollary 3.1.2.** *Let $B : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with $0 < \beta^{-1} < B'(x) < \beta < \infty$ for some constant $\beta > 1$. Then for each $\varepsilon > 0$ and each $\eta \in [-1, 1]$ the operator*

$$\tilde{\mathcal{C}}_{B,\eta,\varepsilon} f(x) := \int_{|x-y|>\varepsilon} \frac{f(y)}{\eta(x - y)i + B(x) - B(y)}\, dy \tag{3.1.41}$$

eq1.3

satisfies

$$\|\tilde{\mathcal{C}}_{B,\eta,\varepsilon}\|_{\mathcal{L}(L^p \to L^p)} \leq C\beta^4. \tag{3.1.42}$$

eq1.4

To prove this, a simple technical result is going to be of importance. Recall that $\mathcal{M}$ stands for the usual Hardy-Littlewood maximal operator.

l1.1 **Lemma 3.1.3.** *Assume that*

$$C_1|x - y| \leq \rho(x, y) \leq C_2|x - y|, \qquad x, y \in \mathbb{R}^n, \tag{3.1.43}$$

eq1.5

and

$$|k(x, y)| \leq \frac{C_0}{|x - y|^n}, \qquad x, y \in \mathbb{R}^n, \tag{3.1.44}$$

eq1.6

for some constants C_0, C_1, C_2. Then

$$\begin{aligned}\Delta(x) &:= \left| \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\mathbb{R}^n}} k(x,y)f(y)dy - \int\limits_{\substack{\rho(x-y)>\varepsilon \\ y\in\mathbb{R}^n}} k(x,y)f(y)dy \right| \\ &\leq C_0(C_1^{-n}+C_2^n)\mathcal{M}f(x) \qquad (3.1.45)\end{aligned}$$ eq1.7

uniformly for $x \in \mathbb{R}^n$.

Proof. A direct size estimate gives

$$\begin{aligned}\Delta(x) &\leq \int_{\substack{|x-y|>\varepsilon,\, \rho(x,y)<\varepsilon \\ y\in\mathbb{R}^n}} \frac{C_0}{|x-y|^n}|f(y)|dy + \int_{\substack{|x-y|<\varepsilon,\, \rho(x,y)>\varepsilon \\ y\in\mathbb{R}^n}} \frac{C_0}{|x-y|^n}|f(y)|dy \\ &:= I + II, \qquad (3.1.46)\end{aligned}$$ eq1.8

where the last equality defines I, II. We have:

$$I \leq \frac{C_0}{\varepsilon^n}\int_{C_1|x-y|<\varepsilon}|f(y)|dy \leq \frac{C_0}{C_1^n}\mathcal{M}f(x), \qquad (3.1.47)$$ eq1.9

and

$$II \leq \frac{C_0C_2^n}{\varepsilon^n}\int_{|x-y|<\varepsilon}|f(y)|dy \leq C_0C_2^n\mathcal{M}f(x). \qquad (3.1.48)$$ eq1.10

The desired conclusion follows. □

To prove Corollary 3.1.2, first, make a change of variables $s = B(x)$, $t = B(y)$. Then we may write

$$(\tilde{\mathcal{C}}_{B,\eta,\varepsilon}f)(B^{-1}(s)) = \int_{|B^{-1}(s)-B^{-1}(t)|>\varepsilon} \frac{f(B^{-1}(t))[B'(B^{-1}(t))]^{-1}}{s-t+i\eta(B^{-1}(s)-B^{-1}(t))}\,dt. \qquad (3.1.49)$$ eq1.62

In particular,

$$|(\tilde{\mathcal{C}}_{B,\eta,\varepsilon}f)(B^{-1}(s))| \leq |\mathcal{C}_{\eta B^{-1},\varepsilon}((f/B')\circ B^{-1})(s)| + C\beta^3\mathcal{M}f(B^{-1}(s)), \qquad (3.1.50)$$ eq1.63

uniformly for $s \in \mathbb{R}$. The desired conclusion can now be deduced from (3.1.50) with the help of (3.1.2). □

t1.2 **Theorem 3.1.4.** *Suppose* $F(z)$ *is an analytic function in the open strip* $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < 2\}$. *Let* $A : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function with* $\|A'\|_{L^\infty} \leq M$. *Then there exists a universal constant* C *such that for each* $\varepsilon > 0$, *the operator*

$$K_{A,F,\varepsilon}f(x) := \int_{|x-y|>\varepsilon} \frac{1}{x-y}F\Big(\frac{A(x)-A(y)}{x-y}\Big)f(y)\,dy, \qquad x \in \mathbb{R}, \qquad (3.1.51)$$ eq1.64

satisfies

$$\|K_{A,F,\varepsilon}\|_{\mathcal{L}(L^p\to L^p)} \leq C(1+M^4)\, sup\,\{|F(z)|;\ z\in\mathbb{C},\ |\mathrm{Im}\, z|<2\}. \tag{3.1.52}$$
eq1.65

Proof. Let $\gamma^1_\pm := \{\zeta = u \pm i:\ |u|\leq 2M\}$, $\gamma^2_\pm := \{\zeta = \pm 2M + iv:\ |v|\leq 1\}$, and set $\gamma := \gamma^1_+ \cup \gamma^2_+ \cup \gamma^1_- \cup \gamma^2_-$. Since F is analytic for $z\in\mathbb{C}$ with $|\mathrm{Im}\, z|<2$, it follows that

$$F(s) = \frac{1}{2\pi\, i}\int_\gamma \frac{F(\zeta)}{\zeta - s}\, d\zeta = \frac{1}{2\pi\, i}\int_{\gamma^1_+\cup\gamma^1_-} \frac{F(\zeta)}{\zeta - s}\, d\zeta + \frac{1}{2\pi\, i}\int_{\gamma^2_+\cup\gamma^2_-} \frac{F(\zeta)}{\zeta - s}\, d\zeta. \tag{3.1.53}$$
eq1.66

Accordingly,

$$\begin{aligned} K_{A,F,\varepsilon} f(x) &= \frac{1}{2\pi\, i}\int_{\gamma^1_+\cup\gamma^1_-} F(\zeta)\int_{|x-y|>\varepsilon} \frac{1}{x-y}\frac{f(y)}{\zeta - \frac{A(x)-A(y)}{x-y}}\, dy\, d\zeta \\ &\quad + \frac{1}{2\pi\, i}\int_{\gamma^2_+\cup\gamma^2_-} F(\zeta)\int_{|x-y|>\varepsilon} \frac{1}{x-y}\frac{f(y)}{\zeta - \frac{A(x)-A(y)}{x-y}}\, dy\, d\zeta \\ &= I_+ + I_- + II_+ + II_-, \end{aligned} \tag{3.1.54}$$
eq1.67

where

$$I_\pm := \mp\frac{1}{2\pi}\int_{\gamma^1_\pm} F(\zeta)\int_{|x-y|>\varepsilon} \frac{f(y)}{x-y+i[A^\pm_\zeta(x) - A^\pm_\zeta(y)]}\, dy\, d\zeta \tag{3.1.55}$$
eq1.68

with $A^\pm_\zeta(x) := \mp[A(x) - (\mathrm{Re}\,\zeta)x]$, and

$$II_\pm := \frac{1}{2\pi\, i}\int_{\gamma^2_\pm} F(\zeta)\int_{|x-y|>\varepsilon} \frac{f(y)}{(\mathrm{Im}\,\zeta)(x-y)i + [B^\pm(x) - B^\pm(y)]}\, dy\, d\zeta \tag{3.1.56}$$
eq1.69

with $B^\pm(x) := -[A(x) \mp 2Mx]$.

At this point, the proof is concluded by invoking Theorem 3.1.1. □

The main result as far as the *one-dimensional* theory of Calderón-Zygmund operators is as follows.

t1.3 **Theorem 3.1.5.** *Suppose $F\in C^N(\mathbb{R})$, $N\geq 6$ and assume that $A:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function with $\|A'\|_{L^\infty(\mathbb{R}^n)}\leq M$. Then for each $1<p<\infty$ there exists a constant C_p such that the operator (3.1.51) satisfies*

$$\|K_{A,F,\varepsilon}\|_{\mathcal{L}(L^p\to L^p)} \leq C_p(1+M^4)\sup\{|F^{(k)}(x)|;\ |x|\leq M+1,\ 0\leq k\leq 6\}, \tag{3.1.57}$$
eq1.70

uniformly in $\varepsilon>0$.

Proof. There is no loss of generality in assuming that F is supported in the interval $[-M-1, M+1]$. With 'hat' denoting the Fourier transform we have

$$F(s) = \int_{\mathbb{R}} e^{is\xi} \hat{F}(\xi)\, d\xi \tag{3.1.58}$$ eq1.71

so that

$$K_{A,F,\varepsilon} f(x) = \int_{\mathbb{R}} \hat{F}(\xi) \Big(\int_{|x-y|>\varepsilon} \frac{1}{x-y} e^{i\xi \frac{A(x)-A(y)}{x-y}} f(y)\, dy \Big) d\xi. \tag{3.1.59}$$ eq1.72

Note that the inner integral is the Cauchy operator (3.1.1) corresponding to the choice $F(z) = \exp(iz)$ and with A replaced by ξA. Consequently, by (3.1.52),

$$\begin{aligned} \|K_{A,F,\varepsilon} f\|_{\mathcal{L}(L^p \to L^p)} &\leq C \int_{\mathbb{R}} (1 + (M|\xi|)^4) |\hat{F}(\xi)|\, d\xi \\ &\leq C(1+M^4) \\ &\quad \sup \{ |F^{(k)}(x)|;\, |x| \leq M+1,\, 0 \leq k \leq 6 \}, \end{aligned} \tag{3.1.60}$$ eq1.73

as desired. □

3.2 Higher dimensional singular integrals

We now discuss a *higher-dimensional* version of the previous theorem.

t1.4 **Theorem 3.2.1.** *Suppose $F \in C^N(\mathbb{R}^m)$, $|N| \geq 5+m$, F is odd, and assume that $A : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function with $\|\nabla A\|_{L^\infty(\mathbb{R}^n, \mathbb{R}^m)} \leq M$. Then for each $1 < p < \infty$ there exists a constant C such that the operator*

$$K_{A,F,\varepsilon} f(x) := \int_{|x-y|>\varepsilon} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|} \Big) f(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{3.2.1}$$ eq1.74

satisfies

$$\|K_{A,F,\varepsilon}\|_{\mathcal{L}(L^p \to L^p)} \leq C_p (1+M^4) \sup \{ |D^\alpha F(z)| :\, |z| \leq M+1,\, |\alpha| \leq 5+m \}, \tag{3.2.2}$$ eq1.75

uniformly in $\varepsilon > 0$.

Proof. In the case $n = 1$,

$$\frac{1}{|x-y|} F\Big(\frac{A(x)-A(y)}{|x-y|} \Big) = \frac{1}{x-y} F\Big(\frac{A(x)-A(y)}{x-y} \Big) \tag{3.2.3}$$ eq1.76

since F is odd, so that (3.2.2) follows from an argument similar to the one used in the proof of Theorem 3.1.5, i.e. writing

$$K_{A,F,\varepsilon}f(x) = \int_{\mathbb{R}^m} \hat{F}(\xi)\Big(\int_{|x-y|>\varepsilon} \frac{1}{x-y} e^{i\langle \xi, \frac{A(x)-A(y)}{x-y}\rangle} f(y)\,dy\Big)d\xi \tag{3.2.4}$$
eq1.77

and invoking Theorem 3.1.4.

For $n > 1$ we can reduce the problem to the 1-dimensional case by the *method of rotation*. First, assume that $f \in C_0^\infty(\mathbb{R}^n)$. Then the integral operator in (3.2.1) can be considered as

$$\begin{aligned} K_{A,F,\varepsilon}f(x) &= \int_{\mathbb{R}^n} k(x,y)f(y)\,dy \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \big(k(x,x+z)f(x+z) + k(x,x-z)f(x-z)\big)\,dz \end{aligned} \tag{3.2.5}$$
eq1.79

with kernel

$$k(x,y) := \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\chi_{\{y\in\mathbb{R}^n:\,|x-y|>\varepsilon\}}. \tag{3.2.6}$$
eq1.78

Next, in polar coordinates

$$K_{A,F,\varepsilon}f(x) = \frac{1}{2}\int_{S^{n-1}} K_\omega f(x)\,d\omega, \text{ where } K_\omega f(x) = \int_{\mathbb{R}} k(x,x+r\omega)f(x+r\omega)|r|^{n-1}\,dr. \tag{3.2.7}$$
eq1.80

Then the goal is to prove

$$\|K_\omega f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.2.8}$$
eq1.81

for some constant C independent of ω.

Every $x \in \mathbb{R}^n$ can be uniquely represented as $x = t\omega + y$, where y belongs to the orthogonal complement of the line $\{t\omega;\ t \in \mathbb{R}\}$ (ω is fixed). Then

$$\begin{aligned} K_\omega f(t\omega+y) &= \int_{\mathbb{R}} k(t\omega+y,(t+r)\omega+y)f((t+r)\omega+y)|r|^{n-1}\,dr \\ &= \int_{\mathbb{R}} k(t\omega+y,s\omega+y)f(s\omega+y)|s-t|^{n-1}\,ds \end{aligned} \tag{3.2.9}$$
eq1.82

can be considered as a 1-dimensional Calderón-Zygmund operator in (3.2.1) and, therefore, is bounded on $L^p(\mathbb{R})$ by the argument presented above. Hence,

$$\|K_\omega f\|^p_{L^p(\mathbb{R}^n)} \le C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} |f(t\omega+y)|^p\,dtdy = C\|f\|^p_{L^p(\mathbb{R}^n)}, \tag{3.2.10}$$
eq1.84

since $dtdy$ is equivalent to the Lebesgue measure in $\mathbb{R}^n$ for each fixed $\omega \in S^{n-1}$. The last bound concludes the proof of Theorem 3.2.1 since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, for $1 < p < \infty$. □

Handling boundary singular integral operators on Lipschitz boundaries and L^p-based function spaces requires the use of the rather sophisticated machinery known as Calderón-Zygmund theory. Below we record the variants which are best suited for the applications we have in mind.

t2.5 **Theorem 3.2.2.** *Let $A:\mathbb{R}^n\to\mathbb{R}^m$ be a Lipschitz function with Lipschitz contant M, and assume that $F:\mathbb{R}^m\to\mathbb{R}$, $F\in C^1(\mathbb{R}^m)$, F is odd function. For $x,y\in\mathbb{R}^n$ with $x\neq y$ we set $K(x,y):=\frac{1}{|x-y|^n}F\left(\frac{A(x)-A(y)}{|x-y|}\right)$, and for $\varepsilon>0$, $f\in Lip_{comp}(\mathbb{R}^n)$, we define the truncated operator*

$$T_\varepsilon f(x):=\int_{|x-y|>\varepsilon}K(x,y)f(y)\,dy. \tag{3.2.11}$$ eq2.1-3

Then, for each $1<p<\infty$, the following assertions hold:

1. *The maximal operator $T_*f(x):=\sup\{|T_\varepsilon f(x)|:\varepsilon>0\}$ is bounded on $L^p(\mathbb{R}^n)$. Moreover,*

$$\|T_*\|_{\mathcal{L}(L^p\to L^p)}\leq C_p+C_p(1+M^4)\sup\{|D^\alpha F(z)|:\,|z|\leq M+1,\,|\alpha|\leq 5+m\}. \tag{3.2.12}$$ maxlp

2. *If $1<p<\infty$ and $f\in L^p(\mathbb{R}^n)$ then the limit $\lim_{\varepsilon\to 0}T_\varepsilon f(x)$ exists for almost every $x\in\mathbb{R}^n$ and the operator*

$$Tf(x):=\lim_{\varepsilon\to 0}T_\varepsilon f(x) \tag{3.2.13}$$ operatorT

   is bounded on $L^p(\mathbb{R}^n)$.

3. *The operator (3.2.13) is bounded from $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$.*

4. *The operator (3.2.13) is bounded from $L^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$.*

Let us start the proof of Theorem 3.2.2 with the following results.

l1-3 **Lemma 3.2.3.** *Let $A:\mathbb{R}^n\to\mathbb{R}^m$ and $B:\mathbb{R}^n\to\mathbb{R}^{m'}$ be functions satisfying*

$$|A(x)-A(y)|\leq M|x-y|,\qquad and \tag{3.2.14}$$ e3.2.14

$$M^{-1}|x-y|\leq|B(x)-B(y)|\leq M|x-y|,\quad\forall\,x,y\in\mathbb{R}^n, \tag{3.2.15}$$ e3.2.15

for some positive constant M. Also let $F:\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}$ be a C^1, odd function. Fix $x\in\mathbb{R}^n$ and for each $\varepsilon>0$ consider

$$U(\varepsilon):=\{y\in\mathbb{R}^n:\,1>|x-y|>\varepsilon\}, \tag{3.2.16}$$ e3.2.16

$$V(\varepsilon):=\{y\in\mathbb{R}^n:\,|\nabla B(x)\cdot(x-y)|>\varepsilon,\,|x-y|<1\}, \tag{3.2.17}$$

$$W(\varepsilon) := \{y \in \mathbb{R}^n : |B(x) - B(y)| > \varepsilon, |x - y| < 1\}. \tag{3.2.18}$$

Then

$$\lim_{\varepsilon \searrow 0} \int_{U(\varepsilon)} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\, dy = \lim_{\varepsilon \downarrow 0} \int_{V(\varepsilon)} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\, dy$$
$$= \lim_{\varepsilon \downarrow 0} \int_{W(\varepsilon)} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\, dy, \tag{3.2.19}$$

provided $\nabla A(x)$ *and* $\nabla B(x)$ *exist, and any one of the above three limits exists and is finite.*

Proof. Without loss of generality we can take $x = 0$, $A(0) = 0$, $B(0) = 0$. By Lemma 2.1.1 there exist nonnegative functions $\eta_A(t)$ and $\eta_B(t)$ defined for $t > 0$, so that $\eta_A(t) \downarrow 0$, $\eta_B(t) \downarrow 0$ as $t \downarrow 0$ and

$$|A(y) - \nabla A(0) \cdot y| \le |y|\, \eta_A(|y|), \tag{3.2.20}$$
$$|B(y) - \nabla B(0) \cdot y| \le |y|\, \eta_B(|y|). \tag{3.2.21}$$

If we let $D(\varepsilon) := \{y \in \mathbb{R}^n : \varepsilon > |y| > \varepsilon/M\}$ then $V(\varepsilon/M) \setminus U(\varepsilon) \subset D(\varepsilon)$. Employing the properties of F and (3.2.20), the difference of the first two limits in (3.2.19) becomes

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \Big| \int_{V(\varepsilon/M)\setminus U(\varepsilon)} \frac{1}{|y|^n} F\Big(\frac{A(y)}{|y|}\Big)\, dy \Big| \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{2} \Big| \int_{V(\varepsilon/M)\setminus U(\varepsilon)} \frac{1}{|y|^n} \Big[F\Big(\frac{A(y)}{|y|}\Big) + F\Big(\frac{A(-y)}{|y|}\Big)\Big]\, dy \Big| \\
&= \lim_{\varepsilon \downarrow 0} \frac{1}{2} \Big| \int_{V(\varepsilon/M)\setminus U(\varepsilon)} \frac{1}{|y|^n} \Big[F\Big(\frac{A(y)}{|y|}\Big) - F\Big(-\frac{A(-y)}{|y|}\Big)\Big]\, dy \Big| \\
&\le \Big[\sup_{|\xi|\le M} |\nabla F(\xi)|\Big] \lim_{\varepsilon \downarrow 0} \int_{D(\varepsilon)} \eta_A(|y|)|y|^{-n}\, dy \\
&\le C \lim_{\varepsilon \downarrow 0} \eta_A(\varepsilon) = 0,
\end{aligned}$$

which proves the first equality in (3.2.19).

In order to prove the second one, observe that for $y \in V(\varepsilon)\setminus W(\varepsilon)$ we have $M^{-1}|y| \le |B(y)| < \varepsilon$, so that $|y| < \varepsilon M$. In turn, this forces

$$|\nabla B(0) \cdot y| \le |\nabla B(0) \cdot y - B(y)| + |B(y)| \le \varepsilon M\, \eta_B(\varepsilon M) + \varepsilon, \tag{3.2.22}$$

so that, for $y \in V(\varepsilon)\setminus W(\varepsilon)$,

$$\varepsilon < |\nabla B(0)\cdot y| \le \varepsilon M\, \eta_B(\varepsilon M) + \varepsilon. \tag{3.2.23}$$ e3.2.23

Hence, the measure of $V(\varepsilon)\backslash W(\varepsilon)$ is of the order $\varepsilon^n \eta(\varepsilon M)$ (cf. Exercise ???). Since the integrand restricted to $V(\varepsilon)\backslash W(\varepsilon)$ is of the order ε^{-n} we conclude, by letting $\varepsilon \downarrow 0$, that the integral over the $V(\varepsilon)\backslash W(\varepsilon)$ vanishes.

Resoning in a similar way it follows that

$$\varepsilon - \varepsilon M\, \eta_B(\varepsilon M) < |\nabla B(0)\cdot y| \le \varepsilon, \tag{3.2.24}$$ e3.2.24

uniformly for $y \in W(\varepsilon)\backslash V(\varepsilon)$. Thus, for the same reasons as before, the integral over $W(\varepsilon)\backslash V(\varepsilon)$ also vanishes as $\varepsilon \downarrow 0$, which completes the proof of the lemma. □

l2.2 **Lemma 3.2.4.** *For every function $f \in L^1(\mathbb{R}^n)$ and every positive number λ there is a decomposition $f = g + \sum_{k=1}^{\infty} b_k$ into a "good" function $g \in L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$ and series of "bad" terms b_k with the following properties:*

$$|g(x)| \le 2^n \lambda \ \textit{a.e. in } \mathbb{R}^n, \tag{3.2.25}$$ cd1

$$\textit{supp } b_k \subset Q_k \ \textit{and} \ \int_{\mathbb{R}^n} b_k(x)\, dx = 0 \tag{3.2.26}$$ cd2

for some family of disjoint cubes $\{Q_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} |Q_k| \le \lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)}. \tag{3.2.27}$$ cd3

Moreover,

$$\|g\|_{L^1(\mathbb{R}^n)} + \sum_{k=1}^{\infty} \|b_k\|_{L^1(\mathbb{R}^n)} \le 3\|f\|_{L^1(\mathbb{R}^n)}, \tag{3.2.28}$$ cd4

and

$$\|g\|_{L^2(\mathbb{R}^n)} \le 2^n \lambda^{1/2} \|f\|^{1/2}_{L^1(\mathbb{R}^n)}. \tag{3.2.29}$$ cd4.1

l2.3 **Lemma 3.2.5.** *Let T_ε be the operator defined in (3.2.11)* [eq2.1-3]*. Then there is a constant C such that*

$$|\{x \in \mathbb{R}^n : |T_\varepsilon f(x)| > \lambda\}| \le \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R}^n)} \tag{3.2.30}$$ w2

for every function $f \in L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$ and every positive number λ, i.e., T_ε is "weak L^1" bounded.

We start with the Calderón-Zygmund decomposition of function f (Lemma ??). Denoting by Q_k^* the cube with the same center as Q_k and twice bigger side-length, and by Ω^* the union $\bigcup_{k=1}^{\infty} Q_k^*$, we have

$$|\Omega^*| \leq \sum_{k=1}^{\infty} |Q_k^*| \leq 2^n \sum_{k=1}^{\infty} |Q_k| \leq 2^n \lambda^{-1} \|f\|_{L^1(\mathbb{R}^n)} \tag{3.2.31}$$

by (3.2.27).

Let us set $E := \{x \in \mathbb{R}^n \setminus \Omega^* : |T_\varepsilon f(x) > \lambda\}$. Then our goal is to prove that $|E| \leq C\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}$, because $\{x \in \mathbb{R}^n : |T_\varepsilon f(x) > \lambda\} \subset E \bigcup \Omega^*$ and $|\Omega^*|$ is bounded by $2^n\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}$ (see (3.2.31)).

Observe that $E \subset E_1 \bigcup E_2$ where we define $E_1 := \{x \in E : |T_\varepsilon g(x)| > \lambda/2\}$ and $E_2 := \{x \in E : |T_\varepsilon(\sum_{k=1}^\infty b_k)(x)| > \lambda/2\}$. The measure of E_1 can be controlled using Chebyshev's inequality and the L^2 boundedness of operator T_ε proved in Theorem 3.2.2. Specifically,

$$|E_1| \leq \left(\frac{2}{\lambda}\right)^2 \|T_\varepsilon g\|^2_{L^2(\mathbb{R}^n)} \leq \left(\frac{2}{\lambda}\right)^2 \|g\|^2_{L^2(\mathbb{R}^n)} \leq 4^{n+1}\lambda^{-1}\|f\|_{L^1(\mathbb{R}^n)}. \tag{3.2.32}$$

Next, consider the set E_2. Since $f \in L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$, the series $\sum_{k=1}^\infty b_k$ converges in $L^2(\mathbb{R}^n)$ and $T_\varepsilon(\sum_{k=1}^\infty b_k)(x) = \sum_{k=1}^\infty T_\varepsilon b_k(x)$ for every $x \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n \setminus \Omega^*} |T_\varepsilon(\sum_{k=1}^\infty b_k(x))|\, dx \tag{3.2.33}$$

$$\leq \sum_{k=1}^\infty \int_{\mathbb{R}^n \setminus Q_k^*} \int_{Q_k} |K(x,y) - K(x,y_k)||b_k(y)|\, dydx, \tag{3.2.34}$$

where y_k denotes the center of cube Q_k and the last inequality invokes vanishing moment condition imposed on b_k's (3.2.26). It is not hard to see that for the kernel $K(x,y)$ introduced in (3.5.3) the following holds:

$$\int_{\{x \in \mathbb{R}^n;\, |x-y'| \geq 2|y-y'|\}} |K(x,y') - K(x,y)|\, dx \leq C \tag{3.2.35}$$

for some constant C. Then changing the order of integration in (3.2.34) along with estimate (3.2.35) and (3.2.28) allows us to bound the integral above by the multiple of $\|f\|_{L^1(\mathbb{R}^n)}$.

This completes the proof of Lemma 3.2.5.

□

STATE THE LEMMA BELOW FOR MORE GENERAL KERNELS!!

Cotlar **Lemma 3.2.6.** [Cotlar's inequality] *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function and $F : \mathbb{R}^m \to \mathbb{R}$ be a smooth and odd function. For $x, y \in \mathbb{R}^n$ with $x \neq y$ we set $K(x,y) := \frac{1}{|x-y|^n} F\left(\frac{A(x)-A(y)}{|x-y|}\right)$, and for $\varepsilon > 0$, $f \in Lip_{comp}(\mathbb{R}^n)$, define the truncated operator*

$$T_\varepsilon f(x) := \int_{|x-y|>\varepsilon} K(x,y) f(y)\, dy. \tag{3.2.36}$$

Then, the following estimate holds

$$T_{*,\varepsilon}f(x) \le C\mathcal{M}f(x) + 2\mathcal{M}(T_{\varepsilon_0})f(x), \qquad \forall\, \varepsilon > \varepsilon_0, \tag{3.2.37}$$ `tstarepsilon`

where C is a constant depending on the dimension only, $\varepsilon_0 > 0$ is fixed, $\mathcal{M}$ denotes the usual Hardy-Littlewood maximal operator, and

$$T_{*,\varepsilon}f(x) := \sup_{\varepsilon' > \varepsilon} |T_{\varepsilon'}f(x)|.$$

Proof. Without loss of generality, it suffices to prove (3.2.37) for $x = 0$. Then our goal is to show that

$$|T_\varepsilon f(0)| \le C\mathcal{M}f(0) + 2\mathcal{M}(T_{\varepsilon_0})f(0), \qquad \forall\, \varepsilon > \varepsilon_0. \tag{3.2.38}$$ `eq2.3-3`

It is clear that the estimate (3.2.38) implies (3.2.37) by suitably taking the supremum.

The first step is to show that $\forall x \in \mathbb{R}^n$ and $\forall \varepsilon > 0$,

$$|T_\varepsilon f(x') - T_\varepsilon f(x)| \le C\mathcal{M}f(0), \tag{3.2.39}$$ `eq2.5-3`

provided $|x - x'| \le \varepsilon/2$. This can be done as follows:

$$\begin{aligned} \Big|T_\varepsilon f(x') - T_\varepsilon f(x)\Big| &\le \Big|\int_{|x-y|\ge\varepsilon} (K(x',y) - K(x,y))f(y)dy\Big| \\ &\quad + \Big|\int_{|x'-y|\ge\varepsilon} K(x',y)f(y)dy - \int_{|x-y|\ge\varepsilon} K(x',y)f(y)dy\Big| \\ &:= I + II. \end{aligned} \tag{3.2.40}$$ `eq2.6-3`

The term II can be bounded by a multiple of $\mathcal{M}f(0)$ using the argument similar to that in Lemma 3.1.3. The estimate for I follows from the restrictions imposed on the kernel $K(x,y)$ and the inequality

$$\varepsilon \int_{|y|\ge\varepsilon} |y|^{-n-1}|f(y)|dy \le C\mathcal{M}f(0), \qquad \forall \varepsilon > 0, \tag{3.2.41}$$ `eq2.9-3`

the proof of which is straightforward and we leave it to motivated reader (see Exercise 19).

Let us turn our attention to the proof of (3.2.38). Set $f_1 := f\chi_{B(0,\varepsilon)}$ and $f_2 := f - f_1$, which entails $T_\varepsilon f(0) = T_{\varepsilon_0} f_2(0)$ for every $\varepsilon > \varepsilon_0$. Then for each $x \in B(0, \varepsilon/2)$, by (3.2.39), we have

$$|T_{\varepsilon_0} f_2(x) - T_{\varepsilon_0} f_2(0)| \le C\mathcal{M}f(0), \tag{3.2.42}$$ `eq2.10-3`

and therefore,

$$|T_{\varepsilon_0} f_2(0)| \le |T_{\varepsilon_0} f(x)| + |T_{\varepsilon_0} f_1(x)| + C\mathcal{M}f(0), \qquad \text{for } a.e.\ x \in B(0, \varepsilon/2). \tag{3.2.43}$$ `eq2.11-3`

We finish the proof by analyzing the weak L^1 norms of the above functions. Set

$$N(f) := \sup_{\lambda>0} (\lambda\, \mu(\{x \in B : |f(x)| > \lambda\})), \tag{3.2.44}$$ eq2.12-3

where $B = B(0, \varepsilon/2)$, and μ stands for the measure supported on the ball B of constant density $|B|^{-1}$. Observe that $f(x) = \alpha$ on B implies $N(f) = \alpha$ for any constant α and $N(f_1 + f_2 + f_3) \leq 2N(f_1) + 4N(f_2) + 4N(f_3)$ for every functions f_1, f_2 and f_3.

Then the estimate

$$|T_\varepsilon f(0)| = |T_{\varepsilon_0} f_2(0)| \leq 2N(T_{\varepsilon_0} f) + 4N(T_{\varepsilon_0} f_1) + 4C\mathcal{M}f(0) \tag{3.2.45}$$ eq2.13-3

follows from (3.2.43).

Going further, the right hand-side above can be bounded using Chebyshev's inequality, which yields $N(T_{\varepsilon_0} f) \leq C\mathcal{M}(T_{\varepsilon_0} f)(0)$, and the "weak L^1" boundedness of the operator T (Lemma 7.1.35), which eventually gives $N(T_{\varepsilon_0} f_1) \leq 2^n C\mathcal{M}f(0)$.

□

fromdensetoLp **Lemma 3.2.7.** *Let $\{T_\varepsilon\}_\varepsilon$ be a family of operators with the following properties.*

1. *There exists a dense subspace $\mathcal{V}$ in $L^p(\mathbb{R}^n)$ such that for any $f \in \mathcal{V}$ the limit $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ exists for almost every $x \in \mathbb{R}^n$.*

2. *The maximal operator $T_* f(x) := \sup\{|T_\varepsilon f(x)| : \varepsilon > 0\}$ is bounded on $L^p(\mathbb{R}^n)$.*

Then, the limit $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ exists for any $f \in L^p(\mathbb{R}^n)$ at almost any $x \in \mathbb{R}^n$ and the operator

$$Tf(x) := \lim_{\varepsilon\to 0} T_\varepsilon f(x)$$

is bounded on $L^p(\mathbb{R}^n)$.

SEE WHAT THE WEAKEST HYPOTHESES ARE !!!!!!!!!!!!!!!!

Proof. The boundedness of the operator T is immediate once we prove its existence. In turn, the pointwise convergence in $L^p(\mathbb{R}^n)$ will follow if we show that

$$|\{x \in \mathbb{R}^n : \limsup_{\varepsilon\to 0} T_\varepsilon f(x) \neq \liminf_{\varepsilon\to 0} T_\varepsilon f(x)\}| = 0. \tag{3.2.46}$$ eq2.17-3

Fix $\theta > 0$ and consider

$$S = \{x \in \mathbb{R}^n : \ |\limsup_{\varepsilon\to 0} T_\varepsilon f(x) - \liminf_{\varepsilon\to 0} T_\varepsilon f(x)| > \theta\}. \tag{3.2.47}$$ eq2.18-3

Fix $\delta > 0$ and select $h \in \mathcal{V}$ such that $\|f - h\|_{L^p(\mathbb{R}^n)} < \delta$.

Then $S \subset S_1 \bigcup S_2$ where

$$S_1 = \{x \in \mathbb{R}^n : \ |\limsup_{\varepsilon\to 0} T_\varepsilon f(x) - \lim_{\varepsilon\to 0} T_\varepsilon h(x)| > \theta/2\}, \tag{3.2.48}$$ eq2.19-3

$$S_2 = \{x \in \mathbb{R}^n : \ |\liminf_{\varepsilon\to 0} T_\varepsilon f(x) - \lim_{\varepsilon\to 0} T_\varepsilon h(x)| > \theta/2\}. \tag{3.2.49}$$ eq2.20-3

Then $|S_1|$, the measure of the set S_1, can be estimated in the following way:

$$|S_1| \leq |\{x \in \mathbb{R}^n : T_*(f-h) > \theta/2\}| \leq \left(\frac{2}{\theta}\right)^p \int_{\mathbb{R}^n} |T_*(f-h)|^p d\sigma \qquad (3.2.50)$$

eq2.21-3

$$\leq \left(\frac{2}{\theta}\right)^p \|f-h\|^p_{L^p(\mathbb{R}^n)} \leq C\left(\frac{2}{\theta}\right)^p \delta^p, \qquad (3.2.51)$$

eq2.22-3

which vanishes as $\delta \to 0$. The same consideration works for the set S_2. This concludes the proof of Lemma 3.2.7. □

Proof of Theorem 3.2.2. The first part of the theorem is a consequence of the Cotlar's inequality as stated in Lemma 3.2.6. Making use of (3.2.37), we observe that

$$T_*f(x) = \lim_{\varepsilon\to 0} T_{*,\varepsilon} f(x), \qquad (3.2.52)$$

eq2.14-3

and, therefore, by Lebesgue's Monotone Convergence Theorem,

$$\begin{aligned} \|T_*f\|_{L^p(\mathbb{R}^n)} &= \lim_{\varepsilon\to 0} \|T_{*,\varepsilon} f\|_{L^p(\mathbb{R}^n)} \\ &\leq C \lim_{\varepsilon\to 0}\Big(\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} + \|\mathcal{M}T_{\varepsilon/2}f\|_{L^p(\mathbb{R}^n)}\Big) \leq C\|f\|_{L^p(\mathbb{R}^n)} \qquad (3.2.53) \end{aligned}$$

eq2.15-3

This completes the proof of (1) in Theorem 3.2.2.

In order to show the pointwise convergence stated in (2), Theorem 3.2.2, we will return to particular operators discussed in previous chapters and show, one at a time, that pointwise convergence holds for those operators when applyed to functions in L^p for almost every $x \in \mathbb{R}^n$, this way "building up" the proof of the pointwise convergence for T_ε as in Theorem 3.2.2.

Step 1. *Pointwise convergence for the Cauchy operator from (3.1.1).*

Let $\mathcal{V} := (1+iA')Lip_{\text{comp}}(\mathbb{R})$, which is a dense subclass of $L^p(\mathbb{R})$, $1<p<\infty$, due to A being Lipschitz.

Claim 1. For any $h \in \mathcal{V}$ the limit $\lim_{\varepsilon\to 0} \mathcal{C}_{A,\varepsilon} h(x)$ exists for almost every $x \in \mathbb{R}$.

Indeed, if $h = (1+iA')f$, $f \in Lip_{\text{comp}}(\mathbb{R})$, then we can write

$$\begin{aligned} \mathcal{C}_{A,\varepsilon} h(x) &= \int_{1>|x-y|>\varepsilon} \frac{1+iA'(y)}{x-y+i(A(x)-A(y))}(f(y)-f(x))\,dy \\ &\quad -f(x)\int_{1>|x-y|>\varepsilon} \frac{-(1+iA'(y))}{x-y+i(A(x)-A(y))}\,dy \\ &\quad +\int_{|x-y|>1} \frac{1+iA'(y)}{x-y+i(A(x)-A(y))} f(y)\,dy \\ &=: I + II + III. \qquad (3.2.54) \end{aligned}$$

limitCauchy

Using the fact that f is a compactly supported Lipschitz function, it is immediate that $\lim_{\varepsilon\to 0}(I+III)$ exists. Furthermore, when computing the integral in II, the part depending on ε can be writen as

$$\ln \frac{-1+i\frac{A(x)-A(x+\varepsilon)}{\varepsilon}}{1+i\frac{A(x)-A(x-\varepsilon)}{\varepsilon}}, \tag{3.2.55}$$

whose limit as $\varepsilon \to 0$ exists for almost every $x \in \mathbb{R}$, since A is Lipschitz. This concludes the proof of the Claim 1.

Finally, a combination of Claim 1, Lemma 3.2.7, and part (1) of Theorem 3.2.2 gives that for $f \in L^p(\mathbb{R})$ the limit $\lim_{\varepsilon\to 0}\mathcal{C}_{A,\varepsilon}f(x)$ exists for almost every $x \in \mathbb{R}$.

Step 2. *Pointwise convergence for the Cauchy operator (3.1.41) from Corollary 3.1.2.*

Using Step 1 it follows that, for $f \in L^p(\mathbb{R})$, the limit $\lim_{\varepsilon\to 0}\tilde{\mathcal{C}}_{B,\eta,\varepsilon}f(x)$ exists for almost every $x \in \mathbb{R}$.

Step 3. *Pointwise convergence for the operator (3.1.51) from Theorem 3.1.4.*

Claim 2. If $f \in L^p(\mathbb{R})$, the limit $\lim_{\varepsilon\to 0}K_{A,F,\varepsilon}f(x)$ exists for almost every $x \in \mathbb{R}$. In order to prove Claim 2 fix $f \in L^p(\mathbb{R})$ and recall $I_\pm$, $II_\pm$ as defined in (3.1.54). The goal is to first show that $\lim_{\varepsilon\to 0}I_+$ exists for almost every $x \in \mathbb{R}$. To this end, for x, $\zeta \in \mathbb{R}$ set

$$F_\varepsilon^{\zeta,x} := F(\zeta)\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y+i[A_\zeta^\pm(x)-A_\zeta^\pm(y)]}\,dy. \tag{3.2.56}$$

Then imploying Step 2 it follows that for each $\zeta \in \gamma_+^1$ the limit

$$\lim_{\varepsilon\to 0}F_\varepsilon^{\zeta,x} \tag{3.2.57}$$

exists for almost every $x \in \mathbb{R}$. Next we want to prove that $\sup_{\varepsilon>0}|F_\varepsilon^{\cdot;x}| \in L^1(\gamma_+^1)$ for almost every $x \in \mathbb{R}$. To see the latter we write

$$\begin{aligned}\int_{\mathbb{R}}\left|\int_{\gamma_+^1}\sup_{\varepsilon>0}|F_\varepsilon^{\zeta,x}|\,d\zeta\right|^2 dx &\leq \int_{\gamma_+^1}\int_{\mathbb{R}}(\sup_{\varepsilon>0}|F_\varepsilon^{\zeta,x}|)^2\,dx\,d\zeta \\ &\leq C(M)\|f\|_{L^2(\mathbb{R})}.\end{aligned} \tag{3.2.58}$$

The first inequality in (3.2.58) is standard, while for the second one we have used (3.2.12). Now combining the above we have all the ingredients in place for applying Lebesque's Dominated Convergence Theorem to conclude that $\lim_{\varepsilon\to 0}I_+ = \lim_{\varepsilon\to 0}(-\frac{1}{2\pi}\int_{\gamma_+^1}F_\varepsilon^{\zeta,x}\,d\zeta)$ exists for almost every $x \in \mathbb{R}$. Similarly, one shows that $\lim_{\varepsilon\to 0}I_-$, $\lim_{\varepsilon\to 0}II_\pm$ exist for almost every $x \in \mathbb{R}$, and thus Claim 2 is proved.

Step 4. *Pointwise convergence for the operator (3.1.57) from Theorem 3.1.5.*

The fact that for $f \in L^p(\mathbb{R})$, the limit $\lim_{\varepsilon\to 0} K_{A,F,\varepsilon} f(x)$ exists for almost every $x \in \mathbb{R}$, follows by a reasoning similar to the one in Step 3. This time the identity (3.1.59) replaces the expressions in (3.1.54) and the decay properties of the Fourier transform $\hat{F}(\xi)$ in (3.1.58) are used when applying Lebesque's Dominated Convergence Theorem.

Step 5. *Pointwise convergence for our operator (3.2.11) from Theorem 3.2.2.*

The case $n = 1$ has been proved in Step 4. If we now assume that $n > 1$, we first look at the action of T_ε on smooth, compactly supported functions. If we fix $f \in C_0^\infty(\mathbb{R}^n)$ and consider $K_{\omega,\varepsilon}$ as defined in (3.2.7), then Step 4 gives that $\lim_{\varepsilon\to 0} K_{\omega,\varepsilon} f(x)$ exists for almost every $x \in \mathbb{R}^n$. Hence, for our operator $K_{A,F,\varepsilon}$ in (3.2.7), we can make use of Lebesque's Dominated convergence Theorem to conclude that $\lim_{\varepsilon\to 0} K_{A,F,\varepsilon} f(x)$ exists for almost every $x \in \mathbb{R}^n$; here we recall the estimate (3.2.8). The latter, together with Lemma 3.2.7 and (3.2.12), finally yields the pointwise convergence stated in (2) of Theorem 3.2.2, since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.

The fact that T maps L^1 into weak-L^1, is implicit in what we have proved so far (see Lemma 3.2.5).

Turning to part (3) of Theorem 3.2.2, observe that the concept of Tf for an arbitrary function $f \in L^\infty(\mathbb{R}^n)$ (not necessarily compactly supported) is not quite clarified for the operator at hand. Following the same pattern as in the case of the Cauchy operator, for every pair of points $x_1, x_2 \in \mathbb{R}^n$ we choose a cube $Q \in \mathbb{R}^n$ such that $x_1, x_2 \in Q$ and denote by $2Q$ the cube centered at the same point as Q with the sidelength twice longer. Next, consider the functions $f_1 := f\chi_{2Q}$, $f_2 := f - f_1$, and

$$F(x_1, x_2) := (Tf_1)(x_1) - (Tf_1)(x_2) + \int_{\mathbb{R}^n} (K(x_1, y) - K(x_2, y)) f_2(y)\, dy. \quad (3.2.59)$$ eq1.48,49,50

Note that the definition of F given above does not depend on the choice of cube Q. Moreover, for every $x_1, x_2 \in \mathbb{R}^n$, the function $F(x, x_1) - F(x, x_2)$ is constant with respect to x. This observation allows to define the action of the operator T on the function $f \in L^\infty(\mathbb{R}^n)$ as the equivalence class (modulo constants) of $x \mapsto F(x_1, x)$, for a.e. $x_1 \in \mathbb{R}^n$.

It remains to show that T is bounded from $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$. Owing to L^2-boundedness of the operator T, we can write

$$\begin{aligned} \frac{1}{|Q|}\int_Q |Tf_1(x)|\, dx \;&\le\; \left(\frac{1}{|Q|}\int_Q |Tf_1(x)|^2\, dx\right)^{1/2} \le \left(\frac{1}{|Q|}\int_{2Q} |f_1(x)|^2\, dx\right)^{1/2} \\ &\le\; C\|f\|_{L^\infty(\mathbb{R}^n)}. \end{aligned} \quad (3.2.60)$$ T12

Going further, for every $x \in Q$

$$\Big|\int_{\mathbb{R}^n} (K(x, y) - K(x_Q, y)) f_2(y)\, dy\Big|$$

$$\leq C\|f\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}\setminus 2Q} \frac{|x-x_Q|}{|y-x_Q|^{n+1}}\, dy \leq C\|f\|_{L^\infty(\mathbb{R}^n)}. \tag{3.2.61}$$

e3.2.61

for arbitrary cube $Q \in \mathbb{R}^n$ centered at point x_Q. Therefore,

$$\frac{1}{|Q|}\int_Q \left| \int_{\mathbb{R}^n} (K(x,y) - K(x_Q,y)) f_2(y)\, dy \right| dx \leq C\|f\|_{L^\infty(\mathbb{R}^n)}. \tag{3.2.62}$$

T14

Finally, combining (3.2.60) and (3.2.62), we establish the estimate

$$\frac{1}{|Q|}\int_Q |F(x,x_Q) - Tf_1(x_Q)|\, dx \leq C\|f\|_{L^\infty(\mathbb{R}^n)} \tag{3.2.63}$$

e3.2.63

for every $Q \in \mathbb{R}^n$ and hence, the boundedness of operator T in $BMO(\mathbb{R}^n)$.

It is also possible to define Tf for $f \in L^\infty(\mathbb{R}^n)$ in an alternative way. It can be viewed as a distribution known modulo an additive constant, i.e. a continuous linear functional defined on the space of functions $g \in H^1(\mathbb{R}^n)$. The continuity would amount to $Tf \in BMO(\mathbb{R}^n)$. Assume that g is an H^1 – atom supported in $B_r(x_0)$. We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{B_{2r}(x_0)}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus B_{2r}(x_0)}$ and then define

$$\langle Tf, g\rangle := \langle Tf_1, g\rangle + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (K(x,y) - K(x_0,y)) f_2(y) g(x)\, dy dx. \tag{3.2.64}$$

e3.2.64

Note that subtracting $K(x_0,y)$ from the kernel above makes the integral well-defined without spoiling the nature of the operator, since $\int_{\mathbb{R}^n} g(x)\, dx = 0$. Next,

$$|\langle Tf_1, g\rangle| \leq \|Tf_1\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}, \tag{3.2.65}$$

T16

which, owing to the boundedness of the operator T on $L^2(\mathbb{R}^n)$ and the size condition imposed on the atom, is controlled by

$$C\|f_1\|_{L^2(\mathbb{R}^n)} r^{-n/2} \leq C. \tag{3.2.66}$$

T17

On the other hand,

$$\left| \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (K(x,y) - K(x_0,y)) f_2(y) g(x)\, dy dx \right|$$
$$\leq C\|f_2\|_{L^\infty(\mathbb{R}^n)} \int_{B_r(x_0)} \int_{\substack{y\in\mathbb{R}^n \\ |x-y|\geq 2|x-x_0|}} |K(x,y) - K(x_0,y)|\, dy\, |g(x)|\, dx. \tag{3.2.67}$$

T15

It is not hard to show that the inside integral above and therefore, the whole expression (3.2.67), is bounded by a finite constant. Combined with (3.2.65) – (3.2.66), this observation concludes the proof.

THERE ARE TWO ARGUMENTS HERE. WHICH ONE TO LEAVE?

The following part was in the same file after enddocument:

....

Verify that the definition given above does not depend on the choice of ball B and that the integral in (eq1.51 3.1.33) converges. Also, check that the definition reduces to the usual one in the case of compactly supported L^∞-function f and the case when T is given by an integrable kernel.

....

3.3 Applications of Calderón-Zygmund Theory

The nontangential behavior of integral operators is modeled in abstract in the next theorem.

t2.6 **Theorem 3.3.1.** *Let $A : \mathbb{R}^n \to \mathbb{R}^m$, $B : \mathbb{R}^n \to \mathbb{R}$ be two Lipschitz functions and let $F : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a C^1, odd function which satisfies the decay conditions*

$$|F(a,b)| \leq C(1+|b|)^{-n}, \tag{3.3.1}$$ eq2.23

$$|\nabla_I F(a,b)| \leq, C \tag{3.3.2}$$ eq2.24

$$|\nabla_{II} F(a,b)| \leq C(1+|b|)^{-1} \tag{3.3.3}$$ eq2.25

uniformly for a in compact subsets of $\mathbb{R}^n$ and arbitrary $b \in \mathbb{R}$. For $x, y \in \mathbb{R}^n$ with $x \neq y$ and $t > 0$ we set

$$K^t(x,y) := \frac{1}{|x-y|^n} F\left(\frac{A(x)-A(y)}{|x-y|}, \frac{B(x)-B(y)+t}{|x-y|}\right). \tag{3.3.4}$$ eq2.26

Also, for each $t > 0$ introduce

$$T^t f(x) := \int_{\mathbb{R}^n} K^t(x,y) f(y)\, dy, \qquad x \in \mathbb{R}^n, \tag{3.3.5}$$ e3.3.5

and, for some fixed, positive λ,

$$T_{**} f(x) := \sup\{|T^t f(z)| : |x-z| < \lambda t\}, \qquad x \in \mathbb{R}^n. \tag{3.3.6}$$ e3.3.6

Then, for each $1 < p < \infty$, the following assertions are valid:

1. *The nontangential maximal operator T_{**} is bounded on $L^p(\mathbb{R}^n)$. Furthermore,*

$$\|T_{**}\|_{\mathcal{L}(L^p \to L^p)} \leq C(1+M^4)\Big[1 + \sup\{|D^\alpha F(z)|;\ |z| < M+1,\ |\alpha| < 5+n\}\Big]. \tag{3.3.7}$$ e3.3.7

2. *For each $f \in L^p(\mathbb{R}^n)$, the limit*

$$\mathcal{T} f(x) := \lim_{\substack{|x-z|<\lambda t \\ z\to x,\, t\to 0}} T^t f(z) \tag{3.3.8}$$ eq2.27

exists at almost every $x \in \mathbb{R}^n$ and the operator $\mathcal{T}$ is bounded on $L^p(\mathbb{R}^n)$.

CAN WE RELEASE THE ASSUMPTION THAT THE LIMIT IN (3.3.8) EXISTS ??? eq2.27

Proof. For $x, y \in \mathbb{R}^n$ with $x \neq y$ consider the kernel

$$K(x,y) := \frac{1}{|x-y|^n} F\left(\frac{A(x)-A(y)}{|x-y|}, \frac{B(x)-B(y)}{|x-y|}\right), \tag{3.3.9}$$ eq2.28

and let T, T_* be the operators canonically associated with this integral kernel as in Theorem A.

The crux of the matter is establishing the a.e. pointwise estimate

$$T_{**}f \leq CT_*f + C\mathcal{M}f \tag{3.3.10}$$ eq2.29

uniformly for $f \in L^p(\mathbb{R}^n)$. Then the statement of the theorem follows from Theorem A and (by now) standard arguments.

To this end, fix $x, z \in \mathbb{R}^n$, $t > 0$ such that $|x-z| < \lambda t$, and let $\alpha > 0$ be a large constant, to be specified later. Then

$$\left|\int_{\mathbb{R}^n} K^t(z,y)f(y)dy - \int_{|x-y|>\alpha t} K(x,y)f(y)dy\right| \tag{3.3.11}$$ eq2.30

$$\leq \int_{|x-y|<\alpha t} |K^t(z,y)||f(y)|dy + \int_{|x-y|>\alpha t} |K^t(z,y) - K(x,y)||f(y)|dy =: I + II. \tag{3.3.12}$$ eq2.31

Clearly, it suffices to show that $|I|$, $|II| \leq C\mathcal{M}f$. To see this, first observe that

$$|K^t(z,y)| \leq Ct^{-n} \quad \text{uniformly for any } z, y \in \mathbb{R}^n,\ z \neq y \tag{3.3.13}$$ eq2.32

(in fact, this also justifies the well definiteness of T^t). Indeed, using the fact that $Ct \leq |B(z) - B(y) + t| + |z-y|$ (easily seen by analyzing the cases $|z-y| \geq \frac{t}{2\|B'\|_{L^\infty}}$ and $|z-y| \leq \frac{t}{2\|B'\|_{L^\infty}}$), we may infer that

$$\left(1 + \frac{|B(z)-B(y)+t|}{|z-y|}\right)^{-n} \leq C\left(\frac{t}{|z-y|}\right)^{-n}. \tag{3.3.14}$$ eq2.33

With this at hand, the estimate (3.3.13) is a direct consequence of (3.3.1). Returning to I, from (3.3.13), we deduce that $|I| \leq C\mathcal{M}f$.

Thus, we are left with analyzing II. First of all, we need to prove that

$$|K^t(z,y) - K(x,y)| \leq Ct|x-y|^{-n-1} \quad \text{for } |x-y| > \alpha t. \tag{3.3.15}$$ eq2.34

Let $G_y(x,t) := K^t(x,y)$. Then

$$|K^t(z,y) - K(x,y)| = |G_y(z,t) - G_y(x,0)| \tag{3.3.16}$$ eq2.35

can be bounded using Mean Value Theorem by

$$Ct(|\nabla_I G_y(w,s)| + |\nabla_{II} G_y(w,s)|), \tag{3.3.17}$$ eq2.36

where $w = (1-\theta)z + \theta x$, $s = (1-\theta)t$ for some $0 < \theta < 1$. Next,

$$
\begin{aligned}
|\nabla_I G_y(w,s)| \approx\ & \frac{1}{|w-y|^{n+1}} \left| F\left(\frac{A(w)-A(y)}{|w-y|}, \frac{B(w)-B(y)+s}{|w-y|} \right) \right| \\
& + \frac{1}{|w-y|^{n}} \left| \nabla_I F\left(\frac{A(w)-A(y)}{|w-y|}, \frac{B(w)-B(y)+s}{|w-y|} \right) \right| \frac{1}{|w-y|} \\
& + \frac{1}{|w-y|^{n}} \left| \nabla_{II} F\left(\frac{A(w)-A(y)}{|w-y|}, \frac{B(w)-B(y)+s}{|w-y|} \right) \right| \\
& \left(\frac{1}{|w-y|} + \frac{|B(w)-B(y)+s|}{|w-y|^2} \right)
\end{aligned}
$$

Keeping in mind the restrictions on size of derivatives of function F stated in the Theorem t2.6 3.3.1, we conclude that the above expression is bounded by $\frac{1}{|w-y|^{n+1}}$.

Similarly it can be shown that

$$|\nabla_{II} G_y(w,s)| \leq C \frac{1}{|w-y|^{n+1}}. \tag{3.3.18}$$ eq2.38

However, $|w-y| \geq C \min\{|x-y|, |y-z|\}$ and

$$|y-z| \geq |x-y| - |z-x| \geq \left(1 - \frac{\lambda}{\alpha}\right) |x-y|. \tag{3.3.19}$$ eq2.40

Therefore, if we choose $\alpha > \lambda$, then $|w-y| \geq C|x-y|$ and

$$|K^t(z,y) - K(x,y)| \leq Ct|x-y|^{-n-1}. \tag{3.3.20}$$ eq2.41

Now let us split the domain of integration of II into dyadic annuli of the form $2^j \alpha t \leq |x-y| \leq 2^{j+1} \alpha t$, $j = 0, 1, \ldots$. Then

$$
\begin{aligned}
& \int_{|x-y|>\alpha t} |K^t(z,y) - K(x,y)||f(y)|dy \\
& \qquad \leq \sum_{j=0}^{\infty} \int_{2^j \alpha t < |x-y| < 2^{j+1} \alpha t} \frac{t}{|x-y|^{n+1}} |f(y)| dy \\
& \qquad \leq C \sum_{j=0}^{\infty} 2^{-j} \mathcal{M} f(x).
\end{aligned} \tag{3.3.21}
$$ eq2.42

This immediately gives the desired inequality, i.e. $|II| \leq C\mathcal{M}f$.

Finally, what we have proved so far, much as in the

The idea is that pointwise convergence for a dense class along with the boundedness of the maximal operator associated with the type of convergence in question always entails a.e. convergence for the entire L^p class.

The proof of last part of the statement of the theorem utilizes the same basic principle as in case of the Theorem t??? ??. More specifically, the idea is to identify a dense subspace $\mathcal{V}$ in $L^p(\mathbb{R}^n)$ such that for any $f \in \mathcal{V}$ the limit (eq??? 4.3.10) exists for almost every $x \in \mathbb{R}^n$. Then, much as before, the boundedness of the maximal operator associated with the type of convergence under discussion ensures that this limit exists for any $f \in L^p(\mathbb{R}^n)$ at almost every $x \in \mathbb{R}^n$.

In our situation, we may take $\mathcal{V} := C_0^1(\mathbb{R}^n)$ and observe that

$$\lim_{\substack{|x-z|<\lambda t \\ z\to x,\, t\to 0}} T^t f(z) = \lim_{\varepsilon\to 0} \lim_{\substack{|x-z|<\lambda t \\ z\to x,\, t\to 0}} \Big[I + II + III \Big], \tag{3.3.22}$$ e3.3.23

where

$$\begin{aligned} I &:= \int_{|x-y|>1} K^t(z,y) f(y)\, dy, \\ II &:= \int_{1>|x-y|>\varepsilon} K^t(z,y)[f(y)-f(x)]\, dy, \qquad (3.3.23) \\ III &:= f(x) \int_{1>|x-y|>\varepsilon} K^t(z,y)\, dy. \end{aligned}$$ e3.3.24

Consequently,

$$\begin{aligned} \lim_{\varepsilon\to 0} \lim_{\substack{|x-z|<\lambda t \\ z\to x,\, t\to 0}} I &= \int_{|x-y|>1} K(x,y) f(y)\, dy \\ \lim_{\varepsilon\to 0} \lim_{\substack{|x-z|<\lambda t \\ z\to x,\, t\to 0}} II &= \int_{1>|x-y|} K(x,y)[f(y)-f(x)]\, dy, \end{aligned}$$

whereas

$$\lim_{\varepsilon\to 0} \lim_{\substack{|x-z|<\lambda t \\ z\to x,\, t\to 0}} III = \lim_{\varepsilon\to 0} \int_{1>|x-y|>\varepsilon} K(x,y)\, dy. \tag{3.3.24}$$ e3.3.27

Now, this last limit is known to exists at a.e. $x \in \mathbb{R}^n$, cf. Exercise ???.

Once the pointwise definition of the operator $\mathcal{T}$ has been established, its boundedness on $L^p(\mathbb{R}^n)$ is implied by that of T_{**}. □

Remark. Applying the above theorems to concrete situations (e.g. the case of layer potential operators on Lipschitz domains) requires a number of small adjustments which we will only silently assume. Instead, for details, we direct the reader to the excellent treatment in FJR [FJR].

Next we state a technical result whose proof is given in the Appendices.

l2-3 **Proposition 3.3.2.** *Let $A : \mathbb{R}^n \to \mathbb{R}^{m+1}$, $B : \mathbb{R}^n \to \mathbb{R}^{m'}$ be two functions such that*

$$|A(x) - A(y)| \leq M|x-y|, \quad M^{-1}|x-y| \leq |B(x) - B(y)| \leq M|x-y|, \tag{3.3.25}$$ e3.3.26

for some positive constant M, and all $x, y \in \mathbb{R}^n$. Also, let $F : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be a C^1, odd function which satisfies the decay conditions

$$|F(a,b)| \leq C(1+|b|)^{-n} \tag{3.3.26}$$ eq2.23bis

$$|\nabla_I F(a,b)| + |\nabla_{II} F(a,b)| \leq C(1+|b|)^{-1}, \tag{3.3.27}$$ eq2.24bis

uniformly for a in compact subsets of $\mathbb{R}^n$ and arbitrary $b \in \mathbb{R}$.

Fix $x \in \mathbb{R}^n$, $\varepsilon > 0$ and introduce $G(\varepsilon) = \{y \in \mathbb{R}^n : |B(x) - B(y)| < \varepsilon\}$ and $H(\varepsilon) = \{y \in \mathbb{R}^n : |\nabla B(x) \cdot (x-y)| < \varepsilon\}$. Also, fix a constant vector $\omega \in S^m \subset \mathbf{R}^{m+1}$ with $\omega_{m+1} > 0$. Then

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \lim_{t \downarrow 0} \int_{G(\varepsilon)} \frac{1}{|x-y|^n} F\Big(\frac{A(x) - A(y) + t\omega}{|x-y|}\Big)\, dy \\ &= \lim_{r \to \infty} \int_{H(r)} \frac{1}{|x-y|^n} F\Big(\frac{\nabla A(x) \cdot (x-y) + \omega}{|x-y|}\Big)\, dy \\ &= \lim_{r \to \infty} \int_{|y|<r} \frac{1}{|x-y|^n} F\Big(\frac{\nabla A(x) \cdot (x-y) + \omega}{|x-y|}\Big)\, dy, \end{aligned} \tag{3.3.28}$$ eq6-3

provided $\nabla A(x)$, $\nabla B(x)$ exist, and $\lim_{\varepsilon \downarrow 0} \int_{\varepsilon < |x-y| < 1} \frac{1}{|x-y|^n} F\Big(\frac{A(x)-A(y)}{|x-y|}\Big)\, dy$ exists and is finite.

Note that since F is odd, the last two integrands in (eq??? 4.3.10) may be replaced by

$$\frac{1}{|x-y|^n} F\Big(\frac{\nabla A(x) \cdot (x-y) + \omega}{|x-y|}\Big) - \frac{1}{|x-y|^n} F\Big(\frac{\nabla A(x) \cdot (x-y) - \omega}{|x-y|}\Big)$$

which leads to absolutely convergent integrals. In particular, the (last two, hence all three) limits in (eq6-3 3.3.28) exist. This can also be used to show that, as functions of x, these expressions are bounded.

MAKE AN EXERCISE OUT OF THIS !!!!

Proof. We shall prove only the first equality in (eq?? ??), the justification of the second one is virtually the same. Without loss of generality we can take $x = 0$, $A(0) = 0$ and $B(0) = 0$. Observe that

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \lim_{t \downarrow 0} \int_{G(\varepsilon)} \frac{1}{|y|^n} F\Big(\frac{-A(y) + t\omega}{|y|}\Big)\, dy - \lim_{\varepsilon \downarrow 0} \lim_{t \downarrow 0} \int_{H(\varepsilon)} \frac{1}{|y|^n} F\Big(\frac{-A(y) + t\omega}{|y|}\Big)\, dy \\ &= \lim_{\varepsilon \downarrow 0} \Big[\int_{W(\varepsilon) \setminus V(\varepsilon)} \frac{1}{|y|^n} F\Big(\frac{A(y)}{|y|}\Big)\, dy - \int_{V(\varepsilon) \setminus W(\varepsilon)} \frac{1}{|y|^n} F\Big(\frac{A(y)}{|y|}\Big)\, dy \Big] = 0, \end{aligned} \tag{3.3.29}$$ efbc

where $V(\varepsilon)$ and $W(\varepsilon)$ are as in Lemma 3.2.3. Consequently, the domain of integration in the first integral in (3.3.28) may be replaced with $H(\varepsilon)$. Thus, we have to compute

$$\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_I \frac{1}{|y|^n}F\Big(\frac{-A(y)+t\omega}{|y|}\Big)\,dy+\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{II} \frac{1}{|y|^n}F\Big(\frac{-A(y)+t\omega}{|y|}\Big)\,dy, \tag{3.3.30}$$

where $I=I(t):=\{y\in\mathbb{R}^n:\,|\nabla B(0)\cdot y|<t\}$ and $II=II(\varepsilon,t):=\{y\in\mathbb{R}^n:\,t<|\nabla B(0)\cdot y|<\varepsilon\}$. Invoking Lemma 3.2.3 and our hypotheses, we obtain that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{II(\varepsilon,t)} \frac{1}{|y|^n}F\Big(\frac{A(y)}{|y|}\Big)\,dy\\
&\quad=\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{II(\varepsilon,t)} \frac{1}{|y|^n}\Big[F\Big(\frac{A(y)}{|y|}\Big)-F\Big(\frac{\nabla A(0)\cdot y}{|y|}\Big)\Big]dy\\
&\quad=\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{V(t)} \frac{1}{|y|^n}\Big[F\Big(\frac{A(y)}{|y|}\Big)-F\Big(\frac{\nabla A(0)\cdot y}{|y|}\Big)\Big]\,dy\\
&\quad-\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{V(\varepsilon)} \frac{1}{|y|^n}\Big[F\Big(\frac{A(y)}{|y|}\Big)-F\Big(\frac{\nabla A(0)\cdot y}{|y|}\Big)\Big]\,dy=0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{II(\varepsilon,t)} \frac{1}{|y|^n}F\Big(\frac{-A(y)+t\omega}{|y|}\Big)\,dy,\\
&\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{II(\varepsilon,t)} \frac{1}{|y|^n}\Big[F\Big(\frac{-A(y)+t\omega}{|y|}\Big)-F\Big(\frac{-A(y)}{|y|}\Big)\Big]\,dy\\
&\quad=\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{II(\varepsilon/t,1)} \frac{1}{|y|^n}\Big[F\Big(\frac{-A_t(y)+\omega}{|y|}\Big)-F\Big(\frac{-A_t(y)}{|y|}\Big)\Big]\,dy,
\end{aligned}\tag{3.3.31}$$

where we have set $A_t(y):=t^{-1}A(ty)$. An application of the Mean Value Theorem shows that the last integrand above is bounded by $|y|^{-n-1}$ times a constant independent of y and t, while it is not hard to see that if $y\in II(\frac{\varepsilon}{t},1)$ then $|y|>M^{-1}$. Furthermore, (3.2.20) implies that $A_t(y)\to\nabla A(0)\cdot y$ as $t\to 0$. By Lebesgue's Dominated Convergence Theorem and the fact that $F(\frac{\nabla A(0)\cdot y}{|y|})$ is odd in $y\in\mathbf{R}^n$, we get that

$$\lim_{\varepsilon\downarrow 0}\lim_{t\downarrow 0}\int_{II(\varepsilon,t)} \frac{1}{|y|^n}F\Big(\frac{-A(y)+t\omega}{|y|}\Big)\,dy=\lim_{r\to\infty}\int_{II(r,1)} \frac{1}{|y|^n}F\Big(\frac{-\nabla A(0)\cdot y+\omega}{|y|}\Big)\,dy. \tag{3.3.32}$$

Finally, we treat the first integral in (3.3.30). In this case, thanks to (3.3.27) and the fact that $\omega_{N+1}>0$, we get that

$$\begin{aligned}&\int_I \frac{1}{|y|^n}\Big[F\Big(\frac{-A(y)+t\omega}{|y|}\Big)-F\Big(\frac{-\nabla A(0)\cdot y+t\omega}{|y|}\Big)\Big]\,dy\\ &\leq C\eta_A(t)\int_{|y|\leq t}\frac{1}{|y|^n}\frac{1}{1+\Big|\xi(y)+\frac{t\omega_{N+1}}{|y|}\Big|}\,dy,\end{aligned} \tag{3.3.33}$$ It1

where $\xi:\mathbb{R}^n\to\mathbb{R}$ is a bounded function, $|\xi|\leq CM$. The right hand side of (3.3.33) is bounded by $C\eta_A(t)$, thus its limit, as $t\to 0$, is zero. This allows us to replace the original integrand in the first integral in (3.3.30) by $\frac{1}{|y|^n}F(\frac{-\nabla A(0)\cdot y+t\omega}{|y|})$. Dilating in t further yields

$$\lim_{t\downarrow 0}\int_{I(t)}\frac{1}{|y|^n}F\Big(\frac{-A(y)+t\omega}{|y|}\Big)\,dy=\int_{I(1)}\frac{1}{|y|^n}F\Big(\frac{-\nabla A(0)\cdot y+\omega}{|y|}\Big)\,dy. \tag{3.3.34}$$ eq8-3

This and (3.3.32) then conclude the proof of the proposition since, clearly, $H(r)=I(1)\cup II(r,1)$. □

For $k:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ and Ω Lipschitz domain in $\mathbb{R}^n$ define the potential

$$\mathcal{K}f(x)=\int_{\partial\Omega}k(Q-x)f(Q)\,d\sigma(Q),\qquad x\in\mathbb{R}^n\setminus\partial\Omega. \tag{3.3.35}$$ potential

For $\beta>1$ and $Q\in\partial\Omega$ the non-tangential cones $\Gamma_\pm(Q)$ are defined as

$$\Gamma_\pm(Q)=\Gamma^\beta_\pm(Q):=\{x\in\Omega_\pm:\,|x-Q|<\beta\,\mathrm{dist}(x,\partial\Omega)\}, \tag{3.3.36}$$ e3.3.37

where, recall that $\Omega_+=\Omega$ and $\Omega_-=\mathbb{R}^n\setminus\bar{\Omega}$.

p1-3 **Proposition 3.3.3.** *There exists $N=N(n)$ positive integer such that if the kernel $k\in C^N(\mathbb{R}^n\setminus\{0\})$ is odd and homogeneous of degree $-(n-1)$, i.e.*

$$k(x)=-k(-x),\quad k(tx)=t^{1-n}k(x), \tag{3.3.37}$$ eq0-3

for all $x\in\mathbb{R}^n\setminus\{0\}$, $t>0$, the following are true. Whenever $1<p<\infty$, the limit

$$Kf(P):=\lim_{\varepsilon\downarrow 0}\int_{\partial\Omega\setminus B(P,\varepsilon)}k(Q-P)f(Q)d\sigma(Q) \tag{3.3.38}$$ eq???

exists at a.e. $P\in\partial\Omega$ for every $f\in L^p(\partial\Omega)$. Also,

$$\|Kf\|_{L^p(\partial\Omega)}\leq C(p,\partial\Omega)\|k|_{S^{n-1}}\|_{C^N}\|f\|_{L^p(\partial\Omega)}. \tag{3.3.39}$$ eq1-3

Furthermore, assuming that β is large enough,

$$\|N(\mathcal{K}f)\|_{L^p(\partial\Omega)}\leq C(p,\beta,\partial\Omega)\|k|_{S^{n-1}}\|_{C^N}\|f\|_{L^p(\partial\Omega)}, \tag{3.3.40}$$ eq???

and, for each $f\in L^p(\partial\Omega)$,

$$\lim_{x\to P, x\in\Gamma_\pm(P)} \mathcal{K}f(x) = \pm\frac{(2\pi)^{n/2}}{2i}\hat{k}\big(\nu(P)\big)f(P) + Kf(P) \tag{3.3.41}$$ eq???

for a.e. $P\in\partial\Omega$, where $\nu(P)$ is the outward unit normal to Ω at P.

For smoother domains we refer the reader to Mir [Mir] and SeSi [SeSi].

Proof. For starters, we will show that

$$\begin{aligned}\lim_{x\to P, x\in\Gamma_\pm(P)} Kf(x) = &\lim_{\varepsilon\downarrow 0}\int_{\partial\Omega\setminus B(P,\varepsilon)} k(Q-P)f(Q)d\sigma(Q)\\ &+ f(P)\lim_{r\to\infty}\int_{\prod(P)\cap B(P,r)} k(y-(P\mp\nu(P))dy\end{aligned} \tag{3.3.42}$$ eq2-3

for a.e. $P\in\partial\Omega$, where $\prod(P)$ denotes the tangent plane to $\partial\Omega$. The tangent plane $\prod(P)$ to $\partial\Omega$ at P is well defined for almost every $P\in\partial\Omega$ (see Theorem 4.1). Using a partition of unity, matters are reduced to the case when Ω is a domain above a compactly supported Lipschitz function φ. Furthermore, since $d\sigma(y')=\sqrt{1+|\nabla\varphi(y)|^2}dy'$ and $\tilde{f}\in L^p(\partial\Omega)$ if and only if $f(y')=\tilde{f}(y',\varphi(y'))\sqrt{1+|\nabla\varphi(y')|^2}\in L^p(\mathbf{R}^{n-1})$, it suffices to analyze the operator

$$Kf(z)=\int_{R^{n-1}} k(y'-z',\varphi(y')-z_n)f(y')dy',\quad z\in\mathbf{R}^n\backslash\partial\Omega,$$

for $f\in L^p(\mathbf{R}^{n-1})$, $z=(z',z_n)$, $y=(y',\varphi(y'))\in\partial\Omega$. In particular, if $x=(x',\varphi(x'))\in\partial\Omega$ is such that $z\in\Gamma_+(x)$, then we can define the unit vector $\omega:=\frac{z-x}{|z-x|}$. The points $x+t\omega$, $t>0$ approach z within $\Gamma_+(z)$ as $t\to 0$. With these identifications in mind, the kernel of our operator can be writen as

$$\frac{1}{|x'-y'|^{n-1}}k\Big(\frac{A(x')-A(y')+t\omega}{|x'-y'|}\Big), \tag{3.3.43}$$ kernel

where $A(y'):=(y',\varphi(y'))$. In order to apply Proposition 12-3 3.3.2 we need to remove the singularity of k at the origin. For this purpose let θ be a smooth even function function defined on $\mathbb{R}^{n-1}$, such that $\theta=0$ near the origin and $\theta(x')=1$ for $|x'|\geq\frac{1}{2}$. Then, if we set the function $F:=-\theta k$, we have

$$Kf(x+t\omega)=\int_{R^{n-1}}\frac{1}{|x'-y'|^{n-1}}F\Big(\frac{A(x')-A(y')+t\omega}{|x'-y'|}\Big)\,dy' := Tf(x',t). \tag{3.3.44}$$ operator

We will first analyze the limit $\lim_{t\to 0+}Tf(x',t)$. Without loss of generality we can assume that $x'=0$, $A(0)=0$. For fixed $R>0$ we take $f\in C_0^1(\mathbf{R}^{n-1})$ with supp$f\subseteq B_R(0)$. Then for $\varepsilon>0$ small we can write

$$\lim_{t\to 0+} Tf(x',t) = \lim_{t\to 0+}\lim_{\varepsilon\to 0+}\int_{\varepsilon<|y'|<R}\frac{1}{|y'|^{n-1}}F\Big(\frac{-A(y')+t\omega}{|y'|}\Big)(f(y')-f(0))\,dy'$$
$$+ f(0)\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\int_{\varepsilon<|y'|<R}\frac{1}{|y'|^{n-1}}F\Big(\frac{-A(y')+t\omega}{|y'|}\Big)\,dy'$$
$$+ f(0)\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\int_{|y'|<\varepsilon}\frac{1}{|y'|^{n-1}}F\Big(\frac{-A(y')+t\omega}{|y'|}\Big)\,dy' =: I+II+III. \tag{3.3.45}$$

IIIIII

It is not difficult to see that one can take $\lim_{t\to 0}$ in I and II which leads to the "removal" of $t\omega$ under the integral. Further combining the resulting integrals we obtain that

$$I+II = \lim_{\varepsilon\to 0+}\int_{|y'|>\varepsilon}\frac{1}{|y'|^{n-1}}F\Big(\frac{-A(y')}{|y'|}\Big)f(y')\,dy = \lim_{\varepsilon\to 0+}\int_{|y'|>\varepsilon}k(y',\varphi(y'))f(y')\,dy'. \tag{3.3.46}$$

IandII

To treat II we recall Proposition 3.3.2 [12-3] and conclude that

$$III = f(0)\lim_{r\to\infty}\int_{|y'|^2+|\langle\nabla\varphi(0),y'\rangle|^2<r^2}k((y',\nabla\varphi(0)\cdot y')-\omega)dy'. \tag{3.3.47}$$

termIII

Next we claim that if ω is replaced in (3.3.47) [termIII] by its normal component the value of III does not change. In order to prove this claim, let $\omega=\omega_{\rm nor}+\omega_{\rm tan}$, where $\omega_{\rm nor}$ and $\omega_{\rm tan}$ are the normal and tangential, respectively, components of ω. Then we can write

$$\lim_{r\to\infty}\int_{|y'|^2+|\langle\nabla\varphi(0),y'\rangle|^2<r^2}\Big[k((y',\nabla\varphi(0)\cdot y')-\omega)-k((y',\nabla\varphi(0)\cdot y')-\omega_{\rm nor})\Big]\,dy'$$
$$=\lim_{r\to\infty}\int_{|y'|^2+|\langle\nabla\varphi(0),y'\rangle|^2<r^2}\int_0^1\frac{d}{d\theta}\Big[k((y',\nabla\varphi(0)\cdot y')-\omega_{\rm nor}-\theta\omega_{\rm tan})\Big]\,d\theta\,dy'$$
$$=\lim_{r\to\infty}\tfrac{1}{\sqrt{1+|\nabla\varphi(0)|^2}}\int_{Q\in\prod(0),\,Q\in B_r(0)}\int_0^1\frac{d}{d\theta}\Big[k(Q-\omega_{\rm nor}-\theta\omega_{\rm tan})\Big]\,d\theta\,dS(Q)$$
$$=\tfrac{1}{\sqrt{1+|\nabla\varphi(0)|^2}}\int_0^1\int_{\prod(0)}-\omega_{\rm tan}\cdot\nabla_Q(k(Q-\omega_{\rm nor}-\theta\omega_{\rm tan}))\,dS(Q)\,d\theta. \tag{3.3.48}$$

replaceomega

Above, $dS(Q)$ denotes the measure on $\prod(0)$, and we have that $dS(Q)=\sqrt{1+|\nabla\varphi(0)|^2}\,dy'$. We point out that we have used the fact that $z\in\Gamma_+(0)$, and thus the direction ω itself stays in a fixed cone, in order to be able to pass to the limit as $r\to\infty$. To continue, observe that the integrand in the last term in (3.3.48) [replaceomega] is a constant multiple of the tangential derivative of k. As such, its integral over $\prod(0)$ is zero. To see this, let $\psi : \prod(0) \to \mathbb{R}$ be a smooth, compactly supported function, identically 1 near 0. Then, if $\nabla_{\tan} k(Q)$ denotes a tangential derivative of k, integration by parts gives

$$\begin{aligned}\int_{\prod(0)} \nabla_{\tan} k(Q)\, dS(Q) &= \lim_{R\to\infty} \int_{\prod(0)} \Psi\Big(\frac{Q}{R}\Big) \nabla_{\tan} k(Q)\, dS(Q) \\ &= \lim_{R\to\infty} -\frac{1}{R} \int_{\prod(0)} (\nabla_{\tan}\psi)\Big(\frac{Q}{R}\Big) k(Q)\, dS(Q) \\ &\le \lim_{R\to\infty} \frac{C}{R} \int_{Q\in\prod(0), |Q|\le R} \frac{1}{|Q|^{n-1}}\, dS(Q) = 0\end{aligned} \tag{3.3.49}$$

tangderivk

For the inequality in (3.3.49) we have used (3.3.37).

This proves the claim that in the expression of III one can replace ω with its normal component. Furthermore, it is immediate that the right hand side of (3.3.47) stays the same if ω is replaced by $c\omega$, $c > 0$ constant. Summing up, we can conclude that in fact (3.3.47) holds with ω replaced by $-\nu(0) = (-\frac{\nabla\varphi(0)}{\sqrt{1+\nabla\varphi(0)^2}}, \frac{1}{\sqrt{1+\nabla\varphi(0)^2}})$, with $\nu(0)$ being the outward unit normal to Ω. More precisely, one has that

$$III = f(0) \lim_{r\to\infty} \int_{|y'|^2+|\langle\nabla\varphi(0),y'\rangle|^2<r^2} k\Big(y' + \frac{\nabla\varphi(0)}{\sqrt{1+\nabla\varphi(0)^2}}, \nabla\varphi(0)\cdot y' - \frac{1}{\sqrt{1+\nabla\varphi(0)^2}}\Big) dy'. \tag{3.3.50}$$

termIIIb

Now a combination of (3.3.45), (3.3.46), and (3.3.50), yields

$$\begin{aligned}\lim_{t\to 0+} Kf(t\omega) &= \lim_{\varepsilon\to 0+} \int_{|y'|>\varepsilon} k(y', \varphi(y')) f(y')\, dy' \\ &+ f(0) \lim_{r\to\infty} \int_{|y'|^2+|\langle\nabla\varphi(0),y'\rangle|^2<r^2} k\Big(y' + \frac{\nabla\varphi(0)}{\sqrt{1+\nabla\varphi(0)^2}}, \nabla\varphi(0)\cdot y' - \frac{1}{\sqrt{1+\nabla\varphi(0)^2}}\Big)\, dy'.\end{aligned} \tag{3.3.51}$$

This proves (3.3.42) for $\Gamma_+(0)$. The proof for $z \in \Gamma_-(0)$ is similar and we omit it.

To summarize, so far we have shown that, if $f \in L^p(\Gamma)$, $1 < p < \infty$, then at almost every point $x \in \Gamma$ we have nontangential limits

$$(\mathcal{K}f)_\pm(x) = \alpha_\pm(x) f(x) + \lim_{\varepsilon\to 0} \int\limits_{|y-x|>\varepsilon} k(x-y) f(y)\, d\sigma(y), \tag{3.3.52}$$

C.1

where

$$\alpha_\pm(x) = \lim_{R\to\infty} \int\limits_{\Pi_x\cap B_R(x)} k\big(x - y \pm n(x)\big)\, dy. \tag{3.3.53}$$

C.2

Here $B_R(x)$ stands for the ball in $\mathbb{R}^n$ with center at x and radius R. Given this, our formula (1.22) is equivalent to the statement that

$$\lim_{R\to\infty} \int_{\Pi_x \cap B_R(x)} k\big(x - y \pm n(x)\big)\, dy = \pm c_n \hat{k}\big(n(x)\big), \tag{3.3.54}$$ C.3

where c_n is a constant we will identify below. Verifying (3.3.54) can be tackled directly, as an exercise in Fourier analysis, but we will take a slightly different route.

Namely, it is clear from (3.3.52)–(3.3.53) that $\alpha_+(x) = -\alpha_-(x)$, and that the jump of $\mathcal{K}f$ across Γ is equal a.e. to $2\alpha_+(x)f(x)$. Also, let us observe from (C.2) that the jump coefficient $\alpha_+(x)$ depends exclusively on k and the tangent hyperplane Π_x. Thus, to prove (3.3.54) we need merely show that, in the case $\Gamma = \Pi$, a hyperplane in $\mathbb{R}^n$, with unit normal n, the jump of $\mathcal{K}f$ across Π is equal a.e. to $2c_n\hat{k}(n)f$. Also, there is no loss of generality in checking this when $\Pi = \{(x', 0) : x' \in \mathbb{R}^{n-1}\},\ x = 0$, and $n = (0, \dots, 0, -1)$. Furthermore, it suffices to consider a single $f \in \mathcal{S}(\Pi)$, such that $f(0) \neq 0$. Then, for $x \in \mathbb{R}^n \setminus \Pi$,

$$\mathcal{K}f(x) = P_{-1}(D)(f \otimes \delta)(x), \tag{3.3.55}$$ C.4

where we write $P_{-1}(D)u = k * u$ for $u \in \mathcal{E}'(\mathbb{R}^n)$, so $P_{-1}(\xi) = (2\pi)^{n/2}\hat{k}(\xi)$. Now

$$(\mathcal{K}f)\hat{}(\xi) = (2\pi)^{-1/2} P_{-1}(\xi)\hat{f}(\xi'), \tag{3.3.56}$$ C.5

and if $\hat{f} \in C_0^\infty(\mathbb{R}^{n-1})$, this differs from $(2\pi)^{-1/2}P_{-1}(0,\xi_n)\hat{f}(\xi')$ by an element of $\mathcal{S}'(\mathbb{R}^n)$ that is integrable outside a sufficiently large ball. Thus $\mathcal{K}f - F$ is continuous on $\mathbb{R}^n$, where $\hat{F}(\xi) = (2\pi)^{-1/2}P_{-1}(0,\xi_n)\hat{f}(\xi')$, i.e.,

$$F(x) = \tfrac{1}{2} i\, P_{-1}(n)\, \mathrm{sgn}(x_n)\, f(x'). \tag{3.3.57}$$ C.6

The nature of the jump of this function across Π is clear; it is equal to $iP_{-1}(n)$. Hence

$$\alpha_\pm(0) = \mp \tfrac{1}{2} i P_{-1}(n). \tag{3.3.58}$$ C.7

This establishes (3.3.54), with $c_n = (2\pi)^{n/2}/2i$, and it also proves (??).

FINISH THE PROOF OF NT-ESTIMATES!!!

□

3.4 Variable coefficient singular integrals

....

3.5 Pseudodifferential operators

We recall that a pseudodifferential operator $P(x, D)$ with (total) symbol $p(x, \xi)$ is given by

$$P(x, D)u \;=\; (2\pi)^{-n/2} \int p(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}\, d\xi$$

$$= \quad (2\pi)^{-n} \iint p(x,\xi) e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi. \tag{3.5.1}$$ e1.6

There are several symbol classes of importance. One class, denoted $S^m_{1,0}$, is defined by

$$p(x,\xi) \in S^m_{1,0} \Longleftrightarrow |D^\beta_x D^\alpha_\xi p(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|^2)^{(m-|\alpha|)/2}, \tag{3.5.2}$$ e1.7

uniformly for x in compact subsets of $\mathbb{R}^n$. Here, we are concerned with a smaller class of symbols. We say $p(x,\xi) \in S^m_{\mathrm{cl}}$ if

$$p(x,\xi) \sim p_m(x,\xi) + p_{m-1}(x,\xi) + \cdots , \tag{3.5.3}$$ e1.8

with p_j smooth in x and ξ and homogeneous of degree j in ξ (for $|\xi| \geq 1$). The meaning of (1.8) is that, for each $k \geq 1$, the difference between the left side and the sum of the first k terms on the right belong to $S^{m-k}_{1,0}$. The term $p_m(x,\xi)$ in (1.8) is called the *principal symbol* of $P(x,D)$.

The expression in the left-hand side of (eq??? 4.3.10) is an oscillatory integral, defined in the following sense. If $L_{x,y} := (1+|x-y|^2)^{-1}(I-\Delta_\xi)$ then $L^t_{x,y} = L_{x,y}$ and $L_{x,y}\Big(e^{i(x-y)\cdot\xi}\Big) = e^{i\langle x-y,\xi\rangle}$. Then we set

$$\int p(x,\xi) e^{i(x-y)\cdot\xi}\, d\xi := \int \Big[L^N_{x,y} p(x,\xi)\Big] e^{i(x-y)\cdot\xi}\, d\xi, \tag{3.5.4}$$ eq???

where the integral in the right-hand side converges absolutely if N is large enough.

According to a celebrated theorem of L. Schwartz, any linear, continuous operator T from the space of test functions $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}'(\mathbb{R}^n)$, the space of disstributions is uniquely defined by a distribution $k \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, in the sense that

$$\langle Tu, v\rangle = \langle k, u\otimes v\rangle, \qquad \forall\, u,v \in \mathcal{D}(\mathbb{R}^n). \tag{3.5.5}$$ eq???

For a classical pseudodifferential operator $P(x,D) \in \mathrm{OP}S^m_{\mathrm{cl}}$ we set

$$k_P(x,y) =\in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n) \tag{3.5.6}$$ e1.3

for its Schwartz kernel, and let $\mathrm{Symb}_P \in S^m_{\mathrm{cl}}$ stand for its principal symbol. For later reference, let us point out here that, as is well-known,

$$k_P(x,y) = (2\pi)^{-n} \int p(x,\xi) e^{i(x-y)\cdot\xi}\, d\xi = (2\pi)^{-n/2}[\mathcal{F}_\xi p](x, y-x). \tag{3.5.7}$$ kernel-symb

This entails $k_P \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \mathrm{diag})$ and

$$|k_p(x,y)| \leq C|x-y|^{-(n+m)}, \qquad x \neq y. \tag{3.5.8}$$ eq???

DERIVATIVES??

In it useful to note here that

$$P_j(x,D) \in \mathrm{OP}S^{m_j}_{\mathrm{cl}},\ j=1,2 \Longrightarrow P_1(x,D)P_2(x,D) \in \mathrm{OP}S^{m_1+m_2}_{\mathrm{cl}}, \tag{3.5.9}$$ eq???

$$\mathrm{Symb}_{P_1P_2}(x,\xi) = \mathrm{Symb}_{P_1}(x,\xi)\mathrm{Symb}_{P_2}(x,\xi). \tag{3.5.10}$$

The class of formal adjoints of (4.3.10) are pseudodifferential operators of the form

$$Q(D,x)v(x) := (2\pi)^{-n}\iint q(\xi,y)e^{i(x-y)\cdot\xi}v(y)\,dy\,d\xi. \tag{3.5.11}$$ eq???

It can then be shown that $Q(D,x) \in \mathrm{OP}S^m_{\mathrm{cl}}$ if $q \in S^m_{1,0}$. The Schwartz kernel of (4.3.10) is then given by

$$k_Q(x,y) = \int q(\xi,y)e^{i(x-y)\cdot\xi}\,dxi = [\mathcal{F}_x q](y-x,y). \tag{3.5.12}$$ eq???

The composition between $P(x,D)$ as in (4.3.10) and $Q(D,x)$ as in (4.3.10) is given by

$$P(x,D)Q(D,x)v(x) = (2\pi)^{-n}\iint p(x,\xi)q(\xi,x)e^{i(x-y)\cdot\xi}v(y)\,dy\,d\xi. \tag{3.5.13}$$ eq???

....

Fix an arbitrary Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ and denote by $\nu(x)$ the outward unit conormal (well defined) at almost every $x \in \partial\Omega$. Also, $d\sigma$ is the surface measure on $\partial\Omega$.

For two fixed nonnegative integers N, M, consider the (trivial) vector bundles

$$\mathcal{E} := [C^\infty(\bar{\Omega})]^N, \qquad \mathcal{F} := [C^\infty(\bar{\Omega})]^M. \tag{3.5.14}$$???

In the sequel, we let

$$\langle u,v\rangle := \sum_j u_j v_j \tag{3.5.15}$$???

denote the pointwise inner product in such bundles. This pairing is bilinear since it does not involve complex conjugation.

We shall agree that $L^1(\mathbb{R}^n,\mathcal{E}) \hookrightarrow \mathcal{D}'(\mathbb{R}^n,\mathcal{E})$, the embedding of integrable sections into the space of distributions with coefficients in $\mathcal{E}$ is made in such a way that any integrable section f is identified with the functional

$$C^\infty(\mathbb{R}^n,\mathcal{E}) \ni \phi \mapsto \int_{\mathbb{R}^n}\langle f(x),\phi(x)\rangle\,dx. \tag{3.5.16}$$ 1.1

Next, if P is a pseudodifferential operator of order -1 with principal symbol

$$\mathrm{Symb}\,(P) : \mathbb{R}^n\setminus\{0\} \longrightarrow \mathrm{Hom}\,(\mathcal{E},\mathcal{F}) \tag{3.5.17}$$ 1.4

we introduce layer potential operators by formally writing, for $f : \partial\Omega \to \mathcal{E}$,

$$K_P f(x) : \;=\; \mathrm{p.v.}\int_{\partial\Omega}\langle k_P(x,y), f(y)\rangle\,d\sigma(y) \tag{3.5.18}$$ 1.5

$$= \lim_{\varepsilon\to 0} \int\limits_{y\in\partial\Omega\,,\,|x-y|>\varepsilon} \langle k_P(x,y), f(y)\rangle\, d\sigma(y), \qquad x\in\partial\Omega. \quad (3.5.19)$$

Also, we set

$$\mathcal{K}_P f(x) := \int_{\partial\Omega} \langle k_P(x,y), f(y)\rangle\, d\sigma(y), \qquad x\in\mathbb{R}^n\setminus\partial\Omega. \quad (3.5.20)$$ 1.6

The main result of this section is as follows.

1.1 **Theorem 3.5.1.** *Let $\mathcal{E},\mathcal{F}$ be as above and let $P(x,D)\in OPS^{-1}_{cl}(\mathcal{E},\mathcal{F})$ be such that* $\mathrm{Symb}_P(x,\xi)$ *is odd in $\xi\in\mathbb{R}^n$. Then, for each $f\in L^p(\partial\Omega,\mathcal{E})$, with $1<p<\infty$, $K_P f(x)$ exists pointwise at almost every boundary point $x\in\partial\Omega$ and*

$$K_P : L^p(\partial\Omega,\mathcal{E}) \to L^p(\partial\Omega,\mathcal{F}) \quad (3.5.21)$$ 1.7

is a bounded operator. Also, there exists $C>0$ so that

$$\|N(\mathcal{K}_P f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega,\mathcal{E})}, \qquad \forall f\in L^p(\partial\Omega,\mathcal{E}), \quad (3.5.22)$$ 1.8

and $\mathcal{K}_P f$ has a nontangential boundary trace at almost every boundary point. More specifically,

$$\mathcal{K}_P f\Big|_{\partial\Omega_\pm} = \pm\tfrac{1}{2i}\,\mathrm{Symb}_P(\cdot,\nu) f + K_P f \ \ a.e.\ on\ \partial\Omega. \quad (3.5.23)$$ 1.9

Furthermore,

$$\mathcal{K}_P : L^2(\partial\Omega,\mathcal{E}) \to H^{1/2,2}(\Omega,\mathcal{F}) \quad (3.5.24)$$ 1.10

is a bounded operator.

We debut with a "variable coefficient" version of Proposition ???3.5.3.

Let Γ be a Lipschitz graph in $\mathbb{R}^n$, of the form $x_n=\phi(x_1,\dots,x_{n-1})$ for some Lipschitz function $\phi:\mathbb{R}^{n-1}\to\mathbb{R}$. Here, $L^p(\Gamma)$ is defined using surface measure (i.e., $(n-1)$-dimensional Hausdorff measure) on Γ.

1.2 **Lemma 3.5.2.** *There exists $M=M(n)$ such that the following holds. Let $b(x,z)$ be odd in z and homogeneous of degree $-(n-1)$ in z, and assume $D^\alpha_z b(x,z)$ is continuous and bounded on $\mathbb{R}^n\times S^{n-1}$, for $|\alpha|\le M$. Then $b(x,x-y)$ is the kernel of an operator*

$$Bf(x) := \mathrm{p.v.}\int\limits_{\Gamma} b(x,x-y) f(y)\, d\sigma(y), \quad x\in\Gamma, \quad (3.5.25)$$ e1.16

bounded on $L^p(\Gamma)$, for $1<p<\infty$.

Furthermore, if

$$\mathcal{B}f(x) := \int\limits_{\Gamma} b(x,x-y) f(y)\, d\sigma(y), \quad x\in\mathbb{R}^n\setminus\Gamma, \quad (3.5.26)$$ e1.16

then

$$\|N(\mathcal{B}f)\| \leq C\|f\|_{L^p(\Gamma)}, \tag{3.5.27}$$ e1.19

and, for any $f \in L^p(\Gamma)$, $1 < p < \infty$,

$$\mathcal{B}\Big|_{\Gamma} f(x) = \pm\tfrac{1}{2i}\mathcal{F}_z[b(x,z)](\nu(x))f(x) + Bf(x), \tag{3.5.28}$$ e1.23

depending on whether the (non-tangential) approach is from above or below Γ.

Proof. We shall employ the classical method of spherical harmonic decomposition, due to Calderón and Zygmund. The idea is to write

$$b(x,z) = \sum_{j\geq 1} b_j(x)\varphi_j\big(z/|z|\big)\,|z|^{-(n-1)}, \tag{3.5.29}$$ e1.2

where $\{\varphi_j : j \geq 1\}$ is an orthonormal basis of $L^2(S^{n-1})$ consisting of eigenfunctions of the Laplace operator on the sphere S^{n-1}. Furthermore, we can assume that φ_j is odd whenever $b_j \neq 0$. With N as in Theorem 1.1 and M sufficiently larger than N, the regularity hypothesis implies

$$\|b_j\|_{L^\infty}\|\varphi_j\|_{C^N} \leq Cj^{-2}. \tag{3.5.30}$$ e1.3

Note that, if $k_j(x) = \varphi_j(x/|x|)|x|^{-(n-1)}$ with φ_j odd, then the operator

$$K_j f(x) := \int_\Gamma k_j(x-y)f(y)\,d\sigma(y), \qquad x \in \Gamma, \tag{3.5.31}$$ e1.15

is estimable by Theorem 1.1, and, for $f \in L^p(\Gamma)$,

$$Bf(x) = \sum_{j\geq 1} b_j(x)K_j f(x), \qquad x \in \Gamma. \tag{3.5.32}$$ e1.4

Hence,

$$\begin{aligned}\|B\|_{\mathcal{L}(L^p)} &\leq C(p,\Gamma)\sum_{j\geq 1}\|b_j\|_{L^\infty}\|\varphi_j\|_{C^N} \\ &\leq C(p,\Gamma)\sup_{|\alpha|\leq M}\|D_z^\alpha b(x,z)\|_{L^\infty(\mathbb{R}^n\times S^{n-1})},\end{aligned} \tag{3.5.33}$$ e1.5

which justifies (???). Similarly, if

$$\mathcal{K}_j f(x) := \int_\Gamma k_j(x-y)f(y)\,d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \Gamma, \tag{3.5.34}$$ e1.15

then

$$\mathcal{B}f(x) = \sum_{j\geq 1} b_j(x)\mathcal{K}_j f(x), \qquad x \in \mathbb{R}^N \setminus \Gamma. \tag{3.5.35}$$ e1.4

Thus,

$$N(\mathcal{B}f)(x) \leq \sum_{j\geq 1} \|b_j\|_{L^\infty} N(\mathcal{K}_j f)(x), \qquad x \in \Gamma, \tag{3.5.36}$$ e1.19

so that, using estimates of the form (3.5.30),

$$\|N(\mathcal{B}f)\|_{L^p(\Gamma_0)} \leq C_p \|f\|_{L^p(\Gamma_0)}, \tag{3.5.37}$$ e1.21

uniformly in f, for each $1 < p < \infty$.

Finally, from (3.5.3),

$$\begin{aligned} \mathcal{B}f\Big|_\Gamma(x) &= \sum_{j\geq 1} b_j(x)\mathcal{K}_j f\Big|_\Gamma(x) \\ &= \sum_{j\geq 1}\Big[\pm\frac{(2\pi)^{n/2}}{2i} b_j(x)\hat{k}_j(\nu(x))f(x) + b_j(x)K_j f(x)\Big] \\ &= \pm\frac{(2\pi)^{n/2}}{2i}\mathcal{F}_z[b(x,z)](\nu(x))f(x) + Bf(x), \end{aligned} \tag{3.5.38}$$ e1.23

where the sign depends on which side of Γ the (non-tangential) approach takes place. This finishes the proof of the lemma. □

After these preliminaries, it is straightforward to carry out the

Proof of Theorem 3.5.3. The problem localizes, so there is no loss of generality in assuming that $\partial\Omega$ is a Lipschitz hypersurface Γ in $\mathbb{R}^n$. Also, the pseudodifferential operator $P(x,D)$ can be canonically identified (locally) with a matrix of classical pseudodifferential operators $(P_{jk}(x,D))_{j,k}$ of order -1. Since lower order terms in the asymptotic expansion (3.5.3) yield only weakly singular integrals, it can be further assumed that each $P_{jk}(x,D)$ is associated with an odd, homogeneous *total* symbol. In this context, by working componentwise, Lemma 3.5.3 yields the desired result, given (3.5.7). □

??? **Corollary 3.5.3.** *Let* $Q(x,D) \in OPS^{-(m+1)}_{cl}(\mathcal{E},\mathcal{F})$ *be a pseudodifferential operator whose principal symbol* $\mathrm{Symb}_Q(x,\xi)$ *has an opposite parity, in* $\xi \in \mathbb{R}^n$*, to that of the nonnegative integer* m*. Also, assume that* $P : \mathcal{E} \to \mathcal{G}$*,* $\tilde{P} : \mathcal{F} \to \mathcal{G}$ *are differential operators of order* m*. Set*

$$Tf(x) := p.v.\int_{\partial\Omega} \langle P_x k_Q(x,y), f(y)\rangle\, d\sigma(y) \quad x \in \partial\Omega, \tag{3.5.39}$$ 1.5

$$\mathcal{T}f(x) := \int_{\partial\Omega} \langle P_x k_Q(x,y), f(y)\rangle\, d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega, \tag{3.5.40}$$ 1.5

and

$$\tilde{T}f(x) \;:=\; p.v.\int_{\partial\Omega} \langle \tilde{P}_y k_Q(x,y), f(y)\rangle \, d\sigma(y), \quad x \in \partial\Omega, \tag{3.5.41}$$ 1.5

$$\tilde{\mathcal{T}}f(x) \;:=\; \int_{\partial\Omega} \langle \tilde{P}_y k_Q(x,y), f(y)\rangle \, d\sigma(y), \qquad x \in \mathbb{R}^n \setminus \partial\Omega. \tag{3.5.42}$$ 1.5

Then the above principal value integrals exist a.e. on $\partial\Omega$*, the operators*

$$T : L^p(\partial\Omega, \mathcal{E}) \to L^p(\partial\Omega, \mathcal{G}), \quad \tilde{T} : L^p(\partial\Omega, \mathcal{F}) \to L^p(\partial\Omega, \mathcal{G}), \tag{3.5.43}$$ 1.6

are bounded, and

$$\|N(\mathcal{T}f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega,\mathcal{G})}, \quad \|N(\tilde{\mathcal{T}}f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega,\mathcal{G})}. \tag{3.5.44}$$ 1.8

Furthermore, for each L^p *function* f, $1 < p < \infty$,

$$\mathcal{T}f\Big|_{\partial\Omega_\pm} = \pm\tfrac{1}{2i}\,\mathrm{Symb}_P(\cdot,\nu)\mathrm{Symb}_Q(\cdot,\nu)f + Tf \;\; a.e.\ on\ \partial\Omega, \tag{3.5.45}$$ 1.9

and

$$\tilde{\mathcal{T}}f\Big|_{\partial\Omega_\pm} = \pm\tfrac{1}{2i}\,\mathrm{Symb}_Q(\cdot,\nu)\mathrm{Symb}_{\tilde{P}}(\cdot,\nu)f + \tilde{T}f \;\; a.e.\ on\ \partial\Omega. \tag{3.5.46}$$ 1.9

3.6 Further results

6.1

Remark. Call a linear, bounded operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ of *Calderón-Zygmund type* if it extends to a bounded operator in $L^2(\mathbb{R}^n)$ and if its Schwartz kenrel $k(x,y)$ satisfies

$$|k(x,y)| + |x-y||\nabla_I k(x,y)| + |x-y||\nabla_{II} k(x,y)| \le \frac{C}{|x-y|^n}, \qquad x,y \in \mathbb{R}^n.$$

It follows that any Calderón-Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, maps $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$. Furthermore,

$$T^*1 = 0 \Longrightarrow T : H^p(\mathbb{R}^n) \longrightarrow H^p(\mathbb{R}^n), \quad n/(n+1) < p \le 1,$$

and

$$T1 = 0 \Longrightarrow T : C^\alpha(\mathbb{R}^n) \longrightarrow C^\alpha(\mathbb{R}^n), \quad 0 < \alpha < 1.$$

...... boundedness on H^p and C^α to be finished later ....

6.2

Cotlar's Lemma Cotlar 3.2.6 allows to control the L^p norm of maximal operator T_*f by the L^p norm of Hardy-Littlewood maximal function $\mathcal{M}$ and, as a result, to establish boundedness of T_*f on L^p spaces for $1 < p < \infty$. $\mathcal{M}$ is also bounded on $BMO(\mathbb{R}^n)$ (see BDeVS [BDeVS]).

The question to be asked here is what happens in weighted L^p spaces. The main result in this theory is the so-called "good λ" inequalities. For Calderón-type operator T and A_∞ weight ω there exist $\delta > 0$, $C > 0$, $\gamma_0 > 0$ such that for every $f \in C_0^\infty(\mathbb{R}^n)$, every $\lambda > 0$ and every $\gamma \in (0, \gamma_0)$

$$\omega\{x \in \mathbb{R}^n : T_*f(x) > 2\lambda \text{ and } \mathcal{M}f(x) \leq \gamma\lambda\} \leq C\gamma^\delta \omega\{x \in \mathbb{R}^n : T_*f(x) > \lambda\}. \tag{3.6.1}$$

eq4.1

This result, established by Burkholder in the context of martingales ??? [?], provides the following estimate. For T and ω as above and $0 < p < \infty$ there exists $C > 0$ such that for every locally integrable function f

$$\int_{\mathbb{R}^n} (T_*f(x))^p \omega(x)\, dx \leq C \int_{\mathbb{R}^n} \mathcal{M}f(x)^p \omega(x)\, dx. \tag{3.6.2}$$

eq4.2

Next, and most importantly, if $1 < p < \infty$ and $\omega \in A_p$, then every Calderón-Zygmund operator T can be extended to a continuous linear operator from $L^p(\mathbb{R}^n, \omega\, dx)$ to itself.

6.3

In the case of Calderón-type singular integral operators of the special form

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} E\Big(\frac{A(x) - A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^n} f(y)\, dy, \tag{3.6.3}$$

eq4.3

(where $A \in I_1(\text{BMO})$ and Ω, $E(t)$ are smooth functions of opposite parities, the following L^p and weighted L^p estimates are available for every $1 < p < \infty$??? [?]:

DOES OMEGA HAVE TO BE SMOOTH?

$$\|Tf\|_{p,\omega} \leq C(n, p, A_p)(1 + \|\nabla A\|_{\text{BMO}})^\mu \|\Omega\|_\infty \|f\|_{p,\omega} \qquad \text{for some } \mu > 0. \tag{3.6.4}$$

eq4.4

Ω is assumed to be homogeneous of degree zero and essentially bounded, $\omega \in A_p$ and a bound on $p.v.$ is understood in the sence that the estimate (eq4.4 3.6.4) holds for every truncation of the operator T with the constant independent of truncation.

By the same techniques one can treat singular integral operators with the structure

$$\tilde{T}f(x) = P.V. \int_{\mathbb{R}^n} (B(x) - B(y)) E\Big(\frac{A(x) - A(y)}{|x-y|}\Big) \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y)\, dy, \tag{3.6.5}$$

eq4.5

for Lipschitz function A, B and with E, Ω of same parity. Then for every $1 < p < \infty$

$$\|\tilde{T}f\|_{p,\omega} \leq C(n,p,A_p)(1+\|\nabla A\|_{\text{BMO}})^{\mu}\|\Omega\|_\infty\|\nabla B\|_\infty\|f\|_{p,\omega}. \tag{3.6.6}$$ eq4.6

It is of particular significance for our discussion that the results mentioned above provide the proof for boundedness of the boundary double layer potential operator K on the BMO_1 domains. Specifically,

$$\|Kf\|_{L^p(\partial\Omega)} \leq C(n,p,A_p)\|f\|_{L^p(\partial\Omega)}, \tag{3.6.7}$$ eq4.7

with $\partial\Omega$ parametrized by $(x, A(x))$ and $C(n,p,A_p) \to 0$ as $\|\nabla A\|_{\text{BMO}} \to 0$.

Another important application is the compactness of the operator $Kf = \lim_{\varepsilon\to 0} K_\varepsilon f$, where

$$K_\varepsilon f(x) = c_n \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\partial\Omega}} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} f(y)\, d\sigma(y), \tag{3.6.8}$$ eq4.8

on $L^p(\partial\Omega)$, $1 < p < \infty$ for bounded VMO_1 domain with boundary $\partial\Omega$.

For details, the reader is referred to Steve [H].

6.4

INCLUDE HERE THE SQUARE-FUNCTION ESTIMATE OF LEWIS AND HOFMANN, WITH THE COMMENT THAT IT RELIES ON THE T(1) THEOREM OF CHRIST. THIS SHOULD BE USEFUL FOR S-B MAPPING PROP'S OF SIO IN THE NEXT CHAPTER

Let $\Delta = \{(x,y) : x \in \mathbb{R}^n\}$. Then $K : (\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta \to \mathbb{C}$ is called *standard* if $\exists\, C < \infty$, $\delta > 0$ such that for all $x, y \in \mathbb{R}^n$, $x \neq y$, and $x', y' \in \mathbb{R}^n$ it holds

$$|K(x,y)| \leq \frac{C}{|x-y|^n}$$

and

$$|K(x,y) - K(x',y')| \leq \frac{C|x-x'|^\delta}{|x-y|^{n+\delta}}$$
$$|K(y,x) - K(y,x')| \leq \frac{C|x-x'|^\delta}{|x-y|^{n+\delta}}$$

provided $|x-x'| < \frac{1}{2}|x-y|$.

Definition 3.6.1. An operator $T : \mathcal{D} \to \mathcal{D}'$ is said to be associated to a kernel $K \in L'_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n\backslash\Delta)$, provided for any $f, g \in \mathcal{D}$ with $\operatorname{supp} f \cap \operatorname{supp} g =$

$$\langle Tf, g\rangle = \iint K(x,y) f(y) g(x)\, dx\, dy$$

We will use the notation $T \sim K$ if the operator T is associated to K.

For a function φ we set $\varphi^{x,t}(y) := \varphi\left(\frac{y-x}{t}\right)$.

Definition 3.6.2. An operator T is said to be weakly bounded if there exist positive constants C, N such that $\forall\, x \in \mathbb{R}^n$, $\forall\, t \in \mathbb{R}_+$, and

$$\varphi, \psi \in B_N := \{\varphi \in C^\infty : \operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n : \|x\| \leq 1\},\ ||\partial^\alpha \varphi||_{\forall|\alpha|\leq N} \leq 1\}$$

there holds

$$|\langle T\varphi^{x,t}, \psi^{x,t}\rangle| \leq Ct^n.$$

Theorem 3.6.3. *If $T : \mathcal{S} \to \mathcal{S}'$ is a linear and continuous operator associated to a standard kernel, then*

$$T : L^2 \to L^2 \text{ is bounded } \Leftrightarrow \begin{cases} T(1) \in \mathrm{BMO} \\ T^*(1) \in \mathrm{BMO} \\ T \text{ is weakly bounded} \end{cases}$$

Theorem 3.6.4. *If $T : \mathcal{S} \to \mathcal{S}'$ is a linear, continuous, operator, associated to a standard and antisymmetric kernel, then*

$$T : L^2 \to L^2 \text{ is bounded } \Leftrightarrow T(1) \in \mathrm{BMO}.$$

Theorem 3.6.5. *Let $T : \mathcal{S} \to \mathcal{S}'$ be a linear and continuous operator, associated to a standard kernel. Then $T : L^2 \to L^2$ is bounded if and only if there exists $A > 0$ such that $\forall\, \phi \in C^\infty$, $\operatorname{supp}\phi \subset B(0,1)$ with $|\partial^\alpha \phi| \leq 1$, $\forall\, 0 \leq |\alpha| \leq N$ there hold*

$$\|T(\phi^{X_0,R})\|_{L^2} \leq A\, R^{\frac{n}{2}}, \quad \|T^*(\phi^{X_0,R})\|_{L^2} \leq A\, R^{\frac{n}{2}}. \tag{3.6.9}$$

Theorem 3.6.6. *Let $T : \mathcal{S} \to \mathcal{S}'$ be a linear, continuous operator associated to a standard kernel. Then $T : L^2 \to L^2$ is bounded if and only if there exists $C > 0$ such that $\|T(X_B)\|_{L^1(B)}$, $\|T^*(X_B)\|_{L^1(B)} \leq C_{B_1}$ for all balls $B \subset \mathbb{R}^n$.*

Theorem 3.6.7. *Let $T : \mathcal{D} \to \mathcal{D}$ be a linear, continuous operator associated to a standard kernel and having the property that there exists $\epsilon > 0$ such that*

$$|\langle T\psi^{x,t}, \phi^{y,t}\rangle| \leq \frac{1}{t}\frac{1}{1 + \left|\frac{u-v}{t}\right|^{1+\epsilon}}, \qquad |\langle T^*\psi^{x,t}, \phi^{y,t}\rangle| \leq \frac{1}{t}\frac{1}{1 + \left|\frac{u-v}{t}\right|^{1+\epsilon}} \tag{3.6.10}$$

Then

$$T : L^2 \to L^2 \text{ is bounded } \Leftrightarrow T(1) \in \mathrm{BMO},\ T^*(1) \in \mathrm{BMO}.$$

Theorem 3.6.8 (David, Journé, Semmers). *Suppose $T : \mathcal{S} \to \mathcal{S}'$ is a linear, continuous operator, associated to a standard kernel. Then,*

$$T : L^2 \to L^2 \text{ is bounded } \Leftrightarrow \exists\, b \in L^\infty \text{ such that } T(b) \in \mathrm{BMO},$$
$$T^*(b) \in \mathrm{BMO},\ M_b T M_b \text{ is weakly bounded,}$$

where M_b is the multiplication by b operator.

6.5. Spaces of homogeneous type and singular integral operators on such spaces

Definition 3.6.9. A topological space X is called *space of homogeneous type* if it is endowed with a Borel measure μ and a quasi-metric (or quasi-distance) $d : X \times X \to \mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$ satisfying
(a) $d(x, y) = d(y, x)$,
(b) $d(x, y) > 0$ if and only if $x \neq y$, and
(c) there exists a constant c such that $d(x, y) \leq c[d(x, z) + d(z, y)]$
for all $x, y, z \in X$.

In this setting the spheres $B_r(x) = \{y \in X : d(x, y) < r\}$ centered at x and of radius r form a basis of open neighborhoods of the point x and $\mu(B_r(x)) > 0$ whenever $r > 0$. Coifman and Weiss (in [CW1]) showed the existence of an absolute constant C such that

$$\mu(B_r(x)) \leq C\, \mu(B_{r/2}(x)).$$

An example of space of homogeneous type would be $X = \mathbb{R}^n$ with μ the Lebesgue measure and $d(x, y) = \sum_{j=1}^n |x_j - y_j|^{\alpha_j}$, where $\alpha_j > 0$. A list of other examples can be found in [CW2].

The following theorem of Coifman and de Guzmán (see [CG]) is particularly important for us.

Theorem 3.6.10. *Let X be a space of homogeneous type endowed with the Borel measure μ and the quasi-metric d. Consider $k(x, y) : X \times X \longrightarrow \mathbb{R}$ a measurable function satisfying the integral Hörmander condition: there exist constants $c_1 > 1$, $c_2 > 0$ such that, for every $x_0, x \in X$,*

$$\int_{d(x_0,y) \geq c_1\, d(x_0,x)} |k(x_0, y) - k(x, y)| d\mu(y) \leq c_2. \tag{3.6.11}$$

Assume that the operator

$$Tf(x) = \int_X k(x, y) f(y) d\mu(y)$$

is continuous from $L^2(X)$ to $L^2(X)$. Then for every $1 \leq p \leq 2$ there exists a constant $c = c(p)$ depending on p, such that for every $f \in L^2(X) \cap L^p(X)$

$$\|Tf\|_{L^p(X)} \leq c\, \|f\|_{L^p(X)} \quad \text{if } 1 < p \leq 2,$$

and

$$\mu\{x \in X : |Tf(x)| > t\} \leq c\, \frac{\|f\|_{L^1(X)}}{t} \quad \text{for every } t > 0, \text{ if } p = 1.$$

If k^* satisfies condition (3.6.11), then by duality, T is also bounded on $L^p(X)$ for $2 < p < \infty$.

A detailed proof is exposed in [CW1].

3.7 Exercises

1. Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz function. There exists $N = N(n)$ such that, if $k \in C^N(\mathbb{R}^n \setminus 0)$ is even (i.e., $k(-x) = k(x)$) and homogeneous of degree $-n$, then $k(x-y)(A(x)-A(y))$ is the kernel of an operator K bounded on $L^p(\Gamma)$, for $1 < p < \infty$, of norm

$$\|K\|_{\mathcal{L}(L^p)} \leq C(p, \|\nabla\phi\|_{L^\infty})\|\nabla A\|_{L^\infty}\|k|_{S^{n-1}}\|_{C^N}. \tag{3.7.1}$$

e1.1a

2. Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a Lipschitz function. There exists $M = M(n)$ such that the following holds. Let $b(x,z)$ be even in z and homogeneous of degree $-n$ in z, and assume $D_z^\alpha b(x,z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$, for $|\alpha| \leq M$. Then $b(x, x-y)(A(x) - A(y))$ is the kernel of an operator B, bounded on $L^p(\Gamma)$, for $1 < p < \infty$.

3. If $p(x,\xi) \in S^{-2}_{\text{cl}}$ has principal symbol that is even in ξ, then the Schwartz kernels of $\partial_j p(x,D)$, $p(x,D)\partial_j$, $1 \leq j \leq n$, are all kernels of operators bounded on $L^p(\partial\Omega)$ for $1 < p < \infty$.

4. If b is as in Lemma ??? 3.5.3, then $b(y, x-y)$ is the kernel of an operator $\tilde{B}$, bounded on $L^p(\Gamma)$, for $1 < p < \infty$.

Hint: One argues similarly as in .... using this time $\tilde{B}f(x) = \sum_{j\geq 1} K_j(b_j f)(x)$.

5. Consider $p(x,\xi) \in S^{-1}_{\text{cl}}$ which has a principal part odd in ξ, and let $b(x, x-y)$ be the Schwartz kernel of $p(x,D) \in OPS^{-1}_{\text{cl}}$. Let g be a fixed C^1 Riemannian metric on $\mathbb{R}^n$ and let $r(x,y)$ denote the geodesic distance (in the metric g) between two points x, y. Further, let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and denote by $d\sigma$ the area element of $\partial\Omega$ induced from the Euclidean structure of $\mathbb{R}^n$. Then, for each $1 < p < \infty$, the principal-value singular integral operators

$$\begin{aligned} Bf(x) &= \lim_{\varepsilon\to 0} \int_{\{y\in\Gamma:|x-y|>\varepsilon\}} b(x,x-y)f(y)\,d\sigma(y),\ \ x \in \partial\Omega, \\ B^g f(x) &= \lim_{\varepsilon\to 0} \int_{\{y\in\Gamma:r(x,y)>\varepsilon\}} b(x,x-y)f(y)\,d\sigma(y),\ \ x \in \partial\Omega. \end{aligned} \tag{3.7.2}$$

B.1

coincide pointwise a.e. for any $f \in L^p(\partial\Omega)$.

6. Prove a similar result for integral operators associated with the kernel $b(y, x-y)$, where b is as in Lemma ??? 3.5.3...

7. Consider

$$\widetilde{\mathcal{B}}f(x) := \int_\Gamma b(y, x-y)f(y)\,d\sigma(y), \quad x \in \mathbb{R}^n \setminus \Gamma, \tag{3.7.3}$$

e1.16

where b is as in Lemma ??? 3.5.3. Using the same superposition arguments as in the proof of ...., show that

$$\|N(\widetilde{\mathcal{B}}f)\| \le C\|f\|_{L^p(\Gamma)}, \qquad (3.7.4)$$ e1.19

and

$$(\widetilde{\mathcal{B}}f)_\pm(x) = \mp\tfrac{1}{2}iq_{-1}\big(n(x),x\big)f(x) + \widetilde{B}f(x), \qquad (3.7.5)$$ e1.23

for any $f \in L^p(\Gamma_0)$, $1 < p < \infty$.

8. As pointed out in ...., the operators treated in Theorem [???] 3.5.3 are bounded on $L^p(\Gamma, \omega\, d\sigma)$ for every A_p weight ω, when $p \in (1,\infty)$. Extend the arguments used in the proofs of .... to show that for any $p \in (1,\infty)$ and any A_p weight ω on Γ, the operators treated in .... are all bounded on $L^p(\Gamma_0, \omega\, d\sigma)$.

9. Extend Theorem [???] 3.5.3 to the case when the codimension of Γ is > 1. Concretely, prove that if Γ be an m-dimensional Lipschitz graph in $\mathbb{R}^n$ and $p(x,\xi) \in S^{-(n-m)}_{\rm cl}$ has principal symbol that is odd in ξ, then the Schwartz kernel of $p(x,D)$ is the kernel of an operator bounded on $L^p(\Gamma_0)$, for any $p \in (1,\infty)$ and any compact $\Gamma_0 \subset \Gamma$.

10. Let T^A denote the operator defined in 2 of Theorem [t2.5] 3.2.2. Prove that the mapping $[0,1] \ni t \mapsto T^{tA} \in \mathcal{L}(L^p(\mathbb{R}^n))$ is Lipschitz continuous.

Hint: Show that the operator $\frac{d}{dt}[T^{tA}]$ falls under the scope of Theorem [???] 3.5.3.

11. A more powerful version of the above statement is as follows. Let A, F, and K be as in Theorem [t2.5] 3.2.2. Then, the mapping $A \mapsto T$ is continuous from the set of Lipschitz functions (from $\mathbb{R}^n$ into $\mathbb{R}^m$) into the set of bounded linear operators on $L^p(\mathbb{R}^n)$.

Hint: Prove that $\|T^A - T^B\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \le C\|\nabla A - \nabla B\|_{L^\infty(\mathbb{R}^n)}$. To show this, write $T^A - T^B = \int_0^1 \frac{d}{dt}[T^{(1-t)A+tB}]\, dt$ and invoke Theorem [Hofmann] ??.

12. Supply a detailed proof of Corollary [???] 3.5.3.

13. Let $\mathcal{Q}$ be a $m \times n$ matrix with real entries so that $\mathcal{Q}x = 0 \Leftrightarrow x = 0$. Prove that there exists a constant $C > 0$ so that the (Euclidean) measure of the set

$$\{x \in \mathbb{R}^n;\ r < |\mathcal{Q}x| < R\}$$

is $\le C(R^n - r^n)$, for each $0 < r < R < \infty$.

Hint: Write $|\mathcal{Q}x| = |(\mathcal{Q}^t\mathcal{Q})^{1/2}x|$ then make a change of variables.

14. Let $B : \mathbb{R}^n \to \mathbb{R}^m$, $M > 0$, be such that $|x - y| \le M|B(x) - B(y)|$ for each $x, y \in \mathbb{R}^n$. Show that if B is differentiable at $x^* \in \mathbb{R}^n$ then $|\nabla B(x^*) \cdot v| \ge M^{-1}|v|$ for each $v \in \mathbb{R}^n$.

Hint: Consider the directional derivative of B at x^* along $v/|v|$.

15. (Jean-Lin Journe)
Suppose that $K : \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \ne y\} \to \mathbb{C}$ is a continuous function for which

there exists a positive constant c such that for all $x, y \in \mathbb{R}^n$, $x \neq y$,

$$K(x,y) \leq \frac{c}{|x-y|^n}$$

and

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leq \frac{c}{|x-y|^{n+1}}.$$

Let Q be a cube in $\mathbb{R}^n$, and $f \in L^1_{loc}(\mathbb{R}^n)$. Show the following estimates.

(a) For $x \in Q$,

$$\int\limits_{2Q\setminus Q} |K(x,y)|\,|f(y)|\,dy \leq c\,f^*(x).$$

(b) For $x, x_0 \in Q$,

$$\int\limits_{(2Q)^c} |K(x,y) - K(x_0,y)|\,|f(y)|\,dy \leq c\,f^*(x_0).$$

(c) For $y_0 \in Q$,

$$\int\limits_{(2Q)^c} \int\limits_{Q} |K(x,y) - K(x,y_0)|\,|f(y)|\,dy\,dx \leq c\,|Q|\,f^*(y_0).$$

Here

$$f^*(x) = \sup_{r>0} \frac{1}{Q(x,r)} \int\limits_{Q(x,r)} f(y)dy,$$

and $Q(x,r)$ is the cube in $\mathbb{R}^n$, centered at x and with radius r.

16. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function and let $F : \mathbb{R}^m \to \mathbb{R}$ be a smooth, odd function. Prove that the limit

$$\lim_{\varepsilon\to 0} \int_{1>|x-y|>\varepsilon} \frac{1}{|x-y|^n} F\left(\frac{A(x)-A(y)}{|x-y|}\right) dy \qquad (3.7.6)$$ eq???

exists for a.e. $x \in \mathbb{R}^n$.

Hint: For a fixed $f \in C^1(\mathbb{R}^n)$ which decays sufficiently fast at infinity and such that $f(x) > 0$ at every point $x \in \mathbb{R}^n$, write

$$\begin{aligned} T_\varepsilon f(x) &= \int\limits_{|x-y|>1} K(x,y)f(y)\,dy + \int\limits_{1>|x-y|>\varepsilon} K(x,y)[f(y)-f(x)]\,dy \\ &\quad + f(x) \int\limits_{1>|x-y|>\varepsilon} K(x,y)\,dy \end{aligned}$$

and invoke Theorem t.??? ??.

17. (After boundedness of Cauchy operator on L^p)

Let A be a Lipschitz function on the real line. Set $K_n(x,y) := \frac{(A(x)-A(y))^n}{(x-y)^{n+1}}$, and let T_n be the principal value operator defined by K_n. Prove that T_n is bounded on $L^2(\mathbb{R}^n)$ with a norm controlled by $C^{n+1}\|A'\|_{L^\infty(\mathbb{R}^n)}$.

Hint: Do this by induction. Show that $T_n(1) = T_{n-1}(A')$ and then proceed, using the corresponding computations in Theorem ??? [Boundedness of Cauchy operator].

18. (After boundedness of Cauchy operator on L^p)

A rectifiable, connected curve $\Gamma \subset \mathbb{C}$ is called chord-arc, if for some constant $C \geq 0$ and any two points $a, b \in \Gamma$, the length of the piece of Γ connecting a to b is less than C times $|a-b|$. In terms of the arclength parametrization $z(t)$ of Γ, the chord-arc condition is $|t-s| \leq C|z(t) - z(s)|$ for every $t, s \in \mathbb{R}$. Define a Cauchy integral operator on Γ as $\mathcal{C}_\Gamma f(x) = \text{p.v.} \int_\Gamma \frac{f(w)\,ds(w)}{z-w}$, where ds stands for the arc-length measure on Γ. Show that $\mathcal{C}_\Gamma$ is a bounded operator on $L^2(\Gamma, ds)$.

19. Consider $F \in C^\infty(\mathbb{R}^m \setminus \{0\})$ odd, $A : \mathbb{R}^n \to \mathbb{R}^m$ a Lipschitz function, and, for $x \in \mathbb{R}^n$, define

$$Tf(x) = \lim_{\epsilon \to 0} \int\limits_{\substack{|x-y|>\epsilon \\ y \in \mathbb{R}^n}} F\left(\frac{A(x) - A(y)}{|x-y|}\right) k(x, x-y)\, f(y)\, dy,$$

where $k(x,y)$ is homogeneous of degree $-n$. If $T(1) = \mathit{constant}$, then

$$T : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

is bounded for $1 < p < \infty$.

Hint. Use the method of spherical harmonics to separate x and $x-y$.

20. Prove that for every function $f \in L^2(\mathbb{R}^n)$, each $\gamma > 0$ and all $\varepsilon > 0$,

$$\varepsilon^\gamma \int_{|y|\geq\varepsilon} |y|^{-n-\gamma} |f(y)| dy \leq C\mathcal{M}f(0), \tag{3.7.7}$$ eq2.9-3

where the constant $C = C(n, \gamma)$ depends only on $\gamma > 0$ and the dimension.

Hint: To establish this inequality, split the range of integration into the dyadic shells $D_k = \{y \in \mathbb{R}^n : 2^k\varepsilon \leq |y| < 2^{k+1}\varepsilon\}$, $k \in \mathbb{N}$. Next, observe that each D_k is a subset of the ball $B_k = \{y \in \mathbb{R}^n : |y| \leq 2^{k+1}\varepsilon\}$ and apply the definition of the maximal function to the ball B_k.

Chapter 4

Singular Integrals on Sobolev-Besov spaces

4.1 Sobolev and Besov spaces in Lipschitz domains

Let us start by recalling the Littlewood-Paley definition of Triebel-Lizorkin ($F^{p,q}_s$) and Besov ($B^{p,q}_s$) spaces. Let Φ be the collection of all systems $\{\phi_j\}_{j=0}^{\infty} \subset \mathcal{S}$, the Schwartz class, such that

(i) there exist positive constants A, B, C such that

$$\begin{cases} supp\ \phi_0 \subset \{x;\ |x| \leq A\}, \\ supp\ \phi_j \subset \{x;\ B2^{j-1} \leq |x| \leq C2^{j+1}\} \quad \text{if } j = 1, 2, 3..., \end{cases} \tag{4.1.1}$$ eq2.1

(ii) for every multi-index α there exists a positive number c_α such that

$$\sup_{x\in\mathbb{R}^n} \sup_{j\in\mathbb{N}} 2^{j|\alpha|} |\partial^\alpha \phi_j(x)| \leq c_\alpha, \tag{4.1.2}$$ eq2.2

(iii)

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \text{ for every } x \in \mathbb{R}^n. \tag{4.1.3}$$ eq2.3

Let $s \in \mathbb{R}$ and $0 < q \leq \infty$ and fix some family $\{\phi_j\}_{j=0}^{\infty} \in \Phi$. Also, let $\mathcal{F}$ denote the Fourier transform in $\mathbb{R}^n$.

(i) If $0 < p < \infty$ then the Triebel-Lizorkin scale is defined by

$$F^{p,q}_s(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}';\quad \|f\|_{F^{p,q}_s(\mathbb{R}^n)} := \Big\| \Big(\sum_{j=0}^{\infty} |2^{sj} \mathcal{F}^{-1}(\phi_j \mathcal{F} f)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\}. \tag{4.1.4}$$ eq2.4

(ii) If $0 < p \leq \infty$ then the Besov scale is

$$B_s^{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'; \quad \|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \Big(\sum_{j=0}^{\infty} \|2^{sj}\mathcal{F}^{-1}(\phi_j \mathcal{F} f)\|^q_{L^p(\mathbb{R}^n)} \Big)^{1/q} < \infty \Big\}. \tag{4.1.5}$$

eq2.5

As is well-known, a different choice of the system $\{\phi_j\}_{j=0}^{\infty} \in \Phi$ yields the same spaces (6.1.4)-(6.1.5) [eq2.4, eq2.5], albeit equipped with equivalent norms.

For the range of indices $n/(n+1) < p,q < \infty$ and $n(1/p-1)_+ < s < 1$, an intrinsic definition for membership to $B_s^{p,q}(\mathbb{R}^n)$ is obtained by requiring that

$$\|f\|_{B_s^{p,q}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} \frac{\|f(\cdot+t) - f(\cdot)\|^q_{L^p(\mathbb{R}^n)}}{|t|^{n+sq}}\, dt \right)^{1/q} < +\infty. \tag{4.1.6}$$

eq2.7

It has been long known that many classical smoothness spaces are encompassed by the above two scales. For example,

$$\begin{array}{lll} C^s(\mathbb{R}^n) = B_s^{\infty,\infty}(\mathbb{R}^n), & 0 < s \notin \mathbf{Z}, & \\ L^p(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n), & 1 < p < \infty, & \\ L^p_k(\mathbb{R}^n) = F_k^{p,2}(\mathbb{R}^n), & 1 < p < \infty, & k = 1, 2..., \\ H^p_{at}(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n), & 0 < p \le 1, & \\ H^{1,p}_{at}(\mathbb{R}^n) = F_1^{p,2}(\mathbb{R}^n), & 0 < p \le 1. & \end{array}$$

We let $d\sigma$ denote the canonical surface measure on $\partial\Omega$ so that ν, the outward unit normal to $\partial\Omega$, is well-defined $d\sigma$-a.e.

The scales of Besov and Triebel-Lizorkin spaces can be naturally transported from $\mathbb{R}^{n-1}$ to the boundary of a (bounded) Lipschitz domain Ω via pull-back and a partition of unity. We denote them by $B_s^{p,q}(\partial\Omega)$ and by $F_s^{p,q}(\partial\Omega)$, respectively. More specifically, if $(n-1)/n < p,q < \infty$, $(n-1)(1/p-1)_+ < s < 1$, when Ω is the region from $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, we define $B_s^{p,q}(\partial\Omega)$ as the space of functions f for which the assignment $x \mapsto f(x, \phi(x))$ belongs to $B_s^{p,q}(\mathbb{R}^{n-1})$. This definition then readily extends to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument. The case when $p = q = \infty$ corresponds to the usual (non-homogeneous) Hölder spaces $C^s(\partial\Omega)$. Similar considerations apply to $F_s^{p,q}(\partial\Omega)$.

Denote by $L^p_1(\partial\Omega)$ the Sobolev space of functions in $L^p(\partial\Omega)$ whose tangential gradients are in $L^p(\partial\Omega)$, $1 < p < \infty$. Sobolev and Besov spaces with positive, fractional smoothness can then be defined via complex and real interpolation methods, respectively, i.e.

$$L^p_\theta(\partial\Omega) \;:=\; [L^p(\partial\Omega), L^p_1(\partial\Omega)]_\theta,\ 0 < \theta < 1,\ 1 < p < \infty, \tag{4.1.7}$$

eqLcomplexIP

$$B^{p,q}_\theta(\partial\Omega) \;:=\; (L^p(\partial\Omega), L^p_1(\partial\Omega))_{\theta,q}, \quad \text{with } 0 < \theta < 1,\ 1 < p,q < \infty. \tag{4.1.8}$$

eqLrealIP

Next, for a Lipschitz domain $\Omega \subset \mathbb{R}^n$, we let $B_s^{p,q}(\Omega)$, $1 \le p,q \le \infty$, $s > 0$, consist of restrictions to Ω of functions from $B_s^{p,q}(\mathbb{R}^n)$. Recall that the trace operator

$$\mathrm{Tr}: B^{p,p}_s(\Omega) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \tag{4.1.9}$$ eq2.9

is well defined, bounded and onto if $1 \le p \le \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$. This also has a bounded right inverse whose operator norm is controlled exclusively in terms of p, s and the Lipschitz character of Ω.

Regarding Besov spaces with a negative amount of smoothness, if Ω is the domain in $\mathbb{R}^n$ above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, we agree that

$$f \in B^{p,q}_{-s}(\partial\Omega) \Longleftrightarrow f(x,\phi(x))\sqrt{1+|\nabla\phi(x)|^2} \in B^{p,q}_{-s}(\mathbb{R}^{n-1}),$$

whenever $(n-1)/n < p,q < \infty$, $(n-1)(1/p-1)_+ < 1-s < 1$. As before, this definition is then extended to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a simple partition of unity argument. In particular, for $-1 < s < 0$ and $1 < p,q < \infty$,

$$B^{p,q}_s(\partial\Omega) = (B^{p',q'}_{-s}(\partial\Omega))^*, \qquad 1/p + 1/p' = 1,\ 1/q + 1/q' = 1, \tag{4.1.10}$$ eq2.8

where the duality pairing between $f \in B^{p,q}_s(\partial\Omega)$ and $g \in B^{p',q'}_{-s}(\partial\Omega)$ is (a natural extension of) $\int_{\partial\Omega} fg\,d\sigma$. We also refer to [**RS**], [**Tr1**], [**BL**], [**JK4**], [**MT3**], for a more detailed exposition of these and other related matters, such as embedding theorems.

In the sequel, we shall also use atomic characterization of the Besov spaces $B^{p,p}_s(\partial\Omega)$. These are a straightforward adaptation of the Euclidean results from [**FJ2**], given the definitions we adopt in this paper. Specifically, let us assume that $(n-1)/n < p < \infty$ and that $(n-1)(\frac{1}{p}-1)_+ < s < 1$. A function a is called an atom for $B^{p,p}_s(\partial\Omega)$ if

$$\begin{aligned}
&(1)\ \exists S_r - \text{surface ball} : supp(a) \subseteq S_r,\\
&(2)\ \|a\|_{L^\infty(\partial\Omega)} \le r^{s-\frac{n-1}{p}},\\
&(3)\ \|\nabla_{tan} a\|_{L^\infty(\partial\Omega)} \le r^{s-\frac{n-1}{p}-1}.
\end{aligned}$$

Here and elsewhere, $\nabla_{tan} = \nabla - \nu\partial_\nu$ will denote the tangential gradient on $\partial\Omega$. Also, a surface ball $S_r(x)$ is any set of the form $B_r(x)\cap\partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}\Omega$. Parenthetically, let us note that (1) & (3) $\Rightarrow$ (2). Then

$$\|f\|_{B^{p,p}_s(\partial\Omega)} \approx \inf\Big\{\Big(\sum_j |\mu_j|^p\Big)^{1/p};\ f = \sum_j \mu_j a_j,\ a_j \text{ are } B^{p,p}_s(\partial\Omega) \text{ atoms},\ \{\mu_j\}_j \in \ell^p\Big\}. \tag{4.1.11}$$ eq2.10

Similarly, there are atomic decompositions for $B^{p,p}_{-s}(\partial\Omega)$. Concretely, let us assume that $(n-1)/n < p < \infty$ and $(n-1)(\frac{1}{p}-1)_+ < 1-s < 1$, and call a an atom for $B^{p,p}_{-s}(\partial\Omega)$ if

$$(1)\ \exists S_r - \text{surface ball} : supp(a) \subseteq S_r,$$

$$(2)\ \|a\|_{L^\infty(\partial\Omega)} \leq r^{-s-\frac{n-1}{p}},$$
$$(3)\ \int_{\partial\Omega} a\, d\sigma = 0.$$

Then the Euclidean results from FrJa [**FJ2**] lifted to $\partial\Omega$ give

$$\|f\|_{B^{p,p}_{-s}(\partial\Omega)} \approx \inf\Big\{ \Big(\sum_j |\mu_j|^p\Big)^{1/p};\ f = \sum_j \mu_j a_j,\ a_j \text{ are } B^{p,p}_{-s}(\partial\Omega) \text{ atoms},\ \{\mu_j\}_j \in \ell^p \Big\}. \tag{4.1.12}$$ eq2.11

As far as the Hardy spaces $H^p_{at}(\partial\Omega)$, $\frac{n-1}{n} < p \leq 1$, are concerned, call a an atom for $H^p_{at}(\partial\Omega)$, if

$$(i)\ \exists S_r - \text{surface ball} : supp(a) \subseteq S_r,$$
$$(ii)\ \|a\|_{L^\infty(\partial\Omega)} \leq r^{-\frac{n-1}{p}},$$
$$(iii)\ \int_{\partial\Omega} a\, d\sigma = 0.$$

Then

$$H^p_{at}(\partial\Omega) := \Big\{ \sum_j \lambda_j a_j;\ a_j \text{ is an } H^p_{at}(\partial\Omega) \text{ atom},\ \{\lambda_j\}_j \in \ell^p \Big\}. \tag{4.1.13}$$ eq2.12

See, e.g., CW [**CW2**] for the more general setting of homogeneous spaces.

We now proceed to record some important interpolation results. By $[\cdot,\cdot]_\theta$ and $(\cdot,\cdot)_{\theta,q}$ we will denote the complex and real methods of interpolation, respectively.

t2.3 **Theorem 4.1.1.** *(cf.* MeMi [**MeMi2**]*) Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 \leq \infty$, $p_0 + q_0 < \infty$, $p_1 + q_1 < \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Then*

$$[F^{p_0,q_0}_{\alpha_0}(\mathbb{R}^n), F^{p_1,q_1}_{\alpha_1}(\mathbb{R}^n)]_\theta = F^{p,q}_\alpha(\mathbb{R}^n). \tag{4.1.14}$$ eq2.19

t2.4 **Theorem 4.1.2.** *(cf.* Tr83 [**Tr1**]*) Let $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < p < \infty$, $0 < q_0, q_1 \leq \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. Then*

$$(F^{p,q_0}_{\alpha_0}(\mathbb{R}^n), F^{p,q_1}_{\alpha_1}(\mathbb{R}^n))_{\theta,q} = B^{p,q}_\alpha(\mathbb{R}^n). \tag{4.1.15}$$ eq2.20

An important remark is that appropriate versions of the last two theorems above continue to hold in the case when the underlying Euclidean space is replaced by the boundary of a Lipschitz domain –this is more or less immediate from the various definitions adopted in this paper. In the sequence, we shall tacitly use this without any special mention. The interested reader is further referred to Za [**Za**], Trnew [**Tr2**], for a more detailed discussion in this regard.

tJeKecriterion **Theorem 4.1.3.** *(cf. [JK4]) Suppose u is a function defined in the Lipschitz domain Ω. For $0 < \alpha < 1$, a nonnegative integer k, and $1 \leq p \leq \infty$, consider the following statements:*

(a) u belongs to $B^{p,p}_{k+\alpha}(\Omega)$;

(b) $\delta^{1-\alpha}|\nabla^{k+1}u| + |\nabla^k u| + |u|$ belongs to $L^p(\Omega)$.

Then

(i) (b) $\Rightarrow$ (a), for any $u \in L^1_{loc}(\Omega)$ such that $\nabla^j u \in L^1_{loc}(\Omega)$, $1 \leq j \leq k+1$;

(ii) (b) $\Leftrightarrow$ (a), if u is harmonic in Ω.

The result similar to (ii) in the theorem above holds for Sobolev scale $L^p_{k+\alpha}(\Omega)$ under the hypotheses $0 \leq \alpha \leq 1$ and $1 < p < \infty$ (cf. [JK4]). However, the proof is rather elaborate and since we have no use for results of this kind in the present monograph, we shall not pursue the subject further.

Proof of Theorem 4.1.3. To start, observe that the matters can be reduced to the case $k = 0$ (see Proposition 4.1.4 for details).

Next, assume that u is locally integrable function on Ω as well as its derivatives of order less than or equal to $k+1$. Suppose, u belongs to the space $B^{p,p}_{\alpha}(\Omega)$. By virtue of interpolation formula (4.1.8) the norm of u in $B^{p,p}_{\alpha}(\Omega)$ can be regarded as the infimum of

$$\left(\int_0^\infty \|t^{1-\alpha}f(t)\|^p_{L^p_1(\Omega)}t^{-1}\,dt\right)^{1/p} + \left(\int_0^\infty \|t^{1-\alpha}f'(t)\|^p_{L^p(\Omega)}t^{-1}\,dt\right)^{1/p}. \qquad (4.1.16)$$ eqIPNorm

over all functions $f : [0,\infty) \to L^p(\Omega) + L^p_1(\Omega)$ such that $f(0) = u$. Since the interior estimates are straightforward, we shall restrict our attention to a small neighborhood of the boundary, where Ω can be locally represented as a domain above the graph of Lipschitz function $\phi(x)$. More precisely, consider a ball $B(x', r)$ centered at boundary point x' such that

$$B \cap \Omega = \{(x', y) \in B(x', r);\, y > \phi(x')\} \qquad \text{where} \qquad \phi(0) = 0. \qquad (4.1.17)$$ eq???

Going further, we introduce functions $\eta \in C^\infty_0(B(x', r))$ and $\theta \in C^\infty_0(-r, r)$ such that $\eta(x)|_{B(x',r/2)} = \theta(y)|_{(-r/2,r/2)} = 1$. Also, set $g(t) := \eta(x)u(x, y+t)\theta(t)$. Then $g(0) = \eta u$ and the expression in (4.1.16) can be controlled modulo finite constant by

$$\left(\int_0^\infty \|t^{1-\alpha}g(t)\|^p_{L^p(\Omega)}t^{-1}\,dt\right)^{1/p} + \left(\int_0^\infty \int_{B\cap\Omega} |t^{1-\alpha}\theta(t)\eta(x)\nabla u(x+te_n)|^p\,dx t^{-1}\,dt\right)^{1/p}.$$

The first summand above is majorized by

$$\left(\int_0^\infty \int_{B\cap\Omega} |u(x', y)|^p\,dx'dy\, t^{p-\alpha p-1}\,dt\right)^{1/p} \leq C\|u\|_{L^p(\Omega)}. \qquad (4.1.18)$$ eq???

As for the second one, it does not exceed

$$\left(\int_{B\cap\Omega}|\nabla u(x)|^p\int_0^{C\delta(x)}t^{p-\alpha p-1}\,dtdx\right)^{1/p}\leq C\left(\int_{B\cap\Omega}|\delta(x)^{1-\alpha}\nabla u(x)|^p\,dx\right)^{1/p}. \tag{4.1.19}$$ eq???

Now it remains to establish the implication $(a)\Rightarrow(b)$ for arbitrary harmonic function u in Ω. According to the Mean Value property of harmonic functions and Holder's inequality,

$$\begin{aligned}|\nabla u(x)| &\leq C\delta(x)^{-1}\left(\frac{1}{\delta(x)^n}\int_{B(x,\delta(x)/2)}|u(z)-u(x)|^p\,dz\right)^{1/p}\\ &\leq C\delta(x)^{\alpha-1}\left(\int_{B(x,\delta(x)/2)}\frac{|u(z)-u(x)|^p}{|x-z|^{n+\alpha p}}\,dz\right)^{1/p}. \end{aligned}\tag{4.1.20}$$ eq???

With this in mind, one can use an intrinsic characterization of Besov spaces to complete the argument.

□

pJeKe2.18 **Proposition 4.1.4.** *(cf. [JK4] (JeKe)) Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Then*

(a) for $\alpha>0$ and $1<p<\infty$, $u\in L^p_{\alpha+1}(\Omega)$ if and only if $u\in L^p(\Omega)$ and $\nabla u\in L^p_\alpha(\Omega)$,

(b) for $\alpha>0$ and $1\leq p\leq\infty$, $u\in B^{p,p}_{\alpha+1}(\Omega)$ if and only if $u\in L^p(\Omega)$ and $\nabla u\in B^{p,p}_\alpha(\Omega)$ for $\alpha>0$ and $1\leq p\leq\infty$.

THE PROOF?????????????????

4.2 Mapping properties of SIO, I

To start, denote by $C^\mu S^m_{\rm cl}$ the class of classical symbols $q(\xi,x)$ of order m which are C^μ in x, for some $\mu\in[0,\infty]$, while still smooth in $\xi\in\mathbb{R}^n\setminus 0$. Also, given any two numbers a,b we set $a\vee b:=max\{a,b\}$.

pmp1 **Proposition 4.2.1.** *Assume that $K(x-y,y)$ is the Schwartz kernel for some $q(\xi,x)\in C^\mu S^{-2}_{cl}$, $\mu\geq 1$, whose principal symbol is even in ξ. Consider the corresponding integral operator*

$$Rf(x):=\int_{\partial\Omega}K(x-y,y)f(y)\,d\sigma(y),\qquad x\in\Omega. \tag{4.2.1}$$ mp1

Then for $1<p<\infty$, $0\leq s\leq 1$, the operator

$$R:L^p_{-s}(\partial\Omega)\longrightarrow B^{p,p\vee 2}_{1-s+1/p}(\Omega) \tag{4.2.2}$$ mp2

is well defined and bounded.

Proof. By performing a decomposition in spherical harmonics (cf. [MT3] for details in similar circumstances), there is no loss of generality in assuming that $\mu = \infty$.

Note that if we treat only the cases when $s = 0$ and $s = 1$, the rest follows by complex interpolation (cf. [BL]). In fact, we shall only consider the situation when $s = 0$ (see Exercise ???). It suffices to show the following. If $q(\xi, x) \in C^\mu S^{-2}_{\rm cl}$, $\mu \geq 1$, has a principal symbol that is even in ξ, then the Schwartz kernels of $\partial_j q(D, x)$, $q(D, x)\partial_j \in \emptyset {\rm P} C^0 S^{-1}_{\rm cl}$ are all kernels of operators mapping $L^p(\partial\Omega)$ boundedly into $B^{p,p\vee 2}_{1/p}(\Omega)$ for each $1 < p < \infty$.

Take for instance the case of the Schwartz kernel $k(x - y, y)$ of $\nabla_x q(D, x) \in \emptyset {\rm P} C^0 S^{-1}_{\rm cl}$, for some $q(\xi, x) \in C^\mu S^{-2}_{\rm cl}$, $\mu \geq 1$, whose principal symbol is even in ξ and denote by $\mathcal{K}$ the corresponding integral operator, i.e.,

$$\mathcal{K} f(x) := \int_{\partial\Omega} k(x - y, y) f(y)\, d\sigma(y), \qquad x \in \Omega. \tag{4.2.3}$$ mp2

Fix $f \in L^p(\partial\Omega)$ and set $u := \mathcal{K} f$ in Ω. Analogously to [JK4], [Ve3], we use the fact that $\|u\|_{B^{p,q}_{1/p}(\Omega)}$ is controlled by a finite sum of expressions of the type

$$\begin{aligned} &\Big(\int_0^r t^{q-q/p}\Big(\int_{S_r}\int_t^r |\nabla u(x', \varphi(x') + s)|^p\, dx' ds\Big)^{q/p}\frac{dt}{t}\Big)^{1/q} \\ &+\Big(\int_0^r t^{q-q/p}\Big(\int_{S_r}\int_t^r |u(x', \varphi(x') + s)|^p\, dx' ds\Big)^{q/p}\frac{dt}{t}\Big)^{1/q} =: I + II. \end{aligned} \tag{4.2.4}$$ mp3

Here $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function used to describe $\partial\Omega$ locally and $S_r \subseteq \partial\Omega$ is a surface ball of fixed radius $r > 0$.

Our aim is to bound I and II by $\|f\|_{L^p(\partial\Omega)}$ in the case when $1 < p < \infty$ and $q := p \vee 2$. The first observation is that II in (4.2.4) can easily be controlled using

$$\|u^*\|_{L^p(\partial\Omega)} = \|(\mathcal{K} f)^*\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \tag{4.2.5}$$ mp4

where the last estimate is proved in [MT3] (recall that $(\cdot)^*$ stands for the nontangential maximal function of u). As for I, following [JK4] we invoke Hardy's inequality (cf., e.g., Appendix A in [St]) in the case $1 < p \leq 2$ plus Minkowski's inequality in order to write

$$\begin{aligned} |I| &\leq C\Big(\int_0^r \Big(\int_{S_r} |s\nabla u(x', \varphi(x') + s)|^p\, dx'\Big)^{q/p}\frac{ds}{s}\Big)^{1/q} \\ &\leq C\Big(\int_{S_r}\Big(\int_0^r |s\nabla u(x', \varphi(x') + s)|^q\, \frac{ds}{s}\Big)^{p/q} dx'\Big)^{1/p}. \end{aligned} \tag{4.2.6}$$ mp5

Note that we can arrive at same majorand for I as above in the case $2 \leq p < \infty$ simply by using Fubini's theorem since, in this case, $p = q$.

At this point observe that matters are reduced to proving the L^p-boundedness, $1 < p < \infty$, of the $L^q((0,r), ds/s)$-valued operator T given by the assignment:

$$L^p(S_r) \ni f \mapsto s\nabla\mathcal{K}f(x', \varphi(x') + s) \in L^p(S_r, L^q((0,r), ds/s)). \tag{4.2.7}$$ mp6

It is preferable to deal first with the Hilbert space setting, i.e., when $q = 2$, since in this case the vector-valued Calderón-Zygmund theory works. Concretely, setting

$$\tilde{k}_s(x', y') := s\,\nabla_1 k((x' - y', \varphi(x') - \varphi(y') + s), (y', \varphi(y'))), \quad x', y' \in \mathbb{R}^{n-1}, \tag{4.2.8}$$ mp7

the estimates

$$|\nabla_1^{i+1}\nabla_2^j k(x,y)| \leq C|x - y|^{-(n-i-j)}, \qquad i, j \geq 0, \tag{4.2.9}$$ mp8

readily imply that

$$\Big(\int_0^r |\nabla_{x'}^i \nabla_{y'}^j \tilde{k}_s(x', y')|^2 \frac{ds}{s}\Big)^{1/2} \leq C|x' - y'|^{-(n-1+i+j)}, \qquad 0 \leq i + j \leq 1. \tag{4.2.10}$$ mp9

In turn, these express the fact that the kernel of T in (4.2.8) is standard. The boundedness of the operator T when $p = 2$ follows from

$$\begin{aligned} \|Tf\|_{L^2(S_r, L^2((0,r), ds/s))} &\leq C\Big(\int_{S_r}\int_0^r s|\nabla u(x', \varphi(x') + s)|^2\, ds\, dx'\Big)^{1/2} \\ &\leq C\Big(\int_\Omega \delta(x)|\nabla u(x)|^2\, dx\Big)^{1/2} \\ &\leq C\|f\|_{L^2(\partial\Omega)}. \end{aligned} \tag{4.2.11}$$ mp10

The crucial step in (4.2.11) is the last inequality and this has been proved in Theorem 1.1 of [MMT]. This finishes the proof of the L^p-boundedness of T when $q = 2$ and takes care of the $1 < p \leq 2$ part in the statement of the theorem.

Next, consider the case when $p \geq 2$. When f has small support, contained in an open subset of $\partial\Omega$ where $\partial\Omega$ is given as the graph of $\varphi : \{x' \in \mathbb{R}^{n-1} : |x'| < r\} \to \mathbb{R}$ and $x_0 = (x_0', \varphi(x_0')) \in \partial\Omega$ is an arbitrary boundary point, then

$$\begin{aligned} & s|\nabla u(x_0', \varphi(x_0') + s)| \\ \leq & \int_{|y'|<r} s|\nabla k((x_0', \varphi(x_0') + s), (y', \varphi(y')))| \cdot |f(y', \varphi(y'))|\, dy' \\ \leq & \ C\mathcal{M}f(x_0), \end{aligned} \tag{4.2.12}$$ mp11

uniformly in s, where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator on $\partial\Omega$. The last inequality in (4.2.12) is a consequence of fact that the expression

$$s|\nabla k((x', \varphi(x') + s), (y', \varphi(y'))| \tag{4.2.13}$$ mp12

behaves like the Poisson kernel for the upper-half space; see, e.g., Theorem 2, pp. 62–63 in [St]. Thus,

$$\sup_{s\in(0,r)} s|\nabla u(x_0', \varphi(x_0') + s)| \leq C\mathcal{M}f(x_0). \tag{4.2.14}$$ mp13

Using this and the fact that $\mathcal{M}$ is bounded on $L^p(\partial\Omega)$, $1 < p < \infty$, it follows that

$$\Big(\int_{S_r} \Big(\sup_{s\in(0,r)}|s\nabla u(x', \varphi(x') + s)|\Big)^p\,dx'\Big)^{1/p} \leq C\|f\|_{L^p(\partial\Omega)}, \tag{4.2.15}$$ mp14

i.e., that T in (4.2.7) is bounded when $q = \infty$.

The case when $p = q > 2$ now follows by interpolating the end-point results corresponding to $q = 2$ and $q = \infty$. (Note that here we use the fact that $B_s^{p,q_1} \hookrightarrow B_s^{p,q_2}$ for $1 \leq q_1 < q_2 \leq \infty$.) This completes the proof of the proposition. □

We now analyze the action of the double layer potential on the scale of Sobolev spaces.

pmp2 **Proposition 4.2.2.** *For $1 < p < \infty$ and $0 \leq s \leq 1$, the operator*

$$\mathcal{D} : L_s^p(\partial\Omega) \longrightarrow B_{s+1/p}^{p,p\vee 2}(\Omega) \tag{4.2.16}$$ mp15

is bounded.

Proof. For (4.2.16) with $s = 0$, arguments similar to the ones used in the proof of the previous proposition apply. Matters can again be reduced to the same pattern in the case $s = 1$, thanks to the identity

$$\begin{aligned}
\partial_j\mathcal{D}f(x) &= \sum_{k=1}^n \int_{\partial\Omega} \nu_k(y)\partial_{x_j}\partial_{y_k}E(x-y)f(y)\,d\sigma_y \\
&= \sum_{k=1}^n \int_{\partial\Omega} (\nu_j(y)\partial_{y_k} - \nu_k(y)\partial_{y_j})[\partial_{y_k}E(x-y)]f(y)\,d\sigma_y \\
&= \sum_{k=1}^n \int_{\partial\Omega} \partial_{y_k}E(x-y)[(\nu_j\partial_k - \nu_k\partial_j)f](y)\,d\sigma_y.
\end{aligned} \tag{4.2.17}$$ mp16

valid for each $1 \leq j \leq n$ and $x \in \Omega$. The important aspect of the above identity is that it allows to write

$$\nabla\mathcal{D}f = \mathcal{T}(\nabla_{tan}f) \tag{4.2.18}$$ mp17

where ∇_{tan} stands for the tangential gradient on $\partial\Omega$, and $\mathcal{T}$ is a singular integral operator of the type $\partial_j\mathcal{S}$. Now the claim about (4.2.16) is obtained by interpolation. □

4.2.1 Mapping properties of layer potentials, II

Theorem 4.2.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Consider the integral operator*

$$Tf(x) = \int_{\partial\Omega} k(x,y)f(y)d\sigma(y), \qquad x \in \Omega, \tag{4.2.19}$$

satisfying the following conditions:

$$(1) \quad T1 = const, \tag{4.2.20}$$

$$(2) \quad |\nabla_x^k k(x,y)| \leq C|x-y|^{-(n+k+\gamma-1)}, \quad k = 1, 2, .., N, \tag{4.2.21}$$

for some positive integer N and $\gamma \geq 0$. Then

$$\|\delta^{k+\gamma-\frac{1}{p}-s}|\nabla^k Tf|\|_{L^p(\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)}, \tag{4.2.22}$$

granted that $\frac{n-1}{n} < p \leq \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$.

It should be noted that the class of operators satisfying (4.2.20) -(4.2.21) contains the double layer potential as well as the Cauchy operator.

Proof. We proceed by analyzing several cases starting with:

Case 1. $1 \leq p \leq \infty$**.** The estimate (4.2.22) will be proved in three steps. The idea is to obtain the result for $p = 1$, then for $p = \infty$ and, finally, use Stein's interpolation theorem for analytic families of operators (Theorem 3.2.2) to cover the range in between.

Consider first the case $p = 1$. We shall prove that under the current assumptions on the operator

$$\|\delta^{k+\gamma-1-s}|\nabla^k Tf|\,\|_{L^1(\Omega)} \leq C\|f\|_{B_s^{1,1}(\partial\Omega)}, \tag{4.2.23}$$

First recall the intrinsic characterization of the Besov space $B_s^{1,1}(\partial\Omega)$

$$\|f\|_{B_s^{1,1}(\partial\Omega)} = \|f\|_{L^1(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x)-f(y)|}{|x-y|^{n-1+s}} d\sigma(x)d\sigma(y). \tag{4.2.24}$$

The estimate we seek has local character. Thus, using a partition of unity, we may assume that the support of f is included in a coordinate patch where $\partial\Omega$ is represented by the graph of the Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$. Assuming that this is the case, we make a change of variables and set $\tilde{f}(x) := f(x, \phi(x))$, extended by zero outside of the support. In particular, $\tilde{f} \in B_s^{1,1}(\mathbb{R}^{n-1})$.

Thanks to (4.2.20) we have that $\nabla^k T$ annihilates constants. In concert with the assumption (4.2.21) on the kernel, this implies that

$$\int_\Omega \delta^{k+\gamma-1-s}(x)|\nabla^k Tf(x)|dx \tag{4.2.25}$$

can be controlled by a multiple of

$$\int_0^\infty t^{k+\gamma-1-s}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|\tilde f(x)-\tilde f(y)|}{(|x-y|^2+t^2)^{\frac{n+k+\gamma-1}{2}}}\,dxdy. \tag{4.2.26}$$ eq3.8

In turn, this can be majorized by

$$C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}|\tilde f(x)-\tilde f(y)|\Big(\int_0^\infty\frac{t^{k+\gamma-1-s}}{(|x-y|+t)^{n+k+\gamma-1}}dt\Big)\,dxdy. \tag{4.2.27}$$ eq3.9

Making the change of variables $r=\frac{t}{|x-y|}$, $r\in(0,\infty)$, $dt=|x-y|dr$, we can further bound the innermost integral above by

$$\int_0^\infty\frac{t^{k+\gamma-1-s}}{(|x-y|+t)^{n+k+\gamma-1}}dt=|x-y|^{-s+1-n}\int_0^\infty\frac{r^{k+\gamma-1-s}}{(1+r)^{n+k+\gamma-1}}dr\leq C|x-y|^{-s+1-n} \tag{4.2.28}$$ 3.10

for some finite constant $C=C(k,\gamma,s,n)$. Thus,

$$\begin{aligned}\int_\Omega\delta^{k+\gamma-1-s}(x)|\nabla^k Tf(x)|\,dx &\leq C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\frac{|\tilde f(x)-\tilde f(y)|}{|x-y|^{n+s-1}}\,dxdy\\ &\leq C\|\tilde f\|_{B^{1,1}_s(\mathbb{R}^{n-1})}\leq C\|f\|_{B^{1,1}_s(\partial\Omega)}\end{aligned} \tag{4.2.29}$$ eq3.11

as desired; this completes the proof of (eq3.5 7.1.39).

Next we turn our attention to the case $p=\infty$. The goal is to show that $\delta^{k+\gamma-s}|\nabla^k Tf|\in L^\infty(\Omega)$ for $f\in B^{\infty,\infty}_s(\partial\Omega)$, with appropriate control of the norms. To this end, let x^* denote the point on $\partial\Omega$ such that $|x-x^*|=\delta(x)$. Then we may write

$$\nabla^k Tf(x)=\int_{\partial\Omega}\nabla^k_x k(x,y)(f(y)-f(x^*))\,d\sigma(y). \tag{4.2.30}$$ eq3.12

Since $f\in B^{\infty,\infty}_s(\partial\Omega)=C^s(\partial\Omega)$ we have $|f(y)-f(x^*)|\leq\|f\|_{B^{\infty,\infty}_s(\partial\Omega)}|y-x^*|^s$, $y\in\partial\Omega$. With this in mind and recalling the assumptions (eqmp46 4.2.21) on the kernel, we split the last integral into two parts, I_1, I_2, corresponding to $y\in S_{100r}(x^*)$ and $y\in\partial\Omega\setminus S_{100r}(x^*)$, respectively, where $r:=|x-x^*|$. We have

$$\begin{aligned}|I_1| &\leq C\|f\|_{B^{\infty,\infty}_s(\partial\Omega)}\int_{S_{100r}(x^*)}\frac{|y-x^*|^s}{|x-y|^{n-1+k+\gamma}}\,d\sigma(y)\\ &\leq C\|f\|_{B^{\infty,\infty}_s(\partial\Omega)}\int_{S_{100r}(x^*)}|x-y|^{s-n+1-k-\gamma}\,d\sigma(y)\\ &\leq C\|f\|_{B^{\infty,\infty}_s(\partial\Omega)}\int_{S_{100r}(x^*)}r^{s-n+1-k-\gamma}d\sigma(y)\leq C\,r^{s-k-\gamma}\,\|f\|_{B^{\infty,\infty}_s(\partial\Omega)}\end{aligned} \tag{4.2.31}$$ eq3.13

On the other hand,

$$
\begin{aligned}
|I_2| &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{\partial\Omega\setminus S_{100r}(x^*)} \frac{|y-x^*|^s}{|x-y|^{n-1+k+\gamma}}\, d\sigma(y) \\
&\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{\partial\Omega\setminus S_{100r}(x^*)} |x^*-y|^{s-n+1-k-\gamma}\, d\sigma(y) \\
&\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{100r}^{\infty} \rho^{s-1-k-\gamma} d\rho \leq C\, r^{s-k-\gamma}\, \|f\|_{B_s^{\infty,\infty}(\partial\Omega)}. \qquad (4.2.32)
\end{aligned}
$$
eq3.14

These inequalities complete the argument for the case $p=\infty$.

Now the estimate

$$
\delta^{k+\gamma-\frac{1}{p}-s}|\nabla^k Tf| \in L^p(\Omega) \quad \text{for} \quad f \in B_s^{p,p}(\partial\Omega), \qquad (4.2.33)
$$
$$
k=1,2,...,N, \quad s\in(0,1), \quad p\in[1,\infty],
$$
eq3.15

will follow from Theorem ??? 3.5.3 [Stein's IP]. The details are as follows. Consider the family of operators

$$
L_z f := \delta^{k+\gamma-1+z-[(1-z)s_0+zs_1]}|\nabla^k Tf|, \qquad (4.2.34)
$$
eq3.16

so that

$$
\begin{aligned}
\Re e z = 0 &\Rightarrow |L_0 f| = \delta^{k+\gamma-1-s_0}|\nabla^k Tf|,\\
\Re e z = 1 &\Rightarrow |L_1 f| = \delta^{k+\gamma-s_1}|\nabla^k Tf|.
\end{aligned}
$$

Our results for $p=1$ and $p=\infty$ lead to the conclusion that the operators

$$
\begin{aligned}
L_0 &: B_{s_0}^{1,1}(\partial\Omega) \to L^1(\Omega),\\
L_1 &: B_{s_1}^{\infty,\infty}(\partial\Omega) \to L^\infty(\Omega),
\end{aligned}
$$

are well-defined and bounded. In turn, thanks to Theorem ??? 3.5.3 [Stein's IP] , and standard complex interpolation results for the Besov scale (cf. BL [**BL**]; here $s_0 \neq s_1$ is needed), we may conclude that

$$
\delta^{k+\gamma-\frac{1}{p}-s}|\nabla^k Tf| : B_s^{p,p}(\partial\Omega) \to L^p(\Omega) \qquad (4.2.35)
$$
eq3.17

is also well defined and bounded granted that $s\in(0,1)$ and $p\in[1,\infty]$.

Case 2. $(n-1)/n < p \leq 1$**.** In this situation we use the atomic characterization eq2.10 (6.1.10) of Besov spaces. First, we observe that it suffices to show

$$
\int_\Omega \left(\delta^{k+\gamma-\frac{1}{p}-s}(x)|\nabla^k Ta|\right)^p dx \leq C, \qquad (4.2.36)
$$
eqmp49

for every $B_s^{p,p}(\partial\Omega)$-atom a.

Given *supp* $a \subseteq S_r$, we first treat the case $x \in B_{2r} \subset \Omega$, where B_{2r} denotes an n-dimensional ball whose center coincides with that of S_r. In this scenario, we introduce a rescaling $\tilde{a} = r^{\#}a$ with the value of $\#$ to be clarified later. Then

$$\int_{B_{2r}} \left(\delta^{k+\gamma-\frac{1}{p}-s}(x)|\nabla^k Ta|\right)^p dx \tag{4.2.37}$$ eqmp50

can be viewed as

$$r^{-\#p}\int_{B_{2r}} \delta^{kp+\gamma p-1-sp}(x)|\nabla^k T\tilde{a}|^p\, dx. \tag{4.2.38}$$ eqmp51

Now we can exploit Holder's inequality and majorize the integral above by

$$r^{-\#p}\left(\int_{B_{2r}} \delta^{k+\gamma-s-\frac{1}{p}+\frac{z}{p}}(x)|\nabla^k T\tilde{a}|\, dx\right)^p \left(\int_{B_{2r}} \delta^{\frac{-z}{1-p}}(x)\, dx\right)^{1-p} := r^{-\#p}\ I_1^p\ I_2^{1-p}. \tag{4.2.39}$$ eqmp52

Under an assumption $z < 1-p$ one can see that I_2 is convergent and dominated by $C\, r^{\frac{-z}{1-p}+n+2}$.

Concerning I_1, we can enlarge the domain of integration to write

$$I_1 = \int_{\Omega} \delta^{k+\gamma-s-\frac{1}{p}+\frac{z}{p}}(x)|\nabla^k T\tilde{a}|\, dx. \tag{4.2.40}$$ eqmp53

According to estimate (eq3.6 7.1.40) the expression (eqmp53 4.2.40) is controlled by

$$C\|\tilde{a}\|_{B^{1,1}_{-1+s+\frac{1-z}{p}}(\partial\Omega)}, \tag{4.2.41}$$ eqmp55

whenever $-1+s+\frac{1-z}{p} \in (0,1)$. In view of all restrictions imposed on z, we discover that the argument described above makes sense for every z that belongs to the nondegenerate interval $(1-p(2-s), 1-p)$.

Going further, we intend to bound (eqmp55 4.2.41) by finite constant and therefore select $\#$ so that $\tilde{a} = r^{\#p}\, a$ is $B^{1,1}_{-1+s+\frac{1-z}{p}}(\partial\Omega)$-atom. An appropriate choice would be $\# = \frac{n-z}{p} - n$. Then it follows that

$$\int_{B_{2r}} \left(\delta^{k+\gamma-\frac{1}{p}-s}(x)|\nabla^k Ta|\right)^p dx$$
$$\leq r^{-\#p}\cdot C\cdot r^{(1-p)\left(\frac{-z}{1-p}+n\right)} \leq C. \tag{4.2.42}$$ eqmp56

Now let us turn our attention to the contribution away from the support of the atom. For notational simplicity we will further assume that *supp* a is centered at 0.

To start,

$$|\nabla^k Ta(x)| \leq C\int_{S_r(0)} \frac{|a(y)|}{|x-y|^{n+k+\gamma-1}}\, d\sigma(y). \tag{4.2.43}$$ eqmp57

According to (eqmp46 4.2.21), for every $x \in \Omega \setminus B_{2r}$ the last expression is majorized by

$$\frac{r^{s+(n-1)(1-\frac{1}{p})}}{|x|^{n+k+\gamma-1}}. \tag{4.2.44}$$

eqmp58

At this point, we pull-back everything to Euclidean model to obtain

$$\begin{aligned}&\int_{\Omega\setminus B_{2r}} \delta^{(k+\gamma-s)p-1}|\nabla^k Ta(x)|^p\,dx\\ \leq\ & C\,r^{(s+(n-1)\left(1-\frac{1}{p}\right))p}\int_{|x'|^2+t^2\geq 4r^2}\frac{t^{(k+\gamma-s)p-1}}{(|x'|+t)^{(n+k+\gamma-1)p}}\,dx'dt.\end{aligned} \tag{4.2.45}$$

eqmp59

This, in turn, can be written (after rescaling) as

$$\int_{|x'|^2+t^2\geq 1}\frac{t^{(k+\gamma-s)p-1}}{(|x'|+t)^{(n+k+\gamma-1)p}}\,dx'dt. \tag{4.2.46}$$

eqmp60

Now the desired conclusion follows from convergence of the integral (eqmp60 4.2.46), which we treat below.

Let us first handle the case $|x'| \leq t$, $|x'|^2+t^2 \geq 1$. It corresponds to $I_1 := I|_{|x'|\leq t}$ with domain of integration restricted by the condition $t \geq 1/2$. Making the change of variables $x' = ty$ ($t \in [\frac{1}{2},\infty)$, $y \in \mathbb{R}^{n-1}$), we compute

$$I_1 = \int_{1/2}^{\infty} t^{-sp+(n-1)(1-p)-1}\,dt\,\cdot\,\int_{\mathbb{R}^{n-1}}\frac{1}{(|y|+1)^{(n+k+\gamma-1)p}}\,dy. \tag{4.2.47}$$

eqmp61

Both integrals in the expression above are convergent under the current assumptions on p and s.

As for the case $t \leq |x|$, $|x'|^2+t^2 \geq 1$, that forces $|x| \geq 1/2$. Denoting $t := \rho|x|$, we obtain

$$I_2 := I|_{t<|x|} = \int_0^{\infty}\frac{\rho^{(k+\gamma-s)p-1}}{(\rho+1)^{(n+k+\gamma-1)p}}\,d\rho\,\cdot\,\int_{|x'|>1/2}|x'|^{-sp-1-np+p}\,dx', \tag{4.2.48}$$

eqmp62

which is finite as well. This, finally, completes the proof of eqmp47 4.2.22.

□

c??? **Corollary 4.2.4.** *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Then for every $\frac{n-1}{n} < p < \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$*

$$\mathcal{D} : B_s^{p,p}(\partial\Omega) \to B^{p,p}_{s+\frac{1}{p}}(\Omega) \cap F^{p,2}_{s+\frac{1}{p}}(\Omega). \tag{4.2.49}$$

eqmp48

Proof To establish (eqmp48 4.2.81), we first invoke the estimate from Theorem tJeKecriterion 4.1.3 to the effect that

$$\|u\|_{B_s^{p,p}(\partial\Omega)} \leq C\,\|\delta^{k(s)-\frac{1}{p}-s}|\nabla^{k(s)}\mathcal{D}f|\|_{L^p(\Omega)}, \tag{4.2.50}$$

eqmp69

where $k(s)$ denotes the smallest nonnegative integer greater than or equal to $1/p+s$ (see also [Further Results in Section on Function Spaces], where the case $p<1$ was discussed). In view of Theorem 4.2.3, the inequality (4.2.50) yields

$$\mathcal{D}: B_s^{p,p}(\partial\Omega) \to B^{p,p}_{s+\frac{1}{p}}(\Omega). \tag{4.2.51}$$ eqmp70

Turning to Triebel-Lizorkin spaces, the classical embedding theorems ascertain

$$B^{p,p}_{s+\frac{1}{p}}(\Omega) \hookrightarrow F^{p,2}_{s+\frac{1}{p}}(\Omega), \qquad \text{for } p\leq 2. \tag{4.2.52}$$ eqmp71

Combined with the discussion in [MT3] for $1<p<\infty$, this readily implies (4.2.81) and finishes the argument.

□

tmp3.2 **Theorem 4.2.5.** *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Consider the integral operator*

$$Rf(x) = \int_{\partial\Omega} k(x,y)f(y)d\sigma(y), \qquad x\in\Omega, \tag{4.2.53}$$ eqmp72

satisfying the conditions

$$|\nabla_x^k \nabla_y^j k(x,y)| \leq C|x-y|^{-(n-2+k+j+\gamma)}, \quad j=0,1, \tag{4.2.54}$$ eqmp73

where $k=1,2,..,N$, for some positive integer N and $\gamma\geq 0$. Then

$$\|\delta^{k+\gamma-1-\frac{1}{p}+s}|\nabla^k Rf|\|_{L^p(\Omega)} \leq C\|f\|_{B^{p,p}_{-s}(\partial\Omega)}, \quad k=1,2...N, \tag{4.2.55}$$ eqmp74

granted that $\frac{n-1}{n}<p\leq\infty$, $(n-1)(\frac{1}{p}-1)_+ < 1-s < 1$.

Proof. Again, we proceed in a sequence of steps.

Case 1. $1\leq p\leq\infty$. We shall focus on the case $p=1$ first, that is, the estimate

$$\|\delta(\cdot)^{s-2+k+\gamma}|\nabla^k Rf|\,\|_{L^1(\Omega)} \leq C(\Omega,\kappa,s)\|f\|_{B^{1,1}_{-s}(\partial\Omega)}, \qquad \forall\, k=0,1,...,N \tag{4.2.56}$$ mp20

By duality, it suffices to verify

$$\Big\|\int_\Omega \delta(x)^{s-2+k+\gamma}\,\nabla_x^k k(x,\cdot)g(x)\,dx\Big\|_{B_s^{\infty,\infty}(\partial\Omega)} \leq C(\Omega,\kappa,s)\|g\|_{L^\infty(\Omega)}, \tag{4.2.57}$$ mp21

uniformly for $g\in L^\infty_{\text{comp}}(\Omega)$. To this end, for a fixed, arbitrary $g\in L^\infty_{\text{comp}}(\Omega)$ of norm ≤ 1, we shall focus on establishing

$$\Big|\int_\Omega g(x)\,\delta(x)^{s-2+k+\gamma}(\nabla_x^k k(x,p)-\nabla_x^k k(x,q))\,dx\Big| \leq C\,|p-q|^s, \tag{4.2.58}$$ mp22

uniformly for $p,q\in\partial\Omega$. Now, fix two arbitrary boundary points $p,q\in\partial\Omega$ and, for a large constant C, bound the integral above by

$$\begin{aligned}
&\int_{|x-p|<C|p-q|} |g(x)|\,\delta(x)^{s-2+k+\gamma}|\nabla_x^k k(x,p)|\,dx \\
&+\int_{|x-p|<C|p-q|} |g(x)|\,\delta(x)^{s-2+k+\gamma}|\nabla_x^k k(x,q)|\,dx \\
&+\int_{|x-p|>C|p-q|} |g(x)|\,\delta(x)^{s-2+k+\gamma}|\nabla_x^k k(x,p)-\nabla_x^k k(x,q)|\,dx \\
&=: I+II+III.
\end{aligned} \tag{4.2.59}$$

mp23

To deal with I, in the light of (4.4.1), by localizing and pulling back to $\mathbb{R}^n_+$, it suffices to consider

$$\int_{|x'-p'|+|t+\varphi(x')-\varphi(p')|<C|p'-q'|} \frac{t^{s-2+k+\gamma}}{(|x'-p'|+|t+\varphi(x')-\varphi(p')|)^{n-2+k+\gamma}}\,dt\,dx', \tag{4.2.60}$$

mp24

where $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ is a Lipschitz function and $x=(x',t+\varphi(x'))$, $p=(p',\varphi(p'))$, $q=(q',\varphi(q'))$. Accordingly, we seek a bound of order $|p'-q'|^s$. To this effect, we note that the integral (4.2.60) is majorized by

$$\begin{aligned}
&C\int_{|x'-p'|+t<C|p'-q'|} \frac{t^{s-2+k+\gamma}}{(|x'-p'|+t)^{n-2+k+\gamma}}\,dt\,dx' \\
&\leq C\Big(\int_0^\infty \frac{t^{s-2+k+\gamma}}{(1+t)^{n-2+k+\gamma}}\,dt\Big)\Big(\int_{|x'|<C|p'-q'|}\frac{1}{|x'|^{n-1-s}}\,dx'\Big) \\
&\leq C_{n,s}|p'-q'|^s,
\end{aligned} \tag{4.2.61}$$

mp25

and the last bound has the right order. Similar arguments also apply to II in (4.2.59) since $|x-q|<|x-p|+|p-q|<C\,|p-q|$. Thus, we are left with estimating III. For this, an application of the mean-value theorem together with (4.4.1) and a change of variables allow us to write

$$\begin{aligned}
III\quad &\leq C\int_{|x'-p'|+|t+\varphi(x')-\varphi(p')|>C|p'-q'|} \frac{t^{s-2+k+\gamma}|p'-q'|}{(|x'-p'|+|t+\varphi(x')-\varphi(p')|)^{n+k+\gamma-1}}\,dx'dt \\
&\leq C\int_{|x'-p'|+t>C|p'-q'|} \frac{t^{s-2+k+\gamma}|p'-q'|}{(|x'-p'|+t)^{n+k+\gamma-1}}\,dx'dt.
\end{aligned} \tag{4.2.62}$$

mp26

Making $x'\mapsto x'':=(x'-p')/|p'-q'|$ and $t\mapsto t':=t/|p'-q'|$ in the last integral above readily leads to a bound of order $|p'-q'|^s$, i.e., the proper size. This finishes the proof of (4.2.56).

The next order of business is to demonstrate the inequality (4.2.55) assuming that $p=\infty$. Here the idea is prove that

$$\|\nabla_x^k k(x,\cdot)\|_{B_s^{1,1}(\partial\Omega)} \leq C(\Omega,s)\,\delta(x)^{-s-k-\gamma+1}, \qquad \forall\, i=0,1,...,N-1, \tag{4.2.63}$$ mp28

uniformly in $x \in \Omega$. Clearly, this suffices in order to conclude (4.2.55) with $p=\infty$. The remainder of the proof, modeled upon [FMM], consists of a verification of (4.2.63).

The problem localizes and, hence, it suffices to prove the estimate

$$\int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \frac{|\nabla_x^k k(x,p) - \nabla_x^k k(x,q)|}{|p-q|^{n-1+s}}\,d\sigma(p)\,d\sigma(q) \leq C\,\delta(x)^{-s-k-\gamma+1}, \tag{4.2.64}$$ mp29

uniformly for $x \in \Omega$, in the case when Ω is the domain above the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$.

Now, for a fix, sufficiently large $C>0$, split the inner integral according to whether $|x-p| < C|p-q|$ or $|x-p| > C|p-q|$. Thus, it suffices to treat $\int_{\partial\Omega} |I|\,d\sigma$, $\int_{\partial\Omega} |II|\,d\sigma$ and $\int_{\partial\Omega} |III|\,d\sigma$, where

$$I := \int_{|x-p|<C|p-q|} \frac{|\nabla_x^k k(x,p)|}{|p-q|^{n-1+s}}\,d\sigma(p), \tag{4.2.65}$$ mp30

$$II := \int_{|x-p|<C|p-q|} \frac{|\nabla_x^k k(x,q)|}{|p-q|^{n-1+s}}\,d\sigma(p), \tag{4.2.66}$$ mp31

$$III := \int_{|x-p|>C|p-q|} \frac{|\nabla_x^k k(x,p) - \nabla_x^k k(x,q)|}{|p-q|^{n-1+s}}\,d\sigma(p). \tag{4.2.67}$$ mp32

To this end, note first that a change of variables based on the representations $x = (x', \varphi(x')+t)$, $p = (p', \varphi(p'))$ and $q = (q', \varphi(q'))$ gives

$$\begin{aligned} |I| &\leq C \int_{\substack{|x'-p'|+|t+\varphi(x')-\varphi(p')| \\ <C|p'-q'|}} \frac{|p'-q'|^{-n+1-s}}{(|t+\varphi(x')-\varphi(p')|+|x'-p'|)^{n-2+k+\gamma}}\,dp' \\ &\leq C \int_{|x'-p'|+t<C|p'-q'|} \frac{dp'}{|p'-q'|^{n-1+s}(t+|x'-p'|)^{n-2+k+\gamma}}. \end{aligned} \tag{4.2.68}$$ mp33

Substituting $x'-p' = th$ in the last integral above and then integrating against $\int_{\mathbb{R}^{n-1}} dq'$ yields

$$\int\limits_{\partial\Omega} |I|\,d\sigma \leq C\frac{1}{t^{k+\gamma-1}} \int\limits_{\mathbb{R}^{n-1}} \Big(\int_{|h|+1\leq C|x'-th-q'|/t} \frac{dh}{(|h|+1)^{n-2+k+\gamma}|x'-th-q'|^{n-1+s}} \Big) dq'. \tag{4.2.69}$$ mp34

Substituting again, this time first $x'-q' = tw$ and then $w-h = r\omega$, $r>0$, $\omega \in S^{n-2}$, we may further bound the last integral in (4.2.69) by

$$
\begin{aligned}
&C\frac{1}{t^{s+k+\gamma-1}}\int\limits_{\mathbb{R}^{n-1}}\int_{|h|+1\leq C|w-h|}\frac{dh\,dw}{(|h|+1)^{n-2+k+\gamma}|w-h|^{n-1+s}}\\
&=\frac{C}{t^{s+k+\gamma-1}}\int\frac{1}{(|h|+1)^{n-2+k+\gamma}}\Big(\int_{|h|+1}^{\infty}\frac{dr}{r^{s+1}}\Big)dh=C_{n,s,k,\gamma}\,t^{-s-k-\gamma+1},
\end{aligned}
\tag{4.2.70}
$$
mp35

which is a bound of the right order for $\int_{\partial\Omega}|I|\,d\sigma$. The same arguments work to bound $\int_{\partial\Omega}|II|\,d\sigma$, by observing that $|x-p|<C|p-q|\Rightarrow|x-q|<C'|p-q|$ and using Fubini's theorem.

As for $\int_{\partial\Omega}|III|\,d\sigma$, we note first that since $|x-z|\geq C|x-p|$ uniformly for $z\in[p,q]$,

$$
\int\limits_{\partial\Omega}|III|\,d\sigma\leq C\int\limits_{\partial\Omega}\int_{|x-p|>C|p-q|}\frac{1}{|p-q|^{n-2+s}|x-p|^{n+k+\gamma-1}}\,d\sigma(p)\,d\sigma(q). \tag{4.2.71}
$$
mp36

As before, pulling back everything to $\mathbb{R}^{n-1}$, it is enough to bound

$$
\int\limits_{\mathbb{R}^{n-1}}\int_{|x'-p'|+t>C|p'-q'|}\frac{1}{|p'-q'|^{n-2+s}(|x'-p'|+t)^{n+k+\gamma-1}}\,dp'\,dq'. \tag{4.2.72}
$$
mp37

Substituting $x'-p'=th$ and then $x'-q'=tw$ gives

$$
\begin{aligned}
&\frac{1}{t^{k+\gamma}}\int\limits_{\mathbb{R}^{n-1}}\Big(\int_{|h|+1>C|p'-q'|/t}\frac{1}{(|h|+1)^{n+k+\gamma-1}|q'-x'+ht|^{n-2+s}}\,dh\Big)dq'\\
&\qquad=\frac{1}{t^{s+k+\gamma-1}}\int\limits_{\mathbb{R}^{n-1}}\frac{1}{(|h|+1)^{n+k+\gamma-1}}\Big(\int_{|h|+1>C|w-h|}\frac{dw}{|w-h|^{n-2+s}}\Big)\,dh\\
&\qquad=C_{n,s,k,\gamma}t^{-s-k-\gamma+1},
\end{aligned}
\tag{4.2.73}
$$
mp38

as desired. This finishes the proof of (4.2.63).

Using the estimates for $p=1$ and $p=\infty$ which we handled above and Theorem 3.2.2, we conclude that

$$
\delta^{k+\gamma-\frac{1}{p}+s-1}|\nabla^k Rf|:B^{p,p}_{-s}(\partial\Omega)\to L^p(\Omega), \tag{4.2.74}
$$
eq3.32

whenever $1\leq p\leq\infty$, $0<s<1$.

Case 2. $(n-1)/n<p\leq 1$**.** Similarly to Theorem 4.2.3, our goal is to establish a bound

$$
\int_{\Omega}\Big(\delta^{k+\gamma-\frac{1}{p}-1+s}(x)|\nabla^k Ra|\Big)^p\,dx\leq C, \tag{4.2.75}
$$
eqmp76

where a is a $B^{p,p}_{-s}(\partial\Omega)$-atom and constant C does not depend on a. Then the proof can be finished following the same lines.

In the neighborhood of support of the atom, arguing as before, we obtain

$$\begin{aligned}
&\int_{B_{2r}} \Big(\delta^{k+\gamma-\frac{1}{p}-1+s}(x)|\nabla^k Ra|\Big)^p\,dx \\
\leq\;& C\,r^{-\#p}\Big(\int_{B_{2r}} \delta^{k+\gamma-1+s-\frac{1}{p}+\frac{z}{p}}(x)|\nabla^k R\tilde{a}|\,dx\Big)^p\Big(\int_{B_{2r}}\delta^{\frac{-z}{1-p}}(x)\,dx\Big)^{1-p} \\
\leq\;& C\,\|\tilde{a}\|_{B^{1,1}_{-1-s+\frac{1-z}{p}}(\partial\Omega)} \leq C,
\end{aligned} \tag{4.2.76}$$ eqmp77

where we employed the rescaling $\tilde{a} = r^{\#}a$ and Holder's inequality.. For this program to work, we assign $\# = n(\frac{1}{p}-1) - \frac{z}{p}$ and allow $1-p(s+1) < z < 1-p$.

Going further, away from *supp* $a \subseteq S_r$ (which we assume to be centered at 0) one sees that $\forall x \in \Omega \setminus B_{2r}$

$$|\nabla^k Ra(x)| = \Big|\int_{S_r(0)} [\nabla^k_x k(x,y) - \nabla^k_x k(x,0)]a(y)\,d\sigma(y)\Big|, \tag{4.2.77}$$ eqmp78

owing to the vanishing moment condition imposed on $B^{p,p}_{-s}(\partial\Omega)$-atom a. The expression in the brackets can be written as

$$\int_0^1 \frac{d}{d\theta}\nabla^k_x k(x,(1-\theta)y)\,d\theta, \tag{4.2.78}$$ eqmp79

which is controlled by

$$|y|\max_{\theta\in(0,1)}|\nabla^k_x\nabla_y k(x,(1-\theta)y)|. \tag{4.2.79}$$ eqmp80

In the current scenario $|x-(1-\theta)y| \approx |x|$, $|y| \leq r$, and therefore, (eqmp78 4.2.77) is majorized by

$$C\frac{r^{1-s+(n-1)(1-\frac{1}{p})}}{|x|^{n+k+\gamma-1}}. \tag{4.2.80}$$ eqmp81

With this set-up, the proof requires only minor alterations compared to that of Theorem tmp3.1 4.2.3. We omit the details.

□

Much as in the case of harmonic double layer, Theorem tmp3.2 4.2.5 allows us to conclude:

c??? **Corollary 4.2.6.** *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Then for every $\frac{n-1}{n} < p < \infty$, $(n-1)(\frac{1}{p}-1)_+ < 1-s < 1$*

$$\mathcal{S} : B^{p,p}_{-s}(\partial\Omega) \to B^{p,p}_{1-s+\frac{1}{p}}(\Omega) \cap F^{p,2}_{1-s+\frac{1}{p}}(\Omega). \tag{4.2.81}$$ eqmp48

....

Theorem 4.2.7. *([CCFJR]) Let $F \in C^{\infty}(\mathbb{R}^m \setminus \{0\})$ be an even function, and $g_i \in \dot{L}^{p_i}(\mathbb{R}^n)$, $1 < p_i \leq \infty$, $i = 1, \ldots, N$. Consider $\varphi_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$ functions with bounded derivatives, and $k(x, y)$ homogeneous kernel of degree $-n$, with $\int_{|y|=1} |k(x, y)|^s d\sigma(y) < \infty$. Then, if $f \in L^q(\mathbb{R}^n)$, $1 < q < \infty$, the operator defined by*

$$T(f,g)(x) = \sup_{\varepsilon>0} \left| \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\mathbb{R}^n}} \prod_{i=1}^{N} \frac{g_i(x) - g_i(y)}{|x-y|} F\left(\frac{\varphi(x) - \varphi(y)}{|x-y|}\right) k(x, x-y)\, f(y)\, dy \right|,$$

for $x \in \mathbb{R}^n$, is finite a.e. and belongs to $L^{\gamma}(\mathbb{R}^n)$, $1/q + \sum_{i=1}^{N} 1/p_i = 1/\gamma$ provided that:
(a) $k(x, -y) = (-1)^{N+1} k(x, y)$,
(b)

$$0 < \frac{1}{q} + \sum_{i=1}^{N} \frac{1}{p_i} \leq \frac{s-1}{s} + \frac{N}{n}, \quad q \geq \frac{s}{s-1},\ p_i > 1.$$

If $k(x, y) = k(y)$, we may take $s = 1$ and condition (b) is replaced by
(b')

$$0 < \frac{1}{q} + \sum_{i=1}^{N} \frac{1}{p_i} \leq 1 + \frac{N}{n}, \quad q > 1,\ p_i > 1.$$

We also have the natural estimates

$$\|T(f,g)\|_{L^{\gamma}(\mathbb{R}^n)} \leq c\, \|F\|_{C^{\infty}(\mathbb{R}^m\setminus\{0\})} \prod_{i=1}^{N} \|g_i\|_{\dot{L}_1^{p_i}(\mathbb{R}^n)}\, \|f\|_{L^q(\mathbb{R}^n)}.$$

Corollary 4.2.8. *([Din]) There exists $N = N(m)$ such that the following holds. Let $F \in C^N(\mathbb{R}^m \setminus \{0\})$ be an even function, $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ a Lipschitz function, $g \in L^{\infty}(\mathbb{R}^n)$, and $f \in L_1^p(\mathbb{R}^n)$, $1 < p < \infty$. For $x \in \mathbb{R}^n$ define*

$$T(f,g)(x) = \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\mathbb{R}^n}} \frac{f(x) - f(y)}{|x-y|} F\left(\frac{A(x) - A(y)}{|x-y|}\right) \frac{1}{|x-y|^n}\, g(y)\, dy.$$

Then

$$\|T(f,g)\|_{L^p(\mathbb{R}^n)} \leq c\, \|F\|_{C^N(\mathbb{R}^m\setminus\{0\})}\, \|g\|_{L^{\infty}(\mathbb{R}^n)}\, \|f\|_{L_1^{\infty}(\mathbb{R}^n)}.$$

Corollary 4.2.9. *Let $F \in C^{\infty}(\mathbb{R}^m \setminus \{0\})$ be an odd function, and $A : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. If, for $x \in \mathbb{R}^n$, we define*

$$Tf(x) = \lim_{\varepsilon\to 0} \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\mathbb{R}^n}} F\left(\frac{A(x) - A(y)}{|x-y|}\right) \frac{1}{|x-y|^n}\, f(y)\, dy,$$

and further assume that $T(1) = constant$, *then*

$$T : \dot{L}^p_1(\mathbb{R}^n) \longrightarrow \dot{L}^p_1(\mathbb{R}^n)$$

is bounded for $1 < p < \infty$.

Corollary 4.2.10. *Let* $F \in C^\infty(\mathbb{R}^m \setminus \{0\})$ *be an odd function, and* $A : \mathbb{R}^n \to \mathbb{R}^m$ *be Lipschitz. If, for* $x \in \mathbb{R}^n$, *we define*

$$Tf(x) = \lim_{\varepsilon \to 0} \int\limits_{\substack{|x-y|>\varepsilon \\ y \in \mathbb{R}^n}} F\left(\frac{A(x) - A(y)}{|x-y|}\right) \frac{1}{|x-y|^n} f(y)\, dy,$$

and further assume that $T(1) = constant$, *then*

$$T : \dot{L}^p_\alpha(\mathbb{R}^n) \longrightarrow \dot{L}^p_\alpha(\mathbb{R}^n)$$

is bounded for $1 < p < \infty$ *and* $-1 \leq \alpha \leq 1$.

4.3 Further results

It should be noted that the criterion of membership of harmonic function to Besov spaces continues to hold for the values of integrability exponent p below 1 (cf. [MaMi]). More specifically, for every $\frac{n-1}{n} < p \leq \infty$, $(n-1)(1/p-1)_+ < \alpha < 1$, a harmonic function u belongs to $B^{p,p}_{k+\alpha}(\Omega)$ if and only if $\delta^{1-\alpha}|\nabla^{k+1}u| + |\nabla^k u| + |u|$ belongs to $L^p(\Omega)$. The similar statement holds for Triebel-Lizorkin spaces $F^{p,2}_{k+\alpha}(\Omega)$, which coincide with the class of Sobolev functions as long as $1 < p < \infty$.

....

In order to formulate one of Meyer and Coifman's result (see [MeCo]) we need to introduce the class of continuous linear operators $\mathcal{L}_\gamma$. For $0 < \gamma \leq 1$, the operator $T : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ belongs to the class $\mathcal{L}_\gamma$, if the kernel $K(x,y)$ of T satisfies

$$|K(x,y)| \leq c\,|x-y|^n, \tag{4.3.1}$$ Me1

and

$$|K(x',y) - K(x,y)| \leq c\,|x'-x|^\gamma\,|x-y|^{-n-\gamma} \quad \text{for } |x'-x| \leq \frac{|x-y|}{2}. \tag{4.3.2}$$ Me2

When $\gamma > 1$, let $\gamma = m + r$, where $m \in \mathbb{N}$ and $0 < r \leq 1$. We write $T \in \mathcal{L}_\gamma$ if

$$|\partial^\alpha_x K(x,y)| \leq c\,|x-y|^{-n-|\alpha|} \quad \text{for } |\alpha| \leq m, \tag{4.3.3}$$ Me3

and

$$|\partial^\alpha_x K(x',y) - \partial^\alpha_x K(x,y)| \leq c\,|x'-x|^r\,|x-y|^{-n-\gamma} \quad \text{for } |\alpha| = m \text{ and } |x'-x| \leq \frac{|x-y|}{2}. \tag{4.3.4}$$ Me4

Now let $T \in \mathcal{L}_\gamma$, $\gamma > 0$. Suppose that the kernel $K(x,y)$ of T satisfies the following estimates for $|x-y| \geq 1$.

$$|\partial_x^\alpha K(x,y)| \leq c_N \, |x-y|^{-N} \quad \text{for } |\alpha| \leq \gamma \text{ and } N \geq 1, \tag{4.3.5}$$

Me5

$$|\partial_x^\alpha K(x',y) - \partial_x^\alpha K(x,y)| \leq c_N \, |x'-y|^r \, |x-y|^{-N} \quad \text{for } |\alpha| = m \text{ and } |x'-x| \leq \frac{|x-y|}{2}. \tag{4.3.6}$$

Me6

If T is weakly continuous on $L^2(\mathbb{R}^n)$ and if $T(x^\alpha) \in C^\gamma(\mathbb{R}^n)$ for $|\alpha| \leq m$, then T can be extended as a continuous linear operator on C^s and L_s^p, where $1 < p < \infty$ and $0 < s < \gamma$.

Calderón-Zygmund Theory for Operator-Valued kernels.

Assume that A and B are two Banach spaces. Recall that by $L^p(A) = L^p(\mathbb{R}^n; A)$ we denote the space of all strongly measurable functions f such that

$$\|f\|_{L^p(\mathbb{R}^n;A)} = \int_{\mathbb{R}^n} \|f(x)\|_A^p \, d\sigma(x) < \infty, \tag{4.3.7}$$

eq???

for $1 \leq p < \infty$ with usual modifications for the case $p = \infty$. Similarly, we can define $BMO(\mathbb{R}^n; A)$.

In turn, the space $H^1(\mathbb{R}^n; A)$ can be viewed as l^p-span of A-valued atoms, which satisfy the following set of conditions:

$$\exists\, S_r \in \mathbb{R}^n - \text{a ball with } supp\,(a) \subseteq S_r, \ \|a\|_A \leq r^{-n}, \ \int_{S_r} a(x)\, dx = 0. \tag{4.3.8}$$

eq???

With this set-up, consider the kernel $k(x,y)$ with values in $\mathcal{L}(A,B)$ such that for every $x \in \mathbb{R}^n$, the function $\|k(x,\cdot)\|_{\mathcal{L}(A,B)}$ is locally integrable away from x and therefore, the operator

$$Tf(x) := \int_{\mathbb{R}^n} k(x,y) f(y)\, dy \tag{4.3.9}$$

eqVV_def_oper

is well-defined for every compactly supported $f \in L_A^1(\mathbb{R}^n)$ and for a.e. $x \notin \text{supp}\,(f)$.

We say that k satisfies Hörmander's condition if for every $y, z \in \mathbb{R}^n$

$$\int_{\substack{|x-z|>2|y-z| \\ x \in \mathbb{R}^n}} \|k(x,y) - k(x,z)\|_{\mathcal{L}(A,B)}\, dx < \infty. \tag{4.3.10}$$

eq???

Suppose an operator T, which is associated with the kernel k as in formula (eqVV_def_oper 4.3.9), is bounded from $L^p(\mathbb{R}^n; A)$ to $L^p(\mathbb{R}^n; B)$ for some *fixed* $1 \leq p_0 \leq \infty$.

If k satisfies Hörmander's condition, then T can be extended to an operator defined in $L^p(\mathbb{R}^n; A)$, with $1 \leq p \leq p_0$, and satisfying

$$(i) \quad \|Tf\|_{L^p(\mathbb{R}^n;B)} \leq C\|f\|_{L^p(\mathbb{R}^n;A)}, \qquad 1 < p \leq p_0,$$

$$(ii) \quad \|Tf\|_{L^1(\mathbb{R}^n;B)} \leq C\|f\|_{H^1(\mathbb{R}^n;A)}.$$

Going further, if the dual kernel $k'(x,y) = k(y,x)$ satisfies Hörmander's condition, then T can be extended to an operator defined in $L^p(\mathbb{R}^n; A)$, with $p_0 \le p < \infty$, and satisfying

$$(i) \quad \|Tf\|_{L^p(\mathbb{R}^n;B)} \le C\|f\|_{L^p(\mathbb{R}^n;A)}, \qquad p_0 \le p < \infty,$$
$$(ii) \quad \|Tf\|_{BMO(\mathbb{R}^n;B)} \le C\|f\|_{L^\infty(\mathbb{R}^n;A)}.$$

For the proofs and more detailed versions of aforementioned results we refer to [RRT]. We would also like to single out in connection with the Calderón-Zygmund theory in the vector-valued setting the work by A. Benedek, A.P. Calderón, R. Panzone [BCP], as well as the related discussion in [St3].

4.4 Exercises

1. Prove that it is sufficient to establish the mapping properties of operator R defined in Proposition 4.2.1 for the case $s = 0$ only.

Hint: Consider $s = 1$. Show that for every function $f \in L^p_{-1}(\partial\Omega)$ there is some finite family of functions $\{g_{ik}\}_{i,k=1}^n$ belonging to $L^p(\partial\Omega)$ and $g_0 \in \text{Ł}^p(\mathbb{R}^n)$ such that $f = g_0 + \sum_{i,k=1}^n \partial_{\tau_{ik}} g_{ik}$, where $\partial_{\tau_{ik}}$ denotes the corresponding tangential derivative. Next, invoke integration by parts in the representation formula for operator R to reduce the matters to situation considered in the case $s = 0$.

Then the range $0 \le s \le 1$ can be covered by interpolation.

2. Let $F \in C^\infty(\mathbb{R}^m \setminus \{0\})$ be an odd function, and $A : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. If, for $x \in \mathbb{R}^n$, we define

$$Tf(x) = \lim_{\epsilon \to 0} \int\limits_{\substack{|x-y|>\epsilon \\ y \in \mathbb{R}^n}} F\left(\frac{A(x) - A(y)}{|x-y|}\right) \frac{1}{|x-y|^n} f(y)\, dy,$$

and further assume that $T(1) = constant$, then

$$T : \dot{L}^p_\alpha(\mathbb{R}^n) \longrightarrow \dot{L}^p_\alpha(\mathbb{R}^n)$$

is bounded for $1 < p < \infty$ and $-1 \le \alpha \le 1$.

3. Call a linear, bounded operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ of *Calderón-Zygmund type* if it extends to a bounded operator in $L^2(\mathbb{R}^n)$ and if its Schwartz kenrel $k(x,y)$ satisfies

$$|k(x,y)| + |x-y||\nabla_I k(x,y)| + |x-y||\nabla_{II} k(x,y)| \le \frac{C}{|x-y|^n}, \qquad x, y \in \mathbb{R}^n.$$

Prove that any Calderón-Zygmund operator T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, maps $L^1(\mathbb{R}^n)$ into weak-$L^1(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$. Furthermore,

$$T^*1 = 0 \Longrightarrow T : H^p(\mathbb{R}^n) \longrightarrow H^p(\mathbb{R}^n), \quad n/(n+1) < p \le 1,$$

and

$$T1 = 0 \Longrightarrow T : C^\alpha(\mathbb{R}^n) \longrightarrow C^\alpha(\mathbb{R}^n), \quad 0 < \alpha < 1.$$

4. Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$. Prove that for every $\frac{n-1}{n} < p < \infty$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$,

$$\mathcal{D} : B_s^{p,q}(\partial\Omega) \to B^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \text{if } 0 < q \leq \infty$$

$$\mathcal{D} : B_s^{p,q}(\partial\Omega) \to F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \text{if } 0 < q \leq p,$$

$$\mathcal{D} : B_s^{p,p}(\partial\Omega) \to F^{p,q}_{s+\frac{1}{p}}(\Omega), \quad \text{if } \min\{p,2\} \leq q \leq \infty,$$

where $\mathcal{D}$ stands for the harmonic double layer.

5. Prove that the conclusion of Theorem tmp3.2 4.2.5 remains valid under weaker assumptions on the kernel

$$|\nabla_x^k k(x,y)| \leq C|x-y|^{-(n-2+k+\gamma)}, \tag{4.4.1}$$ eqmp73

$$|\nabla_x^k k(x,y_1) - \nabla_x^k k(x,y_2)| \leq C\frac{|y_1-y_2|}{|x-y_1|^{(n-1+k+\gamma)}}, \quad \text{for } |y_1-y_2| \leq c|x-y_1|, \tag{4.4.2}$$ eqmp731

where $k = 1, 2, .., N$, for some positive integer N and $\gamma \geq 0$.

Chapter 5

Differential operators

5.1 General systems

5.2 Laplacian, Helmholtz, Lamé, Stokes, Maxvell

Lamé

5.2.1 The derivation of the Lamé operator on manifolds

One way of understanding the genesis of the Laplace-Beltrami operator (eq2.3 7.1.37) is to consider the energy functional

$$\mathcal{E}[u] := \int_\Omega \|\nabla\, u(x)\|^2\, dx, \qquad u \in C^\infty(\Omega), \quad \Omega \subseteq \mathbb{R}^n. \tag{5.2.1}$$

eq3.1

Then any minimizer u of (eq3.1 7.1.35) should satisfy

$$\frac{d}{dt}\mathcal{E}[u+tv]\Big|_{t=0} = 0, \qquad \forall\, v \in C_o^\infty(\Omega), \tag{5.2.2}$$

eq3.2

thus, after an integration by parts,

$$\Delta u = 0 \quad \text{ on } \quad \Omega. \tag{5.2.3}$$

eq3.3

In other words, (eq3.3 6.1.3) is the Euler-Lagrange equation associated with the integral functional (eq3.1 7.1.35).

Our aim is to adopt a similar point of view in the case of the Lamé system of elasticity on M. The departure point is to consider the total free (elastic) energy

$$\mathcal{E}[u] := -\frac{1}{2}\int_\Omega E(x, \nabla u(x))\, dx, \qquad u : \Omega \longrightarrow \mathbb{R}^n, \tag{5.2.4}$$

eq3.6

ignoring at the moment the displacement boundary conditions. As before, equilibria states correspond to minimizers of the above variational integral. The first order of business is to identify the correct form of the stored energy density $E(x, \nabla u(x))$. We shall restrict attention to the case of linear elasticity. In this scenario, E depends