Transmission problems and spectral theory for singular integral operators on Lipschitz domains

Luis Escauriaza and Marius Mitrea*

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be the (unbounded) domain lying above the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$. This paper is concerned with the study of transmission boundary problems of the type

$$\text{(TBVP-Laplace)} \quad \begin{cases} \Delta u^{\pm} = 0 \text{ in } \Omega_{\pm}, \\ M(\nabla u^{\pm}) \in L^p(\partial\Omega), \\ u^+\Big|_{\partial\Omega} - u^-\Big|_{\partial\Omega} = f \in \dot{L}^p_1(\partial\Omega), \\ \partial_\nu u^+ - \mu\, \partial_\nu u^- = g \in L^p(\partial\Omega). \end{cases} \tag{1.1}$$

Here, Δ is the Laplacian, $\mu \in \mathbb{R}$ is a fixed parameter, ν is the outward unit normal to Ω, and $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$. For $1 < p < \infty$, $\dot{L}^p_1(\partial\Omega)$ is the classical homogeneous L^p-based Sobolev spaces of order one on $\partial\Omega$, M denotes the non-tangential maximal operator, ∂_ν is the normal derivative and all restrictions to the boundary are taken in the non-tangential limit sense; detailed definitions are given in the body of the paper (cf. §2).

Two closely related boundary problems are the Neumann problem and the Dirichlet problem with (maximally) regular data:

$$\text{(N)} \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ M(\nabla u) \in L^p(\partial\Omega), \\ \partial_\nu u = g \in L^p(\partial\Omega), \end{cases} \qquad \text{(R)} \begin{cases} \Delta u = 0 \text{ in } \Omega, \\ M(\nabla u) \in L^p(\partial\Omega), \\ u\Big|_{\partial\Omega} = f \in \dot{L}^p_1(\partial\Omega). \end{cases} \tag{1.2}$$

From the work of G. Verchota [**37**], and B. Dahlberg and C. Kenig [**6**], it is now understood that $1 < p < 2 + \varepsilon$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$, is the (asymptotically) sharp well-posedness range for both (N) and (R). In connection with (1.2), let

$$\Lambda : \dot{L}^p_1(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad \Lambda(g) := f \tag{1.3}$$

*The research of L. Escauriaza was supported by the Spanish Government grant BFM 2001-0458 and by the European Commission via the network Harmonic Analysis and Related Problems, project number RTN2-2001-00315. M. Mitrea's work was supported in part by NSF
2000 *Mathematics Subject Classification.* 35J25, 58J32, 31B10, 31B15, 31A10, 45B05, 47G10, 78A30
Key words: Transmission problems, Lipschitz domains, layer potentials, atomic estimates, spectral radius

be the so-called Dirichlet-to-Neumann map (well-defined for $1 < p < 2+\varepsilon$). Then (1.1) contains both (N) and (R) in the following sense. A function u solves (N) for the datum g if and only if $(u,0)$ solves (T) for the data $(\Lambda^{-1}(g), g)$. Furthermore, u solves (R) for the datum f if and only if $(u,0)$ solves (T) for the data $(f, \Lambda(f))$.

Another observation highlighting the connections between these three boundary value problems is that (1.1) decouples into a Neumann problem and a Regularity problem when $\mu = 0$. More specifically, in order to solve (1.1) when $\mu = 0$, one simply takes u^+ to be the solution of (1.2)-(N) in Ω_+ with datum g, then let u^- solve (1.2)-(R) in Ω_- with boundary datum $-f + u^+|_{\partial\Omega}$. In fact, as a simple perturbation argument shows, there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that well-posedness for the problems (1.2) entails well-posedness for (1.1) if $|\mu| < \varepsilon$.

When $\mu = 1$ the problem (1.1) is well-posed for any $1 < p < \infty$; cf. the discussion in §2. Furthermore, simple algebraic manipulations always allow one to reduce the case $\mu > 1$ to the case $\mu < 1$. Finally, it follows from the location of the point-spectrum of the harmonic double layer (cf. [**2**]) that for each $\mu < 0$ and each $1 < p < \infty$ there exists a smooth, bounded domain Ω for which (the bounded domain version of) the problem (1.1) is not well-posed. For these reasons, in the sequel we shall restrict attention to the case when $\mu \in (0,1)$. Our main result is as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be the unbounded domain lying above the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$, and let $\mu \in (0,1)$. Then there exists $\varepsilon = \varepsilon(\partial\Omega, \mu) > 0$ such that the transmission boundary value problem (1.1) has a unique (modulo constants) solution provided that $1 < p < 2+\varepsilon$. In addition, this solution satisfies*

$$\|M(\nabla u^+)\|_{L^p(\partial\Omega)} + \|M(\nabla u^-)\|_{L^p(\partial\Omega)} \leq C\Big(\|\nabla_{tan} f\|_{L^p(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)}\Big) \tag{1.4}$$

granted that $1 < p < 2+\varepsilon$. Moreover, there are integral representation formulas for the solution in terms of harmonic layer potentials.

Similar considerations apply to the case of a bounded Lipschitz interface, with the additional decay condition $u^-(x) = \mathcal{O}(|x|^{2-n})$ as $|x| \to \infty$ (this time, uniqueness holds without the addendum 'modulo constants'). When $n = 2$, the above decay condition at infinity should be replaced by

$$u^-(x) = q \log|x| + \mathcal{O}(1) \ \ as\ |x| \to \infty, \quad q = \text{ constant}. \tag{1.5}$$

The strategy for proving this result is to interpolate between the end-point cases $p = 1$ and $p = 2$. The latter situation has been largely dealt with in [**11**], while the former requires establishing new atomic estimates. This idea has been first used by Dahlberg and Kenig in their ground-breaking work on the Neumann problem for the Laplacian ([**6**]). Implementing this program in the context of the transmission problem constitutes the main technical novelty of the current paper. Our key estimates in this regard are as follows.

Theorem 1.2 *Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is the unbounded domain lying above the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$, and fix $\mu \in (0,1)$. Then there exists $\varepsilon = \varepsilon(\partial\Omega, \mu) > 0$ such that the transmission boundary value problem*

$$(\mathit{TBVP\text{-}atomic}) \quad \begin{cases} \Delta u^\pm = 0 \ \ in\ \Omega_\pm, \\ M(\nabla u^\pm) \in L^p(\partial\Omega), \\ u^+\Big|_{\partial\Omega} - u^-\Big|_{\partial\Omega} = f \in \dot{H}^{1,p}_{at}(\partial\Omega), \\ \partial_\nu u^+ - \mu\,\partial_\nu u^- = g \in \dot{H}^p_{at}(\partial\Omega), \end{cases} \tag{1.6}$$

has a unique (modulo constants) solution provided that $1-\varepsilon<p\leq 1$ *if* $n\geq 3$, *and* $\frac{2}{3}-\varepsilon<p\leq 1$ *if* $n=2$. *In each case, the solution satisfies*

$$\|M(\nabla u^+)\|_{L^p(\partial\Omega)}+\|M(\nabla u^-)\|_{L^p(\partial\Omega)}\leq C\Big(\|\nabla_{tan}f\|_{\dot{H}^p_{at}(\partial\Omega)}+\|g\|_{\dot{H}^p_{at}(\partial\Omega)}\Big). \tag{1.7}$$

Finally, appropriate versions of these estimates hold (for the same ranges of p's) in the case of bounded Lipschitz domains, granted that the boundary data belong to inhomogeneous Hardy spaces.

One key technical point in the proof of Theorem 1.2 is as follows. Following Dahlberg and Kenig, we perform a dyadic decomposition of the boundary $\partial\Omega=\cup\Lambda_j$, with the aim of deriving estimates in each Carleson box D_j associated with the dyadic piece Λ_j, $j=1,2,\ldots$. In **[6]**, where the case of the Neumann problem is treated, the authors use the L^2-theory for the *local* version of this problem in each Carleson box D_j in order to control $\int_{\Lambda_j}M(\nabla u)^2\,d\sigma$ by $C2^{-j}\int_{D_j}|\nabla u|^2\,dx$. This is a crucial step in establishing appropriate decay in j. In our situation, no local transmission L^2-estimates are available, but we overcome this difficulty by using a local, scale-adapted Rellich type estimate, well-suited for the problem at hand.

As was the case with (1.2), there is a close correlation between the well-posedness of (1.1) and the invertibility properties of certain boundary layer potential operators. The relevant boundary integral operators for our transmission problem are (anticipating notation to be introduced later)

$$\lambda I+K^*:L^p(\partial\Omega)\longrightarrow L^p(\partial\Omega),\quad \lambda I+K:\dot{L}^p_1(\partial\Omega)\longrightarrow\dot{L}^p_1(\partial\Omega). \tag{1.8}$$

These are shown to be invertible for each $\lambda\in\mathbb{R}$ with $|\lambda|>1/2$ whenever Ω is a Lipschitz domain and $1<p<2+\varepsilon$, where $\varepsilon=\varepsilon(\partial\Omega)>0$. In fact, this range extends below $p=1$ (when Hardy spaces are employed).

A closely related issue, the so-called *Spectral Radius Conjecture* (**SRC** in short), is the statement that $\lambda I+K$ is in fact invertible on $L^p(\partial\Omega)$, $2\leq p<\infty$, for any λ *complex* with $|\lambda|>1/2$. This has been singled out as an open problem by C. Kenig in **[23]** and G. Verchota in **[9]**. While the **SRC** has long been known to be true in a number of particular cases (such as Lipschitz domains whose unit normal has vanishing mean oscillations, or two dimensional polygonal domains), the problem remains open in full generality. More progress has been made by E. Fabes, M. Sand and K. Seo who have proved in **[14]** that the **SRC** is true in $L^2(\partial\Omega)$ in any bounded *convex* domain Ω in $\mathbb{R}^n$ (while this is automatically Lipschitz, it may fail to be of class C^1).

As a byproduct of our invertibility results for layer potentials, here we are able to extend the aforementioned result by Fabes, Sand and Seo, by proving the following.

Theorem 1.3 *For any convex domain* $\Omega\subset\mathbb{R}^n$, $n\geq 2$, *we have*

$$r\Big(K;L^p_s(\partial\Omega)/\mathbb{R}\Big)<\frac{1}{2} \tag{1.9}$$

if

$$\Big(\tfrac{1-\varepsilon}{2}\Big)s<\tfrac{1}{p}<\Big(\tfrac{1-\varepsilon}{2}\Big)s+\Big(\tfrac{1+\varepsilon}{2}\Big),\quad 0<s<1,\ 1<p<\infty. \tag{1.10}$$

Here and elsewhere, $r(T;X)$ stands for the *spectral radius* of the operator T on the Banach space X, i.e. the radius of the smallest disk (centered at the origin) containing its spectrum. Also, $L^p_s(\partial\Omega)$, $0\leq s\leq 1$, $1<p<\infty$, denotes the classical, L^p-based Sobolev space of order s on $\partial\Omega$.

The main result in **[14]** corresponds to (1.9) for $p=2$, $s=0$. Geometrically, the conditions (1.10) amount to the membership of the point with coordinates $(s,1/p)$ to the parallelogram with vertices at $(0,0)$, $(0,(1+\varepsilon)/2)$, $(1,1)$ and $(1,(1-\varepsilon)/2)$.

The organization of the paper is as follows. In §2 we collect basic definitions and deal with (1.1) in the case when $|p-2|$ is small and Ω is the domain above the graph of a real-valued Lipschitz function. In §3 we simultaneously deal with the case $n \geq 3$ of Theorems 1.1 and 1.2 via an approach based on the De Giorgi-Nash-Moser and Serrin-Weinberger theory for elliptic operators in divergence form, with bounded, measurable coefficients. The two dimensional case is treated separately in §4. Finally, in §5, we present the proof of Theorem 1.3.

Acknowledgments. This collaboration grew out of conversations between the authors during the 2002 Fabes-Chiarenza Lectures in Siracusa, Italy. We are grateful to the organizers, Professors M. Frasca, M.A. Ragusa, S. Salsa and R. Serapioni for putting together such an inspiring meeting.

2 Preliminaries and review of the L^2-theory

2.1 Function spaces in Lipschitz domains

We start by collecting a number of basic definitions. An *unbounded Lipschitz* domain $\Omega \subset \mathbb{R}^n$ is simply the domain lying above the graph of a real-valued Lipschitz function. That is,

$$\begin{aligned} &\Omega := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\, x_n > \varphi(x')\}, \text{ where } x' = (x_1, ..., x_{n-1}),\\ &\varphi : \mathbb{R}^{n-1} \to \mathbb{R} \text{ is Lipschitz, i.e., } \nabla\varphi \text{ exists and belongs to } L^\infty(\mathbb{R}^{n-1}). \end{aligned} \tag{2.1}$$

We denote by $d\sigma$ the surface measure on $\partial\Omega$, and by ν the outward unit normal defined a.e. (with respect to $d\sigma$) on $\partial\Omega$. Also, throughout the paper, we set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$.

Recall that a *bounded* domain $\Omega \subset \mathbb{R}^n$ (no topological assumption made) is called Lipschitz if:

i) $\partial\Omega$ can be covered by a finite family of open (appropriately rotated) cylinders $\{Z_i\}_{i=1}^m$ in $\mathbb{R}^n$;

ii) for each i, there exists a Lipschitz function $\varphi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ so that $2\|\varphi_i\|_{L^\infty}$ is less than the height of Z_i and, if $2Z_i$ denotes the concentric double of Z_i, in the rectangular coordinate system defined by Z_i one has

$$\begin{aligned} \Omega \cap 2Z_i &= \{x = (x', x_n);\, \varphi_i(x') < x_n\} \cap 2Z_i,\\ \partial\Omega \cap 2Z_i &= \{x = (x', x_n);\, \varphi_i(x') = x_n\} \cap 2Z_i, \end{aligned} \tag{2.2}$$

see e.g. **[30]**, **[37]** for more details. In the sequel, we shall say that a constant depends on the Lipschitz character of Ω if its size is controlled in terms of m, the number of cylinders $\{Z_i\}_i$, the size of these cylinders and $\sup\{\|\nabla\varphi_i\|_{L^\infty};\, 1 \leq i \leq m\}$.

In order to introduce the classical non-tangential maximal operator M, fix some $\kappa = \kappa(\partial\Omega) > 1$, sufficiently large. For an arbitrary $u : \Omega_\pm \to \mathbb{R}$, we then set

$$M(u)(x) := \sup\{|u(y)|;\, y \in \Gamma^\pm(x)\}, \qquad x \in \partial\Omega, \tag{2.3}$$

where

$$\Gamma^\pm(x) := \{y \in \Omega_\pm;\, \mathrm{dist}\,(x, y) < \kappa\, \mathrm{dist}\,(y, \partial\Omega)\}, \qquad x \in \partial\Omega, \tag{2.4}$$

are cone-like regions (lying in Ω_+ and Ω_-, respectively) with vertex at boundary points. These regions also play a fundamental role in defining non-tangential restrictions to the boundary. Set

$$u\Big|_{\partial\Omega}(x) := \lim_{y\in\Gamma^\pm(x)} u(y), \qquad \text{for a.e. } x \in \partial\Omega, \tag{2.5}$$

the choice of the sign depending on whether the function u is defined in Ω_+ or Ω_-. Similarly,

$$\partial_\nu u(x) := \nu(x)\cdot\Big(\lim_{y\in\Gamma^\pm(x)}(\nabla u)(y)\Big), \qquad \text{for a.e. } x\in\partial\Omega. \tag{2.6}$$

By $L^p(\partial\Omega)$ we denote the Lebesgue space of measurable, p-th power integrable functions on $\partial\Omega$, with respect to the surface measure $d\sigma$. For an unbounded Lipschitz domain $\Omega\subset\mathbb{R}^n$, the *homogeneous* L^p-Sobolev space of order one is defined as

$$\dot{L}^p_1(\partial\Omega) := \{f\in L^p_{loc}(\partial\Omega);\ |\nabla_{tan}f|\in L^p(\partial\Omega)\}. \tag{2.7}$$

Here and elsewhere, $\nabla_{tan} := \nabla - \nu\partial_\nu$ stands for the tangential gradient on $\partial\Omega$. Clearly, for each $1<p<\infty$, this becomes a Banach space modulo constants when equipped with the homogeneous norm $\|f\|_{\dot{L}^p_1(\partial\Omega)} := \|\nabla_{tan}f\|_{L^p(\partial\Omega)}$. The corresponding *inhomogeneous* Sobolev space is

$$L^p_1(\partial\Omega) := L^p(\partial\Omega)\cap\dot{L}^p_1(\partial\Omega), \quad \|\cdot\|_{L^p_1(\partial\Omega)} := \|\cdot\|_{L^p(\partial\Omega)} + \|\nabla_{tan}\cdot\|_{L^p(\partial\Omega)}, \tag{2.8}$$

for $1<p<\infty$, which also yields a Banach space on bounded domains.

Let us now once again consider the setting of an unbounded Lipschitz domain Ω in $\mathbb{R}^n$. A surface ball $S_r(x)$ is any set of the form $B_r(x)\cap\partial\Omega$, with $x\in\partial\Omega$ and $0<r<\infty$. As far as the *homogeneous* Hardy spaces $\dot{H}^p_{at}(\partial\Omega)$, $\frac{n-1}{n}<p\le 1$, are concerned, call $a:\partial\Omega\to\mathbb{R}$ an atom for $\dot{H}^p_{at}(\partial\Omega)$ (p-atom for short), if

$$\exists\, S_r - \text{surface ball}:\ \operatorname{supp} a\subseteq S_r, \quad \|a\|_{L^\infty(\partial\Omega)}\le r^{-\frac{n-1}{p}}, \text{ and } \int_{\partial\Omega} a\,d\sigma = 0. \tag{2.9}$$

Then

$$\dot{H}^p_{at}(\partial\Omega) := \Big\{\sum_j \lambda_j a_j;\ a_j\ p\text{-atom},\ (\lambda_j)_j\in\ell^p\Big\}, \tag{2.10}$$

equipped with the usual infimum norm. Here, the series is convergent in the space $(\dot{C}^\alpha(\partial\Omega))^*$ if $\alpha := (n-1)(1/p-1)\in(0,1)$ (where $\dot{C}^\alpha(\partial\Omega)$ stands for the *homogeneous* Hölder space of order α, i.e. the Banach space of functions, modulo constants, subject to the requirement $\sup_{x,y\in\partial\Omega}|f(x)-f(y)|/|x-y|^\alpha<+\infty$), and in $L^1(\partial\Omega)$ if $p=1$.

The inhomogeneous version of (2.10) is then obtained by enlarging the class of atoms to contain, besides functions satisfying (2.9), any $a\in L^\infty(\partial\Omega)$ such that

$$\exists\, S_r - \text{surface ball, with } r\ge 1, \text{ such that } \operatorname{supp} a\subseteq S_r, \quad \|a\|_{L^\infty(\partial\Omega)}\le r^{-\frac{n-1}{p}}. \tag{2.11}$$

Following **[18]** we then set

$$H^p_{at}(\partial\Omega) := \Big\{\sum_j \lambda_j a_j;\ \{\lambda_j\}_j\in\ell^p,\ a_j \text{ satisfies either (2.9) or (2.11)}\Big\}, \tag{2.12}$$

and endowed it with the natural infimum norm. This time, the series is convergent in $(C^\alpha(\partial\Omega))^*$ with $\alpha := (n-1)(1/p-1)\in(0,1)$ and in $L^1(\partial\Omega)$ if $p=1$. This definition also makes sense when Ω is a bounded Lipschitz domain. In fact, in this latter scenario,

$$H^p_{at}(\partial\Omega) = \dot{H}^p_{at}(\partial\Omega) + L^q(\partial\Omega), \quad \forall\, q>1. \tag{2.13}$$

It is not difficult to see that the inhomogeneous Hardy space (2.12) is local in the sense that $H^p_{at}(\partial\Omega)$ is a module over $C^\alpha(\partial\Omega)$ with $\alpha>(n-1)(p^{-1}-1)$.

We shall also work with $\dot{H}^{1,p}_{at}(\partial\Omega)$, $\frac{n-1}{n} < p \le 1$, the ℓ^p-span of 'regular' atoms on $\partial\Omega$. More specifically, define

$$f \in \dot{H}^{1,p}_{at}(\partial\Omega) \;\overset{\text{def}}{\Longleftrightarrow}\; \nabla_{tan} f = \sum_{j=1}^{\infty} \lambda_j \nabla_{tan} a_j,\ (\lambda_j)_j \in \ell^p,\ a_j \text{ regular atom}, \tag{2.14}$$

where the series converges in $\dot{H}^p_{at}(\partial\Omega)$, and set $\|f\|_{\dot{H}^{1,p}_{at}(\partial\Omega)} := \inf\,[\sum|\lambda_j|^p]^{1/p}$, where the infimum is taken over all possible representations. Here, for $(n-1)/n < p \le 1$ and a fixed $\max\{1,p\} < p_o < \infty$, a function $a : \partial\Omega \to \mathbb{R}$ is called a *regular atom* if there exists a surface ball S_r so that

$$\operatorname{supp} a \subseteq S_r, \qquad \|\nabla_{tan} a\|_{L^{p_o}(\partial\Omega)} \le r^{(n-1)\left(\frac{1}{p_o}-\frac{1}{p}\right)}. \tag{2.15}$$

Different choices of the parameter p_o above yield the same topology on $\dot{H}^{1,p}_{at}(\partial\Omega)$. Once again there is a corresponding inhomogeneous version of this space defined, for $1/q := 1/p - 1/(n-1)$, as follows:

$$H^{1,p}_{at}(\partial\Omega) := \Big\{\sum_j \lambda_j a_j \text{ convergent in } L^q(\partial\Omega);\ (\lambda_j)_j \in \ell^p,\ a_j \text{ as in (2.15)}\Big\}. \tag{2.16}$$

This inhomogeneous, regular Hardy space is then a module over $\mathrm{Lip}_{comp}(\partial\Omega)$, the class of Lipschitz, compactly supported functions on $\partial\Omega$. As remarked on p. 456 in **[6]**, if $f \in \dot{H}^{1,p}_{at}(\partial\Omega)$ then there exists $c \in \mathbb{R}$ so that $f - c \in H^{1,p}_{at}(\partial\Omega)$. Also, in the case of a bounded domain, it is not too difficult to check that, for $(n-1)/n < p \le 1$,

$$H^{1,p}_{at}(\partial\Omega) = \dot{H}^{1,p}_{at}(\partial\Omega) + L^q_1(\partial\Omega), \quad \forall\, q > 1. \tag{2.17}$$

2.2 Layer potentials

We continue to review background material by recalling the definitions and some of the most basic properties of the classical harmonic layer potentials for a Lipschitz domain $\Omega \subset \mathbb{R}^n$. With $E(x)$ denoting the canonical radial fundamental solution for the Laplace operator Δ in $\mathbb{R}^n$, i.e.

$$E(x) := \begin{cases} \frac{1}{-(n-2)\alpha_n}\frac{1}{|x|^{n-2}}, & n \ge 3, \\ \frac{1}{2\pi}\log|x|, & n = 2, \end{cases} \qquad x \in \mathbb{R}^n \setminus \{0\}, \tag{2.18}$$

where α_n equals the surface measure of the unit sphere in $\mathbb{R}^n$, we define the single and double layer potential operators by

$$\mathcal{S}f(x) := \int_{\partial\Omega} E(x-y)\, f(y)\, d\sigma_y, \qquad x \notin \partial\Omega, \tag{2.19}$$

and

$$\mathcal{D}f(x) := \int_{\partial\Omega} \partial_{\nu_y}[E(x-y)]\, f(y)\, d\sigma_y, \qquad x \notin \partial\Omega, \tag{2.20}$$

respectively. When Ω is an unbounded Lipschitz domain and $f \in L^p(\partial\Omega)$, $1 < p < \infty$, the integral in the right-side of (2.20) is absolutely convergent, as a simple application of Hölder's inequality shows. However, the integral in (2.19) may diverge if $p \ge n-1$. One remedy is to consider $E(x-y) - E(x_o - y)$ in place of $E(x-y)$ as the integral kernel of the single layer, for some $x_o \notin \partial\Omega$, fixed. We shall tacitly assume this convention throughout the paper.

As is well-known (cf., e.g., **[6]**, **[37]**), if $f \in L^p(\partial\Omega)$, $1 < p < \infty$,

$$\partial_j \mathcal{S} f\Big|_{\partial\Omega_\pm} = \mp \tfrac{1}{2}\nu_j\, f \,+\, T_j f, \qquad j = 1, 2, ..., n, \tag{2.21}$$

where ν_j is the j-th component of ν, and

$$T_j f(x) := p.v. \int_{\partial\Omega} (\partial_j E)(x-y)\, f(y)\, d\sigma_y, \quad x \in \partial\Omega, \tag{2.22}$$

Here $p.v.$ indicates that the integral is taken in the principal value sense. In particular,

$$\partial_\nu \mathcal{S}\Big|_{\partial\Omega_\pm} = \mp\tfrac{1}{2}I + K^*, \quad \nabla_{tan}\mathcal{S}\Big|_{\partial\Omega_+} = \nabla_{tan}\mathcal{S}\Big|_{\partial\Omega_-}, \quad \text{and} \quad \mathcal{D}\Big|_{\partial\Omega_\pm} = \pm\tfrac{1}{2}I + K, \tag{2.23}$$

where I denotes the identity operator,

$$Kf(x) := \text{p.v.} \int\limits_{\partial\Omega} \partial_{\nu_y}[E(x-y)]\, f(y)\, d\sigma_y, \qquad x \in \partial\Omega, \tag{2.24}$$

and K^* is the formal adjoint of K. Also, $\mathcal{S}\Big|_{\partial\Omega_+} = \mathcal{S}\Big|_{\partial\Omega_-} =: S$, is the boundary version of (2.19).

It is then clear from this discussion that, in the case when $\mu = 1$,

$$u^\pm := \mathcal{D}f - \mathcal{S}g \;\text{ in }\; \Omega_\pm \tag{2.25}$$

solve (1.1) for any given $1 < p < \infty$. This clarifies a point made in the Introduction, right before the statement of Theorem 1.1.

The boundedness of the operators $K, K^*, T_j : L^p(\partial\Omega) \to L^p(\partial\Omega)$ as well as that of

$$S : L^p(\partial\Omega) \to L^p_1(\partial\Omega) \text{ if } \Omega \text{ is a bounded Lipschitz domain,} \tag{2.26}$$

$$S : L^p(\partial\Omega) \to \dot{L}^p_1(\partial\Omega) \text{ if } \Omega \text{ is an unbounded Lipschitz domain,} \tag{2.27}$$

along with the estimate

$$\|M(\mathcal{D}f)\|_{L^p(\partial\Omega)} + \|M(\nabla\mathcal{S}f)\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}, \tag{2.28}$$

valid for $1 < p < \infty$, with $C = C(\partial\Omega, p) < +\infty$, follow by combining the techniques of **[13]** with the results in **[5]**. Extensions to variable coefficient operators can be found in **[26]**, **[27]**. For further reference, here we also want to record that

$$\|M(\nabla\mathcal{S}f)\|_{L^p(\partial\Omega)} \le C\|f\|_{\dot{H}^p_{at}(\partial\Omega)}, \tag{2.29}$$

if $(n-1)/n < p \le 1$, where $C = C(\partial\Omega, p) < +\infty$.

Let Ω be an unbounded Lipschitz domain and pick an arbitrary $f \in \dot{L}^p_1(\partial\Omega)$, $1 < p < \infty$. Then, for each $1 \le j \le n$,

$$\partial_j\mathcal{D}f(x) = \sum_{k=1}^{n} \partial_k\mathcal{S}(\partial_{\tau_{jk}}f)(x), \qquad x \notin \partial\Omega, \tag{2.30}$$

via successive integrations by parts, where

$$\partial_{\tau_{jk}} := \nu_j\partial_k - \nu_k\partial_j, \qquad 1 \le j, k \le n, \tag{2.31}$$

are tangential derivative operators. It follows that

$$\|M(\nabla \mathcal{D} f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{\dot{L}^p_1(\partial\Omega)} \quad \text{if} \quad 1 < p < \infty, \tag{2.32}$$

$$\|M(\nabla \mathcal{D} f)\|_{L^p(\partial\Omega)} \leq C\|f\|_{\dot{H}^{1,p}_{at}(\partial\Omega)} \quad \text{if} \quad \tfrac{n-1}{n} < p \leq 1, \tag{2.33}$$

$$\text{and} \quad \partial_\nu \mathcal{D} f\Big|_{\partial\Omega_+} = \partial_\nu \mathcal{D} f\Big|_{\partial\Omega_-}, \tag{2.34}$$

where (2.34) is based on (2.21) and the observation that $\sum_{1\leq j,k\leq n} \nu_j \nu_k \partial_{\tau_{jk}} = 0$. Appropriate analogues are valid in bounded domains, working this time with inhomogeneous spaces.

2.3 The L^p transmission problem with $|p-2|$ small

In this subsection we discuss the well-posedness of (1.1) in unbounded Lipschitz domains when p is near 2. To set the stage, let Ω be as in (2.1), and recall an integral identity due to Rellich (**[33]**, to the effect that for any harmonic function u in Ω with $M(\nabla u) \in L^2(\partial\Omega)$ and for any constant vector $e \in \mathbb{R}^n$,

$$\int_{\partial\Omega} |\nabla u|^2 \langle e, \nu\rangle \, d\sigma = 2\int_{\partial\Omega} \partial_\nu u \, \langle e, \nabla u\rangle \, d\sigma. \tag{2.35}$$

Decomposing $|\nabla u|^2 = |\nabla_{tan} u|^2 + |\nabla_\nu u|^2$ and $e = e_{tan} + \langle e, \nu\rangle \nu$ further yields

$$\int_{\partial\Omega} |\nabla_{tan} u|^2 \langle e, \nu\rangle \, d\sigma - \int_{\partial\Omega} |\partial_\nu u|^2 \langle e, \nu\rangle \, d\sigma = 2\int_{\partial\Omega} \partial_\nu u \, \langle e, \nabla_{\rm tan} u\rangle \, d\sigma. \tag{2.36}$$

When written with $\Omega_\pm$ in place of Ω and $u^\pm := \mathcal{S}f$, $f \in L^2(\partial\Omega)$, in place of u, the above identity becomes

$$\begin{aligned}\int_{\partial\Omega} |\nabla_{tan} Sf|^2 \langle e, \nu\rangle \, d\sigma \quad &- \quad \int_{\partial\Omega} |(\mp\tfrac{1}{2}I + K^*)f|^2 \langle e, \nu\rangle \, d\sigma \\ &= \quad 2\int_{\partial\Omega} \langle e, \nabla_{tan} Sf\rangle (\mp\tfrac{1}{2}I + K^*)f \, d\sigma.\end{aligned} \tag{2.37}$$

For an arbitrary $\lambda \in \mathbb{R}$ we then decompose $(\mp\frac{1}{2}I + K^*)f = (\lambda I + K^*)f + (-\lambda \mp \frac{1}{2})f$. Multiplying the $\pm$-versions of (2.37) by $-\lambda + \frac{1}{2}$ and $\lambda + \frac{1}{2}$, respectively, then adding them up yields

$$\begin{aligned}&\int_{\partial\Omega} |\nabla_{tan} Sf|^2 \langle e, \nu\rangle \, d\sigma + \Big(\lambda^2 - \tfrac{1}{4}\Big) \int_{\partial\Omega} |f|^2 \langle e, \nu\rangle \, d\sigma \\ &= \int_{\partial\Omega} |(\lambda I + K^*)f|^2 \langle e, \nu\rangle \, d\sigma + 2\int_{\partial\Omega} \langle e_{\rm tan}, \nabla_{tan} Sf\rangle (\lambda I + K^*)f \, d\sigma.\end{aligned} \tag{2.38}$$

Let us specialize (2.38) to the case when $e = (0, ..., 0, -1)$, which is transversal to $\partial\Omega$. Then $|e_{tan}| = |\nabla\varphi|/\sqrt{1 + |\nabla\varphi|^2} \leq \kappa\langle e, \nu\rangle$, where φ is as in (2.1) and $\kappa := \|\nabla\varphi\|_{L^\infty}$. Thus, by the Cauchy-Schwartz inequality, the last integral above is majorized by

$$\int_{\partial\Omega} |\nabla_{tan} Sf|^2 \langle e, \nu\rangle \, d\sigma + \kappa^2 \int_{\partial\Omega} |(\lambda I + K^*)f|^2 \langle e, \nu\rangle \, d\sigma. \tag{2.39}$$

Utilizing this back in (2.38) then justifies the estimate

$$\Big(\lambda^2 - \tfrac{1}{4}\Big) \int_{\partial\Omega} |f|^2 \langle e, \nu\rangle \, d\sigma \leq (1 + \kappa^2) \int_{\partial\Omega} |(\lambda I + K^*)f|^2 \langle e, \nu\rangle \, d\sigma. \tag{2.40}$$

In particular, $\{\lambda I + K^*\}_\lambda$, $\lambda \in \mathbb{R}$, $|\lambda| > \frac{1}{2}$, is a continuous, one-parameter family of semi-Fredholm operators. Also, obviously, $\lambda I + K^*$ becomes invertible when $|\lambda|$ is large. It follows from the homotopic invariance of the index that $\lambda I + K^*$ is in fact Fredholm with index zero for each $\lambda \in \mathbb{R}$, $|\lambda| > \frac{1}{2}$. Since, by (2.40), each $\lambda I + K^*$ is one-to-one, we may therefore conclude that, with λ as above, $\lambda I + K^*$ is an isomorphism of $L^2(\partial\Omega)$ for each $\lambda \in \mathbb{R}$, $|\lambda| \geq \frac{1}{2}$ (the case $\lambda = \pm\frac{1}{2}$ is contained in **[6]**).

At this stage, we claim that there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ so that

$$\begin{gathered}\lambda I + K^*,\, \lambda I + K : L^p(\partial\Omega) \xrightarrow{\sim} L^p(\partial\Omega) \text{ isomorphically} \\ \text{whenever } \lambda \in \mathbb{R},\ |\lambda| \geq \tfrac{1}{2},\ 2-\varepsilon < p < 2+\varepsilon.\end{gathered} \tag{2.41}$$

The interested reader is referred to **[20]** for a discussion of such stability results from a broader point of view. See also **[1]** which, in particular, makes it clear that for each $p, q \in (2-\varepsilon, 2+\varepsilon)$, the inverse $(\lambda I + K^*)^{-1}$ considered on the space $L^p(\partial\Omega)$ is compatible with $(\lambda I + K^*)^{-1}$ considered on $L^q(\partial\Omega)$ when both operators are restricted to $L^p(\partial\Omega) \cap L^q(\partial\Omega)$.

Next, if we write (2.36) with $\Omega_\pm$ in place of Ω and $u^\pm := \mathcal{D}f$, $f \in \dot{L}^2_1(\partial\Omega)$, in place of u, we arrive at the identity

$$\begin{aligned}\int_{\partial\Omega} |\nabla_{tan}(\pm\tfrac{1}{2}I + K)f|^2 \langle e, \nu\rangle\, d\sigma - \int_{\partial\Omega} |\partial_\nu \mathcal{D}f|^2 \langle e, \nu\rangle\, d\sigma \\ = 2\int_{\partial\Omega} \partial_\nu \mathcal{D}f\, \langle e, \nabla_{tan}(\pm\tfrac{1}{2}I + K)f\rangle\, d\sigma.\end{aligned} \tag{2.42}$$

Proceeding as before, this identity leads to the estimate

$$\Big(\lambda^2 - \tfrac{1}{4}\Big) \int_{\partial\Omega} |\nabla_{tan} f|^2 \langle e, \nu\rangle\, d\sigma \leq (1+\kappa^2) \int_{\partial\Omega} |\nabla_{tan}(\lambda I + K^*)f|^2 \langle e, \nu\rangle\, d\sigma \tag{2.43}$$

and, further, to the conclusion that

$$\begin{gathered}\lambda I + K : L^p_1(\partial\Omega) \xrightarrow{\sim} L^p_1(\partial\Omega) \text{ and } \lambda I + K : \dot{L}^p_1(\partial\Omega) \xrightarrow{\sim} \dot{L}^p_1(\partial\Omega) \\ \text{isomorphically for each } \lambda \in \mathbb{R},\ |\lambda| \geq \tfrac{1}{2},\ 2-\varepsilon < p < 2+\varepsilon,\end{gathered} \tag{2.44}$$

for some $\varepsilon = \varepsilon(\partial\Omega) > 0$ (once again, the case $\lambda = \pm\frac{1}{2}$ is contained in **[6]**).

Having established the invertibility properties of the relevant operators for the problem under discussion, we now tackle the issue of existence for (1.1) when $2-\varepsilon < p < 2+\varepsilon$. Recalling that, in the context we are considering, the operator (2.27) is an isomorphism for $1 < p < 2+\varepsilon$ (cf. **[6]**), it is possible to find $\psi \in L^p(\partial\Omega)$ and $c \in \mathbb{R}$ so that $S\psi = f + c$. We may then take

$$u^+ := \mathcal{S}h^+ - c \text{ in } \Omega_+, \quad \text{and} \quad u^- := \mathcal{S}h^- \text{ in } \Omega_-, \tag{2.45}$$

where, with $\lambda := -\frac{1}{2}\frac{1+\mu}{1-\mu}$, the functions $h^\pm \in L^p(\partial\Omega)$ are given by

$$h^+ := \tfrac{1}{1-\mu}\Big(\lambda I + K^*\Big)^{-1}\Big[g - \mu(\tfrac{1}{2}I + K^*)\psi\Big], \quad \text{and} \quad h^- := h^+ - \psi. \tag{2.46}$$

There remains uniqueness which we address next. One way to see this is to rely on the well-posedness of the L^p-Neumann problem and the invertibility results (2.41). Another, more direct approach, which uses some ideas of importance for us later on, is as follows. First, we claim that for any harmonic function u in an unbounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, and which satisfies $M(\nabla u) \in L^p(\partial\Omega)$ for some $1 < p < \infty$, there holds

$$(\tfrac{1}{2}I + K^*)(\partial_\nu u) = \sum_{j,k=1}^{n} \nu_j T_k(\partial_{\tau_{jk}} u|_{\partial\Omega}). \tag{2.47}$$

Recall that the operators T_k, $1 \leq k \leq n$, have been introduced in (2.22).

To justify this, we formally write Green's formula for u in Ω,

$$u = \mathcal{D}(u|_{\partial\Omega}) - \mathcal{S}(\partial_\nu u), \tag{2.48}$$

take the gradient of both sides,

$$\nabla u = \nabla\mathcal{D}(u|_{\partial\Omega}) - \nabla\mathcal{S}(\partial_\nu u) = \Big(\sum_{k=1}^{n} \partial_k \mathcal{S}(\partial_{\tau_{jk}} u|_{\partial\Omega})\Big)_{1\leq j\leq n} - \nabla\mathcal{S}(\partial_\nu u), \tag{2.49}$$

where the second step involves an integration by parts, go to the boundary non-tangentially and, finally, then take the inner product with the unit normal, proving (2.47). Now, there are certain technical difficulties in justifying (2.48) in an unbounded domain Ω due to the lack of information on the decay of the function u. However, starting with (2.48) written in a suitable sequence of bounded domains $D_j \nearrow \Omega$, allows us, once (2.49) has been obtained for each D_j, to pass to the limit (note that, as opposed to (2.48)), the identity (2.49) involves only derivatives of u) and establish (2.49) in Ω. This finishes the proof of (2.47).

Let now (u^+, u^-) solve the homogeneous version of (1.1) with $|p-2| < \varepsilon$. Writing (2.47) for $u^\pm$ and using the transmission boundary conditions leads, after some minor algebra, to the conclusion that

$$(\lambda I + K^*)(\partial_\nu u^-) = 0, \qquad \text{where } \lambda := \tfrac{1}{2}\tfrac{\mu+1}{\mu-1}. \tag{2.50}$$

Thus, $\partial_\nu u^\pm = 0$. In particular, the function $u := u^+$ in Ω_+, and u^- in Ω_- becomes harmonic in the whole space $\mathbb{R}^n$. If we now recall a general real-variable result, proved in Lemma 6.1 of **[10]**, to the effect that

$$\begin{aligned} &w \in C^0_{loc}(\Omega), \quad M(w) \in L^p(\partial\Omega), \quad 0 < p < \infty \\ &\Longrightarrow w \in L^{p^*}(\Omega) \text{ where } p^* := np/(n-1), \end{aligned} \tag{2.51}$$

plus a naturally accompanying estimate, it follows that $\nabla u \in L^{np/(n-1)}(\mathbb{R}^n)$. By a standard Liouville theorem then u is a constant, as desired.

Remark. The estimate (2.40) involves a *real* parameter λ. Assume we are interested in a similar estimate but with a *complex* parameter instead. That is, we seek an inequality of the form

$$\|f\|_{L^2(\partial\Omega)} \leq C(\partial\Omega, z)\|(zI + K^*)f\|_{L^2(\partial\Omega)}, \quad \forall\, f \in L^2(\partial\Omega), \tag{2.52}$$

where $z \in \mathbb{C}$. Writing $(\lambda I + K^*)f = (zI + K^*)f + (\lambda - z)f$, elementary estimates give

$$\begin{aligned} \int_{\partial\Omega} |(\lambda I + K^*)f|^2 \langle e, \nu\rangle\, d\sigma \quad &\leq \quad \int_{\partial\Omega} |(zI + K^*)f|^2 \langle e, \nu\rangle\, d\sigma + |z - \lambda|^2 \int_{\partial\Omega} |f|^2 \langle e, \nu\rangle\, d\sigma \\ &\quad + \mathcal{O}\Big(\|f\|_{L^2(\partial\Omega)} \cdot \|(zI + K^*)f\|_{L^2(\partial\Omega)}\Big). \end{aligned} \tag{2.53}$$

The bottom line is that (2.40) implies (2.52) for a given $z \in \mathbb{C}$ if $\lambda \in \mathbb{R} \setminus (-\frac{1}{2}, \frac{1}{2})$ can be chosen so that

$$\Big(\lambda^2 - \tfrac{1}{4}\Big) - (1+\kappa^2)|z-\lambda|^2 > 0. \tag{2.54}$$

A simple inspection further shows that this latter condition holds if and only if z belongs to the 'interior' of the hyperbola $\mathcal{H}_\kappa \subset \mathbb{R}^2 \equiv \mathbb{C}$ (i.e., the component of $\mathbb{C} \setminus \mathcal{H}_\kappa$ containing the imaginary axis), with

$$\text{vertices at } \Big(\pm\tfrac{1}{2}\tfrac{\kappa}{\sqrt{1+\kappa^2}}\,,\,0\Big) \text{ and asymptotes with slopes } \pm\tfrac{1}{\kappa}. \tag{2.55}$$

In fact, starting with (2.43), we see that a similar conclusion holds for the operator K on $\dot{L}^2_1(\partial\Omega)$ and, by combining this with the L^2 result above, for the operator K on $L^2_1(\partial\Omega)$.

To further extend these results, we shall invoke the semi-continuity of the spectrum with respect to the parameter in the complex interpolation method. Recall that, for an operator $T : X \to X$, linear and bounded, $\mathrm{Spec}\,(T; X)$ stands for the collection of all $z \in \mathbb{C}$ so that $zI - T$ is not invertible on X. All in all, this proves the following result

Proposition 2.1 *Suppose that Ω is the domain in $\mathbb{R}^n$ lying above the graph of a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. It is assumed that there exists a finite constant $\kappa > 0$ such that $|\varphi(x') - \varphi(y')| \leq \kappa|x'-y'|$ for any $x', y' \in \mathbb{R}^{n-1}$. Recall the hyperbola $\mathcal{H}_\kappa$ described in (2.55). Then there exists $\varepsilon = \varepsilon(n,\kappa) > 0$ such that*

$$\mathrm{Spec}\Big(K; L^p_s(\partial\Omega)\Big) \subset \mathcal{H}_\kappa, \text{ for any } p \in (2-\varepsilon, 2+\varepsilon) \text{ and } 0 \leq s \leq 1. \tag{2.56}$$

3 The proof of Theorem 1.1 when $n \geq 3$

In this section we prove Theorem 1.1 for Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$. As in **[6]**, we make essential use of the De Giorgi-Nash-Moser and Serrin-Weinberger theory for null-solutions of (scalar) elliptic operators in divergence form, with bounded, measurable coefficients **[28]**, **[29]**, **[8]**, **[24]**, **[35]**.

3.1 Main atomic estimates

To set the stage, let Ω be an unbounded Lipschitz domain as in (2.1) and recall the maximal function operator M from (2.3). For $\mu \in (0,1)$ fixed, consider the following (reduced) transmission problem with atomic data:

$$(\text{TBVP-atomic}) \left\{\begin{array}{l} \Delta u^{\pm} = 0 \text{ in } \Omega_{\pm}, \\ M(\nabla u^{\pm}) \in L^2(\partial\Omega), \\ u^+\Big|_{\partial\Omega} = u^-\Big|_{\partial\Omega}, \\ \partial_\nu u^+ - \mu\,\partial_\nu u^- = a \in \dot{H}^1_{at}(\partial\Omega), \end{array}\right. \tag{3.1}$$

where a satisfies (2.9) with $p = 1$. The fact that this problem is well-posed follows from the discussion in §2. Then the function

$$u := \left\{\begin{array}{l} u^+ \text{ in } \Omega_+, \\ u^- \text{ in } \Omega_-, \end{array}\right. \tag{3.2}$$

belongs to $L^2_{1,loc}(\mathbb{R}^n)$ and satisfies

$$Lu = 0 \text{ in } \mathbb{R}^n \setminus \mathrm{supp}\, a, \tag{3.3}$$

where L is the second-order, formally self-adjoint, divergence form operator

$$L := \mathrm{div}(A\nabla\,), \qquad A := \chi_{\Omega_+} I + \mu\,\chi_{\Omega_-} I \tag{3.4}$$

with bounded, measurable coefficients. In particular, by the De Giorgi-Nash-Moser theory,

$$u \text{ is locally Hölder continuous in } \mathbb{R}^n \setminus \mathrm{supp}\, a. \tag{3.5}$$

The first main objective is to show that there exists a finite constant $C = C(\partial\Omega) > 0$ such that

$$\int_{\partial\Omega} [M(\nabla u^+) + M(\nabla u^-)]\, d\sigma \leq C. \tag{3.6}$$

Given the invariant nature of the estimate we seek under translations and dilations, there is no loss of generality in assuming that $\varphi(0) = 0$, $\mathrm{supp}\, a \subseteq \{(x', \varphi(x'));\ |x'| \leq 1\}$ and $\|a\|_{L^\infty(\partial\Omega)} \leq 1$. We proceed in a sequence of steps starting with

Step I. *There exists a finite constant* $\kappa = \kappa(\partial\Omega) > 0$ *such that*

$$|u(x)| \leq \kappa \quad \text{if} \quad \mathrm{dist}\,(x, \partial\Omega) \geq 1. \tag{3.7}$$

To see this, based on the L^2-theory, we write

$$u^\pm = \mathcal{S}f \text{ in } \Omega_\pm, \quad \text{for some } f \in L^2(\partial\Omega), \tag{3.8}$$

after subtracting a suitable constant from both u^+ and u^-. In fact, in light of (2.41), $f := [-(\frac{1}{2} + \frac{\mu}{2})I + (1-\mu)K^*]^{-1} a \in L^p(\partial\Omega)$ for any $2 - \varepsilon < p < 2 + \varepsilon$. In particular, $\|a\|_{L^p(\partial\Omega)} \leq 1$ entails $\|f\|_{L^p(\partial\Omega)} \leq C$ for each $p \in (2 - \varepsilon, 2 + \varepsilon)$. Consequently, with $1/p + 1/q = 1$,

$$\begin{aligned} |u^\pm(x)| &\leq C \int_{\partial\Omega} \frac{1}{|x-y|^{n-2}} |f(y)|\, d\sigma_y \\ &\leq C\Big(\int_{\mathbb{R}^{n-1}} \frac{1}{(1+|y|)^{(n-2)q}}\, dy\Big)^{1/q} \leq \kappa < \infty \end{aligned} \tag{3.9}$$

if $\mathrm{dist}\,(x, \partial\Omega) \geq 1$, provided that $q > (n-1)/(n-2)$. The latter condition can always be arranged if $n \geq 3$.

Step II. *There exists a finite constant* $C = C(\partial\Omega) > 0$ *such that*

$$|u(x)| \leq C \text{ in } \mathbb{R}^n \setminus \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ \varphi(x') - 1 < x_n < \varphi(x') + 1,\ |x'| \leq 2\}. \tag{3.10}$$

With κ as before, introduce $w := \max\{0, |u| - \kappa\}$ so that $w \in L^2_{1,loc}(\mathbb{R}^n)$, $w \geq 0$ and w is a subsolution for the operator (3.4) in $\mathbb{R}^n \setminus \mathrm{supp}\, a$. As in **[6]**, we then observe that there exist $d > 0$ and $r_o > 0$ such that, for every point x_o in the set $\mathbb{R}^n \setminus \{(x', x_n);\ \varphi(x') - 1 < x_n < \varphi(x') + 1,\ |x'| \leq 2\}$, the Lebesgue measure of $\{x \in B_{r_o}(x_o);\ w(x) = 0\}$ is $\geq d$. Furthermore,

$$\begin{aligned} \int_{B_{r_o}(x_o)} |\nabla w|^2\, dx &\leq C \int_{B_{r_o}(x_o)} |\nabla u|^2\, dx \leq r_o \int_{\partial\Omega} [M(\nabla u^+)^2 + M(\nabla u^-)^2]\, d\sigma \\ &\leq C r_o \|a\|^2_{L^2(\partial\Omega)} \leq C. \end{aligned} \tag{3.11}$$

With this at hand, a semi-standard Poincaré inequality gives

$$\int_{B_{r_o}(x_o)} |w|^2 \, dx \le C. \tag{3.12}$$

Recall next Moser's L^∞ estimate (i.e., the sub-mean inequality for nonnegative sub-solutions of L) to the effect that for each $0 < p < \infty$

$$\sup \{w(x);\, x \in B_{R/2}\} \le C_{\mu,n,p} \Big(R^{-n} \int_{B_R} w^p \, dx \Big)^{1/p}, \tag{3.13}$$

uniformly for any sub-solution $w \ge 0$ of L in B_R. This is proved in Theorem 2 pp. 581-582 of [**28**] when $1 < p < \infty$. The extension to $p \le 1$ uses an argument of B. Dahlberg and C. Kenig which may be found in [**15**], pp. 1004-1005. See also Lemma 1.1.8 in [**23**]. In the sequel, we shall frequently use the fact that if $Lu = 0$ then $|u|$ is a sub-solution of L.

It follows from the estimates (3.12) and (3.13), with $p = 2$, that $|w(x)| \le C$ in the domain $\mathbb{R}^n \setminus \{(x', x_n);\, \varphi(x') - 1 < x_n < \varphi(x') + 1,\ |x'| \le 2\}$. This clearly gives (3.10).

Step III. *There exist finite constants $\alpha > 0$, depending on $\|\nabla\varphi\|_{L^\infty}$, and $\beta \in \mathbb{R}$, $c > 0$, depending on α and κ from (3.7), such that*

$$|u(x) - \beta| \le c|x|^{2-n-\alpha}, \quad \text{uniformly for} \quad |x| \ge 2. \tag{3.14}$$

This is going to be a consequence of the asymptotic expansion of Serrin and Weinberger [**35**], [**24**], [**28**]. A suitable version of their main result (cf. also Theorem 2.2.9 in [**22**]) is as follows:

De Giorgi-Nash-Moser theory at infinity: *Assume that L is an elliptic, divergence form, second-order differential operator with real-valued, L^∞-coefficients in $\mathbb{R}^n$, $n \ge 3$; denote by $\lambda > 0$ its ellipticity constant. Let $E(x)$ be the fundamental solution of L with pole at a fixed arbitrary point $x^* \in \mathbb{R}^n$, so that*

$$C_1|x - x^*|^{n-2} \le |E(x)| \le C_2|x - x^*|^{n-2}, \quad x \in \mathbb{R}^n \setminus \{x^*\}, \tag{3.15}$$

*with constants C_1, C_2 depending only on λ (cf. [**24**]). Fix $R > 0$, and suppose that*

$$u \in L^2_{1,loc}(\mathbb{R}^n \setminus \bar{B}_R(x^*)) \cap L^\infty(\mathbb{R}^n \setminus B_R(x^*)) \cap C^0(\mathbb{R}^n \setminus B_R(x^*)) \tag{3.16}$$

solves $Lu = 0$ weakly in $|x - x^*| > R$. Then there exist constants $u_\infty, \gamma \in \mathbb{R}$ and $c, \alpha > 0$, with c and $1/\alpha$ bounded only in terms of λ and n, such that

$$|u(x) - u_\infty - \gamma\, E(x)| \le cR^{n-2+\alpha} \|u\|_{L^\infty(\mathbb{R}^n \setminus B_R(x^*))} |x|^{2-n-\alpha} \text{ for } |x| \ge R. \tag{3.17}$$

Moreover,

$$\gamma = K[u]/K[E], \text{ where } K[v] := \int_{\mathbb{R}^n} \langle A\nabla v, \nabla\psi \rangle \, dx, \tag{3.18}$$

for a fixed function $\psi \in C^\infty(\mathbb{R}^n)$ which is identically zero for $|x| \le 2R$ and is identically one in a neighborhood of infinity.

In our case, choose $R > 1$ so that $\psi \equiv 0$ on $\operatorname{supp} a$. Integrating by parts twice, and keeping in mind that $\psi - 1$ has compact support, then yields

$$\begin{aligned} K[u] = \int_{\mathbb{R}^n} \langle A\nabla u, \nabla\psi\rangle\, dx &= \int_{\Omega_+} \langle \nabla u^+, \nabla\psi\rangle\, dx + \mu \int_{\Omega_-} \langle \nabla u^-, \nabla\psi\rangle\, dx \\ &= \int_{\Omega_+} \langle \nabla u^+, \nabla(\psi - 1)\rangle\, dx + \mu \int_{\Omega_-} \langle \nabla u^-, \nabla(\psi - 1)\rangle\, dx \\ &= \int_{\partial\Omega} (\psi - 1) a\, d\sigma = -\int_{\partial\Omega} a\, d\sigma = 0. \end{aligned} \tag{3.19}$$

Thus $\gamma = 0$ so that (3.14) follows from (3.17).

To proceed, we need to introduce more notation. Concretely, recall the family of cones $\Gamma^{\pm}(x)$, $x \in \partial\Omega$, and for each fixed $R > 0$ set

$$\Gamma^{\pm}_{1,R}(x) := \{y \in \Gamma^{\pm}(x);\, |x - y| > R\}, \qquad \Gamma^{\pm}_{2,R}(x) := \{y \in \Gamma^{\pm}(x);\, |x - y| \leq R\}. \tag{3.20}$$

Also, for an arbitrary function u, defined in Ω_+ or Ω_-, consider

$$M_{j,R}(u)(x) := \sup\{|u(y)|;\, y \in \Gamma^{\pm}_{j,R}(x)\}, \quad j = 1, 2, \tag{3.21}$$

at every boundary point $x \in \partial\Omega$. Finally, set

$$\Lambda(R) := \{x \in \partial\Omega;\, R \leq |x| \leq 2R\}. \tag{3.22}$$

Step IV. *There exits $C > 0$ such that*

$$\int_{\Lambda(R)} [M_{1,R}(\nabla u^+)^2 + M_{1,R}(\nabla u^-)^2]\, d\sigma \leq C R^{1-n-2\alpha} \tag{3.23}$$

uniformly for $R > 2$.

Indeed, fix $x \in \partial\Omega$, $R \leq |x| \leq 2R$, and $y \in \Gamma^{\pm}_{1,R}(x)$. Since $u^{\pm}$ are harmonic in $B_{\delta R}(y)$ with $\delta > 0$ sufficiently small, it follows that

$$|\nabla u^{\pm}(y)| \leq C R^{-n-1} \int_{B_{\delta R}(y)} |u^{\pm}(z) - \beta|\, dz \leq C R^{1-n-\alpha}, \tag{3.24}$$

since $\sup\{|u(z) - \beta|;\, z \in B_{\delta R}(y)\} \leq C R^{2-n-\alpha}$, by (3.14). We may therefore conclude that $M_{1,R}(\nabla u^{\pm})(x) \leq C R^{1-n-\alpha}$ uniformly for $x \in \Lambda(R)$ which, in turn, readily yields (3.23).

Step V. *There exits $C > 0$ such that*

$$\int_{\Lambda(R)} [M_{2,R}(\nabla u^+)^2 + M_{2,R}(\nabla u^-)^2]\, d\sigma \leq C R^{1-n-2\alpha} \tag{3.25}$$

uniformly for $R > 2$.

The way to handle this part is as follows. Define for $\tau \in [1/4, 1/2]$ and some fixed $\lambda = \lambda(\partial\Omega) > 0$, the domains

$$\begin{aligned} D_{R,\tau} &= \{(x', x_n);\, \varphi(x') - \lambda\tau R < x_n < \varphi(x') + \lambda\tau R,\ \tau R < |x'| < \tau^{-1} R\}, \\ D^+_{R,\tau} &= \{(x', x_n);\, \varphi(x') < x_n < \varphi(x') + \lambda\tau R,\ \tau R < |x'| < \tau^{-1} R\}, \\ D^-_{R,\tau} &= \{(x', x_n);\, \varphi(x') - \lambda\tau R < x_n < \varphi(x'),\ \tau R < |x'| < \tau^{-1} R\}. \end{aligned} \tag{3.26}$$

Recall that in general, if u is sufficiently smooth in a Lipschitz domain D and if θ is a C^1 vector field, the following Rellich identity holds:

$$\int_{\partial D} \theta \cdot \nu \, [|\nabla_{tan} u|^2 - |\partial_\nu u|^2] \, d\sigma \tag{3.27}$$
$$= 2 \int_{\partial D} (\theta \cdot \nabla_{tan} u)(\partial_\nu u) \, d\sigma + \int_D [|\nabla u|^2 \mathrm{div}\, \theta - \nabla\theta(\nabla u) \cdot \nabla u - (\Delta u)(\theta \cdot \nabla u)] \, dx.$$

Now, if θ is a C^1 vector field such that $\|\theta\|_{L^\infty(\mathbb{R}^n)} \leq 1$, $\theta \cdot \nu \geq 1$ on $\partial D^+_{R,\tau} \cap \partial\Omega$, $\theta \cdot \nu \geq 0$ on $\partial\Omega$, $\mathrm{supp}(\theta) \subset D_{R,\tau/2}$ and $|\nabla\theta| \leq C/R$, apply to u^+ the Rellich identity (3.27) in $D^+_{R,\tau}$ and do the same with u^- in $D^-_{R,\tau}$. These yield the formulae

$$\int_{\partial D^+_{R,\tau} \cap \partial\Omega} \theta \cdot \nu^+ \, \left[|\nabla_{tan} u^+|^2 - |\partial_\nu u^+|^2\right] d\sigma = 2 \int_{\partial D^+_{R,\tau} \cap \partial\Omega} (\theta \cdot \nabla_{tan} u^+)(\partial_{\nu^+} u^+) \, d\sigma$$
$$+ \int_{D^+_{R,\tau}} \mathcal{O}(|\nabla u^+|^2 |\nabla\theta|) \, dx + \int_{\partial D^+_{R,\tau} \setminus \partial\Omega} \mathcal{O}(|\nabla u^+|)^2 \, d\sigma \tag{3.28}$$

and

$$\int_{\partial D^-_{R,\tau} \cap \partial\Omega} \theta \cdot \nu^- \, \left[|\nabla_{tan} u^-|^2 - |\partial_{\nu^-} u^-|^2\right] d\sigma = 2 \int_{\partial D^-_{R,\tau} \cap \partial\Omega} (\theta \cdot \nabla_{tan} u^-)(\partial_{\nu^-} u^-) \, d\sigma$$
$$+ \int_{D^-_{R,\tau}} \mathcal{O}(|\nabla u^-|^2 |\nabla\theta|) \, dx + \int_{\partial D^-_{R,\tau} \setminus \partial\Omega} \mathcal{O}(|\nabla u^-|)^2 \, d\sigma, \tag{3.29}$$

where $\nu^\pm$ are the outward unit normals to $\partial D^\pm_{R,\tau}$.

Next, the idea is to combine these two identities in such a way that the 'mixed' terms in the right-hand sides cancel out; this is achieved by multiplying formula (3.29) by μ and then adding it to (3.28). Keeping in mind that $\nu^\pm = \pm\nu$ and $\theta \cdot \nu \geq 1$ on $\partial D^\pm_{R,\tau} \cap \partial\Omega$, as well as $\nabla_{tan} u^+ = \nabla_{tan} u^-$ on $D_{R,\tau} \cap \partial\Omega$, this yields

$$\int_{\Lambda(R)} \left[(1-\mu)|\nabla_{tan} u^+|^2 + (\tfrac{1}{\mu} - 1)|\partial_\nu u^+|^2\right] d\sigma$$
$$\leq C(\partial\Omega) \left\{ \int_{\partial D_{R,\tau}} |\nabla u|^2 \, d\sigma + R^{-1} \int_{D_{R,\tau}} |\nabla u|^2 \, dx \right\}. \tag{3.30}$$

From this and the well-posedness of the L^2 Neumann problem in $D^\pm_{R,\tau}$, we then obtain

$$\int_{\Lambda(R)} M_{2,R}(\nabla u^\pm)^2 \, d\sigma \leq C \int_{\partial D^\pm_{R,\tau}} |\partial_\nu u^\pm|^2 \, d\sigma$$
$$\leq C \int_{\partial D_{R,\tau}} |\nabla u|^2 \, d\sigma + C R^{-1} \int_{D_{R,\tau}} |\nabla u|^2 \, dx. \tag{3.31}$$

Proceeding now as in **[6]** and integrating this inequality for $\tau \in [1/4, 1/2]$ one gets

$$\int_{\Lambda(R)} [M_{2,R}(\nabla u^+)^2 + M_{2,R}(\nabla u^-)^2] \, d\sigma \leq C R^{-1} \int_{D_{R,1/4}} |\nabla u|^2 \, dx, \tag{3.32}$$

since $d\sigma\,d\tau \approx R^{-1}dx$ in this context. By using that u satisfies $Lu = 0$ in $B_{4R} \setminus B_{2R}$, Caccioppoli's inequality (cf., e.g., p. 2 in **[23]**, or **[17]**, p. 24) further gives

$$\begin{aligned}\int_{\Lambda(R)} [M_{2,R}(\nabla u^+)^2 + M_{2,R}(\nabla u^-)^2]\,d\sigma &\leq CR^{-3}\int_{C_1R\leq|x|\leq C_2R} |u|^2\,dx \\ &\leq CR^{1-n-2\alpha}, \qquad (3.33)\end{aligned}$$

completing the proof of (3.25).

Step VI. The last details in the proof of (3.6) are as follows. First,

$$\int_{\Lambda(R)} [M(\nabla u^+)^2 + M(\nabla u^-)^2]\,d\sigma \leq CR^{1-n+2\alpha} \qquad (3.34)$$

from (3.23) and (3.25). With this at hand and relying on the L^2-theory, we may then write

$$\begin{aligned}\int_{\partial\Omega} M(\nabla u^{\pm})\,d\sigma &\leq \int_{\{(x,\varphi(x));\,|x|\leq 2\}} M(\nabla u^{\pm})\,d\sigma + \sum_{j=1}^{\infty}\int_{\Lambda(2^j)} M(\nabla u^{\pm})\,d\sigma \\ &\leq C\Big(\int_{\partial\Omega} M(\nabla u^{\pm})^2\,d\sigma\Big)^{1/2} + C\sum_{j=1}^{\infty} 2^{j(n-1)/2}\Big(\int_{\Lambda(2^j)} M(\nabla u^{\pm})^2\,d\sigma\Big)^{1/2} \\ &\leq C + C\sum_{j=1}^{\infty} 2^{-j\alpha} < +\infty, \qquad (3.35)\end{aligned}$$

as desired.

3.2 Uniqueness

Here we focus on the issue of uniqueness. The goal is to prove that if u^+, u^- solve the homogeneous version of the transmission boundary problem (1.1) with $1 \leq p < n-1$, then there exists a constant $c \in \mathbb{R}$ so that $u^+ \equiv c$ and $u^- \equiv c$ in Ω_+ and Ω_-, respectively.

For starters, we shall find it useful to record a suitable version of the classical fractional integration theorem of Hardy and Littlewood, proved in **[4]** (cf. Lemma 2.2 *loc. cit.*). Specifically, let Ω be the (unbounded) domain above the graph of a Lipschitz function. Then, for every $0 < p < n-1$ there exists $\kappa = \kappa(\partial\Omega, p) > 0$ finite such that, with $1/p^* := 1/p - 1/(n-1)$,

$$\Delta w = 0 \text{ in } \Omega \Longrightarrow \exists c \in \mathbb{R} \text{ such that } \|M(w-c)\|_{L^{p^*}(\partial\Omega)} \leq \kappa\|M(\nabla w)\|_{L^p(\partial\Omega)}. \qquad (3.36)$$

When used in conjunction with the homogeneous PDE satisfied by $u^{\pm}$, this implies that there exist $c_{\pm} \in \mathbb{R}$ so that $M(u^{\pm} - c_{\pm}) \in L^{p^*}(\partial\Omega_{\pm})$. Using the first transmission boundary condition we may therefore write $c_- - c_+ = (u^+ - c_+)|_{\partial\Omega} - (u^- - c_-)|_{\partial\Omega} \in L^{p^*}(\partial\Omega)$, so that $c_+ = c_- =: c \in \mathbb{R}$. Let us re-denote $u^{\pm} - c$ by $u^{\pm}$ so that our goal is to show that $u^{\pm} \equiv 0$ in $\Omega_{\pm}$. Introducing

$$u^{\pm}_{\tau}(x) := u^{\pm}(x \pm (0,\tau)), \quad \tau > 0, \quad x \in \Omega_{\pm}, \qquad (3.37)$$

it follows from (2.51) and (3.36) that

$$\int_{\partial\Omega} |u^{\pm}_{\tau}|^{p^*}\,d\sigma + \int_{\Omega_{\pm}} |u^{\pm}_{\tau}|^{np^*/(n-1)}\,dx \leq C, \quad \text{uniformly for } \tau > 0. \qquad (3.38)$$

Also from (2.51) and our hypotheses,

$$\int_{\Omega_\pm} |\nabla u_\tau^\pm|^{np/(n-1)}\,dx \le C, \quad \text{uniformly for } \tau > 0. \tag{3.39}$$

Next, fix $x_o \in \Omega_+$ and let $E(x_o, y)$ be the fundamental solution for L with pole at x_o. We claim that, if $(n-1)/n < p < n-1$, there holds

$$u_\tau^+(x_o) = \int_{\partial\Omega} E(x_o, y)\,[\partial_\nu u_\tau^+(y) - \mu\partial_\nu u_\tau^-(y)]\,d\sigma_y, \tag{3.40}$$

for each $\tau > 0$. To see this, fix $\psi \in C^\infty(\mathbb{R}^n)$ be such that $\psi \equiv 1$ in $B_1(x_o)$, $\psi \equiv 0$ outside $B_2(x_o)$, and set $\psi_R := \psi(\cdot/R)$. For R large, we write

$$u_\tau^+(x_o) = \int_{\mathbb{R}^n} [L_y E(x_o, y)](\psi_R\, u_\tau)(y)\,dy \tag{3.41}$$

and then integrate by parts successively until all derivatives on E are transferred to the other terms. The resulting identity reads

$$\begin{aligned}
u_\tau^+(x_o) \quad &= \quad \int_{\partial\Omega} E(x_o, y)\,\psi_R(y)\,[\partial_\nu u_\tau^+(y) - \mu\partial_\nu u_\tau^-(y)]\,d\sigma_y \\
&\quad +(1-\mu)\int_{\partial\Omega} E(x_o, y)\,\partial_\nu\psi_R(y)\,u_\tau^+(y)\,d\sigma_y \\
&\quad +\int_{\Omega_+} E(x_o, y)[\Delta\psi_R(y)u_\tau^+(y) + 2\langle\nabla\psi_R(y), \nabla u_\tau^+(y)\rangle]\,dy \\
&\quad +\mu\int_{\Omega_-} E(x_o, y)[\Delta\psi_R(y)u_\tau^-(y) + 2\langle\nabla\psi_R(y), \nabla u_\tau^-(y)\rangle]\,dy \\
&=: \quad I + II + III + IV.
\end{aligned} \tag{3.42}$$

Note that $\nabla\psi_R$ is supported in $B_{2R}(x_o)\setminus B_R(x_o)$ and $R|\nabla\psi_R| + R^2|\nabla^2\psi_R| \le C$. Also, $|E(x_o, y)| \approx R^{2-n}$ for $|y| \approx R$, granted that $|x_o| \le R/2$ which we can assume. Consequently,

$$|II| \le CR^{1-n}\int_{\partial\Omega\cap B_R(x_o)} |u_\tau^+|\,d\sigma \le C\Big(R^{1-n}\int_{\partial\Omega\cap B_R(x_o)} |u_\tau^+|^{p^*}\,d\sigma\Big)^{1/p^*} \tag{3.43}$$

and the last term converges to zero as $R \to \infty$ by (3.38). Going further,

$$\begin{aligned}
|III| \quad &\le \quad C\int_{\Omega_+\cap B_R(x_o)} [R^{1-n}|\nabla u_\tau^+| + R^{-n}|u_\tau^+|]\,dx \qquad (3.44)\\
&\le \quad CR\Big(R^{-n}\int_{\Omega_+} |\nabla u_\tau^+|^{np/(n-1)}\,dx\Big)^{(n-1)/np} + C\Big(R^{-n}\int_{\Omega_+} |u_\tau^+|^{np^*/(n-1)}\Big)^{(n-1)/np^*}
\end{aligned}$$

which once again converges to zero as $R \to \infty$, by assumptions and (3.39). In fact, a similar analysis applies to IV. Finally,

$$I \longrightarrow \int_{\partial\Omega} E(x_o, y)\,[\partial_\nu u_\tau^+(y) - \mu\partial_\nu u_\tau^-(y)]\,d\sigma_y \quad \text{as} \quad R \to \infty, \tag{3.45}$$

justifying (3.40).

In turn, if $1 \leq p < n-1$, (3.40) further yields $u^+(x_o) = 0$ by making $\tau \to 0$, by Lebesgue's Dominated Convergence Theorem. That the latter is applicable in the current context is ensured by our assumptions on $u^\pm$ and the fact that $E(x_o,\cdot) \in L^{p'}(\partial\Omega)$ where $1 < p' \leq \infty$, $1/p + 1/p' = 1$. Since $x_o \in \Omega_+$ was arbitrary, the desired conclusion follows easily.

3.3 Existence and estimates

Having established the well-posedness of (3.1) along with the accompanying estimate (3.6), we are now ready to tackle the issue of existence and estimates for (1.1) in the case when Ω is an unbounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. To start with, consider the sublinear operator

$$Tg := M(\nabla u^+) + M(\nabla u^-) \tag{3.46}$$

where u^+, u^- solve the reduced transmission problem, i.e. (1.1) with $f = 0$. What we have proved so far amounts to the fact that

$$T : \dot{H}^1_{at}(\partial\Omega) \longrightarrow L^1(\partial\Omega) \tag{3.47}$$

is well-defined and bounded. Indeed, for $g = \sum \lambda_j a_j \in \dot{H}^1_{at}(\partial\Omega)$, we set $Tg := M(\sum_j \lambda_j \nabla u_j^+) + M(\sum_j \lambda_j \nabla u_j^-)$ where, for each j, the pair (u_j^+, u_j^-) solves the reduced transmission problem with datum a_j. Then our previous analysis applies to each individual atom.

Also, from §2 we known that

$$T : L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega) \tag{3.48}$$

is well-defined and bounded.

We next prove that the action of T in (3.47) is compatible with that of T in (3.48); that is, if $g \in \dot{H}^1_{at}(\partial\Omega) \cap L^2(\partial\Omega)$, then Tg, considered in the sense of (3.47), coincides with $T(g)$ considered in the sense of (3.48). To see this, we shall invoke an observation made in (6.5) on p. 948 of **[32]**, to the effect that for any $g \in \dot{H}^1_{at}(\partial\Omega) \cap L^2(\partial\Omega)$ there exist a sequence of coefficients $(\lambda_j)_j \in \ell^1$ and a sequence of 1-atoms a_j, such that

$$\begin{gathered} g = \textstyle\sum_{j=1}^\infty \lambda_j\, a_j \text{ in } \dot{H}^1_{at}(\partial\Omega), \quad \sum_{j=1}^\infty |\lambda_j| \leq C\|g\|_{\dot{H}^1_{at}(\partial\Omega)}, \text{ and} \\ g_N := \textstyle\sum_{j=1}^N \lambda_j\, a_j \text{ converges to } g \text{ in } L^2(\partial\Omega) \text{ as } N \to \infty. \end{gathered} \tag{3.49}$$

It follows that $Tg = \lim_{N\to\infty} Tg_N$ in $L^2(\partial\Omega)$ and, if we temporarily denote the operator in (3.48) by $\tilde{T}$, we also have $\tilde{T}g = \lim_{N\to\infty} Tg_N$ in $L^1(\partial\Omega)$. This readily entails $\tilde{T}g = Tg$ a.e. on $\partial\Omega$.

To continue from here, we invoke a general interpolation result for sublinear operators from **[19]**. In concert with (5.1) on p. 156 of **[16]**, this proves that the operator

$$T : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{3.50}$$

is well-defined and bounded for $1 < p \leq 2$. This takes care of existence and estimates for (1.1) with $f = 0$, $g \in L^p(\partial\Omega)$ arbitrary, in the range $1 < p \leq 2$.

In order to pass to the most general case, i.e. when $f \in \dot{L}^p_1(\partial\Omega)$ and $g \in L^p(\partial\Omega)$ are arbitrary, $1 < p \leq 2$, we first let (w^+, w^-) solve the 'reduced' transmission problem

$$\begin{cases} \Delta w^\pm = 0 \text{ in } \Omega_\pm, \\ M(\nabla w^\pm) \in L^p(\partial\Omega), \\ w^+\Big|_{\partial\Omega} = w^-\Big|_{\partial\Omega}, \\ \partial_\nu w^+ - \mu\, \partial_\nu w^- = g + (-\frac{1}{2}I + K^*)(S^{-1}f) \in L^p(\partial\Omega). \end{cases} \tag{3.51}$$

Here we have used the fact that the operator (2.27) is an isomorphism for $1 < p \leq 2$ ([**6**]). Then

$$u^+ := w^+ + \mathcal{S}\Big(S^{-1}f\Big) \;\text{ in }\; \Omega_+ \;\text{ and }\; u^- := w^- \;\text{ in }\; \Omega_- \tag{3.52}$$

solve (1.1) and satisfy the estimate (1.4), as desired.

To summarize, for the problem (1.1) when $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is an unbounded Lipschitz domain, we have at this stage existence and estimates (proved in the reasoning above for $1 < p \leq 2$ and in §2 for $2 - \varepsilon < p < 2 + \varepsilon$), as well as uniqueness (proved in §3.2 for $1 < p < 2$ and in §2 for $2 - \varepsilon < p < 2 + \varepsilon$).

3.4 Invertibility of layer potentials and bounded Lipschitz domains

In order to be able to deal with the case of a *bounded* Lipschitz domain $\Omega \subset \mathbb{R}^n$ we derive some useful invertibility results for the classical harmonic layer potential operators. First, from the work of B. Dahlberg and C. Kenig [**6**] we know that

$$\Delta u = 0 \text{ in } \Omega,\; M(\nabla u) \in L^1(\partial\Omega) \Longrightarrow \partial_\nu u \in \dot{H}^1_{at}(\partial\Omega) \text{ and } u\Big|_{\partial\Omega} \in \dot{H}^{1,1}_{at}(\partial\Omega), \tag{3.53}$$

plus natural estimates. This implies

$$\begin{aligned} &\|M(\nabla u^+)\|_{L^1(\partial\Omega)} + \|M(\nabla u^-)\|_{L^1(\partial\Omega)} \\ &\qquad \leq C\|u^+|_{\partial\Omega} - u^-|_{\partial\Omega}\|_{\dot{H}^{1,1}_{at}(\partial\Omega)} + C\|\partial_\nu u^+ \,-\, \mu\,\partial_\nu u^-\|_{\dot{H}^1_{at}(\partial\Omega)}, \end{aligned} \tag{3.54}$$

uniformly for any harmonic functions $u^\pm$ in $\Omega_\pm$. If, for an arbitrary $f \in \dot{H}^1_{at}(\partial\Omega)$, we use $u^\pm := \mathcal{S}f$ in $\Omega_\pm$, back in (3.54), we arrive at the conclusion that for each $\mu \in (0,1)$,

$$\begin{aligned} \|f\|_{\dot{H}^1_{at}(\partial\Omega)} &= \|\partial_\nu u^+ \,-\, \partial_\nu u^-\|_{\dot{H}^1_{at}(\partial\Omega)} \\ &\leq \|M(\nabla u^+)\|_{L^1(\partial\Omega)} + \|M(\nabla u^-)\|_{L^1(\partial\Omega)} \\ &\leq C\|\partial_\nu u^+ \,-\, \mu\,\partial_\nu u^-\|_{\dot{H}^1_{at}(\partial\Omega)} + C\|u^+|_{\partial\Omega} \,-\, u^-|_{\partial\Omega}\|_{\dot{H}^{1,1}_{at}(\partial\Omega)} \\ &= C\|[-\tfrac{1}{2}(1+\mu)I + (1-\mu)K^*]f\|_{\dot{H}^1_{at}(\partial\Omega)}, \end{aligned} \tag{3.55}$$

since $u^+|_{\partial\Omega} = u^-|_{\partial\Omega}$.

A similar estimate from below holds for the dual operator on $\dot{H}^{1,1}_{at}(\partial\Omega)$. More specifically, for an arbitrary $f \in \dot{H}^{1,1}_{at}(\partial\Omega)$ (strictly speaking, it is convenient to work first with functions which can be represented as a finite linear combination of regular atoms and derive estimates independent on the number of terms in the sum), consider the estimate (3.54) written for $u^+ := \mu\mathcal{D}f$ in Ω_+ and $u^- := \mathcal{D}f$ in Ω_-. Then, keeping (2.34) in mind, we write

$$\begin{aligned} \|f\|_{\dot{H}^{1,1}_{at}(\partial\Omega)} &= \|\mu^{-1}u^+|_{\partial\Omega} - u^-|_{\partial\Omega}\|_{\dot{H}^{1,1}_{at}(\partial\Omega)} \\ &\leq C\|M(\nabla u^+)\|_{L^1(\partial\Omega)} + \|M(\nabla u^-)\|_{L^1(\partial\Omega)} \\ &\leq \|u^+|_{\partial\Omega} - u^-|_{\partial\Omega}\|_{\dot{H}^{1,1}_{at}(\partial\Omega)} \\ &= C\|[\tfrac{1}{2}(1+\mu)I - (1-\mu)K]f\|_{\dot{H}^{1,1}_{at}(\partial\Omega)}, \end{aligned} \tag{3.56}$$

for each $\mu \in (0,1)$.

Granted (3.55) and (3.56), the same type of spectral theoretical argument used in conjunction with the L^2-estimate (2.40) then leads to the conclusion that

$$\lambda I + K^* : \dot{H}^1_{at}(\partial\Omega) \stackrel{\sim}{\longrightarrow} \dot{H}^1_{at}(\partial\Omega), \qquad \lambda I + K : \dot{H}^{1,1}_{at}(\partial\Omega) \stackrel{\sim}{\longrightarrow} \dot{H}^{1,1}_{at}(\partial\Omega) \tag{3.57}$$

are *isomorphisms* for each $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$. There are also L^p-counterparts of (3.57), proved in a very similar fashion, based on the L^p-version of (3.54), i.e.

$$\begin{aligned} &\|M(\nabla u^+)\|_{L^p(\partial\Omega)} + \|M(\nabla u^-)\|_{L^p(\partial\Omega)} \\ &\qquad \leq C\|u^+|_{\partial\Omega} - u^-|_{\partial\Omega}\|_{\dot{L}^p_1(\partial\Omega)} + C\|\partial_\nu u^+ \,-\, \mu\,\partial_\nu u^-\|_{L^p(\partial\Omega)}, \end{aligned} \tag{3.58}$$

for $1 < p < 2 + \varepsilon$, uniformly for any harmonic functions $u^\pm$ in $\Omega_\pm$. Thus, we see that

$$\lambda I + K^* : L^p(\partial\Omega) \stackrel{\sim}{\longrightarrow} L^p(\partial\Omega), \qquad \lambda I + K : \dot{L}^p_1(\partial\Omega) \stackrel{\sim}{\longrightarrow} \dot{L}^p_1(\partial\Omega) \tag{3.59}$$

are *isomorphisms* for each $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, provided $1 < p < 2 + \varepsilon$.

Furthermore, the inverse operators for $1 < p, q < 2 + \varepsilon$ agree on $L^p(\partial\Omega) \cap L^q(\partial\Omega)$ and on $\dot{L}^p_1(\partial\Omega) \cap \dot{L}^q_1(\partial\Omega)$, respectively. This latter assertion follows from jump-relations and a similar compatibility statement at the level of the entire transmission boundary problem, where the solution operator is known to act coherently in the range $1 < p < 2 + \varepsilon$.

The adaptation of (1.1) to the case of a *bounded* Lipschitz domain is done working at the level of layer potentials. Indeed, for a singular integral operator, the property of being bounded from below, modulo compacts, can be localized (cf. §10 in **[26]**). This allows us to conclude that for each *bounded* Lipschitz domain Ω the operator

$$\lambda I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{3.60}$$

is semi-Fredholm for each $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, provided $1 < p < 2 + \varepsilon$. Given that the index is homotopic invariant and that $\lambda I + K^*$ is obviously invertible when $|\lambda|$ is sufficiently large, it follows that the operator (3.60) is actually Fredholm with index zero on $L^p(\partial\Omega)$ for the same parameters λ, p, as above. Next, an inspection of the argument at the beginning of §3 in **[11]** gives that (with no topological assumptions on Ω)

$$\begin{gathered} \text{the operator } \lambda I + K^* \text{ is one-to-one on } L^p(\partial\Omega) \\ \text{for } n \geq 2, \text{ if } p > 2(n-1)/n,\ \lambda \in \mathbb{R},\ |\lambda| > \tfrac{1}{2}. \end{gathered} \tag{3.61}$$

From the above discussion and elementary functional analysis it ultimately follows that, if $1 < p < 2 + \varepsilon$, the operator (3.60) is invertible for each $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$. This allows us to conclude the proof of Theorem 1.1 when $n \geq 3$.

Remark Based on (3.57) and the general stability theory developed in **[20]**, it follows that

$$\lambda I + K^* : \dot{H}^p_{at}(\partial\Omega) \stackrel{\sim}{\longrightarrow} \dot{H}^p_{at}(\partial\Omega), \qquad \lambda I + K : \dot{H}^{1,p}_{at}(\partial\Omega) \stackrel{\sim}{\longrightarrow} \dot{H}^{1,p}_{at}(\partial\Omega) \tag{3.62}$$

isomorphically, for each $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, granted that $1 - \varepsilon < p \leq 1$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$.

This is the key ingredient in the proof of Theorem 1.2 when $n \geq 3$. The case $n = 2$ is dealt with below.

4 The two dimensional case

The case $n = 2$ is special, in the sense that the natural end-point estimate occurs at $p = 2/3$ (instead of $p = 1$ as in higher dimensions). Interestingly enough, this is not as a result of a better asymptotic theory at infinity of Serrin-Weinberger type, although the correlation between the best Hölder exponent in the De Giorgi-Nash-Moser theory and the ellipticity constant of the operator in question is, at the moment, best understood in two dimensions (cf. **[31]**).

Our approach is somewhat akin to **[7]** which deals with the three dimensional Neumann problem for the Lamé system. We shall mostly emphasize the novel technical aspects and, in the interest of brevity, only present a detailed proof for the main atomic estimate in this case.

Let $\Omega \subset \mathbb{R}^2$ be the unbounded domain lying above the graph of a real-valued Lipschitz function defined on $\mathbb{R}$. The crux of the matter is establishing the well-posedness of the transmission boundary problem (1.6) along with the naturally accompanying estimate (1.7) for $2/3 \leq p \leq 1$.

As far as existence is concerned, much as before, matters can be reduced to analyzing the case when $f = 0$ and g is an atom $a \in \dot{H}^p_{at}(\partial\Omega)$ since, in the two-dimensional setting,

$$S : \dot{H}^p_{at}(\partial\Omega) \xrightarrow{\sim} \dot{H}^{1,p}_{at}(\partial\Omega) \tag{4.1}$$

is an isomorphism for $2/3 \leq p \leq 1$ (this is implicit in **[25]**). Going further, given the dilation invariant nature of the problem, there is no loss of generality in assuming that the atom a is supported in a surface ball of radius one. Recall the 'truncated' maximal operators $M_{1,R}$, $M_{2,R}$, $R > 0$ from (3.21) and let (u^+, u^-) be a solution of (1.6) with $f = 0$ and $g = a$ which satisfies $M(\nabla u^\pm) \in L^2(\partial\Omega)$ (which exists and is unique, modulo constants, according to the discussion in §2).

The goal is to show that $\|M(\nabla u^\pm)\|_{L^p(\partial\Omega)} \leq C_p < +\infty$, for each $2/3 \leq p \leq 1$. Regarding the contribution from $M_{2,R}$, the idea is to estimate

$$\int_{D_R\cap\partial\Omega} M_{2,R}(\nabla u^\pm)^p\, d\sigma$$

$$\leq CR^{1-p/2}\Big[\int_{D_R\cap\partial\Omega} M_{2,R}(\nabla u^\pm)^2\, d\sigma\Big]^{p/2}, \qquad \textit{by Hölder's inequality}$$

$$\leq CR^{1-p/2}\Big[\int_{\partial D_{2R}} |\nabla u|^2\, d\sigma + R^{-1}\int_{D_{2R}} |\nabla u|^2\, dx\Big]^{p/2}$$

by the L^2-theory, and the transmission Rellich estimates (3.30)

$$\leq CR^{1-p/2}\Big[R^{-1}\int_{D_{2R}} |\nabla u|^2\, dx\Big]^{p/2},$$

by averaging, as in the derivation of (3.32) from (3.31)

$$\leq CR^{1-p/2}\Big[R^{-3}\int_{D_{2R}} |u-c|^2\, dx\Big]^{p/2},$$

by Caccioppoli's inequality, as in **[17]**, **[23]**

$$\leq CR^{1-p}\Big[R^{-2}\int_{D_{2R}} |u-c|^q\, dx\Big]^{p/q},$$

if $q < 2$, by a reverse Hölder estimate, as, e.g., in Lemma 2.8 of **[36]**

$$\leq CR^{1-p-p/q}\Big[\int_{\partial\Omega}|M(u-c)|^q\,d\sigma\Big]^{p/q}, \quad \textit{from geometrical considerations}$$

$$\leq CR^{1-p-p/q}, \tag{4.2}$$

where c is any fixed constant, and the last step assumes that c can be chosen so that

$$\int_{\partial\Omega}|M(u-c)|^q\,d\sigma<+\infty, \quad \text{for some } q<2. \tag{4.3}$$

In order to justify the existence of a constant $c\in\mathbb{R}$ such that (4.3) holds, we present an approach which works whenever $1/2<p\leq 1$ to begin with. Denote by τ the unit tangent vector to $\partial\Omega$ and let $b:\partial\Omega\to\mathbb{R}$ be an antiderivate for the atom a; i.e.,

$$\operatorname{supp} b\subseteq S_1, \quad \|b\|_{L^\infty(\partial\Omega)}\leq C=C(\partial\Omega), \quad \partial_\tau b=a \text{ on } \partial\Omega. \tag{4.4}$$

That such a function exists is ensured by (2.9) (recall that we are assuming $r=1$). Next, consider the function

$$w:=\mathcal{D}[(\tfrac{1}{2}I+K)^{-1}b] \ \text{ in } \ \Omega_+, \tag{4.5}$$

so that

$$\Delta w=0, \quad w\Big|_{\partial\Omega}=b, \quad M(w),\, M(\nabla w)\in L^q(\partial\Omega) \text{ for each } q \text{ near } 2, \tag{4.6}$$

hold, by virtue of (2.41), (2.44), (2.32) and (2.28). From Theorem 4.1 of **[21]**, there exists a function v such that

$$v+iw \ \text{ is holomorphic in } \ \Omega_+, \quad M(v),\, M(\nabla v)\in L^q(\partial\Omega) \text{ for each } q \text{ near } 2. \tag{4.7}$$

In particular, v solves the Neumann problem

$$\Delta v=0, \quad \partial_\nu v=(\partial_\tau w=)a, \quad M(v),\, M(\nabla v)\in L^q(\partial\Omega) \text{ for each } q \text{ near } 2. \tag{4.8}$$

Since, from the discussion in §2 we know already that (1.6) with $f=0$ and $g=a$, atom in $\dot{H}^1_{at}(\partial\Omega)$, has a unique (modulo constants) solution which satisfies $M(\nabla u^\pm)\in L^2(\partial\Omega)$, we are interested in finding a suitable representation formula which will eventually allows to "read (4.3) off it."

One convenient way to approach this is to look for a function $\psi\in L^q_1(\partial\Omega)$, for each q near 2, so that $u^+=v+\mu\mathcal{D}\psi$ in Ω_+ and $u^-=\mathcal{D}\psi$ in Ω_-. Executing this program leads to the conclusion that any L^2-solution of the problem in question has the form

$$u^+=v+\tfrac{\mu}{1-\mu}\mathcal{D}[(\lambda I+K)^{-1}(v|_{\partial\Omega})]+c \ \text{ in } \ \Omega_+, \tag{4.9}$$

$$u^-=\tfrac{1}{1-\mu}\mathcal{D}[(\lambda I+K)^{-1}(v|_{\partial\Omega})]+c \ \text{ in } \ \Omega_-, \tag{4.10}$$

where $c\in\mathbb{R}$ and $\lambda:=\frac{1}{2}\frac{\mu+1}{\mu-1}\in\mathbb{R}$ satisfies $|\lambda|>\frac{1}{2}$. Since $\lambda I+K$ is invertible in $L^q(\partial\Omega)$ with $|q-2|$ small it follows that $M(u^\pm-c)\in L^q(\partial\Omega)$ and $\|M(u^\pm-c)\|_{L^q(\partial\Omega)}\leq C$, if $|q-2|$ is small. This finishes the justification of (4.3).

As for $M_{1,R}(\nabla u^\pm)$, if $x\in\Lambda(R)$ and $y\in\Gamma^\pm_{1,R}(x)$, then for $\lambda=\lambda(\partial\Omega)>0$ small enough we write

$$\begin{aligned}
|\nabla u^\pm(y)| &\leq CR^{-3}\int_{B_{\lambda R}(y)}|u^\pm(z)-c|\,dz\leq CR^{-2}\int_{\Lambda(\lambda^{-1}R)}M(u^\pm-c)\,d\sigma \\
&\leq CR^{-1}\Big(R^{-1}\int_{\Lambda(\lambda^{-1}R)}M(u^\pm-c)^q\,d\sigma\Big)^{1/q} \\
&\leq CR^{-1-1/q}\|M(u^\pm-c)\|_{L^q(\partial\Omega)}, \qquad (4.11)
\end{aligned}$$

by interior estimates, Hölder's inequality, as well as simple geometrical considerations. Thus,

$$M_{1,R}(\nabla u^{\pm})(x) \leq CR^{-1-1/q}, \quad \text{for } x \in \Lambda(R), \tag{4.12}$$

if $|q-2|$ is sufficiently small. Assuming that this is the case, this estimate further implies

$$\int_{\Lambda(R)} M_{1,R}(\nabla u^{\pm})^p \, d\sigma \leq CR^{1-p-p/q} \tag{4.13}$$

which agrees with (4.2). Hence, all in all,

$$\int_{\Lambda(R)} [M(\nabla u^+)^p + M(\nabla u^-)^p] \, d\sigma \leq C(\partial\Omega, p, q)\, R^{1-p-p/q} \tag{4.14}$$

if $|q-2|$ is sufficiently small, so that, ultimately,

$$\int_{\partial\Omega} [M(\nabla u^+)^p + M(\nabla u^-)^p] \, d\sigma \leq C \sum_{j=0}^{\infty} 2^{j(1-p-p/q)}, \tag{4.15}$$

by taking $R = 2^j$, $j = 0, 1, ...,$ and adding up the resulting terms. The series converges if there exists $q < 2$ such that $1 - p - p/q < 0$. This, in turn, follows from $p \geq 2/3$, which we assume.

Remark Much as before, the above discussion also proves that

$$\lambda I + K^* : \dot{H}^p_{at}(\partial\Omega) \xrightarrow{\sim} \dot{H}^p_{at}(\partial\Omega), \qquad \lambda I + K : \dot{H}^{1,p}_{at}(\partial\Omega) \xrightarrow{\sim} \dot{H}^{1,p}_{at}(\partial\Omega) \tag{4.16}$$

isomorphically, for each $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$, granted that $2/3 - \varepsilon < p \leq 1$. Once again, this is the key ingredient in the proof of the well-posedness of the transmission boundary problem (1.6) with atomic data, from $H^{1,p}_{at}(\partial\Omega) \oplus H^p_{at}(\partial\Omega)$ if $2/3 - \varepsilon < p \leq 1$, for some small $\varepsilon = \varepsilon(\partial\Omega) > 0$, if $n = 2$.

5 The spectral radius conjecture revisited

The departure point is the classical Krein-Rutman Theorem for positive operators. Here we record a version of this result, due to Bonsall **[3]** and Schaefer **[34]**, which is well-suited for the applications we have in mind. Recall that $\mathcal{C} \subseteq X$ is called a *closed cone* (with vertex at zero) if $\lambda\mathcal{C}$ is a closed, convex set such that $\lambda\mathcal{C} \subset \mathcal{C}$ for any $\lambda > 0$, and $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$. We have:

Proposition 5.1 *Let X be a real Banach space and let $\mathcal{C}$ be a closed cone for X which is reproducing (or generating), i.e. $X = \mathcal{C} - \mathcal{C}$, which is also normal, i.e. there exists $\kappa > 0$ such that $x, y, x - y \in \mathcal{C} \Rightarrow \|y\| \leq \kappa\|x\|$. Suppose that $T : X \to X$ is a linear, bounded operator, which is positive, i.e. $T\mathcal{C} \subseteq \mathcal{C}$. Then, with X^c denoting the complexified version of X, we have*

$$r(T; X^c) \in \mathrm{Spec}\,(T; X^c) \cap \mathbb{R}. \tag{5.1}$$

The key observation in **[14]** is that K maps the cone of nonnegative functions in $L^p(\partial\Omega)$ into itself if Ω is convex since, in this case, $\langle \nu(y), y - x\rangle \geq 0$ for any $x, y \in \partial\Omega$. Granted (3.59), the same arguments as in **[14]** then yield

$$r\Big(K^*;\, L^p_0(\partial\Omega)\Big) < \tfrac{1}{2}, \quad 1 < p < 2 + \varepsilon, \tag{5.2}$$

where $L^p_0(\partial\Omega) := \{f \in L^p(\partial\Omega);\ \int_{\partial\Omega} f\,d\sigma = 0\}$. Next, by duality, (5.2) also proves that

$$r\Big(K;\ L^p(\partial\Omega)/\mathbb{R}\Big) < \tfrac{1}{2}, \quad 2-\varepsilon < p < \infty. \tag{5.3}$$

Given that the single layer potential operator

$$S : L^p_0(\partial\Omega) \longrightarrow L^p_1(\partial\Omega)/\mathbb{R}, \quad 1 < p < 2+\varepsilon, \tag{5.4}$$

is an isomorphism, the intertwining identity $SK^* = KS$ and (5.3) further give

$$r\Big(K;\ L^p_1(\partial\Omega)/\mathbb{R}\Big) < \tfrac{1}{2}, \quad 1 < p < 2+\varepsilon. \tag{5.5}$$

The claim made in the statement of Theorem 1.3 now follows, from (5.3), (5.5) and interpolation.

References

[1] E. Albrecht and V. Müller, *Spectrum of interpolated operators*, Proc. Amer. Math. Soc., 129 (2001), 807–814.

[2] J.F. Ahner, V.V. Dyakin and V.Ya. Raevskii, *New spectral results for the electrostatic integral operator*, J. Math. Anal. Appl., 185 (1994), 391–402.

[3] F.F. Bonsall, *Linear operators in complete positive cones*, Proc. London Math. Soc., 8 (1958), 53–75.

[4] R.M. Brown, *The Neumann problem on Lipschitz domains in Hardy spaces of order less than one*, Pacific J. Math., 171 (1995), 389–407.

[5] R.R. Coifman, A. McIntosh and Y. Meyer, *L'intégrale de Cauchy définit un opérateur borné sur L_2 pour les courbes lipschitziennes*, Ann. of Math., 116 (1982), 361–387.

[6] B. Dahlberg and C. Kenig, *Hardy spaces and the L^p–Neumann problem for Laplace's equation in a Lipschitz domain*, Ann. of Math. 125 (1987), 437–465.

[7] B. Dahlberg and C. Kenig, *L^q-Estimates for the three-dimensional system of elastostatics on Lipschitz domains*, Lecture Notes in Pure and Applied Math., Vol. 122, C. Sadosky ed., 1990, 621–634.

[8] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat., 3 (1957), 25–43.

[9] W. Derrick, *Open problems in singular integral theory*, J. Integral Equations Appl., 5 (1993), 23–28.

[10] M. Dindoš and M. Mitrea, *Semilinear Poisson problems in Sobolev-Besov spaces on Lipschitz domains*, Publ. Mat., 46 (2002), 353–403.

[11] L. Escauriaza, E. Fabes and G. Verchota, *On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries*, Proc. Amer. Math. Soc., 115 (1992), 1069–1076.

[12] L. Escauriaza and J.K. Seo, *Regularity properties of solutions to transmission problems*, Trans. Amer. Math. Soc., 338 (1993), 405–430.

[13] E. Fabes, M. Jodeit, and N. Rivière, *Potential techniques for boundary value problems on C^1 domains*, Acta Math., 141 (1978), 165–186.

[14] E. Fabes, M. Sand and J.-K. Seo, *The spectral radius of the classical layer potentials on convex domains*, Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990), 129–137, IMA Vol. Math. Appl., 42, Springer, New York, 1992.

[15] E.B. Fabes and D.W. Stroock, *The L_p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J., 51 (1984), 997–1016.

[16] C. Fefferman and E.M. Stein, *H_p spaces of several variables*, Acta Math., 129 (1972), 137–193.

[17] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1993.

[18] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J., 46 (1979), 27–42.

[19] S. Janson, *On the interpolation of sublinear operators*, Studia Math., 75 (1982), 51–53.

[20] N. Kalton and M. Mitrea, *Stability of Fredholm properties on interpolation scales of quasi-Banach spaces and applications*, Transactions of the Amer. Math. Soc., Vol. 350, No. 10, (1998), 3837–3901.

[21] C. E. Kenig, *Weighted H_p spaces on Lipschitz domains*, Amer. J. Math., 102 (1980), 129–163.

[22] C. E. Kenig, *Elliptic boundary value problems in Lipschitz domains*, pp. 131-183 in Beijing Lectures in Harmonic Analysis (Beijing, 1984), Ann. of Math. Stud., 12, Princeton Univ. Press, Princeton NJ, 1986.

[23] C. E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics, No. 83, AMS, Providence, RI, 1994.

[24] W. Littman, G. Stampacchia and H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa, 17 (1963), 43–77.

[25] D. Mitrea, *Layer potentials and Hodge decompositions in two dimensional Lipschitz domains*, Math. Ann., 322 (2002), 75–101.

[26] D. Mitrea, M. Mitrea and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Mem. Amer. Math. Soc., Vol. 150 No. 713, 2001.

[27] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: L^p, Hardy, and Hölder space results*, Communications in Analysis and Geometry, Vol. 9 No. 2 (2001), 369–421.

[28] J. Moser, *On Harnack's theorem for elliptic differential operators*, Comm. Pure and Appl. Math., 14 (1961), 577-591.

[29] J. Nash, *Continuity of the solutions of parabolic and elliptic equations*, Amer. J. Math., 80 (1957), 931-954.

[30] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967.

[31] L.C. Piccinini and S. Spagnolo, *On the Hölder continuity of solutions of second order elliptic equations in two variables*, Ann. Scuola Norm. Sup. Pisa, 26 (1972), 391–402.

[32] J. Pipher and G. Verchota, *The Dirichlet problem in L_p for the biharmonic equation on Lipschitz domains*, Amer. J. Math., 114 (1992), 923–972.

[33] F. Rellich, *Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral*, Math. Z., 46 (1940), 635–636.

[34] H.H. Schaefer, *Banach Latices and Positive Operators*, Springer-Verlag, 1974.

[35] J. Serrin and H. F. Weinberger, *Isolated singularities of solutions of linear elliptic equations*, Amer. J. Math., 88 (1966), 258–272.

[36] Z. Shen, *Boundary value problems in Morrey spaces for elliptic systems in Lipschitz domains*, Amer. J. Math., 125 (2003), 1079–1115.

[37] G. Verchota, *Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal. 59 (1984), 572–611.

L. Escauriaza: Department of Mathematics, UPV/EHU, Apto. 644, 48080 Bilbao, Spain, *e-mail: mtpeszul@lg.ehu.es*

M. Mitrea: Department of Mathematics, University of Missouri-Columbia, Columbia, MO 65211, USA, *e-mail: marius@math.missouri.edu*