

CONSTRUCTING INFINITE TIGHT FRAMES

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ABSTRACT. For finite and infinite dimensional Hilbert spaces \mathbb{H} we classify the sequences of positive real numbers $\{a_n\}_{n=1}^{\infty}$ so that there is a tight frame $\{\varphi_n\}_{n=1}^{\infty}$ for \mathbb{H} satisfying: $\|\varphi_n\| = a_n$, for all $n = 1, 2, 3, \dots$. In the finite dimensional case we will identify the frames which are closest to being tight (in the sense of minimizing potential energy) for any sequence $\{a_n\}_{n=1}^{\infty}$.

1. INTRODUCTION

If \mathbb{H} is a Hilbert space, a sequence $\{\varphi_n\}_{n=1}^M$ (M is finite or infinite) is a *frame* for \mathbb{H} if there are constants $A, B > 0$ so that for all $\varphi \in \mathbb{H}$,

$$A\|\varphi\|^2 \leq \sum_{n=1}^M |\langle \varphi, \varphi_n \rangle|^2 \leq B\|\varphi\|^2.$$

If $A = B = \lambda$, $\{\varphi_n\}_{n=1}^M$ is a λ -*tight frame*. If $\lambda = 1$, it is a *Parseval frame*; if $\|\varphi_n\| = \|\varphi_m\|$ for all $1 \leq n, m \leq M$ it is a *equal-norm frame*; and if $\|\varphi_n\| = 1$ for all n it is a *unit-norm frame*. The importance of λ -tight frames is that they allow simple *reconstruction* of the elements of \mathbb{H} . It is known that $\{\varphi_n\}_{n=1}^M$ is a frame for \mathbb{H} if and only if

$$S\varphi = \sum_{n=1}^M \langle \varphi, \varphi_n \rangle \varphi_n,$$

is an invertible operator on \mathbb{H} called the *frame operator*. To reconstruct an element $\varphi \in \mathbb{H}$ we write

$$\varphi = SS^{-1}\varphi = \sum_{n=1}^M \langle S^{-1}\varphi, \varphi_n \rangle \varphi_n.$$

So reconstruction requires inverting the frame operator which is often difficult or impossible in practice. It follows that for all $\varphi \in \mathbb{H}$ we have

$$\langle A\varphi, \varphi \rangle = A\|\varphi\|^2 \leq \sum_{n=1}^M |\langle \varphi, \varphi_n \rangle|^2 = \langle S\varphi, \varphi \rangle \leq B\|\varphi\|^2 = \langle B\varphi, \varphi \rangle.$$

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Hence, $AI \leq S \leq BI$ and so our frame is λ -tight if and only if $S = \lambda I$. So if $\{\varphi_n\}_{n=1}^M$ is a λ -tight frame then for all $\varphi \in \mathbb{H}_N$,

$$\varphi = \frac{1}{\lambda} \sum_{n=1}^M \langle \varphi, \varphi_n \rangle \varphi_n.$$

Frames were introduced in 1952 by Duffin and Schaeffer [12] while they were working on some deep problems nonharmonic Fourier series. Since then, frames have been used extensively in signal/image processing where they are called *Gabor frames* or *Weyl-Heisenberg frames* [3, 14, 18, 21]. Recently, many new applications of tight frames have arisen in internet coding [8, 15, 16, 17], wireless communication [20, 22, 23], quantum detection theory [13], and much more. Each new application requires a new class of tight frames. Until recently, it was thought that the class of tight frames was quite sparse and that they probably did not exist for many applications. However, after the introduction of *frame potentials* by Benedetto and Fickus [2], there was an explosion of new results concerning the construction of tight frames for finite dimensional Hilbert spaces [6, 7, 9, 10]. For a survey of all of these important developments see [5]. The importance of [2] is that it gives a geometric interpretation for equal-norm finite tight frames. Building on the work of Benedetto and Fickus, Casazza, Fickus, Kovačević, Leon and Tremain [6] gave a *physical interpretation* for finite tight frames along the lines of Columb's law in Physics. This allows us to anticipate results in frame theory by using results from classical Mechanics. In particular, in [6] the authors have identified the finite frames which are the closest to being tight and having the norms of the frame vectors prescribed in advance.

In this paper we will extend these results to infinite frames for both finite and infinite dimensional Hilbert spaces. Throughout the paper we will use \mathbb{H} to denote a finite or infinite dimensional Hilbert space and \mathbb{H}_N for an N -dimensional Hilbert space (real or complex). For a background on frame theory we refer the reader to [3, 4, 11, 18]

2. FRAME POTENTIALS

Recently, Benedetto and Fickus [2] introduced the notion of frame potentials (see also [6]).

Definition 2.1. *If $\{\varphi_n\}_{n=1}^M$ (M finite or infinite) is a frame for \mathbb{H}_N , the **Frame Potential** of $\{\varphi_n\}_{n=1}^M$ is given by:*

$$FP(\{\varphi_n\}_{n=1}^M) = \sum_{n,m=1}^M |\langle \varphi_n, \varphi_m \rangle|^2.$$

The frame potential is measuring how close a frame is to being orthogonal. In particular, it is shown in [6] that if \mathcal{F} is the family of frames with lower frame bound λ then the λ -tight frames are the minimizers of the frame potential over \mathcal{F} . This theorem gives us a way to identify tight frames. i.e. They are the minimizers of the frame potential on certain families of frames. Our goal is to identify those families of frames which have minimizers of the frame potential and for which these minimizers must be tight. We will need some standard facts about frames (see [6]).

For any frame $\{\varphi_n\}_{n=1}^M$ for \mathbb{H}_N with frame bounds A, B , and frame operator S we have

$$A\|\varphi_n\|^2 \leq \sum_{m=1}^M |\langle \varphi_n, \varphi_m \rangle|^2 \leq B\|\varphi_n\|^2.$$

Hence,

$$A \sum_{n=1}^M \|\varphi_n\|^2 \leq FP(\{\varphi_n\}_{n=1}^M) \leq B \sum_{n=1}^M \|\varphi_n\|^2.$$

It is known that

$$\text{Trace } S = \sum_{n=1}^M \|\varphi_n\|^2.$$

In particular:

$$A \text{Trace } S \leq FP(\{\varphi_n\}_{n=1}^M) \leq B \text{Trace } S.$$

Also,

$$\|\varphi_n\|^4 \leq \sum_{m=1}^M |\langle \varphi_n, \varphi_m \rangle|^2 \leq B\|\varphi_n\|^2,$$

implies that

$$\|\varphi_n\|^2 \leq B, \quad \text{for all } n.$$

For a tight frame, $S = AI$ so $\text{Trace } S = NA$ and

$$FP(\{\varphi_n\}_{n=1}^M) = NA^2.$$

So for a Parseval frame (and hence for an orthonormal basis)

$$FP(\{\varphi_n\}_{n=1}^M) = N,$$

and this is the minimal value of the frame potential for any frame for \mathbb{H}_N with lower frame bound 1 (see [6]).

Relating this to frame potentials, since S^2 is the frame operator for $\{S^{1/2}\varphi_n\}_{n=1}^M$ we have:

$$\begin{aligned} \text{Trace } S^2 &= \sum_{n=1}^M \|S^{1/2}\varphi_n\|^2 = \sum_{n=1}^M \langle S^{1/2}\varphi_n, S^{1/2}\varphi_n \rangle \\ &= \sum_{n=1}^M \langle S\varphi_n, \varphi_n \rangle = \sum_{n=1}^M \left\langle \sum_{m=1}^M \langle \varphi_n, \varphi_m \rangle \varphi_m, \varphi_n \right\rangle \\ &= \sum_{n=1}^M \sum_{m=1}^M |\langle \varphi_n, \varphi_m \rangle|^2 = FP(\{\varphi_n\}_{n=1}^M). \end{aligned}$$

Finally, it follows from the frame inequality if $\{\varphi_n\}_{n=1}^M$ is a frame for \mathbb{H}_N with frame bounds $A, B > 0$ and P is an orthogonal projection on \mathbb{H}_N , then $\{P\varphi_n\}_{n=1}^M$ is a frame for $P\mathbb{H}_N$ with frame bounds A, B . In particular, if $\{\varphi_n\}_{n=1}^M$ is a λ -tight frame for \mathbb{H}_N then $\{P\varphi_n\}_{n=1}^M$ is a λ -tight frame for $P(\mathbb{H})$.

There are two main results from [6] we will need in our work. The first is a result which measures how equally distributed a sequence of nonnegative decreasing sequence of numbers are.

Proposition 2.2. *Given any sequence $\{a_m\}_{m=1}^\infty$ of real numbers with $a_1 \geq a_2 \geq a_3 \geq \dots$, and any natural number \mathbf{N} , there is a unique index $1 \leq d \leq N$, such that the inequality*

$$(N - n)a_n > \sum_{m=n+1}^\infty a_m$$

holds for all $1 \leq n < d$, while the opposite inequality

$$(N - n)a_n \leq \sum_{m=n+1}^\infty a_m$$

holds for $d \leq n \leq N$.

Proposition 2.2 is proved in [6] for finite sequences but the proof works without change for infinite sequences as well. We also need the main results from [6]. First a piece of notation. For $a > 0$ we let $S(a)$ denote the sphere of radius a centered at the origin in \mathbb{H} . For any positive sequence $\{a_m\}_{m=1}^M$ we let $S(a_1, a_2, \dots, a_M)$ denote the Cartesian product of the corresponding sequence of spheres: $S(a_1, a_2, \dots, a_M) = S(a_1) \times S(a_2) \times \dots \times S(a_M)$.

Theorem 2.3. *Given a sequence $a_1 \geq a_2 \geq \dots \geq a_M > 0$ and any $N \leq M$, let d denote the smallest index n for which*

$$a_n^2 \leq \frac{\sum_{m=n+1}^M a_m^2}{N - n}$$

holds (cf. Proposition 2.2). Then, any local minimizer of the frame potential $FP : S(a_1, a_2, \dots, a_M) \rightarrow \mathbf{R}$ is of the form

$$\{f_m\}_{m=1}^M = \{f_m\}_{m=1}^{d-1} \cup \{f_m\}_{m=d}^M,$$

where $\{f_m\}_{m=1}^{d-1}$ is an orthogonal set for whose orthogonal complement $\{f_m\}_{m=d}^M$ forms a tight frame.

3. FINITE DIMENSIONAL HILBERT SPACES

Here we will show that Theorem 2.3 also holds for infinite sets of vectors on finite dimensional spaces. One way to do this would be to systematically show that all of the main results concerning frame potentials holds for infinite frames for finite dimensional spaces. However, this would be exceptionally cumbersome. So instead, we will derive this case from Theorem 2.3. We first need a lemma.

Lemma 3.1. *Let $\{\varphi_n\}_{n=1}^M$ be a frame for \mathbb{H}_N with frame bounds A, B and let $\{\psi_n\}_{n=1}^L$, $L \leq M$ satisfy:*

$$\sum_{n=1}^L \|\psi_n - \varphi_n\|^2 \leq \epsilon,$$

and

$$\sum_{n=L+1}^M \|\varphi_n\|^2 \leq \delta.$$

Then $\{\psi_n\}_{n=1}^L$ is a frame for \mathbb{H}_N with frame bounds $(\sqrt{A} - \sqrt{\epsilon} - \sqrt{\delta})^2$ and $(\sqrt{B} + \sqrt{\epsilon})^2$.

Proof. For all $\varphi \in \mathbb{H}_N$ we have:

$$\begin{aligned} \sqrt{\sum_{n=1}^L |\langle \varphi, \psi_n \rangle|^2} &= \sqrt{\sum_{n=1}^L |\langle \varphi, \varphi_n \rangle + \langle \varphi, \psi_n - \varphi_n \rangle|^2} \\ &\leq \sqrt{\sum_{n=1}^L |\langle \varphi, \varphi_n \rangle|^2} + \sqrt{\sum_{n=1}^L |\langle \varphi, \psi_n - \varphi_n \rangle|^2} \\ &\leq \sqrt{B} \|\varphi\| + \sqrt{\|\varphi\|^2 \sum_{n=1}^L \|\psi_n - \varphi_n\|^2} \\ &\leq \|\varphi\| (\sqrt{B} + \sqrt{\epsilon}). \end{aligned}$$

Similarly,

$$\begin{aligned}
\sqrt{\sum_{n=1}^L |\langle \varphi, \varphi_n \rangle|^2} &\geq \sqrt{\sum_{n=1}^L |\langle \varphi, \varphi_n \rangle|^2} - \sqrt{\sum_{n=1}^L |\langle \varphi, \psi_n - \varphi_n \rangle|^2} \\
&\geq \sqrt{\sum_{n=1}^M |\langle \varphi, \varphi_n \rangle|^2} - \sum_{n=L+1}^M |\langle \varphi, \varphi_n \rangle|^2 - \sqrt{\epsilon} \|\varphi\| \\
&\geq \sqrt{\sum_{n=1}^M |\langle \varphi, \varphi_n \rangle|^2} - \sqrt{\sum_{n=L+1}^M |\langle \varphi, \varphi_n \rangle|^2} - \sqrt{\epsilon} \|\varphi\| \\
&\geq (\sqrt{A} - \sqrt{\epsilon}) \|\varphi\| - \|\varphi\| \sqrt{\sum_{n=L+1}^M \|\varphi_n\|^2} \\
&\geq (\sqrt{A} - \sqrt{\epsilon} - \sqrt{\delta}) \|\varphi\|.
\end{aligned}$$

□

Now we are ready for the main results of this section.

Theorem 3.2. *Let $a_1 \geq a_2 \geq \dots > 0$ be real numbers with $\sum_{n=1}^{\infty} a_n^2 < \infty$. Suppose there exists a $1 \leq d < N$ satisfying Proposition 2.2. Then the frames $\{\varphi_n\}_{n=1}^{\infty}$ which are the closest to being tight (in the sense of minimizing potential energy) with $\|\varphi_n\| = a_n$ are of the form:*

$$\{c_n a_n e_n\}_{n=1}^d \cup \{\varphi_n\}_{n=d+1}^{\infty}$$

where $\{e_i\}_{i=1}^N$ is an orthonormal basis for \mathbb{H}_N , $|c_i| = 1$, for all $1 \leq i \leq d$, and $\{\varphi_n\}_{n=d+1}^{\infty}$ is a tight frame for span $\{e_i\}_{i=d+1}^N$.

Proof. Fix $\epsilon > 0, \delta > 0$ and let $\{e_i\}_{i=1}^N$ be an orthonormal basis for \mathbb{H}_N . We have assumed that $\sum_{n=d+1}^{\infty} a_n^2 > N a_d^2$. Hence, there is a natural number $M_0 > 0$ so that for all $M \geq M_0$ we have

$$\sum_{n=d+1}^M a_n^2 \geq (N - d) a_d^2.$$

Now by Theorem 2.3 for every $M \geq M_0$ there is a λ_M -tight frame $\{\varphi_n^M\}_{n=1}^M$ for \mathbb{H}_N of minimum frame potential with respect to the property $\|\varphi_n^M\| = a_n$, for all $1 \leq n \leq M$. We may assume our frame has the form:

$$\{a_n e_n\}_{n=1}^d \cup \{\varphi_n^M\}_{n=d+1}^M,$$

where $\{\varphi_n^M\}_{n=d+1}^M$ is a λ_M -tight frame for $\mathbb{H} = \text{span} \{e_i\}_{i=d+1}^M$ and

$$\lambda_M = \frac{\sum_{n=d+1}^M a_n^2}{N-d}.$$

By a standard compactness argument, there are natural numbers $M_0 < M_1 < M_2 < \dots$ so that

$$\lim_{i \rightarrow \infty} \varphi_n^{M_i} = \varphi_n.$$

First we will show that $\{\varphi_n\}_{n=d+1}^\infty$ is a tight frame for \mathbb{H}_{N-d} . There is a natural number L_0 so that for all $L \geq L_0$, and all large i we have:

$$\sum_{n=L+1}^{M_i} \|\varphi_n^{M_i}\|^2 \leq \sum_{n=L}^\infty a_n^2 \leq \delta$$

and

$$\sum_{n=1}^L \|\varphi_n^{M_i} - \varphi_n\|^2 \leq \epsilon.$$

By Lemma 3.1, $\{\varphi_n\}_{n=d+1}^L$ is a frame for \mathbb{H}_{N-d} with frame bounds,

$$\left(\sqrt{\frac{\sum_{n=d+1}^{M_i} a_n^2}{N-d}} - \sqrt{\epsilon} - \sqrt{\delta} \right), \quad \left(\sqrt{\frac{\sum_{n=d+1}^{M_i} a_n^2}{N-d}} + \sqrt{\epsilon} \right).$$

Letting $L \rightarrow \infty$ yields that $\{\varphi_n\}$ is a frame for \mathbb{H}_{N-d} with frame bounds

$$\left(\sqrt{\frac{\sum_{n=d+1}^\infty a_n^2}{N-d}} - \sqrt{\epsilon} - \sqrt{\delta} \right), \quad \left(\sqrt{\frac{\sum_{n=d+1}^\infty a_n^2}{N-d}} + \sqrt{\epsilon} \right).$$

Since $\epsilon, \delta > 0$ were arbitrary, we have that $\{\varphi_n\}_{n=1}^\infty$ is a λ -tight frame for \mathbb{H}_N with tight frame bound

$$\lambda = \frac{\sum_{n=d+1}^\infty a_n^2}{N-d}$$

Next we check that $\{\varphi_n\}_{n=1}^\infty$ is of minimal frame potential for those frames $\{\psi_n\}_{n=1}^\infty$ for \mathbb{H}_N with $\|\psi_n\| = a_n$, for all $n = 1, 2, \dots$. We proceed by way of contradiction. So assume there is a frame $\{\psi_n\}_{n=1}^\infty$ for \mathbb{H}_N satisfying the above but

$$FP(\{\psi_n\}_{n=1}^\infty) \leq FP(\{\varphi_n\}_{n=1}^\infty) - \epsilon.$$

Choose a natural number M so that

$$2 \sum_{n=1}^M a_n^2 \sum_{m=M+1}^\infty a_m^2 + \sum_{n,m=M+1}^\infty a_n^2 a_m^2 < \frac{\epsilon}{4}.$$

Choose $M_i > M$ so that

$$FP(\{\varphi_n\}_{n=1}^{M_i}) \geq \sum_{n,m=1}^M |\langle \varphi_n^{M_i}, \varphi_m^{M_i} \rangle|^2 \geq \sum_{n,m=1}^M |\langle \varphi_n, \varphi_m \rangle|^2 - \frac{\epsilon}{4}.$$

Now,

$$\begin{aligned} FP(\{\psi_n\}_{n=1}^{M_i}) &\leq FP(\{\psi_n\}_{n=1}^\infty) \\ &\leq FP(\{\varphi_n\}_{n=1}^\infty) - \epsilon \\ &= FP(\{\varphi_n\}_{n=1}^M) + 2 \sum_{n=1}^M \sum_{m=M+1}^\infty |\langle \varphi_n, \varphi_m \rangle|^2 + \sum_{n,m=M+1}^\infty |\langle \varphi_n, \varphi_m \rangle|^2 - \epsilon \\ &\leq FP(\{\varphi_n\}_{n=1}^M) + 2 \sum_{n=1}^M a_n^2 \sum_{m=M+1}^\infty a_m^2 + \sum_{n,m=M+1}^\infty a_n^2 a_m^2 \\ &\leq FP(\{\varphi_n\}_{n=1}^{M_i}) + \frac{\epsilon}{4} + \frac{\epsilon}{4} - \epsilon \\ &< FP(\{\varphi_n\}_{n=1}^{M_i}). \end{aligned}$$

Since $\|\psi_n\| = \|\varphi_n\| = a_n$, for all $1 \leq n \leq M_i$, we have contradicted the minimality of the frame potential for $\{\varphi_n\}_{n=1}^{M_i}$. \square

Theorem 3.2 yields a classification of infinite tight frames.

Theorem 3.3. *Given $a_1 \geq a_2 \geq \dots > 0$ and a Hilbert space \mathbb{H}_N , the following are equivalent:*

(1) *There is a normalized tight frame $\{\varphi_n\}_{n=1}^\infty$ for \mathbb{H}_N with $\|\varphi_n\| = a_n$, for all $n = 1, 2, 3, \dots$.*

(2) *We have that $\sum_{n=1}^\infty a_n^2 < \infty$ and for all $1 \leq d < N$,*

$$a_d^2 \leq \frac{\sum_{n=d+1}^\infty a_n^2}{N-d}.$$

Proof. (2) \Rightarrow (1): Choose $a_0 > 0$ so that

$$Na_0 > \sum_{n=0}^\infty a_n^2.$$

By Theorem 3.3, the frame $\{\varphi_n\}_{n=0}^\infty$ satisfying $\|\varphi_n\| = a_n$ for all $n = 0, 1, \dots$ and which is the closest to being tight for \mathbb{H}_{N+1} is of the form:

$$\{\varphi_n\}_{n=0}^\infty = \{\varphi_0\} \cup \{\varphi_n\}_{n=1}^\infty,$$

where $\{\varphi_n\}_{n=1}^\infty$ is a tight frame for \mathbb{H}_N .

(1) \Rightarrow (2): Now we want to show that whenever $\{\varphi_n\}_{n=1}^\infty$ is a tight frame for \mathbb{H}_N with $\|\varphi_n\| = a_n$ and $a_1 \geq a_2 \geq \dots > 0$, then $\{a_n\}_{n=1}^\infty$ satisfies (1). Fix $1 \leq d < N$ and choose $d \leq k$ smallest so that $\dim(\text{span}\{\varphi_n\}_{n=1}^k) = d$.

Let P be the orthogonal projection of \mathbb{H}_N onto $[\text{span } \{\varphi_n\}_{n=1}^k]^\perp$. Then both $\{P\varphi_n\}_{n=1}^\infty$ and $\{(I - P)\varphi_n\}_{n=1}^\infty$ are λ -tight frames for their spans. Hence,

$$\frac{\sum_{n=1}^\infty \|P\varphi_n\|^2}{N - d} = \lambda = \frac{\sum_{n=1}^\infty \|(I - P)\varphi_n\|^2}{d}.$$

Since $(I - P)\varphi_n = \varphi_n$ for all $1 \leq n \leq k$ and $P\varphi_n = 0$ for all $1 \leq n \leq k$ we have,

$$\begin{aligned} \frac{\sum_{n=d+1}^\infty a_n^2}{N - d} &\geq \frac{\sum_{n=k+1}^\infty a_n^2}{N - d} \\ &\geq \frac{\sum_{n=k+1}^\infty \|P\varphi_n\|^2}{N - d} \\ &= \frac{\sum_{n=1}^\infty \|P\varphi_n\|^2}{N - d} \\ &= \lambda \\ &= \frac{\sum_{n=1}^M \|(I - P)\varphi_n\|^2}{d} \\ &\geq \frac{\sum_{n=1}^d \|\varphi_n\|^2}{d} \geq \frac{da_d^2}{d} = a_d^2. \end{aligned}$$

□

4. INFINITE DIMENSIONAL HILBERT SPACES

In this section, we will classify the sequence of norms of tight frame vectors for infinite dimensional Hilbert spaces. For our construction we will need a result of Casazza and Leon [10].

Theorem 4.1. *Let S be a positive, self-adjoint, invertible operator on \mathbb{H}_N with an orthonormal basis of eigenvectors $\{e_i\}_{i=1}^N$ and respective eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$. Fix $M \geq N$ and $\{a_n\}_{n=1}^M$ with $a_1 \geq a_2 \geq \dots \geq a_m$. The following are equivalent:*

(1) *There is a frame $\{\psi_n\}_{n=1}^M$ for \mathbb{H}_N with $\|\psi_n\| = a_n$, for $1 \leq n \leq M$ having frame operator S with eigenvectors $\{e_n\}_{n=1}^N$ and respective eigenvalues $\{\lambda_n\}_{n=1}^M$.*

(2) *We have*

$$\sum_{n=1}^k a_n^2 \leq \sum_{n=1}^k \lambda_n, \quad \text{for all } 1 \leq k \leq N.$$

and

$$\sum_{n=1}^M a_n^2 = \sum_{n=1}^N \lambda_n.$$

To simplify our construction, we will single out the main point in the next lemma.

Lemma 4.2. *Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for a Hilbert space \mathbb{H} , $a_n > 0$ such that $\sup_{1 \leq n < \infty} a_n^2 \leq A$ and $\sum_{n=1}^\infty a_n^2 = \infty$. Fix $\epsilon > 0$, $k \in \mathbf{N}$, $N_k \in \mathbf{N}$ and assume $\{\varphi_n\}_{n=1}^{M_k}$ is a λ_k -tight frame for $\mathbb{H}_{N_k} = \text{span}_{1 \leq n \leq N_k} e_n$ with $\|\varphi_n\| = a_n$ and $\lambda_k \leq A + \sum_{n=1}^k \frac{\epsilon}{2^n}$. Then there exists natural numbers $N_{k+1} > N_k$, $M_{k+1} > M_k$ and vectors $\{\varphi_n\}_{n=M_k+1}^{M_{k+1}}$ with $\|\varphi_n\| = a_n$ and $\{\varphi_n\}_{n=1}^{M_{k+1}}$ is a λ_{k+1} -tight frame for $\mathbb{H}_{N_{k+1}} = \text{span}_{1 \leq n \leq N_{k+1}} e_n$ with $\lambda_k < \lambda_{k+1} \leq A + \sum_{n=1}^{k+1} \frac{\epsilon}{2^n}$.*

Proof: Choose N_{k+1} so that

$$\frac{A}{N_{k+1}} \leq \frac{\epsilon}{2^{k+2}}$$

and

$$\left(A + \frac{\epsilon}{2}\right) \left(1 - \frac{N_k}{N_{k+1}}\right) > A.$$

Now choose M_{k+1} smallest so that

$$\sum_{n=1}^{M_{k+1}} a_n^2 > \lambda_k N_k + \left(\lambda_k + \frac{\epsilon}{2^{k+2}}\right) (N_{k+1} - N_k).$$

We now have

$$\begin{aligned} \lambda_k N_k + \left(\lambda_k + \frac{\epsilon}{2^{k+2}}\right) (N_{k+1} - N_k) &\leq \sum_{n=1}^{M_{k+1}} a_n^2 \\ &\leq \lambda_k N_k + \left(\lambda_k + \frac{\epsilon}{2^{k+2}}\right) (N_{k+1} - N_k) + a_{M_{k+1}}^2. \end{aligned}$$

Now choose λ_{k+1} so that

$$\lambda_{k+1} N_{k+1} = \sum_{n=1}^{M_{k+1}} a_n^2.$$

Now we compute,

$$\lambda_k N_k + \left(\lambda_k + \frac{\epsilon}{2^{k+2}}\right) (N_{k+1} - N_k) = \lambda_k N_{k+1} + \frac{\epsilon}{2^{k+2}} (N_{k+1} - N_k) \leq \lambda_{k+1} N_{k+1}.$$

Hence,

$$\lambda_k < \lambda_k + \frac{\epsilon}{N_{k+1} 2^{k+2}} (N_{k+1} - N_k) \leq \lambda_{k+1}.$$

Also,

$$\lambda_{k+1} N_{k+1} \leq \lambda_k N_{k+1} + \frac{\epsilon}{2^{k+2}} (N_{k+1} - N_k) + a_{M_{k+1}}^2.$$

So,

$$\lambda_{k+1} \leq \lambda_k + \frac{\epsilon}{2^{k+2}} \frac{N_{k+1} - N_k}{N_{k+1}} + \frac{a_{M_{k+1}}^2}{N_{k+1}} \leq A + \sum_{n=1}^k \frac{\epsilon}{2^n} + \frac{\epsilon}{2^{k+2}} + \frac{\epsilon}{2^{k+2}} = A + \sum_{n=1}^{k+1} \frac{\epsilon}{2^n}.$$

We finish by choosing a frame $\{\varphi_n\}_{n=M_k+1}^{M_{k+1}}$ with $\|\varphi_n\| = a_n$, for all $M_k + 1 \leq n \leq M_{k+1}$ whose frame operator is

$$S(\varphi) = \sum_{n=1}^{M_k} (\lambda_{k+1} - \lambda_k) \langle \varphi, e_k \rangle e_k + \sum_{n=M_k+1}^{M_{k+1}} \lambda_k \langle \varphi, e_k \rangle e_k.$$

To see that such a frame exists, we must verify that we have the hypotheses of Theorem 4.1. But first, let us observe that this finishes the proof of the lemma since the frame operator S_k for $\{\varphi_n\}_{n=1}^{M_k}$ satisfies

$$S_k(\varphi) = \sum_{n=1}^{M_k} \lambda_k \langle \varphi, e_k \rangle e_k,$$

and so the frame operator S_{k+1} for $\{\varphi_n\}_{n=1}^{M_{k+1}}$ is

$$\begin{aligned} S_{k+1}(\varphi) &= (S_k + S)(\varphi) \\ &= \sum_{n=1}^{M_k} \lambda_k \langle \varphi, e_k \rangle e_k + \sum_{n=1}^{M_k} (\lambda_{k+1} - \lambda_k) \langle \varphi, e_k \rangle e_k + \sum_{n=M_k+1}^{M_{k+1}} \lambda_k \langle \varphi, e_k \rangle e_k \\ &= \sum_{n=1}^{M_{k+1}} \lambda_{k+1} \langle \varphi, e_k \rangle e_k. \end{aligned}$$

That is, $\{\varphi_n\}_{n=1}^{M_{k+1}}$ is a λ_{k+1} -tight frame.

To check the hypotheses of Theorem 4.1, note that by definition we have

$$\lambda_{k+1} N_{k+1} = \sum_{n=1}^{M_{k+1}} a_n^2 = \sum_{n=1}^{M_k} a_n^2 + \sum_{n=M_k+1}^{M_{k+1}} a_n^2 = \lambda_k N_k + \sum_{n=M_k+1}^{M_{k+1}} a_n^2.$$

So,

$$\sum_{n=1}^{M_{k+1}} a_n^2 = \lambda_{k+1} N_{k+1} - \lambda_k N_k = \lambda_{k+1} (N_{k+1} - N_k) + (\lambda_{k+1} - \lambda_k) N_k.$$

The right hand side of this equality is the sum of the eigenvalues for the frame operator S . Also, for $1 \leq m \leq N_k$ we have

$$\sum_{n=m}^{N_{k+1}} a_n^2 \leq A N_{k+1} \leq \left(A + \frac{\epsilon}{2}\right) \left(1 - \frac{N_k}{N_{k+1}}\right) N_{k+1}$$

$$= \left(A + \frac{\epsilon}{2}\right) (N_{k+1} - N_k) \leq \lambda_{k+1} (N_{k+1} - N_k).$$

Also, for $N_k + 1 \leq m \leq N_{k+1}$ we have

$$\sum_{n=m}^{N_{k+1}} a_n^2 \leq (N_{k+1} - m)A \leq (N_{k+1} - m)\lambda_{k+1}.$$

This shows that the hypotheses of Theorem 4.1 are satisfied. \square

Theorem 4.3. *Let \mathbb{H} be an infinite dimensional Hilbert space and $a_n > 0$, for all $n = 1, 2, 3, \dots$. The following are equivalent:*

- (1) *There is a frame $\{\psi_n\}_{n=1}^{\infty}$ for \mathbb{H} with $\|\psi_n\| = a_n$, for all $n = 1, 2, 3, \dots$.*
- (2) *For every $\epsilon > 0$, there is a λ -tight frame $\{\varphi_n\}_{n=1}^{\infty}$ for \mathbb{H} with $\|\varphi_n\| = a_n$, for all $n = 1, 2, 3, \dots$ and $\sup_{1 \leq n < \infty} a_n^2 \leq \lambda \leq \sup_{1 \leq n < \infty} a_n^2 + \epsilon$.*

(3) *The sequence $\{a_n\}_{n=1}^{\infty}$ is bounded and $\sum_{n=1}^{\infty} a_n^2 = \infty$.*

Moreover, if $\{\psi_n\}_{n=1}^{\infty}$ is any λ_1 -tight frame for \mathbb{H} with $\|\psi_n\| = a_n$, for all $n = 1, 2, 3, \dots$, then $\lambda_1 \geq \sup_{1 \leq n < \infty} a_n^2$. Finally, in general we cannot find a λ -tight frame for \mathbb{H} with $\|\varphi_n\| = a_n$, for all $n = 1, 2, 3, \dots$ and satisfying $\lambda = \sup_{1 \leq n < \infty} a_n^2$.

Proof: (2) \Rightarrow (1): Obvious.

(1) \Rightarrow (3): For any frame $\{\varphi_n\}_{n=1}^{\infty}$ with upper frame bound B we have that $\|\varphi_n\| \leq B$ for all $n = 1, 2, \dots$. Hence, $\{a_n\}_{n=1}^{\infty}$ is bounded. Assume $\{\varphi_n\}_{n=1}^{\infty}$ is any frame for \mathbb{H} with frame bounds A, B . We proceed by way of contradiction. If $\sum_{n=1}^{\infty} \|\varphi_n\|^2 < \infty$, then choose M so that

$$\sum_{n=M}^{\infty} \|\varphi_n\|^2 < A.$$

Let P be the orthogonal projection onto $\text{span} \{\varphi_n\}_{n=1}^{M-1}$. Since \mathbb{H} is infinite dimensional, Then $\{(I - P)\varphi_n\}_{n=1}^{\infty}$ is a frame for its span with frame bounds A, B . But $(I - P)\varphi_n = 0$ for $1 \leq n \leq M$. Hence,

$$\sum_{n=1}^{\infty} \|(I - P)\varphi_n\|^2 \leq \sum_{n=M}^{\infty} \|\varphi_n\|^2 \leq A.$$

So the Bessel bound of the frame $\{(I - P)\varphi_n\}_{n=1}^{\infty}$ is less than A and so the upper frame bound (and hence the lower frame bound) is less than A , which is a contradiction.

(3) \Rightarrow (2): Fix $\epsilon > 0$. By induction on Lemma 4.2, there are sequences of natural numbers $N_1 < N_2 < N_3 < \dots$ and $M_1 < M_2 < M_3 < \dots$ and vectors $\varphi_n \in \mathbb{H}$ for $n = 1, 2, 3, \dots$ satisfying:

(1) $\|\varphi_n\| = a_n$, for all $n = 1, 2, 3, \dots$.

(2) $\{\varphi_n\}_{n=1}^{M_k}$ is a λ_k -tight frame for \mathbb{H}_{N_k} with $\lambda \leq \lambda_k \leq \lambda + \sum_{n=1}^k \frac{\epsilon}{2^n}$ and $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$.

It follows that

$$\lim_{k \rightarrow \infty} \lambda_k = B \leq \lambda + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \lambda + \epsilon.$$

Also, if we fix $\varphi = \sum_{n=1}^{M_k} a_n e_n$, then

$$\sum_{n=1}^{\infty} |\langle \varphi, \varphi_n \rangle|^2 = \lim_{k \rightarrow \infty} \sum_{n=1}^{M_k} |\langle \varphi, \varphi_n \rangle|^2 = \lim_{k \rightarrow \infty} \lambda_k \|\varphi\|^2 = B \|\varphi\|^2.$$

Since this equality holds on a dense subspace of \mathbb{H} , it follows that it holds on \mathbb{H} . That is, for all $\varphi \in \mathbb{H}$,

$$\sum_{n=1}^{\infty} |\langle \varphi, \varphi_n \rangle|^2 = B \|\varphi\|^2.$$

So $\{\varphi_n\}_{n=1}^{\infty}$ is a B-tight frame for \mathbb{H} with $\lambda \leq B \leq \lambda + \epsilon$.

For the moreover part of the theorem, we observe that if $\{\psi_n\}_{n=1}^{\infty}$ is a λ_1 -tight frame for \mathbb{H} , then $\|\psi_n\|^2 \leq \lambda_1$, for all $n = 1, 2, 3, \dots$. Hence, $\lambda_1 \geq \sup_{1 \leq n < \infty} a_n^2$.

Finally, (and this simple example was communicated to us by D.R. Larson) assume $a_1 = \frac{1}{\sqrt{2}}$ and $a_n = 1$ for all $n \geq 2$. Suppose, by way of contradiction, there is a Parseval frame $\{\psi_n\}_{n=1}^{\infty}$ for \mathbb{H} with $\|\psi_n\| = a_n$ for all $n = 1, 2, 3, \dots$. Since $\|\psi_n\| = 1$, for all $n \geq 2$ it follows that $\psi_n \perp \text{span}_{k \neq n} \psi_k$, for all $n \geq 2$. Hence, $\psi_1 \perp \text{span}_{2 \leq k} \psi_k$, and so $\{\psi_n\}_{n=1}^{\infty}$ has an optimal lower frame bound of $\frac{1}{2}$ and optimal upper frame bound of 1, and so is not a tight frame. \square

REFERENCES

- [1] N.I. Akhiezer and I.M. Glazman. *Theory of Linear Operators in Hilbert spaces*, Volume 1. Frederick Ungar Publisher, 1966.
- [2] J.J. Benedetto and M. Fickus. *Finite normalized tight frames*. Advances in Computational Math, **vol. 18** Nos. 2-4 (2003) 357-385.
- [3] P.G. Casazza. *Modern tools for Weyl-Heisenberg (Gabor) frame theory*. Adv. in Imag. and Electron Physics, Vol. 115 (2000) 1-127.
- [4] P.G. Casazza. *the art of frame theory*. Taiwanese Journ. of Math. **(4)2** (2000) 129-202.
- [5] P.G. Casazza. *Custom building finite frames*. Preprint.
- [6] P.G. Casazza, M. Fickus, J. Kovačević, M. Leon and J.C. Tremain, *A physical interpretation for finite frames*, Preprint.
- [7] P.G. Casazza, M. Fickus, J. Kovačević, M. Leon and J.C. Tremain, *Representations of frames*. Preprint.
- [8] P.G. Casazza and J. Kovačević. *Equal norm tight frames with erasures*. Advances in Computational Math. **vol. 18** Nos. 2-4 (2003) 387-430.
- [9] P.G. Casazza and M. Leon. *Existence and construction of finite tight frames*. Preprint.
- [10] P.G. Casazza and M. Leon. *Frames with a given frame operator*. Preprint.
- [11] O. Christensen, *An introduction to frames and Riesz bases*. Birkh auser, 2002.
- [12] R.J. Duffin and A.C. Schaeffer. *A class of nonharmonic Fourier series*. Trans. Amer. Math. Soc., Vol 72 (1952) 341-366.

- [13] Y. Eldar and G.D. Forney, Jr. *Optimal tight frames and quantum measurement*. Preprint.
- [14] H. G. Feichtinger and T. Strohmer, eds. *Gabor Analysis and Algorithms - Theory and Applications*. Birkh'user, Boston (1998).
- [15] V.K. Goyal and J. Kovacević. *Optimal multiple description transform coding of Gaussian vectors*. In Proc. Data Compr. Conf. Snowbird, UT, March 1998, 388-397.
- [16] V.K. Goyal, J. Kovacević, and J.A. Kelner. *Quantized frame expansions with erasures*. Journal of Appl. and Comput. Harmonic Analysis, 10 (3), (2001) 203-233.
- [17] V.K. Goyal, J. Kovacević, and M. Vetterli. *Quantized frame expansions as source-channel codes for erasure channels*. In Proc. Data Compr. Conf., Snowbird, UT (1999) 388-397.
- [18] K. Gr'ochenig. *Foundations of time-frequency Analysis*. Birkh'ausser, Boston (2001).
- [19] D. Han and D.R. Larson. *Frames, bases and group representations*. Memoirs AMS, Providence, RI, 2000.
- [20] B. Hassibi, B. Hochwald, A. Shkrollahi, and W. Sweldens. *Representation theory for high-rate multiple-antenna code design*. IEEE trans. Inform. Th. 47 (6) (2001) 2355-2367.
- [21] C.E. Heil and D. Walnut. *Continuous and discrete wavelet transforms*. SIAM Rev. 31 (4) (1989) 628-666.
- [22] B. Hochwald, T. Marzetta, T. Richardson, W. Sweldens, and R. Urbanke. *Systematic design of unitary space-time constellations*. Preprint.
- [23] T. Strohmer. *Approximation of dual Gabor frames, window decay, and wireless communications*. Appl. Comp. Harm. Anal., 11 (2) (2001) 243-262.

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