Does the Period of the Librational Motion of a Pendulum Depend on its Amplitude? Undergraduate Project in Differential Equations

Oksana Bihun
Department of Mathematics
University of Missouri,
Columbia, Missouri 65211, USA

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Consider a point mass m suspended from a rod of length l (a simple pendulum). Denote the acceleration of gravity by g.

The motion of a simple pendulum is modeled by the differential equation

$$\ddot{\theta} + \lambda \sin\theta = 0, \tag{1}$$

where $\theta(t)$ is the angular displacement of the pendulum at time t, and $\lambda = \frac{g}{l}$.

Galileo is said to have deduced that the period of the librational motion of a pendulum does not depend on its amplitude. He made this deduction while sitting in a cathedral and observing a chandelier swinging in the breeze blowing through an open window. Was Galileo right? (See [1], p.506.)

To check Galileo's hypothesis, let us express the period of motion of a simple pendulum as a function of the initial displacement θ_0. With this purpose, let us multiply equation (1) by $\dot{\theta}$ and integrate with respect to t. We obtain that

$$\frac{1}{2}\dot{\theta}^2 - \lambda\cos\theta = C,$$

where C is a constant.

In particular,

$$\frac{1}{2}\dot{\theta}^2 - \lambda\cos\theta = -\lambda\cos\theta_0,$$

where $0 < \theta_0 < \pi$. One of the choices for $\dot{\theta}$ is

$$\dot{\theta} = -\sqrt{2\lambda(\cos\theta - \cos\theta_0)},$$

which yields

$$-\int_0^\tau \frac{1}{\sqrt{2\lambda(\cos\theta - \cos\theta_0)}}\dot{\theta}dt = \int_0^\tau dt.$$

By symmetry, we find that the period of the motion of the simple pendulum is

$$T(\theta_0) = 4\frac{1}{\sqrt{2\lambda}}\int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}d\theta.$$

Using the formula $\cos\theta = 1 - 2\sin^2(\theta/2)$ and performing the change of variables

$$\sin\phi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}},$$

we obtain

$$T(\theta_0) = \frac{4}{\sqrt{\lambda}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}},$$

where $k = \sin\frac{\theta_0}{2}$ (see Exercise 6.7.4 in [2].)

Check that $T(0) = \frac{2\pi}{\sqrt{\lambda}}$. Using a numerical solver, compute $T(\theta_0)$ for $\theta_0 = \pi/12, \pi/6, \pi/3 \ldots$ How do the values of the period T at different initial displacements differ from $\frac{2\pi}{\sqrt{\lambda}}$? Was Galileo right?

References

[1] C. Chicone, *Ordinary Differential Equations with Applications*, Texts in Applied Mathematics, New York: Springer-Verlag, 2006.

[2] S. Strogatz, *Nonlinear Dynamics and Chaos*, Perseus Publishing, 2000.